A Three-Dimensional Analysis of Symmetric Composite Laminates with Damage

C. L. CHOW
Department of Mechanical Engineering
University of Michigan-Dearborn
Dearborn, MI 48128-1491

F. YANG* AND A. ASUNDI
Department of Mechanical Engineering
University of Hong Kong
Hong Kong

ABSTRACT: Damage behavior of a symmetric composite laminate without an initial imperfection or macro-crack is analyzed based on a three-dimensional lamination theory under multi-axial loading. The global response of the laminate during the damaging process is determined from the individual response of its constituent plies and their mutual relations. Some specific results are presented to illustrate the damage characteristics of several typical composite laminates when they are subjected to proportional loading. The application of the method to characterize damage initiation and growth in more complex structures is also discussed.

1. INTRODUCTION

IT IS WELL recognized that in realistic engineering situations, the observable macroscopic failure behavior of composite materials is generally attributed to the result of the initiation and growth of material imperfections or micro-defects. These defects often induce degradation in several global material properties of composite laminate. As the micro-failure events which reduce the strength and stiffness and determine the life of composite laminates, commonly referred to as damage, are complex and intricately related to a variety of fracture modes under different circumstances, any attempt to model exactly the fine details of them is often impractical, if not impossible to achieve [1]. However, as internal damage in laminates is noncatastrophic, it is reasonable to consider the locally averaged effect of the damage on the response of the material, which can be accomplished by an equivalent continuous and homogeneous material element described by a

*Currently on leave from Huazhong University of Science & Technology, Wuhan, China.

1056-7895/93/03 0229-17 $6.00/0

group of internal state variables to replace the damage element that is noncontinuous and nonhomogeneous—so long as the two have the same macroscopic mechanical behaviors. This approach is usually referred to as damage mechanics, in which the effects of micro-cracks are reflected in constitutive relations of the material rather than treated as boundary conditions based on the concept of fracture mechanics. For example, under certain operating conditions, a composite material may exhibit nonlinear properties before its final failure. Instead of assuming linear elastic behavior, this nonlinearity is considered by the theory of damage mechanics to be primarily caused by the influence of micro-structure changes of the material [2–13].

Experience from previous studies suggests that it is difficult to predict quantitatively the mechanical behaviors of a composite laminate in terms of the properties of its individual constituent elements, that is, the fiber and matrix elements. However, it is of value to draw some conclusions that are helpful in formulating the macroscopic theory based on the analysis of constituent layers of the laminate. To this end, it is expedient to utilize a macro-mechanics approach in which the development of a sufficiently simple but accurate mathematical model—taking specific deformation features of composite laminate into account and using the results of a limited number of tests as initial data—comes into focus. With this approach, considerable knowledge and insight into the damage description of composite material has been achieved [4–20].

Schapery proposed an energy approach to model deformation and fracture of elastic, nonlinear composite materials [2,3] based on the experimental observations that the stresses and mechanical work are practically independent of deformation history for certain loading paths. The method was based on thermodynamic principles and an internal state variable description of the microstructural changes responsible for the inelasticity. Consequences of this limited path-independence were investigated and various relationships for stable mechanical responses derived.

Under the assumption that the strain component and the damage magnitudes were insignificant—i.e., small strain and low concentration of damage entities—Talreja proposed a method of analysis in predicting stiffness variations in composite laminates for a given damage state [4–7]. While the method was able to predict satisfactory stiffness degradation, it is limited to thin laminate, in-plane loading, and transverse cracking in the ply.

An internal state variable approach was introduced by Allen et al. to describe the energy dissipation due to the growth of microcracks [8–10]. The approach has been employed to characterize the effects of crack surface displacements on the total strain and to provide analytical solutions to the upper bounds of degraded in-plane stiffness of angle-ply laminates with matrix cracks. It is restricted to the angle-ply laminates under general in-plane loading when the crack density in each layer is known *a priori.*

Allix et al. [11–13] proposed a model based on the concept of effective stress which was incorporated in the formulation of the damaged constitutive equations for composite lamina. The damage evolution equation was derived using the thermodynamic conjugate force to the chosen damage variable and the failure of a material element was postulated to have occurred when the damage variable D reaches its critical value D_c.

The studies described above offer various ways of incorporating the micromechanical failure mechanisms into macromechanical damage models. Other existing laminate-based theoretical studies such as shear-lag model [14], self-consistent scheme [15,16], strain energy approach [17] and complimentary strain energy method [18–20] mainly investigated the effect of matrix cracks on the stiffness variation of composites for a given damage state, while more practical problems of mechanical responses of composite laminates with damage for engineering design have not been addressed.

Recently, the authors have introduced a damage model characterizing nonlinearity of fiber-reinforced composite lamina with damage consideration and associated variation of material properties [21] based on their earlier model for brittle materials [22]. The present study is intended to develop a full three-dimensional lamination theory by extending the damage model limited to unidirectional lamina [21]. It may be expected that after a layer in an nth ply laminate begins to experience a damage process in which the damaged lamina behaves nonlinearly while the others remain elastic, the stresses in each ply will redistribute in order to satisfy the equilibrium condition. Henceforth, the overall laminate will have a new series of stress-strain relations instead of those without damage, which depend upon the orientations of constituent laminae and their mutual constraints. Under certain specific loading situations, the response of damaged composite laminate will be illustrated.

