EXTREME POINT ENUMERATION

Katta G. Murty
Department of Industrial & Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117

Sung-Jin Chung
Department of Industrial Engineering
Seoul National University
Seoul, Korea

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Katta G. Murty*
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117, USA
and
Sung-Jin Chung**
Department of Industrial Engineering
Seoul National University
Seoul, South Korea

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Abstract

We discuss an algorithm for enumerating the vertices of a convex polytope specified by a system of linear constraints. Existing algorithms for this problem are usually based on enumerating the feasible basic vectors for the system, and their worst case computational complexity is exponential in the number of vertices of the polytope. Our algorithm generates an objective coefficient vector in each iteration, such that the optimization of this objective function subject to the specified constraints by linear programming techniques leads to a new vertex; until all the vertices are obtained. When the system of constraints has n variables and size L; the worst case computational complexity of our algorihtm is $0(\ell^5 n^4 L)$ where ℓ is the unknown number of vertices of the polytope.

Key words Convex polytopes, enumeration of extreme points and facets, adjacency, segments, polynomial time algorithms.

*Author for correspondence.
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1 Introduction

The problem of enumerating all the extreme points (or vertices) of a convex polytope specified by a system of linear constraints has been studied extensively in the literature. It is discussed in textbooks (see Section 3.19 in [14]), and a large number of journal articles have discussed a variety of algorithms for it (see [1-6, 10-11, 18]). There are some applications in which this problem appears, but the typical exponential growth of the number of extreme points in terms of the number of variables, n, and the number of constraints ρ in the linear system describing the polytope make this practical only for systems in which both n and ρ are small. In spite of this, there is considerable mathematical interest in developing algorithms for this problem which can be considered efficient from a computational complexity points of view, when this efficiency is assessed relative to the enormity of the task. This is the main issue in this paper.

Without any loss of generality, we consider the polytope $\mathbf{K}$ which is the set of feasible solutions of the system of constraints

$$\begin{aligned} Ax &= b \\ x &\geqq 0 \end{aligned} \tag{1}$$

where A is a matrix of order $m \times n$ and rank m, and $n > m$. For $j = 1$ to n let $A_{.j}$ denote the jth column vector of A, it is the column of the variable x_j in (1). Our method can be extended very directly to handle polytopes defined by more general systems consisting of linear equations, inequalities and/or bounds on variables; or such general systems can be transformed into a system of the form (1) by simple linear transformations that preserve one to one correspondence between extreme points of the original and the transformed systems. We assume that A, b are integer, and that $\mathbf{K}$ is nonempty and bounded.

If max $\{x_j : x \in \mathbf{K}\} = 0$, then the variable x_j is equal to the constant 0 all over $\mathbf{K}$ and can be eliminated. So, we assume that max $\{x_j : x \in \mathbf{K}\} > 0$ for all $j = 1$ to n. This implies that the dimension of $\mathbf{K}$ is $n - m$.

L denotes the total number of binary digits in $(A \vdots b)$. L is the size of the system (1). We denote by ℓ, the unknown number of extreme points of $\mathbf{K}$.

All the methods discussed in the literature for enumerating the extreme points of $\mathbf{K}$ are based on enumerating the feasible bases for (1) (see [14]). Every extreme point of $\mathbf{K}$ is the basic feasible solution (BFS) associated with one or more feasible bases for (1), this is the principle used in these methods. If system (1) is nondegenerate, every extreme point of $\mathbf{K}$ is associated with a unique feasible basis for (1) and vice versa, and after an initial feasible basis is obtained, these methods require at most $\ell(n - m)$ pivot steps on (1) before termination, an effort which grows linearly with ℓ.

However, when (1) is degenerate, there may be several feasible bases associated with an extreme point of $\mathbf{K}$, the number of feasible bases may be strictly greater than ℓ, and may even grow exponentially with ℓ and n. An example of this can be obtained from the class of problems constructed by J. Edmonds [8] (see also [16]) to display the worst case behavior of the primal simplex algorithm on the shortest chain problem. For $r \geqq 2$, the rth problem in this class is a minimum cost flow problem on the network in Figure 1 with $(2r + 1)$ nodes and $(4r - 1)$ arcs. The lower bound and capacity for flow on each arc are $0, \infty$ respectively; the source, sink nodes are $2r$, $2r + 1$; and it is required to ship one unit from the source to the sink at minimum cost. The cost data is not given in Figure 1 since it is not relevant for our discussion. There is a unique feasible flow vector for this problem (this has a flow of one unit on the arc $(2r, 2r + 1)$, and zero flow on all the other arcs), hence $\ell = 1$; but the system of constraints in this problem has at least $3(2^r)$ feasible bases, all corresponding to the single feasible solution. So, when (1) is degenerate, the computational effort in the traditional methods of enumerating extreme points of $\mathbf{K}$ based on enumerating feasible bases for (1) is not polynomially bounded in n, L, ℓ in the worst case.

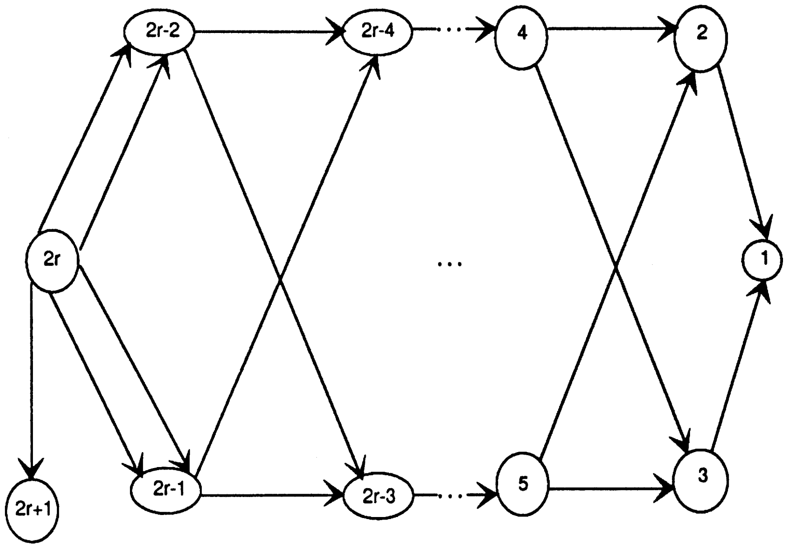

Figure 1 Network for the rth problem in the class for $r \geqq 2$. All lower bounds are 0, and capacities are $+\infty$. Source = node $2r$, with 1 unit available, sink = node $2r+1$ with requirement 1 unit.

The motivation for this study came from the question "is there an algorithm for enumerating the extreme points of $\mathbf{K}$ when (1) is degenerate, with worst case computational effort bounded above by a polynomial in ℓ, L and n?" posed by J. S. Provan [17]. Provan has answered this question in the affirmative for polyhedra associated with network linear programs. Here we answer this question in the affirmative for polytopes specified by systems of general linear constraints, by providing an algorithm.

