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ON AXISYMMETRIC VIBRATIONS OF THIN SHALLOW VISCOELASTIC SPHERICAL SHELLS

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SUMMARY

This paper is concerned with transverse vibrations of shallow viscoelastic spherical shells subjected to axisymmetric loads which are harmonic in time. In particular, the steady state solution is obtained for an unlimited viscoelastic shallow shell subjected to an oscillating load uniformly distributed over a small circular region about the apex. Numerical results for axial displacement and stresses are obtained for two special viscoelastic media (Maxwell and Kelvin) as well as for the elastic shell, and comparison is made with known results.

INTRODUCTION

The method of solution employed in the quasi-static treatment of problems of viscoelasticity rests on the use of Laplace transform (to eliminate the dependence on time), and the correspondence principle—between the field equations and boundary conditions in the linear theories of homogeneous and isotropic elasticity and viscoelasticity—which, in the absence of thermal effect, has been established for incompressible media by Alfrey [1][1] and in general form by Lee [2]. The extension of Lee's analogy to problems involving time-dependent temperature fields has been given by Sternberg [3], where an exposition of the subject may be found. While the quasi-static viscoelastic solutions are useful and have met with increasing success in recent years, nevertheless, in situations where the duration of application of loads is very short, it is necessary to include the inertia effects. Furthermore, in contrast to steady state solutions, only a few transient viscoelastic solutions are available, and in addition to the one-dimensional wave propagation in rods, such as those discussed in [4,5,6,7], we mention the transient solutions for the axisymmetric transverse motion of shallow spherical shells contained in [8][2] to which further reference will be made presently.

Closely related to the scope of the present investigation is the recent work on vibrations of thin shallow elastic shells by E. Reissner [9], who, by utilizing the linear differential equations due to Marguerre [10], has shown that for transverse vibrations of shallow shells the longitudinal inertia terms (with negligible error) may be omitted; and hence the formulation of the elastokinetic problems of shallow elastic shells, as in the case of elastostatics, may be reduced to the determination of axial displacement and an Airy stress function. Subsequently, E. Reissner [11] dealt with transverse vibrations of axisymmetric shallow elastic spherical shells, and in particular obtained the solution for an unlimited shell due to a point load (varying harmonically in time) at the apex.

The response of shallow viscoelastic spherical shells to arbitrary time-dependent axisymmetric load has been studied in [8] by the present authors; the solutions in integral form, utilizing the differential equations of transverse motion of shallow elastic shells, were deduced with the joint use of the Laplace and Hankel transforms. While the solutions for shallow viscoelastic spherical shells subjected to harmonically oscillating loads may in integral form be obtained as a special case of the general results in [8], for such solutions (which include steady state solutions), it is desirable from a practical point of view to obtain an alternative closed representation. Thus, the present paper deals

1. Numbers in brackets designate References at end of paper.

2. The analysis in [8] also includes the vibrations of axisymmetric viscoelastic plates.

with the response of shallow viscoelastic spherical shells to harmonically oscillating axisymmetric loads, where the medium is assumed homogeneous and isotropic. The solution, employing the differential equations for the transverse motion of thin shallow elastic shells, is obtained with the use of Laplace transform in a manner similar to that employed by Lee [2,6] and is expressed in terms of Kelvin functions. Explicit results are deduced for an unlimited shallow viscoelastic shell subjected to a uniformly distributed oscillating load, and are particularized to the cases of elastic, Maxwell, and Kelvin solids. The stresses and displacements are plotted for all three cases and comparison is made with known results [11].

GENERAL BACKGROUND

For future reference, we recall that the Laplace transform with respect to time t of a (suitably restricted) function U(x,t) is given by[3]

$$\hat{U}(x,s) = \mathcal{L}\left\{U(x,t)\,;\,s\right\} \equiv \int_0^\infty e^{-st}\,U(x,t)\,dt \tag{1}$$

and that with reference to rectangular Cartesian coordinates x_i, the stress-strain law for an isotropic and homogeneous viscoelastic medium in the Laplace transform-plane, as shown in [8], may be written as[4]

