HIGHER ORDER SEPARATION THEOREMS
AND
A DESCENT ALGORITHM FOR P-MATRIX LCPS

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January 1984

Technical Report 84-5

ABSTRACT

We discuss some generalizations of the strict separation property in the linear complementarity problem associated with a P-matrix. Using them, we develop a principal pivoting descent method for this problem, in which a distance measure decreases stricity in each step.

KEYWORDS: Linear Complementarity Problem, Complementarity Cones, Separation Property, P-matrix, Nearest Point Problem, Principal Pivoting Method.

1. INTRODUCTION

We consider the LCP (q,M) of order n, which is to find $w = (w_1,\ldots,w_n)^T$, $z = (z_1,\ldots,z_n)^T$ satisfying

$$
\begin{aligned}
w - M z &= q \\
w, z &\geqq 0 \qquad (1) \\
w^T z &= 0
\end{aligned}
$$

where M is a given P-matrix of order n. If $q \geqq 0$, (w=q,z=0) is a solution of (1), so we assume $q \ngeqq 0$. A square matrix is said to be a P-*matrix* if all its principal subdeterminants are strictly positive. In this case it is well known that (1) has a unique solution for each $q \in R^n$ [8,11].

When M is Positive definite (PD) or Positive semi-definite (PSD), Polynomially bounded ellipsoid algorithms have been developed for solving the LCP (q,M) [2,6]. These algorithms use the convexity of the set $E=\{z:z^T (Mz+q) \leqq 0\}$ and the fact that any point in the intersection of E with the polyhedron $\{z: z \geqq 0, Mz+q \geqq 0\}$ is on the boundary of E and leads to the solution of the LCP (q,M). When M is a P-matrix but not even PSD, the set $\{z: z^T (Mz+q) \leqq 0\}$ is not in general convex, and because of this, the ellipoid algorithm cannot be used to solve these problems. Eventhough when M is a P-matrix, the LCP (q,M) is known to have a unique solution, no polynomially bounded algorithm for finding it is known (except when M is also PSD). The general LCP is known to be strongly NP-complete [1], but the complexity of the special case when M is a P-matrix is not known. Developing a polynomially bounded algorithm for the LCP (q,M) when M is a general P-matrix is one of the important unresolved theoretical problems in lenear complementarity. Such an algorithm would seem to require some type of comvexification. Here we discuss an algorithm for the LCP (q,M) when M is a P-matrix, that requires the solution of a finite sequence of LCPs each associated with a positive definite matrix. So each problem in the sequence can be solved in polynomial time using the ellipsoid algorithms of [2,6]. As we move from one problem in the sequence to the next, a distance measure decreases strictly and the method terminates with the solution of the LCP (q,M) when this distance measure is zero. Viewed in terms of this distance measure, this algorithm is therefore a descent algorithm. Its computational complexity is not known. It is being studied, and it is hoped that a polynomially bounded variant of this approach will be found.

2. HIGHER ORDER SEPERATION THEOREMS.

For any matrix D, we denote its jth column by $D_{.j}$. Let I denote the unit matrix of order n. For j=1 to n, $\{I_{.j}, -M_{.j}\}$ is the jth complementary pair of column vectors in (1) and each vector in this pair is the *complement* of the other. A *complementary set of vectors* is an ordered set $\{A_{.1},\ldots,A_{.n}\}$ where $A_{.j} \in \{I_{.j}, -M_{.j}\}$ for j=1 to n, and the matrix A with column vectors $A_{.1},\ldots, A_{.n}$ is called a *complementary matrix* for (1). Since M is a P-matrix, all the complementary sets of vectors are linearly independent, and every complementary matrix is a *complementary basis* for (1) [8,9]. Associated with a complementary matrix A is the *complementary cone* Pos (A) = $\{Ay: y \geq 0\}$. Since M is a P-matrix, the class of complementary cones partitions R^n [8,11]. The complementary basis corresponding to a complementary cone containing q is a *complementary feasible basis* for (1), and the basic solution of (1) corresponding to that basis is the solution of the LCP (q,M) [8,9].

A *subcomplementary set* associated with a nonempty proper subset $J=\{j_1,\ldots,j_r\} \subset \{1,\ldots,n\}$ is a set $\{A_{.j_1},\ldots,A_{.j_r}\}$ with $A_{.j} \in \{I_{.j},-M_{.j}\}$ for each $j \in J$. Given any two sets E,F, let $E \setminus F$ denote the set of all elements in E not in F. If $\{A_{.j}: j \in J\}$ is a subcomplementarity set associated with J, let $B_{.j}$ be the complement of $A_{.j}$, then the subcomplementary set $\{B_{.j}: j \in J\}$ is known as the *complement* of the subcomplementary set $\{A_{.j}: j \in J\}$.

The well known *strict separation property* associated with a P-matrix M states that if $\{A_{.i}: i \in \{1,\ldots,j-1,j+1,\ldots,n\}\}$ is a subcomplementary set, then the unique hyperplane in R^n containing the points 0, and all the vectors in this subcomplementary set, strictly separates the left out complementary pair of vectors $\{I_{.j},-M_{.j}\}$ [8]. Here we will present a generalization of this result for a subcomplementary set associated with any nonempty proper subset of $\{1,\ldots,n\}$.