In pivotal algorithms such as the simplex algorithm, perturbation is a standard technique used to prevent the problem of cycling under degeneracy [14]. This would replace b in the right hand side of (1) by $b(\epsilon) = b+(\epsilon, \epsilon^2, \ldots, \epsilon^m)^T$, where ϵ is not given a specific value, but treated as a small positive parameter whose value is smaller than any other positive number not involving ϵ with which it is compared. Even if the original system (1) is degenerate, the perturbed system is nondegenerate when ϵ is sufficiently small [14]. A feasible basis for the perturbed system is said to be a *lexico feasible basis* for (1). Any BFS for the perturbed system becomes a BFS for the original system (1) when ϵ is set equal to 0 in it. Two BFSs of the perturbed system may be different when ϵ is a small positive number, but may correspond to the same degenerate BFS of the original system obtained by setting $\epsilon = 0$ in them. Since the perturbed system is nondegenerate, its extreme points can be enumerated in time which grows linearly with their number, by enumerating the lexico feasible bases for (1); and then, by setting $\epsilon = 0$ in each of these extreme points we get a list of extreme points of the original system (1); this list may duplicate each degenerate extreme point of (1) several times because of the fact mentioned above. Unfortunately, the number of lexico feasible bases for (1), i.e., the number of extreme points of the perturbed system, could grow exponentially with the number of extreme points of the original system (1). As an example, consider the polytope $\mathbf{K}_1 \subset \mathbb{R}^{2n+1}$ which is the set of feasible solutions of

$$x_1 \;+\; \sum(x_{j+1} : \text{ over } 1 \leqq j \leqq n, j \neq q) + s_q = 1, \quad q = 1 \text{ to } n \tag{2}$$
$$x_j \geqq 0, s_q \geqq 0, \quad \text{for } j = 1 \text{ to } n+1, q = 1 \text{ to } n$$

discussed in [17]. The dimension of $\mathbf{K}_1$ is $n+1$, and it has exactly $n+3$ extreme points which are $(0, 0, e)$,

$(1,0,0)$; $(0, I_{.j}, I_{.j}), j = 1$ to n, and $(0, e/(n-1), 0)$, where $I_{.j}$ is the jth column vector of the unit matrix of order n, and $e \in \mathbb{R}^n$ is the vector of all 1s.

The right hand side constants vector in (2) is e. When it is perturbed into $e + (\epsilon^n, \epsilon^{n-1}, \ldots, \epsilon)^T$, the perturbed system has more than 2^n extreme points, since there are at least 2^n lexico feasible basic vectors with basic variables among x_2 to x_{n+1} and s_1 to s_n only. That is, the number of lexico feasible bases grows exponentially with the number of extreme points for the original system in this example. This indicates that methods based on enumerating the lexico feasible bases do not by themselves provide a solution to our problem.

For any matrix H, we denote by $H_{i.}$, $H_{.j}$, its ith row vector and jth column vector respectively.

If $\{a^1, \ldots, a^t\}$ is a set of points in $\mathbb{R}^n$, we denote its convex hull by $< a^1, \ldots, a^t >$. The affine rank of $\{a^1, \ldots, a^t\}$ is defined to be the rank of the set $\{a^2 - a^1, \ldots, a^t - a^1\}$, it is the dimension of the affine space of $\{a^1, \ldots, a^t\}$.

We will use the abbreviation LP for "linear program".

2 Some Preliminaries

(3)	(4)
Min x_j	Max x_j
over $x \in \mathbf{K}$	over $x \in \mathbf{K}$

If the optimum objective values in (3) and (4) are the same, $= \alpha$ say, then $\mathbf{K}$ lies on the hyperplane $x_j = \alpha$ in $\mathbb{R}^n$. In this case fix x_j at α in (1) and eliminate it from the system. Each extreme point of the reduced system becomes an extreme point of the original system when we include x_j at value α in it. So, in the sequel we assume that the maximum value of any variable in (1) is strictly greater than its minimum value. The operations carried out here do not change the dimension of the set of feasible solutions of (1), which we continue to denote by $\mathbf{K}$.

Let $x_B = (x_1, \ldots, x_m)$ say to be specific, be a basic vector for (1). $x_N = (x_{m+1}, \ldots, x_n)$ is the nonbasic vector when considering this basic vector. Let (B, N) be the partition of A corresponding to the basic, nonbasic partition of x into (x_B, x_N). The equality constraints in (1) are equivalent to

$$x_B = B^{-1}b - B^{-1}Nx_N \tag{5}$$

Using these, the basic variables x_B can be eliminated and system (1) expressed purely in terms of the nonbasic variables x_N as

$$\begin{aligned} -B^{-1}Nx_N + B^{-1}b &\geqq 0 \\ x_N &\geqq 0 \end{aligned} \tag{6}$$

In (6) all the constraints are inequality constraints, and if $\bar{x}_N$ is an extreme point of the set of feasible solutions of (6), then $(\bar{x}_B, \bar{x}_N)$ (where $\bar{x}_B$ is obtained from (5) by substituting $x_N = \bar{x}_N$) is an extreme point of (1), and vice versa. Thus enumerating extreme points of (1), or those of (6), are the same problem. In fact, the set of feasible solutions of (6) is $\mathbf{K}$ itself expressed in the space of nonbasic variables x_N; in this space, $\mathbf{K}$ is a full dimensional convex polytope. We will find it convenient to use this transformation.

How to Check the Adjacency of Two Extreme Points on the Convex Hull of a Given Set of Points

Consider the distinct set of points $\{p^1, \ldots, p^t\} \subset \mathbb{R}^s$ with $t \geqq 3$ and $s \geqq 2$, and $\mathbf{K}_2 = < p^1, \ldots, p^t >$ with each p^k being an extreme point of $\mathbf{K}_2$. The following Theorem 1 provides a result that can be used to check whether two of these points, say p^1 and p^2 are adjacent on $\mathbf{K}_2$.

THEOREM 1 Let $\mathbf{K}_2 = < p^1, \ldots, p^t > \subset \mathbb{R}^s$, where each p^k is an extreme point of $\mathbf{K}_2$ and all the points are distinct. The points p^1 and p^2 are adjacent on $\mathbf{K}_2$ iff there exists a $c = (c_1, \ldots, c_s)$ satisfying

$$\begin{aligned} c(p^1 - p^2) &= 0 \\ c(p^1 - p^k) &> 0, \text{ for all } k = 3 \text{ to } t \end{aligned} \tag{7}$$

PROOF First assume that $t \geqq 4$. From [12]; p^1 and p^2 are adjacent on $\mathbf{K}_2$ iff the following system of constraints in variables $\alpha_1, \ldots, \alpha_t$ is infeasible.

$$\begin{array}{rcccccl} \alpha_1 p^1 & + & \alpha_2 p^2 & - & \sum_{k=3}^t \alpha_k p^k & = & 0 \\ \alpha_1 & + & \alpha_2 & & & = & 1 \\ & & & & \sum_{k=3}^t \alpha_k & = & 1 \\ & & \multicolumn{3}{l}{\alpha_k \geqq 0, k = 1 \text{ to } t} \end{array} \tag{8}$$

By Motzkin's theorem of the alternatives, (7) has no solution c iff the following system in variables $(\delta, \pi_3$ to $\pi_t)$ has a solution.

$$\begin{aligned} \delta(p^1 - p^2) + \sum_{k=3}^t \pi_k (p^1 - p^k) &= 0 \\ \delta \text{ unrestricted; } (\pi_3 \text{ to } \pi_t) &\geqq 0 \text{ and } \neq 0 \end{aligned} \tag{9}$$