2. FORMULATION

For ease of illustration, the contracted notation $\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$, $\sigma_3 = \sigma_{33}$, $\sigma_4 = \sigma_{23}$, $\sigma_5 = \sigma_{13}$, $\sigma_6 = \sigma_{12}$, $\epsilon_1 = \epsilon_{11}$, $\epsilon_2 = \epsilon_{22}$, $\epsilon_3 = \epsilon_{33}$, $\epsilon_4 = 2\epsilon_{23}$, $\epsilon_5 = 2\epsilon_{13}$, $\epsilon_6 = 2\epsilon_{12}$ is used and summation convention is adopted. Moreover, from the discussions in Reference [21], it is convenient to express the formulation in terms of total stress and strain and the concepts of deviator stress and strain are not introduced since the hydrostatic stress will also contribute to damaging process.

A damage surface which also represents initial and subsequent damage surfaces, similar to the yield function in plasticity, is successfully defined and has the following form:

$$F = F(\sigma_i, D) = 0 \tag{1}$$

where σ_i ($i = 1,2,\ldots,6$) are stress components. D is a measure of damage extent, changes only when damage progresses, and thus controls the size of the subsequent damage surfaces. The realization of damage surface is not simply a mathematical convenience but a phenomenon experimentally recorded by means of the acoustic technique [23]. The method also identified the effect of loading and unloading on the damage surface.

For brittle composite lamina with damage, at a certain damage state, the stress and strain relation may be expressed as

$$\epsilon_i^r = C_{ij}\sigma_j \qquad (D \text{ fixed}) \tag{2}$$

where $C_{ij} = C_{ji}$ are the elastic compliance components at current state and will change during the entire loading process when damage in the lamina progresses, and ϵ_i^r are reversible strain components.

During damage evolution, the total reversible strain increment is

$$d\epsilon_i^r = C_{ij}d\sigma_j + dC_{ij}\sigma_j \tag{3}$$

and can be assumed to consist of two parts:

$$d\epsilon_i^r = d\epsilon_i^e + d\epsilon_i^d \tag{4}$$

where superscripts e and d stand for elastic and damaged components, respectively. The elastic strain increment $d\epsilon_i^e$ may be related to the stress increment $d\sigma_j$ through the generalized Hooke's law as

$$d\epsilon_i^e = C_{ij}d\sigma_j \tag{5}$$

The damaged strain increment is expressed as

$$d\epsilon_i^d = dC_{ij}\sigma_j \tag{6}$$

and it can be estimated from the normality rule or flow rule by using the damage function F,

$$d\epsilon_i^d = \frac{1}{A}\frac{\partial F}{\partial \sigma_i}\frac{\partial F}{\partial \sigma_j}d\sigma_j \tag{7}$$

where A is a function of damage state.

The energy per unit volume dissipated during damaging process is selected here as damage variable D which is

$$D = \int_0^{\epsilon_i^r} \sigma_i d\epsilon_i - \frac{1}{2}\sigma_i\epsilon_i^r \tag{8}$$

and its increment is expressed as

$$dD = \frac{1}{2}\sigma_i d\epsilon_i^d \tag{9}$$

When the damage surface is constructed as

$$F(\sigma_i,D) = (R_{ij}\sigma_i\sigma_j)^{1/2} - K(D) = 0 \tag{10}$$

where the components in R_{ij} (without loss generality, it may be assumed that $R_{ij} = R_{ji}$) may be determined from experiments and $K(D)$ is a state function and changes when D grows, and if the equivalent stress σ_o and the equivalent damaged strain increment $d\epsilon^d$ are defined as

$$\sigma_o^2 = R_{ij}\sigma_i\sigma_j \tag{11}$$

$$dD = \frac{1}{2}\sigma_i d\epsilon_i^d = \frac{1}{2}\sigma_o d\epsilon^d \tag{12}$$

the value of A in Equation (7) can be given by σ_o–ϵ^d curve established from experiment as

$$A = \frac{d\sigma_o}{d\epsilon^d} \tag{13}$$

The comparison between Equation (6) and Equation (7) with Equations (10) and (13), gives

$$dC_{ij}\sigma_j = \frac{d\epsilon^d}{\sigma_o}R_{ij}\sigma_j \tag{14}$$

If it is further assumed that C_{ij} is just a function of damage state and independent of processes [2,4] and σ_o–ϵ^d relation

$$\epsilon_o^d = a\sigma_o^b - a\sigma_s^b \tag{15}$$

in which a and b are material constants, the expression for instantaneous compliance may be obtained as

$$C_{ij}(D) = C_{ij}(0) + \frac{ab}{(b-1)}(\sigma_o^{b-1} - \sigma_s^{b-1})R_{ij} \tag{16}$$

$C_{ij}(0)$ is the compliance component of the lamina without damage and

$$D = \frac{ab}{2(b+1)}(\sigma_o^{b+1} - \sigma_s^{b+1}) \qquad (\text{for } d\sigma_o \geq 0) \tag{17}$$

Obviously, Equation (17) relates the damage variable to the current stress state and can be applied to calculate the instantaneous $C_{ij}(D)$ in Equation (16) during a loading process with damage. It can be seen that the coefficients R_{ij} also reflect the anisotropic behavior induced by material damage. In a certain unloading–reloading cycle, $C_{ij}(D)$ are constants as D equals the value determined from σ_o at the start of unloading; upon reloading, D develops again in accordance with Equation (17) and thus the $C_{ij}(D)$ change when σ_o exceeds its proceding maximum value. From the preceding discussions, it is apparent that the Equations (2), (4), (16), and (17) are capable of describing the damage behavior of brittle composite ply, provided that the material coefficients in Equations (10) and (15) are determined from experimental measurements.

3. GOVERNING EQUATIONS

With the formulation of the constitutive relation of composite lamina coupled with damage, the description for damage response of composite laminate can be established by a lamination theory.