$$\begin{aligned} \hat{\sigma}_{ij} &= P_2(s)P_1^{-1}(s)\,\hat{\varepsilon}_{ij} - \tfrac{1}{3}\left[P_2(s)P_1^{-1}(s) - P_4(s)P_3^{-1}(s)\right]\hat{\varepsilon}_{kk}\,\delta_{ij} \\ \hat{\sigma}_{ii} &= P_4(s)P_3^{-1}(s)\,\hat{\varepsilon}_{ii} \end{aligned} \tag{2}$$

where σ_{ij} and ϵ_{ij} are the components of the stress and strain tensor respectively, δ_{ij} is the Kronecker delta, s is the Laplace transform parameter, and the operators $P_m(s)$, (m = 1,2,3,4), are the images of $P_m(\theta)$ in the physical plane defined as[5]

$$\begin{aligned} P_m(\theta) &= \sum_{n=0}^{N_m} c_m^{(n)}\,\theta^n \quad ; \left[\, c_m^{(N_m)} \neq 0 \right] \\ \theta^n &= \frac{\partial^n}{\partial t^n} \end{aligned} \tag{3}$$

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3. See, for example, Churchill [12]; the argument x in U refers to the space variable.

4. The Latin indices have, unless otherwise stated, the range of i,j = 1,2,3, and the repeated indices imply the summation convention.

5. It should be noted that unlike $P_m^{-1}(s)$ the operators $P_m^{-1}(\theta)$ are in general noncommutative [13].

where $C_m^{(n)}$ are constants.

According to the correspondence principle mentioned earlier, the field equations and the boundary conditions governing the original viscoelastic problem are reducible to the field equations and boundary conditions of an associated problem in the linear theory of elasticity, with Young's modulus E and Poisson's ratio ν of the elastic solid replaced by[6]

$$\begin{Bmatrix} E(s) \\ \nu(s) \end{Bmatrix} = \left[P_2(s)P_3(s) + 2P_1(s)P_4(s)\right]^{-1} \begin{Bmatrix} 3P_4(s)P_2(s) \\ \left[P_1(s)P_4(s) - P_2(s)P_3(s)\right] \end{Bmatrix} \tag{4}$$

In (4), the elastic medium may be identified by allowing $E(s) \rightarrow E$ and $\nu(s) \rightarrow \nu$ (corresponding to $P_1(s) = P_3(s) = 1$, $P_2(s) = 2\mu$, $P_4(s) = 3K$, μ and K being the shear and the bulk moduli of the elastic solid respectively), while for special viscoelastic media such as Maxwell and Kelvin (or Voigt), the expressions (4) become respectively[7]

$$\begin{Bmatrix} E(s) \\ \nu(s) \end{Bmatrix} = \left[s + \frac{2}{3}(1+\nu)\tau^{-1}\right]^{-1} \begin{Bmatrix} sE \\ \left(s + \frac{1+\nu}{3\nu}\tau^{-1}\right)\nu \end{Bmatrix} \tag{5a}$$

and

$$\begin{Bmatrix} E(s) \\ \nu(s) \end{Bmatrix} = \left[1 + \frac{1-2\nu}{3}\tau s\right]^{-1} \begin{Bmatrix} (1+\tau s)E \\ \left(1 - \frac{1-2\nu}{3\nu}\tau s\right)\nu \end{Bmatrix} \tag{5b}$$

where τ (defined as $\tau = \eta/\mu$, η being the viscosity) denotes the relaxation time in (5a), and the retardation time in (5b).

We also recall that with reference to cylindrical polar coordinates (r being the polar radius), and with the neglect of the effect of longitudinal inertia, the differential equations for transverse axisymmetric vibrations of thin shallow elastic spherical shells are characterized by [11]

6. The correspondence principle as stated by Sternberg [3] holds also in the presence of inertia forces.

7. A more detailed account of the stress-strain law of the linear theory of viscoelasticity and its specialization to the case of Maxwell and Kelvin solids may be found in [2,3,8].