$(\pi_3$ to $\pi_t) \geqq 0$ and $\neq 0$ implies that in any solution to (9), $\sum_{k=3}^t \pi_k > 0$; then dividing both sides by this $\sum_{k=3}^t \pi_k$ we get a new solution in which this sum is one. Hence (9) has a solution iff the following (10) has a solution.

$$\begin{aligned} \delta(p^1 - p^2) + \sum_{k=3}^t \pi_k (p^1 - p^k) &= 0 \\ \sum_{k=3}^t \pi_k &= 1 \\ \delta \text{ unrestricted; } (\pi_3 \text{ to } \pi_t) &\geqq 0 \end{aligned} \tag{10}$$

We claim that in any solution to (10), δ must be < 0. Suppose not. If $(\delta, \pi_3, \ldots, \pi_t)$ is a solution of (10) with $\delta \geqq 0$, then from (10) we obtain $p^1 = (\delta p^2 + \sum_{k=3}^t \pi_k p^k)/(\delta + 1)$, i.e., p^1 is a convex combination of p^2 to p^k, contradiction to the fact that p^1 is an extreme point of $\mathbf{K}_2$. So, if (10) has a solution, $\delta < 0$ in it. Using a similar argument it can be verified that in any solution to (10), $\delta > -1$.

Thus, if $(\bar{\delta}, \bar{\pi}_3, \ldots, \bar{\pi}_t)$ is a solution of (10), then $0 < \bar{\delta} < 1$, this implies that $\alpha_1 = 1 + \bar{\delta}$, $\alpha_2 = -\bar{\delta}$; $\alpha_k = \bar{\pi}_k$ for $k = 3$ to t, is a solution for (8).

Similarly, if $(\bar{\alpha}_1, \ldots, \bar{\alpha}_t)$ is a solution for (8), then we must have $\bar{\alpha}_1, \bar{\alpha}_2$ both > 0 since p^1, p^2 are extreme points of $\mathbf{K}_2$, and $\pi_k = \bar{\alpha}_k$ for $k = 3$ to t, $\delta = -\bar{\alpha}_2$ is a solution for (10). Thus (10) has a solution $(\delta, \pi_k, k = 3$ to $t)$ iff (8) has a solution $(\alpha_k, k = 1$ to $t)$. So, (8) has no solution $(\alpha_k, k = 1$ to $t)$ iff (7) has a solution c, proving the theorem for the case $t \geqq 4$.

If $t = 3$, the hypothesis implies that p^1, p^2, p^3 are the vertices of a triangle, so, p^1, p^2 are adjacent on $\mathbf{K}_2$ and (7) has a solution; and the result in the theorem holds. ∎

3 Enumeration of Facets of the Convex Hull of a Given Set of Points

Let $\{p^1, \ldots, p^t\}$ be a given set of points in $\mathbb{R}^n$ and $\mathbf{P} = < p^1, \ldots, p^t > =$ convex hull of $\{p^1, \ldots, p^t\}$. Let $\text{rank}\{p^2 - p^1, \ldots, p^t - p^1\} = s$. Then $\mathbf{P}$ has dimension s. If $s = n$, $\mathbf{P}$ is a full dimensional convex polytope in $\mathbb{R}^n$. If $s < n$, a system of $(n - s)$ equations characterizing the affine space of $\mathbf{P}$ can be determined, and by eliminating $(n - s)$ variables using it one goes into the affine space of $\mathbf{P}$ in which $\mathbf{P}$ is a full dimensional convex polytope. In this reduced system the facets of $\mathbf{P}$ can be determined by the procedure discussed below. Each facet leads to a linear inequality constraint for characterizing $\mathbf{P}$ through a system of linear constraints. From these and from the system of equations characterizing the affine space of $\mathbf{P}$ in $\mathbb{R}^n$, we get a linear constraint representation of $\mathbf{P}$.

So, we assume without any loss of generality that $\mathbf{P}$ is a full dimensional convex polytope in $\mathbb{R}^n$. Let $\bar{p} = (p^1 + \ldots + p^t)/t$. The point $\bar{p}$ is an interior point of $\mathbf{P}$. Let $\mathbf{Q}$ be the polytope obtained from $\mathbf{P}$ by translating the origin to $\bar{p}$, i.e., $\mathbf{Q} = < q^1, \ldots, q^t >$ where $q^k = p^k - \bar{p}$, for $k = 1$ to t. Hence, 0 is an interior point of $\mathbf{Q}$, and thus is not contained on any of the facetal hyperplanes of $\mathbf{Q}$. Thus every facetal hyperplane of $\mathbf{Q}$ can be represented by an equation of the form

$$a_1 x_1 + \ldots + a_n x_n = 1 \tag{11}$$

with $\mathbf{Q}$ lying in the halfspace represented by $a_1 x_1 + \ldots + a_n x_n \leqq 1$ (since $0 \in \mathbf{Q}$). We will represent this facetal hyperplane by the vector of coefficients $(a_1, \ldots, a_n)$ in (11). Then $a = (a_1, \ldots, a_n)$ is a vector representing a facetal hyperplane of $\mathbf{Q}$, iff a is an extreme point of the following system of constraints

$$a q^k \leqq 1, \quad k = 1 \text{ to } t \tag{12}$$

in which a is the vector of variables, and $q^1, \ldots, q^t$ are the data.

And, $a_1 x_1 + \ldots + a_n x_n = 1$ represents a facetal hyperplane of $\mathbf{Q}$ with $\mathbf{Q}$ lying in the halfspace defined by $a_1 x_1 + \ldots + a_n x_n \leqq 1$, iff $a_1 x_1 + \ldots + a_n x_n = 1 + a\bar{p}$ is a facetal hyperplane of $\mathbf{P}$ with $\mathbf{P}$ lying in the halfspace defined by $a_1 x_1 + \ldots + a_n x_n \leqq 1 + a\bar{p}$. Therefore, any algorithm for enumerating the extreme points corresponding to a system of linear constraints can be used to enumerate the facets of the convex hull of a given set of points.

4 Segments of a Polytope

DEFINITION Let $\Omega = < p^1, \ldots, p^t >$, where $\{p^1, \ldots, p^t\}$ is a subset of extreme points of a convex polytope Γ. Ω is called a *segment* of Γ if the following conditions are satisfied.

Dimension of Ω = dimension of $\mathbf{\Gamma}$ (13)

For every pair of points in $\{p^1, \dots, p^t\}$, they are adjacent in Ω iff they are adjacent in $\mathbf{\Gamma}$; i.e., adjacency on Ω coincides with that on $\mathbf{\Gamma}$. (14)

As an example consider the polytope of full dimension in $\mathbb{R}^3$ shown in Figure 2, with 5 extreme points. The convex hull of $\{p^1, p^2, p^3, p^4\}$ in this polytope is a segment of it, because it satisfies both (13), and (14).

A polytope is said to be a *simple* (or *regular)* polytope if for each of its vertices, the number of incident edges is equal to the dimension of the polytope. A simple polytope is the set of feasible solutions of a system of the form (1) in which the right hand side constants vector b is nondegenerate as defined in linear programming literature[14]. It is clear that in a simple polytope, the only possible segment is the whole polytope itself.