Consider a symmetrical laminate composed of n unidirectional layers with different fiber orientations. Its representative element is illustrated in Figure 1. All the plies in the laminate have the same physical properties. In Figure 1, t is the total thickness of the laminate, t_m and θ_m ($m = 1,2,\ldots,n$) are the thickness and

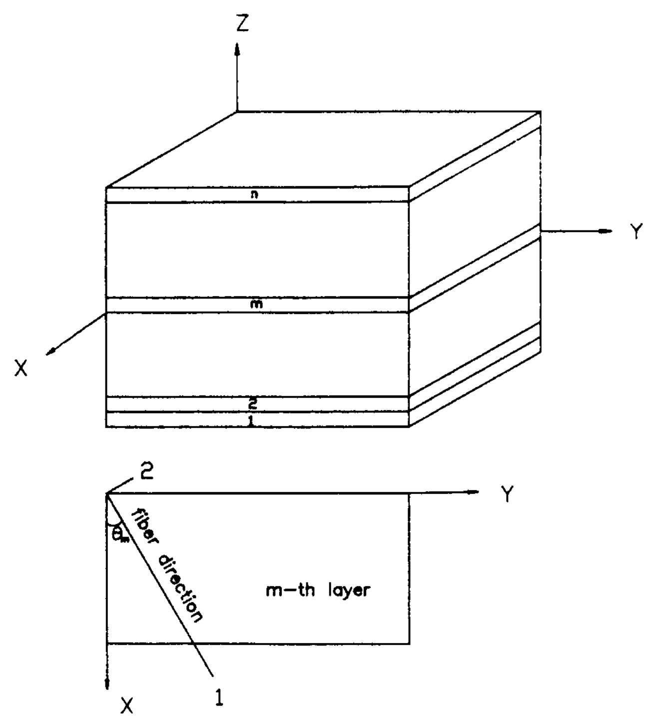

Figure 1. Coordinate systems of laminate.

fiber orientation angle of mth layer. Two coordinate systems, material principal axes and laminate axes, are selected and represented as 1-2-3 and X-Y-Z respectively, where 1- and 2-axes are parallel and perpendicular to the fiber directions, the 3-axis is normal to the ply plane, the X-Y plane is located in the middle plane of the laminate, and the 3-axis coincides with the Z-axis. All the symbols throughout the report with top bars are used to denote in the material principal coordinate system, otherwise, in the laminate coordinate system. In the investigation, the following hypotheses will be employed:

1. Each layer is modeled as a macroscopically homogeneous orthotropic material. Individual fibers and matrix are not considered.
2. Time and rate effects are neglected and environmental effects such as temperature and moisture are excluded.
3. The laminate is in an initially free stress state and body force is ignored.
4. Deformation is small and the material without damage is linearly elastic and obeys generalized Hooke's law.
5. Each ply in the laminate is very thin so that the stress and strain components in each layer are uniform.

The assumptions that the constituent laminae in the laminate are perfectly bonded together and in equilibrium state under applied loads, lead to the condition of continuity of stress and strain components at the ply interfaces; i.e.,

$$\epsilon_i = \epsilon_i^{(m)} \qquad i = 1,2,6;\ m = 1,2,\ldots,n \tag{18}$$

$$\epsilon_i = \sum_{m=1}^{n} V_m \epsilon_i^{(m)} \qquad i = 3,4,5 \tag{19}$$

$$\sigma_i = \sigma_i^{(m)} \qquad i = 3,4,5;\ m = 1,2,\ldots,n \tag{20}$$

$$\sigma_i = \sum_{m=1}^{n} V_m \sigma_i^{(m)} \qquad i = 1,2,6 \tag{21}$$

where σ_i and ϵ_i are the macroscopic stress and strain components applied to the laminate, V_m is the volume fraction of the mth ply, and

$$V_m = t_m/t, \ \sum_{m=1}^{n} V_m = 1 \tag{22}$$

For the mth ply, at any instant, its constitutive relation in material principal coordinate system, from Equation (2), is

$$\bar{\sigma}_i^{(m)} = \bar{S}_{ij}^{(m)} \bar{\epsilon}_j^{(m)}, \bar{\epsilon}_i^{(m)} = \bar{C}_{ij}^{(m)} \bar{\sigma}_j^{(m)} \qquad (m = 1,2,\ldots,n) \tag{23}$$

where $\bar{\sigma}_i^{(m)}$ and $\bar{\epsilon}_i^{(m)}$ are stress and strain components of mth layer, $\bar{S}_{ij}^{(m)}$ and $\bar{C}_{ij}^{(m)}$ $(i,j = 1,2,\ldots,6)$ are the instantaneous stiffness and compliance components of mth lamina in material principal coordinate system corresponding to some certain damage state, respectively. Their values will change during loading which is accompanied with damage. Moreover,

$$\bar{S}_{ij}^{(m)} = \bar{S}_{ji}^{(m)}, \bar{C}_{ij}^{(m)} = \bar{C}_{ji}^{(m)}$$

and

$$[\bar{C}_{ij}^{(m)}]^{-1} = [\bar{S}_{ij}^{(m)}] \tag{24}$$

which can be evaluated from Equation (16).

In the laminate coordinate system, Equation (23) becomes

$$\sigma_i^{(m)} = S_{ij}^{(m)}\epsilon_j^{(m)}, \epsilon_i^{(m)} = C_{ij}^{(m)}\sigma_j^{(m)} \qquad (m = 1,2,\ldots,n) \tag{25}$$

where $\sigma_i^{(m)}$, $\epsilon_i^{(m)}$, $S_{ij}^{(m)}$ and $C_{ij}^{(m)}$ are the components in the laminate coordinate system.