THEOREM 1: Let M be a P-matrix of order n and let J, $\bar{J}$ be a partition of $\{1,\ldots,n\}$ with J, $\bar{J}$ both being nonempty. Let $\{A_{.j}: j \in J\}$, $\{A_{.j}: j \in \bar{J}\}$ be the corresponding partition of a complementary set of vectors. Let $\{B_{.j}: j \in \bar{J}\}$ be the complement of the subcomplementary set $\{A_{.j}: j \in \bar{J}\}$. If H is a hyperplane in R^n satisfying

i) H contains the origin 0 and all the vectors in the subcomplementary sets $\{A_{.j}: j \in J\}$

ii) All the vectors in the subcomplementary set $\{A_{.j}: j \in \bar{J}\}$ lie in one of the closed half-spaces, $H^{\geqq}$, defined by H then at least one of the vectors in $\{B_{.j}: j\varepsilon\bar{J}\}$ lies strictly on the other side of H in the other open half-space $H^{<}$ defined by H.

PROOF: Consider the system (2)

$$w - M z = 0 \quad (2)$$

Perform principal pivot steps in (2) to transform the complementary set of vectors $\{A_{.j}: j\varepsilon J\cup\bar{J}\}$ into the set of unit vectors. This is a non-singular linear transformation that preserves separation properties. If u_j denotes the variable in (2) associated with $A_{.j}$ and v_j denotes its complement, this transforms (2) into

$$u - \bar{M} v = 0 \quad (3)$$

where $\bar{M}$ is also a P-matrix because it is a principal pivot transform of the P-matrix M [12]. Let $\bar{M}_{\bar{J}\bar{J}}$ denote the principal submatrix of $\bar{M}$ corresponding to the subset $\bar{J}$. Let $\bar{H} = \{x: \sum_{j=1}^{n} a_j x_j = 0\}$ be the transform of H. Since $A_{.j}$ is transformed into $I_{.j}$, by (i) we have $a_j = 0$ for each $j\varepsilon J$, and by (ii) we have $a_{\bar{J}} = (a_j: j\varepsilon\bar{J}) \geq 0$. So $a = (a_j) \geqq 0$ and since $\bar{H}$ is a hyperplane, $a \geq 0$, that is $a_{\bar{J}} \geq 0$. (A vector $y = (y_j) \geq 0$ means that each y_j is nonnegative and at least one y_j is strictly positive). For $j\varepsilon\bar{J}$, $B_{.j}$ is now transformed into $-\bar{M}_{.j}$. The vector $(a(-\bar{M}_{.j}) : j\in\bar{J}) = -a_{\bar{J}}\bar{M}_{\bar{J}\bar{J}}$. Since $\bar{M}_{\bar{J}\bar{J}}$ is itself a P-matrix and $a_{\bar{J}} \geq 0$, by a theorem of D. Gale and H.Nikaido [4] at least one of the components of $a_{\bar{J}}\bar{M}_{\bar{J}\bar{J}}$ is strictly positive, that is $a(-\bar{M}_{.j}) < 0$ for at least one $j\in\bar{J}$. That is, at least one of the $-\bar{M}_{.j}$ for $j\varepsilon\bar{J}$ lies in the open half-space $\bar{H}^{<} = \{x: \sum_{j=1}^{m} a_j x_j < 0\}$ not containing the unit vectors. In terms of the original space this implies that at least one of the $B_{.j}$, $j\varepsilon\bar{J}$ is contained in the open half-space $H^{<}$ defined by H not containing the complementary set of vectors $\{A_{.j}: j\varepsilon J\cup\bar{J}\}$

THEOREM 2: Let M be a P-matrix of order n, J a nonempty proper subset of $\{1, \ldots, n\}$ and let $\{A_{.j}: j \epsilon J\}$ be a subcomplementary set of vectors. Let H be a hyperplane in R^n that contains the origin 0 and all the vectors in the set $\{A_{.j}: j \epsilon J\}$. Then H strictly separates at least one pair of the left out complementary pairs of vectors $\{I_{.j}, -M_{.j}\}$ for $j \epsilon \bar{J} = \{1, \ldots, n\} \setminus J$.