$$D\nabla^2\nabla^2 w + \frac{1}{R}\nabla^2 F = -\rho h \frac{\partial^2 w}{\partial t^2} + p(r,t)$$
$$\nabla^2\nabla^2 F - \frac{hE}{R}\nabla^2 w = 0 \tag{6}$$

and the various stress resultants and stress couples are given by

$$N_r = \frac{1}{r}F' \ , \quad N_\theta = F''$$
$$M_r = -D\left[w'' + \frac{\nu}{r}w'\right] , \quad M_\theta = -D\left[\frac{1}{r}w' + \nu w''\right] \tag{7}$$
$$Q = -D(\nabla^2 w)'$$

where w is the axial displacement, F is the Airy stress function, R is the radius of curvature, ρ is the mass density, p(r,t) is the axial component of the surface load, $D = (Eh^3)/[12(1-\nu^2)]$, h is the shell thickness, $\nabla^2(\ ) \equiv (\ )'' + 1/r(\ )'$, and prime denotes differentiation with respect to r.

Let H, L, and Γ, respectively, denote the rise of the shell segment, the characteristic length (which for spherical shells may be taken as the radius of curvature R), and the representative wave length; then, as has been shown by E. Reissner [9], the omission of the effect of longitudinal inertia leading to (6) is justified as long as (Γ/L) is of the order of unity and as long as Γ is characterized by the following classification:[8]

(a) If $\left(\frac{H}{h}\right) = O(1)$ or smaller, then

$$\Gamma^4 = \frac{1}{12(1-\nu^2)}(h\gamma)^2 \tag{8a}$$

(b) If $\frac{H}{h} \gg 1$, then

$$\Gamma = \frac{1}{1-\nu^2}\left(\frac{H}{L}\right)^2\left(\frac{\gamma^2}{L}\right) \tag{8b}$$

where

$$\gamma^{-2} = \frac{\omega^2}{(E/\rho)} \tag{8c}$$

ω being the circular frequency.

8. Actually Reissner in [9] has further pointed out that the neglect of longitudinal inertia terms is justified when $\Gamma/L = O(1)$ or smaller, but not when $\Gamma/L \gg 1$.

With zero initial conditions, i.e.

$$w(r,0) = \frac{\partial w}{\partial t}(r,0) = F(r,0) = 0 \tag{9}$$

and with an appeal to the correspondence principle, the Laplace transform of the differential equations (6) yields

$$\begin{aligned} &D(s)\,\nabla^2\nabla^2\hat{w} + \frac{1}{R}\nabla^2\hat{F} + \rho h s^2 \hat{w} = \hat{p}(r,s) \\ &\nabla^2\nabla^2\hat{F} - \frac{hE(s)}{R}\nabla^2\hat{w} = 0 \end{aligned} \tag{10}$$

Confining attention to a harmonically oscillating load of the type

$$p = p_0\, e^{i\omega t} \tag{11}$$

where $i = (-1)^{1/2}$ and p_0 is constant,[9] the right-hand side of the first of (10) becomes $p_0/(s-i\omega)$. Imposing the condition of vanishing circumferential displacement (which as in the elastostatic solution of shallow elastic spherical shells [14] demands the vanishing of the coefficients of log r in $\hat{w}$ and r^2log r in $\hat{F}$) and excluding, without loss of generality, the term in $\hat{F}$ which involves a function of s alone, the solution of (10) may be written as

$$\begin{aligned} \hat{w}(r,s) = {} & C_1(s)\,J_0[\lambda(s)r] + C_2(s)\,Y_0[\lambda(s)r] \\ & + C_3(s)\,I_0[\lambda(s)r] + C_4(s)\,K_0[\lambda(s)r] \\ & + C_5(s) \end{aligned} \tag{12a}$$

$$\begin{aligned} \hat{F}(r,s) = {} & -\frac{hE(s)}{R\lambda^2(s)}\left\{ \begin{matrix} C_1(s)\,J_0[\lambda(s)r] + C_2(s)\,Y_0[\lambda(s)r] \\ -C_3(s)\,I_0[\lambda(s)r] - C_4(s)\,K_0[\lambda(s)r] \end{matrix} \right\} \\ & + C_6(s)\ln\frac{r}{\ell} + B(s)\,r^2 \end{aligned} \tag{12b}$$

where

$$B(s) = \frac{R}{4}\left[\frac{p_0}{s-i\omega} - \rho h s^2 C_5(s)\right], \tag{12c}$$

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9. The restriction that p be independent of position is not essential to the analysis presented here.