From the geometry relation between the material principal system and the laminate coordinate system, as shown in Figure 1, the following transformation formulae can be derived:

$$\bar{\sigma}_i^{(m)} = T_{ij}^{(m)}\sigma_j^{(m)}, \bar{\epsilon}_i^{(m)} = P_{ij}^{(m)}\epsilon_j^{(m)} \qquad (m = 1,2,\ldots,n) \tag{26}$$

and

$$S_{ij}^{(m)} = [P_{iu}^{(m)}]^T \bar{S}_{uv}^{(m)} P_{vj}^{(m)}, C_{ij}^{(m)} = [T_{iu}^{(m)}]^T \bar{C}_{uv}^{(m)} T_{vj}^{(m)} \tag{27}$$

where $[T_{ij}^{(m)}]^T = [P_{ij}^{(m)}]^{-1}$ and,

$$T_{ij}^{(m)} = \begin{bmatrix} \cos^2\theta_m & \sin^2\theta_m & 0 & 0 & 0 & \sin 2\theta_m \\ \sin^2\theta_m & \cos^2\theta_m & 0 & 0 & 0 & -\sin 2\theta_m \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_m & -\sin\theta_m & 0 \\ 0 & 0 & 0 & \sin\theta_m & \cos\theta_m & 0 \\ -\frac{1}{2}\sin 2\theta_m & \frac{1}{2}\sin 2\theta_m & 0 & 0 & 0 & \cos 2\theta_m \end{bmatrix} \tag{28}$$

Finally, for the composite laminate, its stress-strain relation in laminate coordinate system can be written as

$$\sigma_i = S_{ij}\epsilon_j, \quad \epsilon_i = C_{ij}\sigma_j \tag{29}$$

where S_{ij} and C_{ij} are the instantaneous stiffness and compliance components of the laminate, and $C_{ij} = [S_{ij}]^{-1}$. To determine their expression is the problem of interest here.

4. DAMAGE BEHAVIOR OF SYMMETRIC LAMINATES

When a certain three-dimensional uniform loading state is imposed on the laminate, the present analysis is complicated by the necessity that the full three-dimension effects such as out-of-plane stress components, which are considered negligible based on the plane stress assumption in the classical laminated plate theory (CLPT), must be taken into account in the lamination formulation from constituent ply properties to that of laminate. Accordingly, S_{ij} of the laminate can be expressed as:

$$S_{ij} = \begin{cases} \left[\sum_{m=1}^{n} V_m S_{ip}^{(m)}[S_{pq}^{(m)}]^{-1}\right]\left[\sum_{m=1}^{n} V_m[S_{qr}^{(m)}]^{-1}\right]^{-1}\left[\sum_{m=1}^{n} V_m[S_{ru}^{(m)}]^{-1}S_{uj}^{(m)}\right] \\ \quad + \sum_{m=1}^{n} V_m\left[S_{ij}^{(m)} - S_{ip}^{(m)}[S_{pq}^{(m)}]^{-1}S_{qj}^{(m)}\right] & i,j = 1,2,6 \\ & p,q,r,u = 3,4,5 \\ \left[\sum_{m=1}^{n} V_m S_{ip}^{(m)}[S_{pq}^{(m)}]^{-1}\right]\left[\sum_{m=1}^{n} V_m[S_{qj}^{(m)}]^{-1}\right]^{-1} & i = 1,2,6 \\ & j,p,q = 3,4,5 \\ \left[\sum_{m=1}^{n} V_m[S_{ip}^{(m)}]^{-1}\right]^{-1}\left[\sum_{m=1}^{n} V_m[S_{pq}^{(m)}]^{-1}S_{qj}^{(m)}\right] & i,p,q = 3,4,5 \\ & j = 1,2,6 \\ \left[\sum_{m=1}^{n} V_m[S_{ij}^{(m)}]^{-1}\right]^{-1} & i,j = 3,4,5 \end{cases} \tag{30}$$

and C_{ij} can be obtained from the inversion of S_{ij} as indicated in Equation (29). The relation between the stress components of the mth layer and the overall stress components of the laminate is

$$\sigma_i^{(m)} = G_{ij}^{(m)}\sigma_j \tag{31}$$

where

$$G_{ij}^{(m)} = \begin{cases} [C_{ip}^{(m)}]^{-1}C_{pj} & i,j,p = 1,2,6 \\ [C_{ip}^{(m)}]^{-1}[C_{pj} - C_{pj}^{(m)}] & i,p = 1,2,6,\ j = 3,4,5 \\ \delta_{ij} & i,j = 3,4,5 \\ 0 & \text{others} \end{cases} \tag{32}$$

and δ_{ij} is the Kronecker delta function,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Equations (29)–(32) enable the description of the overall damage response of the laminate. The representations in Equations (31) and (32) can be reduced to those without damage consideration, and under the conditions of plane stress and strain and in-plane loading they can be simplified to the results presented in Reference [13].