A nonsimple polytope is one which has at least one vertex at which the number of incident edges is strictly greater than the dimension of the polytope. It is the set of feasible solutions of a system of form (1) in which the b-vector is degenerate. A nonsimple polytope may have a segment which is a proper subset of it. Figure 2 provides an example of this.

In a simple polytope, every pair of edges with a common vertex defines a unique two dimensional face. This property was used in Murty [13] to characterize (and build using a constructive algorithm) every face of a simple polytope from its two dimensional skeleton. In nonsimple polytopes, there may be pairs of edges with a common vertex which do not lie together on any two dimensional face. The pair of edges, one joining p^4 and p^3, and the other joining p^4 and p^2 in Figure 2 is an example of this.

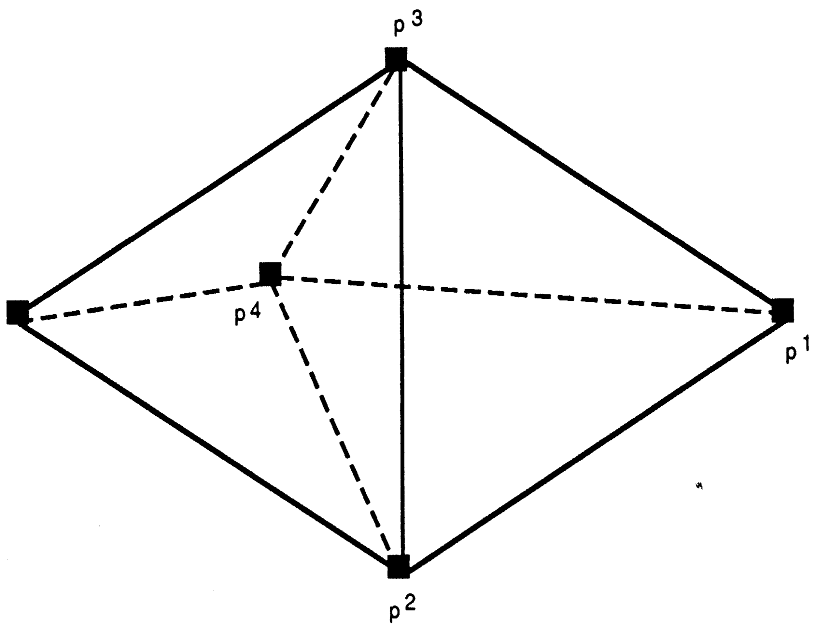

Figure 2 Polytope with 5 extreme points in $\mathbb{R}^3$. The lines are edges. $< p^1, p^2, p^3, p^4 >$ is a segment of this polytope.

A *facetal hyperplane* of a full dimensional convex polytope in $\mathbb{R}^n$ is a hyperplane in $\mathbb{R}^n$ containing one of its facets.

Let Ω be a segment of a full dimensional convex polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$. It is possible that none of the facetal hyperplanes of $\mathbf{\Gamma}$ is a facetal hyperplane of Ω. As an example, let Ω_1 be the unit cube in $\mathbb{R}^3$. Draw the normal to each facet of Ω_1 through the center of that facet, and take a point on it a little bit outside of Ω_1 as a new extreme point. This generates a convex polytope $\mathbf{\Gamma}_1$ in $\mathbb{R}^3$ with 14 extreme points (8 extreme points of Ω_1 and one extreme point on each of the 6 normal lines to the facets of Ω_1). It can be verified that Ω_1 (outlined with thick edges in Figure 3) is a segment of $\mathbf{\Gamma}_1$. Each facet of $\mathbf{\Gamma}_1$ is a 2-dimensional simplex, but none of the facetal hyperplanes of $\mathbf{\Gamma}_1$ is a facetal hyperplane of Ω_1.

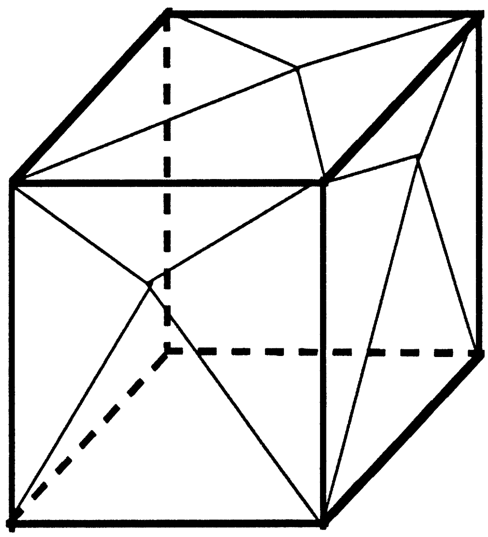

Figure 3 Ω_1 is the unit cube in $\mathbb{R}^3$ with thick edges. $\mathbf{\Gamma}_1$ is the convex polytope containing Ω_1 as a segment, plus 6 new extreme points (only 3 visible in the figure) with one on the outer side of each facet of Ω_1.

There can exist full dimensional convex polytopes $\mathbf{\Gamma}$ in $\mathbb{R}^n$ and segments Ω of $\mathbf{\Gamma}$ such that for every face $\mathbf{F}$ of $\mathbf{\Gamma}$ of dimension r, $3 \leqq r \leqq n-1$, either $\mathbf{F} \cap \Omega = \mathbf{F}$, or $\mathbf{F} \cap \Omega$ has dimension $\leqq r-1$. To construct an example like this, repeat the construction in Figure 3, replacing the unit cube with any full dimensional convex polytope Ω_2 in $\mathbb{R}^n$. That is, draw the normal to each facetal hyperplane of Ω_2 through the center (or any relative interior point) of the corresponding facet, and select a point on this normal just outside Ω_2 as a new extreme point. So, the number of new extreme points added is equal to the number of facets of Ω_2. Let $\mathbf{\Gamma}_2$ be the convex hull of the union of Ω_2 and all these new extreme points. It can be verified that none of the facetal hyperplanes of $\mathbf{\Gamma}_2$ is a facetal hyperplane of Ω_2, and that Ω_2 is a segment of $\mathbf{\Gamma}_2$ which satisfies the properties mentioned above.

THEOREM 2 Let Ω be a segment of a full dimensional convex polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$, $\Omega \neq \mathbf{\Gamma}$. If $\mathbf{\Gamma}\backslash\Omega$ is a convex set, then Ω_1 = the closure of $\mathbf{\Gamma}\backslash\Omega$, is a segment of $\mathbf{\Gamma}$; and $\Omega \cap \Omega_1$ is a facet of Ω.

PROOF Since Ω is a closed subset of $\mathbf{\Gamma}$ of the same dimension as $\mathbf{\Gamma}$, and $\mathbf{\Gamma}\backslash\Omega$ is convex; $\mathbf{\Gamma}\backslash\Omega$ has the same dimension as $\mathbf{\Gamma}$. So, Ω_1 has the same dimension as $\mathbf{\Gamma}$.

Since $\mathbf{\Gamma}\backslash\Omega$ and Ω are disjoint convex sets, they can be separated by a hyperplane. This hyperplane contains $\Omega\cap\Omega_1$, and hence $\Omega\cap\Omega_1$ is a face of both Ω and Ω_1. Two convex polytopes of full dimension with a common face and with disjoint interiors form a union which is also convex, only if that common face is a common facet. So, $\Omega\cap\Omega_1$ is a common facet of both Ω and Ω_1.