J_0, Y_0, I_0, K_0 are Bessel functions of order zero and λ and ℓ are given by

$$\lambda^4(s) = -\frac{h}{D(s)}\left[\frac{E(s)}{R^2} + \rho s^2\right] \tag{13a}$$

$$\ell(s) = \frac{(Rh)^{1/2}}{\left[12\left(1-\nu^2(s)\right)\right]^{1/4}} \tag{13b}$$

It should be noted that the terms involving p_0 in (12) represent the exact particular solution of (10) with p specified by (11), as may be verified by substitution.

Defining $\lambda_0(s)$ through

$$\lambda(s) = e^{i\frac{\pi}{4}}\lambda_0(s) \tag{14}$$

then the Bessel functions may be expressed in terms of Kelvin functions [15] by

$$J_0(e^{i\frac{\pi}{4}}\lambda_0 r) = ber(\lambda_0 r) - i\,bei(\lambda_0 r) \tag{15a}$$

$$I_0(e^{i\frac{\pi}{4}}\lambda_0 r) = ber(\lambda_0 r) + i\,bei(\lambda_0 r) \tag{15b}$$

$$K_0(e^{i\frac{\pi}{4}}\lambda_0 r) = ker(\lambda_0 r) + i\,kei(\lambda_0 r) \tag{15c}$$

In addition, it will prove convenient to introduce the function $\mathcal{Y}_0$ (related to the negative conjugate of K_0), i.e.

$$\mathcal{Y}_0(e^{i\frac{\pi}{4}}\lambda_0 r) = \frac{2}{\pi}\left[-ker(\lambda_0 r) + i\,kei(\lambda_0 r)\right] \tag{15d}$$

whose properties may be deduced from those of the constituent Kelvin functions. Here, also we note that

$$\nabla^2\left\{\begin{array}{c} ker(\lambda_0 r) \\ kei(\lambda_0 r) \\ \mathcal{Y}_0(\lambda r) \end{array}\right\} = \lambda^2\left\{\begin{array}{c} i\,kei(\lambda_0 r) \\ -i\,ker(\lambda_0 r) \\ -\mathcal{Y}_0(\lambda r) \end{array}\right\} \tag{16}$$

While the solution (12) is appropriate to the loaded region of the shell, the solution for the unloaded region [i.e. the homogeneous solutions of the differential equations (10)] may be obtained by merely setting $p_0 = 0$ in (12); and it will be of the same form as that for free vibrations of shallow elastic shells given by Reissner [11].

In the following, the loaded ($0 \leqq r \leqq a$) and the unloaded ($r > a$) regions of the shell will be distinguished by subscripts 1 and 2, respectively. The regularity requirements at r = 0 demand that

$$r = 0; \quad \hat{w}_1 , \hat{w}_1' , \frac{1}{r} \hat{F}_1' , \hat{F}_1'' \tag{17}$$

remain finite and the continuity conditions at r = a are given by

$$r = a; \begin{cases} \hat{w}_1 = \hat{w}_2 , \quad \hat{w}_1' = \hat{w}_2' , \quad \nabla^2 \hat{w}_1 = \nabla^2 \hat{w}_2 \\ (\nabla^2 \hat{w}_1)' = (\nabla^2 \hat{w}_2)' , \quad \hat{F}_1' = \hat{F}_2' , \quad \nabla^2 \hat{F}_1 = \nabla^2 \hat{F}_2 \end{cases} \tag{18}$$

The coefficients C's in (12) are determined from (17), (18), and the boundary conditions at the edge $r = r_0$ for shallow shell segments, or from those for unlimited shells, as $r \to \infty$.

It should be clear that so far the solution (12) is transient in character, and in order to obtain the steady state solution, following Lee [6], we let $\lambda(s) \to \lambda(i\omega)$ and replace $p_0/(s-i\omega)$ by merely p_0 which, here, is equivalent to taking the inverse Laplace transform.