As elucidated earlier, the concept of damage surface which plays the same role as the yield function in the classical theory of hardening plasticity can be introduced to determine the overall stress that induces the nonlinear response of constituent layer of the laminate. The initial and subsequent damage surfaces for the mth ply can be interpreted in the laminate coordinate system in terms of overall stresses

$$F^{(m)}(\sigma_i, D) = [A_{ij}\sigma_i\sigma_j]^{1/2} - K^{(m)}(D) = 0 \tag{33}$$

where

$$A_{ij} = G_{ui}^{(m)} T_{vu}^{(m)} R_{vr} T_{rm}^{(m)} G_{mj}^{(m)}$$

Actually, this equation imposes a restriction on Equation (16). If the laminate is loaded monotonously to current stress σ_i and a stress increment $d\sigma_i$ is applied such that the conditions

$$F^{(m)} = 0 \quad \text{and} \quad \frac{\partial F^{(m)}}{\partial \sigma_i} d\sigma_i > 0 \tag{34}$$

are simultaneously satisfied, in some layers, at least in the mth ply, damaging process will initiate. Otherwise, the constituent laminae, as well as the laminate itself, remain in elastic range without damage initiation, or, after having experienced damage, in a certain unloading-reloading cycle without further damage development, their responses are those whose material coefficients are evaluated from Equation (16) where the value of D determined from the beginning of unloading remains constant.

It can be seen that the overall response of a laminate during a damaging process must be established in conjunction with the solutions based on a series of Equations (29)–(32) describing the damage behavior of each constituent layer and of the laminate itself at each loading step. Obviously, to solve these equations explicitly is difficult and numerical methods may have to be exploited to solve them because of the nonlinear properties of the evolution law of damage and of the stress-strain relations.

5. EXAMPLES

In order to illustrate the application of the proposed method of analysis, the damage response of laminate composite plates under different loading conditions will next be discussed.

First, a group of unidirectional laminates of different orientations is tested in simple tension in order to measure their damage characteristics. The specimens are manufactured from panels made of T300 graphite fiber in 648 epoxy matrix. The fiber volume fraction of the lamina is about 65 percent. As the composite lamina is approximated to be transversely isotropic, the material coefficients in Equation (15) are evaluated as [21]

$$a = 0.114 \times 10^{-4},\ b = 1.24,\ \sigma_s = 9 \text{ MPa},$$

and the properties of initial undamaged lamina are

$$E_{11}^{\circ} = 1.25 \times 10^5 \text{ MPa},\ E_{22}^{\circ} = E_{33}^{\circ} = 1.11 \times 10^4 \text{ MPa}$$

$$\nu_{21}^{\circ} = \nu_{31}^{\circ} = 0.03,\ \nu_{23}^{\circ} = 0.338,\ G_{12}^{\circ} = G_{13}^{\circ} = G_{23}^{\circ} = 3.3 \times 10^3 \text{ MPa}$$

The non-vanishing R_{ij} values for each lamina that are required for the damage analysis are R_{22}, R_{23}, R_{33}, R_{44}, R_{55} and R_{66}. These values can be experimentally determined using a similar method described in Reference [22]. Briefly, the condition under which $R_{22} = R_{33} = 1$ is satisfied is to load an unidirectional composite lamina perpendicular to the fiber direction. The other three values of R_{44}, R_{55} and R_{66} are next evaluated by experimentally establishing the relationship between the R_{ij} and θ angles for which unidirectional composite lamina of three angles, namely 22.5, 45 and 67.5 degrees are loaded in x-direction (see Figure 1). As to the remaining coefficient of R_{23}, the transversely isotropic property commonly assumed for composite lamina readily establishes the relationship,

$$C_{44} = 2[C_{22} - C_{23}],$$

which enables the determination of R_{23} from Equation (16). The R_{ij} values obtained are:

$$R_{22} = R_{33} = 1,\ R_{44} = R_{55} = R_{66} = 1.9,\ R_{23} = 0.05.$$

Figure 2 shows the relation between the measured damage D and the equivalent stress σ_o which may be considered as being equivalent to the evolution law of damage as described in Equation (17).

In the prediction of the overall stress for the angled composite laminates of $[\pm 15]_{2s}$, $[\pm 30]_{2s}$ and $[\pm 45]_{2s}$ subjected to a uniaxial tensile strain, an iterative method is adopted. This is because the stress at mth ply is dependent upon the stiffness $S_{ij}^{(m)}$ shown in Equation (25) which itself requires *a priori* knowledge of the unknown applied stress described in Equation (16). No divergence has been

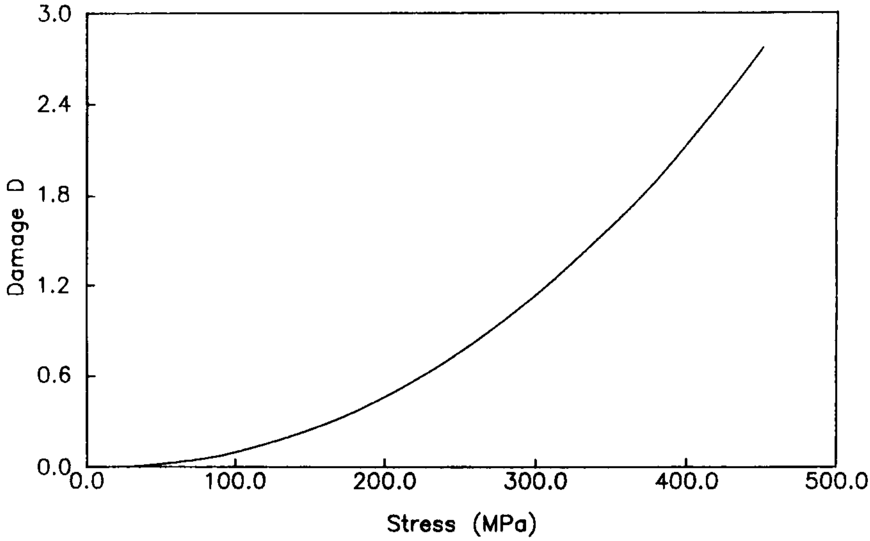

Figure 2. The evolution law of damage.

experienced for the computed stiffnesses and stresses at each ply. The calculated stiffness for each ply can then be used to assess the overall stiffness of a laminate based on the two-dimensional form of Equation (30) which in turn is used to determine the overall stress at a prescribed strain using Equation (29).