If x^1 and x^2 are two extreme points of Ω_1 at least one of which is not in Ω, then x^1 and x^2 are adjacent on Ω_1 iff they are adjacent on $\mathbf{\Gamma}$. And two extreme points of $\Omega\cap\Omega_1$ are adjacent on Ω (and hence on Ω_1) iff they are adjacent on $\mathbf{\Gamma}$. These facts imply that Ω_1 is also a segment of $\mathbf{\Gamma}$. ∎

THEOREM 3 Let Ω be a segment of a full dimensional convex polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$, $\Omega \neq \mathbf{\Gamma}$. If $\mathbf{\Gamma}\backslash\Omega$ is the union of convex sets $\Delta_1,\ldots,\Delta_r$, where each Δ_t for $t=1$ to r is a maximal convex set in the union; let $\bar{\Delta}_t$ be the closure of Δ_t. Then $\bar{\Delta}_t$ is a segment of $\mathbf{\Gamma}$, and $\bar{\Delta}_t\cap\Omega$ is a facet of both Ω and $\bar{\Delta}_t$, for each $t=1$ to r.

PROOF Since Δ_t is a maximal convex subset of $\mathbf{\Gamma}\backslash\Omega$, it is clear that $\mathbf{\Gamma}\backslash\Delta_t$ is convex, and as in the proof of Theorem 2 it can be verified that Δ_t is full dimensional.

Since both Δ_t and $\mathbf{\Gamma}\backslash\Delta_t$ are convex and have disjoint interiors, they can be separated by a hyperplane. So, as in the proof of Theorem 2, it can be concluded that $\bar{\Delta}_t\cap\Omega$ is a common facet of both of them, for $t = 1$ to r.

If x^1 is an extreme point of $\mathbf{\Gamma}$ in Δ_t and x^2 is an extreme point of $\mathbf{\Gamma}$ in $\bar{\Delta}_t$, then the line segment $\{x : x=\alpha x^1+(1-\alpha)x^2, 0<\alpha \leqq 1\}\subset\Delta_t$, and hence x^1 and x^2 are adjacent on $\bar{\Delta}_t$ iff they are adjacent on $\mathbf{\Gamma}$, for $t=1$ to r. Also, two extreme points in $\bar{\Delta}_t\cap\Omega$ are adjacent in $\bar{\Delta}_t$ iff they are adjacent in $\mathbf{\Gamma}$ since Ω is a segment of $\mathbf{\Gamma}$. So, $\bar{\Delta}_t$ is a segment of $\mathbf{\Gamma}$ for $t=1$ to r. ∎

THEOREM 4 Let Ω be a segment of a full dimensional convex polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$. If a two dimensional face $\mathbf{T}$ of $\mathbf{\Gamma}$ has an intersection with Ω that is more than an edge, then $\mathbf{T}$ is entirely contained in Ω.

PROOF Suppose Ω has an intersection with $\mathbf{T}$ that is more than an edge, but does not contain all of $\mathbf{T}$. Then one of the facets of Ω splits $\mathbf{T}$ through its relative interior, and the intersection of $\mathbf{T}$ and that facet must lead to an edge of Ω in the facet, that edge is not an edge of $\mathbf{T}$, contradicting the segment property of Ω. ∎

THEOREM 5 Let $\mathbf{\Gamma}$ be a full dimensional convex polytope in $\mathbb{R}^3$, and $\Omega \neq \mathbf{\Gamma}$ a segment of $\mathbf{\Gamma}$. Then there exists at least one facet $\mathbf{F}$ of $\mathbf{\Gamma}$ such that $\mathbf{F}\cap\Omega$ has dimension $\leqq 1$ (i.e., $\mathbf{F}\cap\Omega$ is either empty or Ω contains at most an edge joining a pair of adjacent vertices on $\mathbf{F}$).

PROOF Since $\Omega \neq \mathbf{\Gamma}$, there must exist at least one facet of $\mathbf{\Gamma}$, $\mathbf{F}$ say, such that Ω does not contain all of $\mathbf{F}$. Since $\mathbf{\Gamma}$ is three dimensional, $\mathbf{F}$ is a two dimensional face of $\mathbf{\Gamma}$, Then Theorem 4 implies that either $\mathbf{F}\cap\Omega=\emptyset$, or Ω contains at most an edge joining a pair of adjacent vertices on $\mathbf{F}$. ∎

How to Check the Adjacency of edges

DEFINITION A pair of edges e_1, e_2 of a convex polytope are said to be *adjacent* on it if they have a common vertex, and if both of them together lie on a two dimensional face of that polytope.

THEOREM 6 Let e_1, e_2 be edges of $\mathbf{K}$ defined by (1) with a common vertex x^1. Let x^2, x^3 be the other vertex on e_1, e_2 respectively. Then e_1, e_2 are adjacent on $\mathbf{K}$ (i.e., they form a two dimensional face of $\mathbf{K}$, see

[14]) iff rank$\{A_{.j} : j$ such that at least one of x_j^1, or x_j^2, or x_j^3 is $> 0\}$ is equal to two less than its cardinality.

PROOF Standard result, see [14]. This is an equivalent way of defining adjacency of edges. ∎

THEOREM 7 Let the s-dimensional convex polytope $\mathbf{K}_2 =< p^1, \ldots, p^t > \subset \mathbb{R}^s$, where each p^k is an extreme point of $\mathbf{K}_2$ and all the points are distinct. Suppose e_1 = the line segment joining p^1 and p^2, and e_2 = the line segment joining p^1 and p^3; are two edges of $\mathbf{K}_2$ with a common extreme point p^1. Then, e_1, e_2 are adjacent edges on $\mathbf{K}_2$ iff there exists a row vector $c \in \mathbb{R}^s$ and a $\beta \in \mathbb{R}^1$, such that

$$\begin{aligned} cp^1 = cp^2 = cp^3 &= \beta \\ cp^t &< \beta \quad \text{if } p^t \text{ is an adjacent extreme point of } p^1 \text{ on } \mathbf{K}_2 \text{ different from } p^2 \text{ and } p^3 \\ cp^k &\leqq \beta \quad \text{if } p^k \text{ is not adjacent to } p^1 \text{ on } \mathbf{K}_2 \end{aligned} \tag{15}$$

PROOF By definition, a face of a convex polytope is its intersection with a supporting hyperplane.So, e_1, e_2 form a two dimensional face of $\mathbf{K}_2$ iff there exists a hyperplane, $\mathbf{H}$ defined by $cy = \beta$ say, which contains e_1 and e_2, and does not contain any other adjacent extreme point of p^1 on $\mathbf{K}_2$, and $\mathbf{K}_2$ lies entirely in one of the half-spaces defined by $\mathbf{H}$. The system (15) is exactly a restatement of these conditions. ∎

Some More Results on Segments

DEFINITIONS Let p^1 be an extreme point of a full dimensional convex polytope $\mathbf{P}$ in $\mathbb{R}^n$. A hyperplane $\mathbf{H}$ which separates p^1 from all the other extreme points of $\mathbf{P}$, and intersects every edge of $\mathbf{P}$ incident at p^1 in its relative interior, is known as an *edge cutting hyperplane* (or *EC hyperplane)* for the extreme point p^1 of $\mathbf{P}$. In this case $\mathbf{P} \cap \mathbf{H}$ is known as an *edge polytope* (or *EP)* of p^1 on $\mathbf{P}$.