UNLIMITED SHALLOW VISCOELASTIC SHELLS

In the remainder of this paper, we will confine attention to unlimited shallow viscoelastic shells ($0 \leqq r \leqq \infty$) subjected to a harmonically oscillating load of the type (11) applied over the region $0 \leqq r \leqq a$ ($a/R \ll 1$). In this case the coefficients C's are determined from (17), (18), and as $r \to \infty$ from

$$r \to \infty; \left\{ \hat{w}_2 , \hat{w}_2' , (\nabla^2 \hat{w}_2)' , \hat{F}_2' , \nabla^2 \hat{F}_2 \right\} \longrightarrow 0 \tag{19}$$

Thus

$$C_1(s) = Q(s)\,\Phi_1^{-1}(s) \quad , \quad C_2(s) = -Q(s)\,\Phi_2^{-1}(s)$$

$$C_3(s) = Q(s)\,\Phi_3^{-1}(s) \quad , \quad C_4(s) = -Q(s)\,\Phi_4^{-1}(s) \tag{20a}$$

$$C_5(s) = 2\,Q(s) \quad , \quad C_6(s) = \frac{a^2 h E(s)}{R}\,Q(s)$$

where

$$Q(s) = p_0\left\{2(s-i\omega)\rho h\left[s^2 + \frac{E(s)}{\rho R^2}\right]\right\}^{-1}$$

$$\Phi_1(s) = \frac{Y_0[\lambda(s)a]}{Y_0'[\lambda(s)a]}\,J_0'[\lambda(s)a] - J_0[\lambda(s)a]$$

$$\Phi_2(s) = \frac{J_0[\lambda(s)a]}{J_0'[\lambda(s)a]}\,Y_0'[\lambda(s)a] - Y_0[\lambda(s)a] \tag{20b}$$

$$\Phi_3(s) = \frac{K_0[\lambda(s)a]}{K_0'[\lambda(s)a]}\,I_0'[\lambda(s)a] - I_0[\lambda(s)a]$$

$$\Phi_4(s) = \frac{I_0[\lambda(s)a]}{I_0'[\lambda(s)a]}\,K_0'[\lambda(s)a] - K_0[\lambda(s)a]$$

With the notation

$$\bar{w} = Re\,\hat{w}(r,s) \quad , \quad \bar{F} = Re\,\hat{F}(r,s)$$

where "Re" stands for "real part of," and omitting the details, we finally obtain the following solutions for the two regions of the shell:

$$\bar{w}_1 = Re\left\{Q(s)\left[J_0(\lambda(s)r)\,\Phi_1^{-1}(s) + I_0(\lambda(s)r)\,\Phi_3^{-1}(s) + 2\right]\right\}$$

$$\bar{F}_1 = Re\left\{\frac{hE(s)}{R\lambda^2(s)}\,Q(s)\left[\frac{r^2\lambda^2(s)}{2} - J_0(\lambda(s)r)\,\Phi_1^{-1}(s) + I_0(\lambda(s)r)\,\Phi_3^{-1}(s)\right]\right\} \tag{21a}$$

$$\bar{w}_2 = R_e\left\{ - Q(s)\left[\mathcal{Y}_0(\lambda(s)r)\,\Phi_2^{-1}(s) + K_0(\lambda(s)r)\,\Phi_4^{-1}(s) \right] \right\}$$

$$\bar{F}_2 = R_e\left\{ \frac{h E(s)}{R\lambda^2(s)}\, Q(s)\left[a^2\lambda^2(s)\ln\frac{r}{l} + \mathcal{Y}_0(\lambda(s)r)\,\Phi_2^{-1}(s) - K_0(\lambda(s)r)\,\Phi_4^{-1}(s) \right] \right\} \tag{21b}$$

where Φ_m, $(m = 1,2,3,4)$, is defined by (20b).

The choice of functions for the representation of the solution in the form (21) is motivated by their simpler properties (when subjected to the Laplacian operator), as seen from (16), as well as by

$$Y_0[\lambda(s)r] = \mathcal{Y}_0[\lambda(s)r] - i\, J_0[\lambda(s)r] \tag{22}$$

The above relation is especially useful in obtaining the real part of (21) in the case of the elastic medium. In particular, as will be seen presently, when $\omega^2 > (E/\rho)/R^2$, then [defined by (13a) with s replaced by $i\omega$], is real and with the aid of (22), the solution (21) may be expressed entirely in terms of Bessel functions. On the other hand, when $\omega^2 < (E/\rho)/R^2$, then [defined by (14)] is real and with (15d), the solution (21) may be conveniently written in terms of Kelvin functions.