Figures 3, 4 and 5 display the predicted and experimental overall axial stresses against the overall strains along the loading directions respectively for symmetric angled-ply laminates of $[\pm 15]_{2s}$, $[\pm 30]_{2s}$ and $[\pm 45]_{2s}$. It can be seen from the figures that the stress and strain relations are linear before the development of damage strain ϵ_d, after which the responses of the laminates exhibiting nonlinear damaged properties become pronounced. As in angle-ply laminates, there are only two ply orientations which have the same magnitudes but opposite signs, i.e., they are symmetric about the loading direction, these layers will simultaneously experience damage processes under uniaxial loading. Close agreement between the predicted and measured curves can also be observed from these figures.

The response of cross-ply laminate $[0/90°]_s$ under uniaxial loads is illustrated in Figure 6. It can be observed from the figure that during loading, after the 90° layer degrades, the line initially straight becomes nonlinear, even though the 0° lamina contains no damage. This may be attributed to stress redistribution between the layers due to the damage of 90° ply.

In order to illustrate the viability of the three-dimensional formulation described in the preceding section, a cross-ply laminate $[0/90°]_s$ was subjected to the multi-axial loading of $\epsilon_x = \epsilon_y = -2\epsilon_z = \epsilon$. A finite element analysis taking into account the constitutive equations of damage is used to predict the stresses

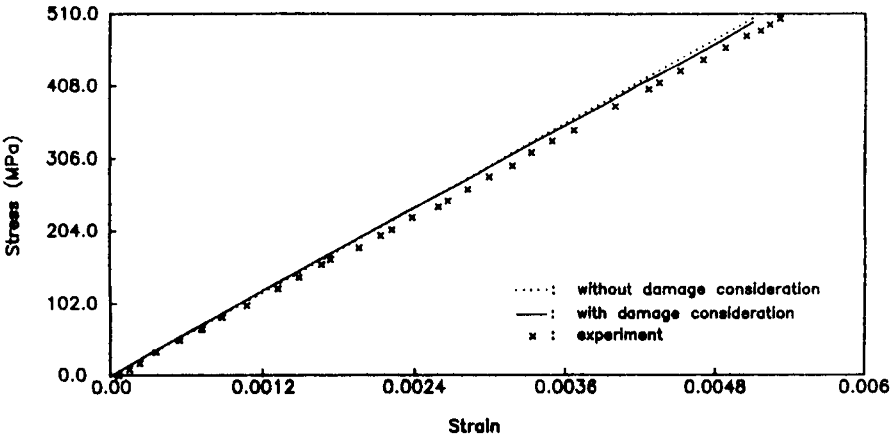

Figure 3. Overall stress-strain curves for $[\pm 15°]_{2s}$.

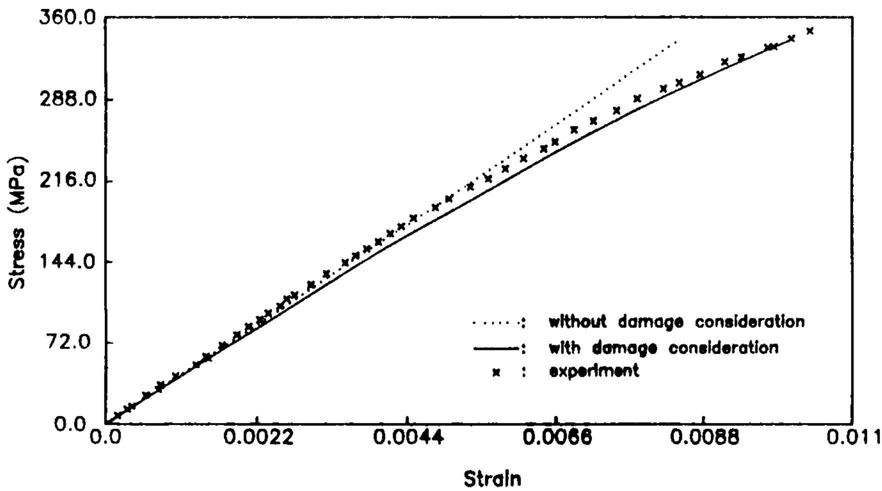

Figure 4. Overall stress-strain curves for $[\pm 30°]_{2s}$.