The combinatorial structure of an EP for an extreme point p^1 on $\mathbf{P}$ is independent of which EC hyperplane for p^1 is selected. So, we will refer to it as *the* edge polytope or EP of p^1 on $\mathbf{P}$. Two vertices on the EP of p^1 on $\mathbf{P}$ are adjacent on this EP iff the edges of $\mathbf{P}$ on which they lie are adjacent edges in $\mathbf{P}$.

THEOREM 8 Let x^1 be an extreme point on a full dimensional convex polytope $\mathbf{P}$ in $\mathbb{R}^n$, $n \geqq 3$. $\mathbf{S}$ is a nonempty subset of edges of $\mathbf{P}$ incident at x^1 satisfying the following properties.

(i) $\mathbf{S}$ is a proper subset of the set of edges of $\mathbf{P}$ incident at x^1, $|\mathbf{S}| \geqq 2$.

(ii) For any face $\mathbf{F}$ of $\mathbf{P}$ containing x^1, $\mathbf{F} \neq \mathbf{P}$, either $\mathbf{S}$ contains all the edges of $\mathbf{F}$ incident at x^1, or the dimension of the convex hull of all the edges of $\mathbf{F}$ incident at x^1 in $\mathbf{S}$ is $<$ dimension of $\mathbf{F}$.

(iii) For at least one face $\mathbf{F}$ of $\mathbf{P}$, $\mathbf{F} \neq \mathbf{P}$, the convex hull of the set of edges of $\mathbf{F}$ incident at x^1 in $\mathbf{S}$ has dimension between 1 and $-1 +$ dimension of $\mathbf{F}$.

Let $\mathbf{H}$ be an EC hyperplane for x^1 on $\mathbf{P}$, and $\mathbf{K}_1 = \mathbf{P} \cap \mathbf{H}$ and $\mathbf{K}_2$ = the convex hull of the intersections of $\mathbf{H}$ with the edges in $\mathbf{S}$. Clearly, $\mathbf{K}_2 \subset \mathbf{K}_1$. Then there exists a couple of extreme points of $\mathbf{K}_2$ which are adjacent on $\mathbf{K}_2$ but not adjacent on $\mathbf{K}_1$.

PROOF Proof is by induction on n. First consider the case $n = 3$. In this case $\mathbf{K}_1$ is a two dimensional polytope, $\mathbf{K}_2$ is the convex hull of a proper subeset of extreme points of $\mathbf{K}_1$, and $\mathbf{K}_2$ contains only one extreme point from some edges of $\mathbf{K}_1$. So, $\mathbf{K}_2$ must contain a pair of extreme points which are adjacent on $\mathbf{K}_2$ but not on $\mathbf{K}_1$, hence the theorem holds for $n = 3$.

Now set up an induction hypothesis that the result in the theorem holds for polytopes of dimension $\leqq n-1$. We will now prove that under the induction hypothesis, the result in the theorem also holds for the polytope $\mathbf{P}$ of dimension n. We consider two cases.

Case 1: $\mathbf{S}$ contains no more than a single edge from any facet containing x^1.

In this case $\mathbf{K}_2$ has at least two extreme points, and no pair of extreme points of $\mathbf{K}_2$ are adjacent on $\mathbf{K}_1$. Hence the result in the theorem holds for $\mathbf{P}$.

Case 2: There exists at least one facet, $\mathbf{F}_1$ say, of $\mathbf{P}$ containing x^1 such that $\mathbf{S}$ contains a proper subset of edges in $\mathbf{F}_1$ incident at x^1, but at least two.

Let $\mathbf{S}_1 = \mathbf{S} \cap \mathbf{F}_1$. So, properties (i), (ii), (iii) hold for the subset $\mathbf{S}_1$ of edges of $\mathbf{F}_1$ incident at x^1. Let $\mathbf{K}_3 = \mathbf{F}_1 \cap \mathbf{H}$, $\mathbf{K}_4$ = convex hull of intersections of $\mathbf{H}$ with the edges in $\mathbf{S}_1$. By the induction hypothesis, there exist a pair of extreme points on $\mathbf{K}_4$, y^1 and y^2 say, which are adfjacent on $\mathbf{K}_4$ but not on $\mathbf{K}_3$. But since $\mathbf{F}_1$ is a facet of $\mathbf{P}$, $\mathbf{K}_4$ is a face of $\mathbf{K}_2$, so, y^1 and y^2 are also adjacent on $\mathbf{K}_2$. However, since they are not adjacent on $\mathbf{K}_3$, they are not adjacent on $\mathbf{K}_1$. Hence the result in the theorem must hold for the convex polytope $\mathbf{P}$ of dimension n.

By induction, the result in the theorem holds in general. ∎

THEOREM 9 Let Ω be a segment of a full dimensional convex polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$, $n \geqq 3$. If $\Omega \neq \mathbf{\Gamma}$, there exists an extreme point x^1 of Ω satisfying the property that a pair of extreme points on the EP for x^1 on Ω are adjacent on it, but not adjacent on the EP for x^1 on $\mathbf{\Gamma}$.

PROOF Assume that $\Omega \neq \mathbf{\Gamma}$. Notice that the set of extreme points on the EP for x^1 on Ω , is a subset of the set of extreme points on the EP for x^1 on $\mathbf{\Gamma}$.

First consider the case $n = 3$. In this case Theorem 5 implies that there exists a vertex x^1 on Ω such that there is a facet $\mathbf{F}$ of $\mathbf{\Gamma}$ containing x^1, and Ω contains only one edge incident at x^1, say e_1 , from $\mathbf{F}$. Since $\mathbf{F}$ is two dimensional, there is a second edge, say e_2, incident at x^1 in $\mathbf{F}$, and this edge is not in Ω. Let p^1, p^2 be the points of intersection of e_1, e_2 with an EC hyperplane $\mathbf{H}$ for x^1 of $\mathbf{\Gamma}$. The points, p^1, p^2 are adjacent extreme points of $\mathbf{H} \cap \mathbf{\Gamma}$, of these p^1 is in $\mathbf{H} \cap \mathbf{\Gamma}$ but p^2 is not. Both $\mathbf{H} \cap \mathbf{\Gamma}$ and $\mathbf{H} \cap \Omega$ are two dimensional, and $\mathbf{H} \cap \Omega \subset \mathbf{H} \cap \mathbf{\Gamma}$. These facts imply that one of the two adjacent extreme points of p^1 on $\mathbf{H} \cap \Omega$ satisfies the property that it and p^1 are adjacent on $\mathbf{H} \cap \Omega$ but not on $\mathbf{H} \cap \mathbf{\Gamma}$. So, the statement in the theorem holds for $n = 3$.

Now set up an induction hypothesis that the statement in the theorem holds for polytopes of dimension $\leqq n-1$, and their segments. We will now show that this implies that the statement in the theorem must also hold for $\mathbf{\Gamma}$ of dimension n and its segment $\Omega \neq \mathbf{\Gamma}$. We consider two cases.

Case 1: There exists a face $\mathbf{F}$ of $\mathbf{\Gamma}$ of dimension r, $3 \leqq r \leqq n-1$, such that $\mathbf{F} \cap \Omega$ is r dimensional and $\neq \mathbf{F}$.