For the viscoelastic medium, the representation of the solution in terms of Kelvin functions is more convenient. To this end, utilizing (15), we introduce the quantities

$$\left.\begin{aligned} \alpha(s) = {} & \mathrm{kei}(\lambda_0 a)\,\mathrm{bei}'(\lambda_0 a) + \mathrm{ber}(\lambda_0 a)\,\mathrm{ker}'(\lambda_0 a) \\ & - \mathrm{ker}(\lambda_0 a)\,\mathrm{ber}'(\lambda_0 a) - \mathrm{bei}(\lambda_0 a)\,\mathrm{kei}'(\lambda_0 a) \\ \beta(s) = {} & \mathrm{kei}(\lambda_0 a)\,\mathrm{ber}'(\lambda_0 a) + \mathrm{ker}(\lambda_0 a)\,\mathrm{bei}'(\lambda_0 a) \\ & - \mathrm{bei}(\lambda_0 a)\,\mathrm{ker}'(\lambda_0 a) - \mathrm{ber}(\lambda_0 a)\,\mathrm{kei}'(\lambda_0 a) \end{aligned}\right\} \tag{23a}$$

$$A_1(r,s) = \mathrm{bei}(\lambda_o r)\,\mathrm{kei}'(\lambda_o a) - \mathrm{ber}(\lambda_o r)\,\mathrm{ker}'(\lambda_o a)$$

$$B_1(r,s) = \mathrm{bei}(\lambda_o r)\,\mathrm{ker}'(\lambda_o a) + \mathrm{ber}(\lambda_o r)\,\mathrm{kei}'(\lambda_o a) \qquad (23b)$$

$$A_2(r,s) = \mathrm{ker}(\lambda_o r)\,\mathrm{ber}'(\lambda_o a) - \mathrm{kei}(\lambda_o r)\,\mathrm{bei}'(\lambda_o a)$$

$$B_2(r,s) = \mathrm{kei}(\lambda_o r)\,\mathrm{ber}'(\lambda_o a) + \mathrm{ker}(\lambda_o r)\,\mathrm{bei}'(\lambda_o a) \qquad (23c)$$

with the aid of which, for the steady state solution, (21) becomes

$$\bar{w}_1 = \mathrm{Re}\left\{2Q(i\omega)\left[1 + \frac{A_1\alpha + B_1\beta}{\alpha^2+\beta^2}\right]\right\}$$

$$\bar{F}_1 = \mathrm{Re}\left\{\frac{hE(i\omega)}{R\lambda_o^2(i\omega)}\,Q(i\omega)\left[\frac{r^2\lambda_o^2(i\omega)}{2} + 2\frac{A_1\beta - B_1\alpha}{\alpha^2+\beta^2}\right]\right\} \qquad (24a)$$

and

$$\bar{w}_2 = \mathrm{Re}\left\{+2\,Q(i\omega)\left[\frac{A_2\alpha - B_2\beta}{\alpha^2+\beta^2}\right]\right\}$$

$$\bar{F}_2 = \mathrm{Re}\left\{\frac{hE}{R\lambda_o^2(i\omega)}\,Q(i\omega)\left[a^2\lambda_o^2(i\omega)\ln\frac{r}{\ell} - 2\frac{A_2\beta + B_2\alpha}{\alpha^2+\beta^2}\right]\right\} \qquad (24b)$$

REDUCTION TO THE ELASTIC SOLUTION

The reduction of the above solution to that for an elastic shell is carried out by allowing $E(s) \to E$ and $\nu(s) \to \nu$; the steady state solution, as noted above, may be obtained by merely replacing $\lambda(s)$ with $\lambda(i\omega)$ and $p_o/(s-i\omega)$ with p_o in the entire solution. In the remainder of this section we consider the transition of the viscoelastic solutions (21) and (24), corresponding respectively to the two frequency ranges $\omega^2 > (E/\rho)/R^2$ and $\omega^2 < (E/\rho)/R^2$, to the known results for an unlimited elastic shell due to a point load given in [11].