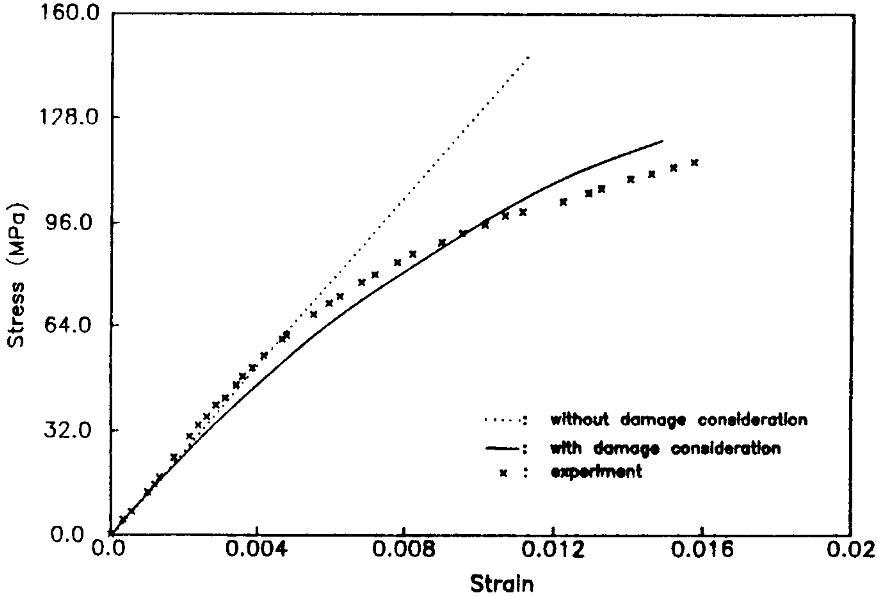

Figure 5. *Overall stress-strain curves for* $[\pm 45°]_{2s}$.

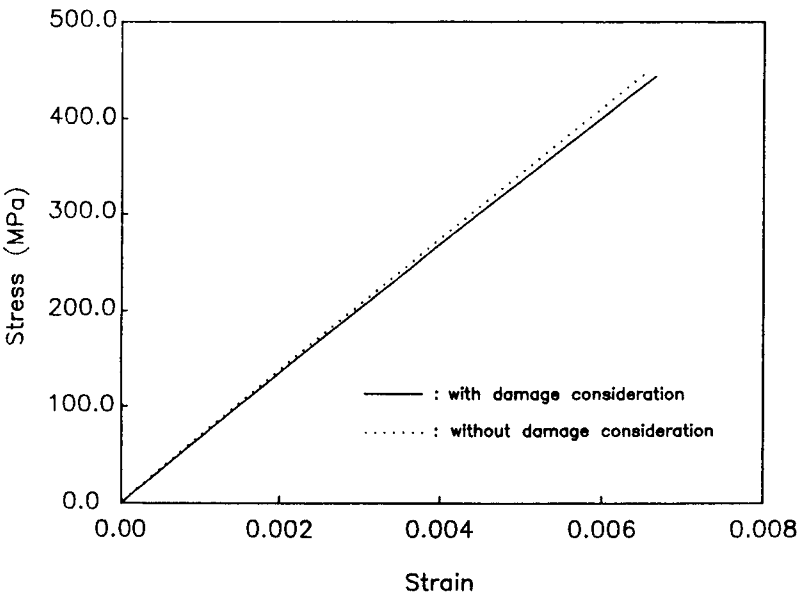

Figure 6. *The* σ–ϵ *curves for* $[0/90°]_s$ *laminate.*

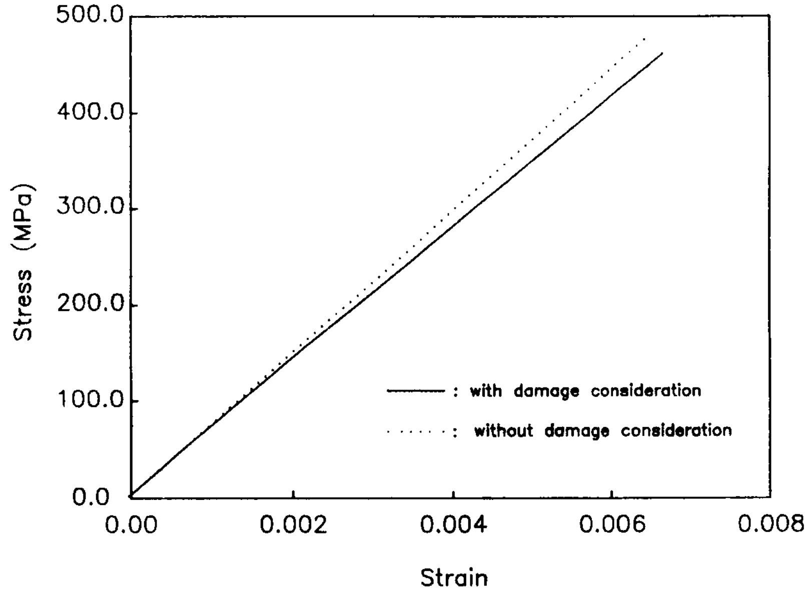

Figure 7. *The σ–ε curves for [0/90°]ₛ laminate.*

and strains in each constituent ply. A 20-node parametric element was adopted to calculate the initial stresses and strains with the stiffness equations shown in Equation (30). An iterative scheme is then employed as in the case of the angled-ply laminate to arrive at satisfactory stress and stiffness which are required to meet the equivalent condition of each ply shown in Equations (25) and (26). The overall stress and strain are finally obtained with the aid of Equations (18)–(21) based on those of the constituent plies and their corresponding stiffnesses.

Figure 7 shows the curve of stress σ as function of different strain ϵ for the $[0/90°]_s$ laminate. In this case, the overall stress component in X-direction is equal to that in Y-direction, $\sigma_x = \sigma_y = \sigma$, and the constituent layers also exhibit simultaneously their nonlinear processes.

Also shown in Figures 3–7 are the predicted results of the overall stress "without damage consideration." This stress may also be known as the "pseudo elastic stress" which is defined as $\sigma_i = S_{ij}(0) \cdot \epsilon_j^r$, while the damage stress, as $\sigma_i = S_{ij}(D) \cdot \epsilon_j^r$. Hence the sole difference between the two stresses is the difference in the stiffness employed. It can be observed from Figures 3–7 that the linear behavior of the "elastic" stiffness does not hold at larger applied strains and that the stiffness degradation becomes most pronounced in $[\pm 45]_{2s}$ laminate but less significant for other angled-ply laminates.