In this case, clearly $\mathbf{F} \cap \Omega$ is a segment of $\mathbf{F}$, both have dimension r; and $\mathbf{F} \cap \Omega$ is an r-dimensional face of Ω. By the induction hypothesis, there exists an extreme point x^1 in $\mathbf{F} \cap \Omega$, sûch that there are two extreme points, p^1, p^2 say, on the EP of x^1 on $\mathbf{F} \cap \Omega$ which are adjacent on it, but not adjacent on the EP of x^1 on $\mathbf{F}$. The set of extreme points on the EP of x^1 on $\mathbf{F}$ is a subset (those on a face) of the set of extreme points on the EP of x^1 on $\mathbf{\Gamma}$. These facts imply that p^1, p^2 are nonadjacent extreme points on the EP of x^1 on $\mathbf{\Gamma}$, but are adjacent on the EP of x^1 on Ω. So, the statement in the theorem holds for $\mathbf{\Gamma}$ and its segment Ω in this case.

Case 2: For every face $\mathbf{F}$ of $\mathbf{\Gamma}$ of dimension r, $3 \leqq r \leqq n-1$, either $\mathbf{F} \cap \Omega = \mathbf{F}$, or $\mathbf{F} \cap \Omega$ has dimension $\leqq r-1$.

Let x^1 be an extreme point on Ω such that Ω does not contain all the adjacent extreme points of x^1 on $\mathbf{\Gamma}$. In this case the result in this theorem can be verified to hold by Theorem 8.

Hence the result in this theorem must hold for the convex polytope $\mathbf{\Gamma}$ of dimension n under the induction hypothesis. It has already been shown to hold for polytopes of dimension 3. Hence, by induction, the result in the theorem holds in general. ∎

THEOREM 10 Let Ω be a segment of a full dimensional polytope $\mathbf{\Gamma}$ in $\mathbb{R}^n$. If $\Omega \neq \mathbf{\Gamma}$, there exists an extreme point x^1 of Ω and a couple of edges e_1, e_2 incident at x^1 in Ω such that e_1, e_2 are adjacent in Ω, but not in $\mathbf{\Gamma}$.

PROOF By Theorem 9, there exists an extreme point, x^1 say, of Ω satisfying the property that a pair of extreme points, say y^1, y^2, on the EP of x^1 on Ω are adjacent on it, but not adjacent on the EP of x^1 on $\mathbf{\Gamma}$. Let e_1, e_2 be the edges of $\mathbf{\Gamma}$ containing y^1, y^2. Then e_1, e_2 satisfy the result in the theorem. ∎

How to Check Whether a Segment is the Whole Polytope

Let $\mathbf{\Gamma}$ be a full dimensional polytope in $\mathbb{R}^n$ specified through a system of linear constraints, and $\{p^1, \ldots, p^t\}$ a subset of extreme points of $\mathbf{\Gamma}$ such that $\Omega =< p^1, \ldots, p^t >$ is a segment of $\mathbf{\Gamma}$. To check whether $\Omega = \mathbf{\Gamma}$, do the following for each r, $1 \leqq r \leqq t$.

Identify all the adjacent extreme points of p^r on Ω using the result in Theorem 1, and hence find all the edges of Ω incident at p^r. For each pair of edges of Ω incident at p^r, check whether they are adjacent on Ω using the result in Theorem 7. For every pair of adjacent edges of Ω incident at p^r check whether they are adjacent on $\mathbf{\Gamma}$ using the result in Theorem 6. If one such pair is not adjacent on $\mathbf{\Gamma}$, $\Omega \neq \mathbf{\Gamma}$ by the result in Theorem 10, terminate. Otherwise continue.

If the above work is completed for all $r = 1$ to t without encountering a pair of adjacent edges of Ω which are not adjacent on $\mathbf{\Gamma}$, $\Omega = \mathbf{\Gamma}$, terminate.

The whole work requires the solving of at most $O(t^3)$ linear programs, and computing the ranks of an equal number of sets of vectors.

5 Algorithm for Enumerating Extreme Points

Consider the convex polytope $\mathbf{K}$ defined by (1). This algorithm is initiated with one extreme point of $\mathbf{K}$ obtained by solving, for example, a Phase I problem using any of the polynomially bounded algorithms for LP. Beginning with this, the algorithm develops a list of extreme points of $\mathbf{K}$, adding at least one new extreme point to the list per iteration, until all of them are in.

At some stage suppose the list is $\{d^1, \ldots, d^r\}$, consisting of r distinct extreme points of $\mathbf{K}$. Each of these extreme points is a rational vector of size at most nL. Let $\mathbf{K}_1 = < d^1, \ldots, d^r >$. At this stage we need to check whether there is an extreme point of $\mathbf{K}$ which is not in $\mathbf{K}_1$. This involves checking: is $\mathbf{K} \subseteq \mathbf{K}_1$? Here $\mathbf{K}$ is defined through a system of linear constraints, and $\mathbf{K}_1$ as the convex hull of a finite set of rational vectors. If $\mathbf{K}_1$ is an arbitrary set of points, B. C. Eaves brought to our attention the paper of R. Freund and J. Orlin [9] which established that the problem of checking whether $\mathbf{K} \nsubseteq \mathbf{K}_1$ is NP-complete. However, in our case every point in $\mathbf{K}_1$ is an extreme point of $\mathbf{K}$, this makes our problem special and we are able to develop an efficient algorithm for it.

Let (x_B, x_N) be a partition of the variables in (1) into basic, nonbasic parts for some basic vector for (1). Without any loss of generality we assume that $x_B = (x_1, \ldots, x_m)^T$, $x_N = (x_{m+1}, \ldots, x_n)^T$. As explained in

Section 2, we will use this partition in the algorithm to look at the representation of **K** through a system of linear inequalities (as in (6)) in the space of the nonbasic variales x_N. This partition is never changed during the algorithm.

General Iteration $\{d^1, \ldots, d^r\}$ is the present list of extreme points of **K**.

STEP 1 For $k = 1$ to r, let (d^k_B, d^k_N) be a partition of the vector d^k into basic, nonbasic parts as in the partition (x_B, x_N) of the variables in (1). Here we check whether the dimension of $\mathbf{K}_1 = < d^1, \ldots, d^r >$ is $< n - m =$ dimension of **K**. This step is carried out only if in the previous iteration this step resulted in the affirmative answer for the corresponding question at that stage, otherwise we go directly to Step 2 in this iteration.

The dimension of $\mathbf{K}_1$ = the rank of $\{d^k_N - d^1_N : k = 1 \text{ to } r\}$. If it is $n - m$, go to Step 2, and in all subsequent iterations omit this step. If it is $\leqq n - m - 1$, there exists an $f_N = (f_{m+1}, \ldots, f_n) \neq 0$ such that $f_N(d^k_N - d^1_N) = 0$ for all $k = 2$ to r; find such a vector f_N. Let $f_N d^1_N = \beta$. Then all the extreme points $d^1, \ldots, d^r$ in the current list correspond to points in the x_N-space lying on the hyperplane defined by $f_N x_N = \beta$. Now solve the two LPs

(16)	(17)
Minimize $f_N x_N$	Maximize $f_N x_N$
subject to (1)	subject to (1)

One or both of the LPs (16), (17) will have as an optimum extreme point a point not in the current list. Call it d^{r+1}, add it to the list and go to the next iteration.