Thus, in the first frequency range, in (21), we replace p_0 with $P/(\pi a^2)$, use (22) and the series representation for Bessel functions with small argument (as indicated in the Appendix) and let $a \to 0$. Omitting details of the algebra, and with the notation W and f for the amplitudes of w and F respectively, we obtain

$$W = \mathcal{R}e\left\{\frac{P}{4\pi D\lambda^2}\left[\frac{\pi}{2}Y_0(\lambda r)+K_0(\lambda r)-i\frac{\pi}{2}J_0(\lambda r)\right]\right\} \tag{25a}$$

$$f = \mathcal{R}e\left\{-\frac{P}{4\pi DR^2\lambda^4}\left[\frac{\pi}{2}Y_0(\lambda r)-K_0(\lambda r)-i\frac{\pi}{2}J_0(\lambda r) -2\ln\frac{r}{\ell}\right]\right\} \tag{25b}$$

which are identical with those given by Reissner [11] for $\omega^2 > E/(\rho R^2)$. In the second frequency range, making use of solution (24), we again replace p_0 by $P/(\pi a^2)$, use the series representation of Kelvin functions with $a \to 0$, and obtain

$$\begin{aligned} W &= \frac{P}{2\pi D\lambda_0^2}\,\mathrm{kei}(\lambda_0 r) \\ f &= -\frac{P}{2\pi D\lambda_0^4}\,\frac{Eh}{R}\left[\mathrm{ker}(\lambda_0 r)+\ln\frac{r}{\ell}\right] \end{aligned} \tag{26}$$

which are the same as the corresponding results in [11][10] for $\omega^2 < E/(\rho R^2)$.

SOLUTIONS FOR SPECIAL VISCOELASTIC MATERIALS

In this section, we deduce explicit solutions for the two viscoelastic Maxwell and Kelvin materials. Rather than set down the expressions for the steady state displacement and stress function amplitudes in their entirety, we instead

10. With the use of known relations of the type (see e.g. [15, p. 20])

$$Y_0(i^{1/2}x) = [\mathrm{bei}\,x + i\,\mathrm{ber}\,x] + \frac{2}{\pi}[-\mathrm{ker}\,x + i\,\mathrm{kei}\,x]$$

solution (25) for the range $\omega^2 < E/(\rho R^2)$ may be reduced to (26); in this connection compare with [11, Eqs. (42) to (45)].

obtain the expressions for the quantity $\lambda_0(i\omega)$ as well as

$$\frac{p_0}{2\rho h\left[s^2+\frac{E(s)}{\rho R^2}\right]} \tag{27}$$

which occur in the solutions (21) and (24). With these results and recalling that for the steady state amplitude the quantity $(s-i\omega)$ is to be replaced by unity, numerical evaluations of the solution may be readily accomplished.

For the Maxwell solid, with the aid of (5a), (13a) and (14), $\lambda_0(i\omega)$ and the expression (27) become, respectively

$$\lambda_0(i\omega)=\left[\frac{(R/\tau)}{(E/\rho)^{1/2}}\right]^{1/2}\left(\frac{1}{\ell}\right)\left\{\frac{\left[\frac{1+\nu}{3(1-\nu)}-(\tau\omega)^2\right]^2+\frac{4}{9}\left(\frac{2-\nu}{1-\nu}\right)^2(\tau\omega)^2}{\left[\frac{4}{9}(1+\nu)^2+(\tau\omega)^2\right]}\times\right.$$

$$\left.\left\langle\left[\frac{(E/\rho)}{(R/\tau)^2}-(\tau\omega)^2\right]^2+\frac{4}{9}(1+\nu)^2(\tau\omega)^2\right\rangle\right\}^{1/8} e^{i\frac{\psi_0}{4}} \tag{28a}$$

and

$$-\frac{1}{2\omega\tau}\,\frac{p_0}{E}\left[\frac{(E/\rho)^{1/2}}{(R/\tau)}\right]^2\left(\frac{R^2}{h}\right)\left\{\left[\frac{4}{9}(1+\nu)^2+(\tau\omega)^2\right]\times\right.$$

$$\left.\left[\left[(\tau\omega)^2-\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2\right]^2+\frac{4}{9}(1+\nu)^2(\tau\omega)^2\right]^{-1}\right\}^{1/2} e^{i\psi_1} \tag{28b}$$