6. CONCLUSIONS

The damage responses of symmetric composite laminated plates without initial discontinuity such as notches and holes have been estimated based on a full 3-dimensional analysis which takes the out-of-plane stress and strain components into account and is applicable to thick laminates. The overall constitutive relations of the laminates at any instant in the course of damaging process can be determined from the damage properties of individual constituent layers described by a relatively simple continuum model and mutual constraints of constituent plies. A few examples have been presented to show the damage characteristics of some composite laminates when they are subjected to proportional loading.

The method can be extended to investigate large composite structures when a finite element program is used. In this case, a typical 3-dimensional element may be composed of many individual composite laminae and the stiffness problem of each single element could be solved by the proposed approach since the determination of the stiffness of the element constitutes the first step in forming the global stiffness matrix necessary for any structural analysis. The present study can also be incorporated into the failure analysis of composite laminates, if appropriate failure criterion is used.

REFERENCES

1. Reifsnider, K. L., ed. 1982. "Damage in Composite Materials: Basic Mechanisms, Accumulation, Tolerance, and Characterization," *ASTM STP-775*.
2. Schapery, R. A. 1988. "A Method for Mechanical State Characterization of Inelastic Composite Laminates with Damage," Report No. MM5762-88-10, Texas A&M Univ.
3. Schapery, R. A. 1990. "A Theory of Mechanical Behavior of Elastic Media with Growing Damage and Other Changes in Structure," *J. Mech. Phys. Solids*, 38:215–253.
4. Talreja, R. 1985. "A Continuum Mechanics Characterization of Damage in Composite Materials," *Proc. R. Soc., Lond.*, A309:195–216.
5. Talreja, R. 1986. "Stiffness Properties of Composite Laminates with Matrix Cracking and Interior Delamination," *Engng. Frac. Mech.*, 25:751–762.
6. Talreja, R. 1987. "Modeling of Damage Development in Composites Using Internal Variables Concepts," *ASME AD–Vol. 12*, "Damage Mechanics in Composites," A. S. D. Wang and G. K. Haritos, eds.
7. Talreja, R. 1989. "Damage Development in Composites: Mechanisms and Modeling," *J. Strain Analysis*, 24:215–222.
8. Allen, D. H., C. E. Harris and S. E. Groves. 1987. "A Thermomechanical Constitutive Theory for Elastic Composites with Distributed Damage, Parts I and II," *Int. J. Solids Structures*, 23:1301–1318 and 1319–1338.
9. Harris, C. E., D. H. Allen and E. W. Nottorf. 1987. "Damage-Induced Changes in the Poisson's Ratio of Cross-Ply Laminates: An Application of a Continuum Damage Mechanics Model for Laminated Composites," *ASME AD–Vol. 12*, "Damage Mechanics in Composites," A. S. D. Wang and G. K. Haritos, eds.
10. Groves, S. E., C. E. Harris, A. L. Highsmith, D. H. Allen and R. G. Novell. 1986. "An Experimental and Analytical Treatment of Matrix Cracking in Cross-Ply Laminates," *Experimental Mech.*, (March):73–79.
11. Ladeveze, P. et al. 1982. "Damage and Fracture of Tridirectional Composite," *Proc. ICCM-IV*, pp. 619–628.
12. Allix, O., D. Gilletta and P. Ladeveze. 1985. "Mechanical Behavior of Elementary Constituents of Laminates," *Proc. ICCM-V, San Diego*.

13. Allix, O. et al. 1987. "Damage Mechanics for Composite Laminates under Complex Loading," *Proc. IUTAM/IUM, Symposium Yielding, Damage and Failure of Anisotropic Solids, Grenoble.*
14. Highsmith, A. L. and K. L. Reifsnider. 1982. "Stiffness-Reduction Mechanisms in Composite Materials," *Damage in Composite Materials, ASTM STP 775*, K. L. Reifsnider, ed., pp. 103–117.
15. Laws, N., G. J. Dvorak and M. Hejazi. 1983. "Stiffness Changes in Unidirectional Composites Caused by Crack Systems," *Mechanics of Materials*, 2:123–137.
16. Dvorak, G. J., N. Laws and M. Hejazi. 1985. "Analysis of Progressive Matrix Cracking in Composite Laminates: Thermoelastic Properties of a Ply with Cracks," *J. of Composite Materials*, 19:216–234.
17. Aboudi, J. 1987. "Stiffness Reduction in Cracked Solids," *Engng. Frac. Mech.*, 26:637–650.
18. Hashin, Z. 1985. "Analysis of Cracked Laminates: A Variational Approach," *Mechanics of Materials*, 4:121–136.
19. Hashin, Z. 1986. "Analysis of Stiffness Reduction of Cracked Cross-Ply Laminates," *Engng. Frac. Mech.*, 25:771–778.
20. Hashin, Z. 1987. "Analysis of Orthogonally Cracked Laminates under Tension," *Trans. ASME, J. of Appl. Mech.*, 54:872–879.
21. Chow, C. L. and F. Yang. 1992. "A Simple Model for Brittle Composite Lamina with Damage," *J. Reinforced Plastics and Composites*, 11:222–242.
22. Chow, C. L. and F. Yang. 1991. "On One-Parameter Characterization of Material Damage State," *Engng. Frac. Mech.*, 40:335–343.
23. Holcomb, D. J. and L. S. Costin. 1986. "Detecting Damage Surfaces in Brittle Materials Using Acoustic Emission," *Trans. ASME, J. of Appl. Mech.*, 53:536–544.