STEP 2 In this step the algorithm tries to find a pair of points in the current list $\{d^1, \ldots, d^r\}$ satisfying

the pair are not adjacent on **K**, but adjacent on the convex hull of the current list (18)

From the results in Section 2, this is done by doing the following for each of the $\binom{r}{2}$ pairs of points in $\{d^1, \ldots, d^r\}$ until one satisfying (18) is found.

If the pair from the current list being examined now is $p = (p_j), q = (q_j)$, they are not adjacent on **K** iff the rank of the set of vectors $\{A_{.j} : j$ such that at least one of p_j or q_j or both are $> 0\}$ is strictly less than its cardinality $- 1$, see [14]. Checking this takes at most $O(m^3)$ effort. The points p, q are adjacent on the convex hull of the current list iff the following system in variables $c = (c_1, \ldots, c_n)$ has a feasible solution, which can be checked by solving an LP.

$$\begin{aligned} c(p - q) &= 0 \\ c(p - d^k) &> 0 \quad \text{for all } k \text{ such that } d^k \neq p \text{ or } q \end{aligned} \tag{19}$$

If a pair of points satisfying (18) is found in the current list, let $\bar{c}$ be the vector obtained as the feasible solution for (19) for that pair. Now solve the LP

$$\text{Maximize } \bar{c}x \quad \text{subject to (1)} \tag{20}$$

The maximum objective value in (20) will be $\geqq \gamma = \bar{c}p = \bar{c}q > \min \{\bar{c}d^k : k = 1 \text{ to } r \text{ such that } d^k \neq p \text{ or } q\}$. If the maximum objective value in (20) is $> \gamma$, an extreme point optimum for it is a new extreme point of **K**, add it to the list and go to the next iteration.

If the maximum objective value in (20) is γ, and an optimum extreme point obtained for it when (20) is solved is either p or q, the set of optimum solutions for (20) is a face **S** of **K** determined by

$$\begin{aligned} Ax &= b \\ cx &= \gamma \\ x &\geqq 0 \end{aligned} \tag{21}$$

In this case, p, q are the only two points from the current list feasible to (21) (this follows from (19) by the choice of $\bar{c}$). Hence, its set of feasible solutions, the face **S** of **K**, contains only p, q from the current list. Since **S** contains p, q which are not adjacent on **K** (and hence not adjacent on **S**, since **S** is a face of **K**), the dimension of **S** must be $\geqq 2$. So, **S**, the set of feasible solutions of (21), has dimension $\geqq 2$ and we only have two extreme points p, q on it in the current list. Therefore, by applying Step 1 discussed earlier, on (21) with p, q as the only known extreme points on it at this stage, we can get a new extreme point of **S**, this will be an extreme point of **K** since **S** is a face of **K**, add it to the list and go to the next iteration.

If there exist no pair of points in the current list satisfying (18), then $< d^1, \dots, d^r >$, the convex hull of the current list, is a segment of **K**, go to Step 3.

STEP 3 When we come to this step, $\mathbf{K}_1 = < d^1, \dots, d^r >$, the convex hull of the current list, is a segment of **K**.

Using the results in Section 2, find all the edges of $\mathbf{K}_1$. For every pair of edges of $\mathbf{K}_1$ with a common vertex, check whether they are adjacent on $\mathbf{K}_1$ using the result in Theorem 7. For every pair of adjacent edges of $\mathbf{K}_1$ check whether they are adjacent on **K** using Theorem 6. If every pair of adjacent edges of $\mathbf{K}_1$ is also adjacent on **K**, $\mathbf{K}_1 = \mathbf{K}$ by Theorem 10, i.e., the present list contains all the extreme points of **K**, terminate the algorithm.

If we find a pair of adjacent edges of $\mathbf{K}_1$, e_1, e_2 say, which are not adjacent on **K**; let d^1 be the common vertex, and d^2, d^3 the other vertices on them. So, in this case we have a row vector c in $\mathbb{R}^n$ and a $\beta \in \mathbb{R}^1$ satisfying

$$\begin{aligned} cd^1 = cd^2 = cd^3 &= \beta \\ cd^t &< \beta \quad \text{if } d^t \text{ is an adjacent extreme point of } d^1 \text{ on } \mathbf{K}_1 \text{ different from } d^2 \text{ and } d^3 \\ cd^k &\leqq \beta \quad \text{if } d^k \text{ is not adjacent to } d^1 \text{ on } \mathbf{K}_1 \end{aligned} \tag{22}$$

Now solve the LP

$$\text{Maximize } cx \quad \text{subject to (1)} \tag{23}$$

The optimum objective value in (23) is $\geqq \beta$. If it is $> \beta$, an extreme point optimum for (23) yields an extreme point of $\mathbf{K}$ not in $\mathbf{K}_1$, add it to the list and go to the next iteration.

If the optimum objective value in (23) is β, and if the extreme point optimum obtained is not in $\mathbf{K}_1$, then again we have a new extreme point, add it to the list and go to the next iteration. On the other hand, suppose an extreme point of $\mathbf{K}_1$ is obtained as an optimum solution of (23). Now consider the LP

$$\begin{aligned} &\text{Maximize } cx \\ &\text{subject to } x \in \mathbf{K}_1 \end{aligned} \tag{24}$$

We know from these conditions that the optimum objective value in (24) is β. Among adjacent extreme points of d^1 on $\mathbf{K}_1$, the only ones which are optimum to (24) are d^2, d^3. This implies that the set of optimum solutions for (24) is the two dimensional face $\mathbf{F}$ say, of $\mathbf{K}_1$ determined by the edges e_1, e_2. However, since e_1, e_2 do not form a two dimensional face of $\mathbf{K}$, the set of optimum solutions for (23) has dimension at least 3, and contains $\mathbf{F}$. That set is the set of feasible solutions of

$$\begin{aligned} Ax &= b \\ cx &= \beta \\ x &\geqq 0 \end{aligned} \tag{25}$$

Let Δ be the set of extreme points on $\mathbf{F}$. The affine rank of Δ is 2, and it is the subset of extreme points of (25) in the present list. By applying Step 1 to the system (25) and the current known set of extreme points of it, Δ, we can get a new extreme point of the set of feasible solutions of (25). This set is the set of optimum solutions of (23), and hence it is a face of $\mathbf{K}$, so, that new extreme point of (25) is a new extreme point of $\mathbf{K}$, add it to the list and go to the next iteration.

Each iteration except the final one generates one or more new extreme points which are added to the list. When the list has r extreme points, the work in Steps 2, 3 requires the solution of at most $O((\max\{r,n\})^3)$ LPs each of size at most $O(rnL)$, and hence requires at most $O(r(\max\{r,n\})^3 n^4 L)$ effort using the best available polynomial time algorithms for solving LPs. Thus when the list has r extreme points, the computational effort needed to either conclude that there are no new extreme points, or finding a new one, is at most $O(r(\max\{r,n\})^3 n^4 L)$. Thus the overall computational complexity of this algorithm to enumerate all the extreme points of $\mathbf{K}$ is at most $O(\ell^5 n^4 L)$.

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