PROOF: Choose the subcomplementary set $\{A_{.j}: j \epsilon \bar{J}\}$ arbitrarily and transform the system (2) into (3) as in the proof of Theorem 1. Using the notation in the proof of Theorem 1, suppose this transforms H into $\bar{H} = \{x: \sum_{j=1}^{n} a_j x_j = 0\}$. Since $A_{.j}$ is transformed into $I_{.j}$ and H contains $A_{.j}$ for $j \epsilon J$, $\bar{H}$ must contain $I_{.j}$ for $j \epsilon J$, that is $a_j = 0$ for all $j \epsilon J$. Since $\bar{H}$ is a hyperplane, we must have $a \neq 0$, that is $a_{\bar{J}} = (a_j: j \epsilon \bar{J}) \neq 0$. Define $\bar{M}_{\bar{J}\bar{J}}$ as in the proof of Theorem 1, it is a P-matrix as noted there. By the sign nonreversal theorem for P-matrices of D. Gale and H. Nikaido [4] if $(y_j: j \epsilon \bar{J}) = a_{\bar{J}} \bar{M}_{\bar{J}\bar{J}}$, $a_j y_j > 0$ for at least one $j \epsilon \bar{J}$. Since $a_j = 0$ for $j \epsilon J$, these facts imply that there exists at least one $j \epsilon \bar{J}$ satisfying the property that $aI_{.j}$ and $a(-\bar{M}_{.j})$ have strictly opposite signs, that is $\bar{H}$ separates the complementary pair of vectors $\{I_{.j}, -\bar{M}_{.j}\}$ strictly. In terms of the original space, this implies that H strictly separates the complementary pair of vectors $\{I_{.j}, -M_{.j}\}$ for that $j \epsilon \bar{J}$.

3 THE ALGORITHM:

Consider the LCP (1), where M is a P-matrix. The algorithm obtains a sequence of complementary bases $A^0, A^1, \ldots$, each one differing from the previous in exactly one column vector. Because of this feature, the method belongs to the class of principal pivoting methods [3,5].

For $d \varepsilon \Gamma = \{I_{.1}, \ldots, I_{.n}\ -M_{.1}, \ldots, -M_{.n}\}$ let $V(d) = 0 \in R^n$ if $d^T q \leqq 0$, or $= d(d^T q)/||d||^2$ if $d^T q > 0$. $V(d)$ is the nearest point on the ray of d, $\{d\delta: \delta \geqq 0\}$, to q. The nearest point in the complementary cone Pos $\{A_{.1}, \ldots, A_{.n}\}$ to q is 0 iff $V(A_{.j}) = 0$ for each $j = 1$ to n. Since q is contained in a complementary cone, there exists a complementary cone Pos (A) such that $V(A_{.j}) \neq 0$ for at least one j. So there exists a $d \varepsilon \Gamma$ for which $V(d) \neq 0$. Find the d that minimizes $||V(d)-q||$ among $d \varepsilon \Gamma$.. Find any complementary basis A° that contains this d as a column vector. Use A° as the initial complementary basis in step 1 of the algorithm.

GENERAL STEP $r + 1$ FOR $r \geqq 0$: This step begins with the complementary basis A^r obtained at the end of the previous step (with A^0 if $r = 0$). Find the nearest point (in terms of the usual Euclidean distance) to q in the complementary cone Pos (A^r). As shown in [10] this nearest point problem can be posed as an LCP associated with a positive definite matrix, and can be solved in polynomial time by the ellipsoid methods discussed in [2,6]. If $q \varepsilon$ Pos (A^r), this nearest point is q itself, A^r is a complementary feasible basis for (1), and the method terminates. Otherwise let x^r be the nearest point in Pos (A^r) to q. Let $x^r = A^r \alpha^r$ where $\alpha^r = (\alpha_1^r, \ldots, \alpha_n^r) \geqq 0$. By the choice of A°, $x^r \neq 0$ for any r and so $\alpha^r \geq 0$, that is $J_r = \{j: \alpha_j^r > 0\} \neq \emptyset$. By the results in [7, 10], x^r is the orthogonal projection of q in the subspace $\{x: x = \sum_{j \in J_r} \beta_j A^r_{.j}, \beta_j$ real numbers for $j \varepsilon J_r\}$. Define the distance in this step to be $\delta^r = || x^r - q||$. Let E_r be the ball $\{x: ||x-v|| \leq \delta^r\}$ and let H_r be the tangent plane to E_r at its boundary point x^r, $H_r = \{x: (x-x^r)(x^r-q) = 0\}$. Clearly Pos $\{A^r_{.j} : j \varepsilon J_r\} \subset H_r$. So H_r is a hyperplane containing the origin, and it separates Pos (A^r) from q. Let $\Delta_r = \{j: J \notin J_r$, $A_{.j}$ and its complement are strictly separated by $H_r\}$. By theorems 1 and 2, $\Delta_r \neq \emptyset$. Select a $j \varepsilon \Delta_r$ and let A^{r+1} be the complementary basis whose columns are $A^r_{.1}, \ldots, A^r_{.j}$, $B^r_{.j}$, $A^r_{.j+1}, \ldots, A^r_{.n}\}$ where $B^r_{.j}$ is the complement of $A^r_{.j}$. By the results in [7,10] the nearest point in the complementary cone Pos (A^{r+1}) to q will be strictly closer to q then x^r. With A^{r+1}, go to the next step.

The distance measure δ^r decreases strictly in each step and since there are only a finite number of complementary bases the algorithm is clearly finite.

One can get different variants of this algorithm by choosing j from Δ^r according to different rules. One can consider the least index rule in which the j chosen from Δ_r is always the least, or a cyclical rule like the least recently considered rule popular in implementations of the simplex algorithm, or some other rule. We can also consider a block principal pivoting method in which A^{r+1} is obtained from A^r by replacing each A^r_j, $j \in \Delta_r$, by its complement in a block principal pivot step. The computational complexity of each of these variants is currently under investigation.

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