where

$$\psi_0=\tan^{-1}\frac{\frac{2}{3}\left(\frac{2-\nu}{1-\nu}\right)\tau\omega}{\frac{1+\nu}{3(1-\nu)}-(\tau\omega)^2}-2\tan^{-1}\frac{\tau\omega}{\frac{2}{3}(1+\nu)}$$

$$+\tan^{-1}\frac{\frac{2}{3}(1+\nu)\tau\omega}{\frac{(E/\rho)}{(R/\tau)^2}-(\tau\omega)^2} \tag{29a}$$

$$\psi_1 = \tan^{-1} \frac{\tau\omega}{\frac{2}{3}(1+\nu)} - \tan^{-1} \frac{(\tau\omega)^2 - \left[\frac{(E/\rho)^{1/2}}{R/\tau} \right]^2}{\frac{2}{3}(1+\nu)\tau\omega} \tag{29b}$$

In the case of Kelvin solid specified by (5b), expressions corresponding to (28) and (29) are given by

$$\lambda_0(i\omega) = \left[\frac{2}{(1-\nu)}\right]^{1/4} \left[\frac{R/\tau}{(E/\rho)^{1/2}} \right]^{\frac{1}{2}} \left(\frac{1}{\ell}\right) \left\{ \frac{\left[\frac{3(1-\nu)}{2(1-2\nu)}\right]^2 + (\tau\omega)^2}{\left[\left(\frac{3}{1-2\nu}\right)^2 + (\tau\omega)^2\right]^2 (1+\tau^2\omega^2)} \times \right.$$

$$\left\langle \left(\frac{3}{1-2\nu}\right)^2 \left[\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2 - (\tau\omega)^2 \right] \right. \tag{30a}$$

$$\left. \left. + (\tau\omega)^2 \left[\frac{3}{1-2\nu} \left(\frac{(E/\rho)^{1/2}}{R/\tau} \right)^2 - (\tau\omega)^2 \right] \right\rangle \right\}^{1/8} e^{i\frac{\psi_2}{4}}$$

and

$$\frac{p_0}{2\rho h} \tau^2 \left\{ \frac{\left(\frac{3}{1-2\nu}\right)^2 \left\{ \left[\frac{(E/\rho)^{1/2}}{R/\tau} \right]^2 - (\tau\omega)^2 \right\} + (\tau\omega)^2 \left[\frac{3}{1-2\nu} \left(\frac{(E/\rho)^{1/2}}{R/\tau} \right)^2 - (\tau\omega)^2 \right]^2}{\left(\frac{3}{1-2\nu}\right)^2 + (\tau\omega)^2} \right\}^{-1/2} e^{-i\psi_3}$$

(30b)

where

$$\psi_2 = \tan^{-1} \frac{\tau\omega}{\left[\frac{3(1-\nu)}{2(1-2\nu)}\right]} + \tan^{-1} \frac{\left[\frac{3}{1-2\nu}\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2 - (\tau\omega)^2\right]\tau\omega}{\frac{3}{1-2\nu}\left[\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2 - (\tau\omega)^2\right]} \tag{31a}$$

$$- 2\tan^{-1} \frac{\tau\omega}{\frac{3}{(1-2\nu)}} - \tan^{-1}(\tau\omega)$$

$$\psi_3 = \tan^{-1} \frac{\left[\frac{3}{1-2\nu}\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2 - (\tau\omega)^2\right]\tau\omega}{\left(\frac{3}{1-2\nu}\right)\left[\left(\frac{(E/\rho)^{1/2}}{R/\tau}\right)^2 - (\tau\omega)^2\right]} - \tan^{-1} \frac{\tau\omega}{\frac{3}{1-2\nu}}$$

(31b)

NUMERICAL RESULTS. DISCUSSION

As an example of the foregoing solution, we consider an unlimited shallow shell with $R/h = 30$ subjected to an oscillating load of the type (11) distributed over the circular region $0 \leqq r/\ell \leqq a/\ell = 0.256$; and then we obtain numerical results for Maxwell and Kelvin materials [characterized by $(R/\tau)/(E/\rho)^{1/2} = 5 \times 10^{-5}$], as well as for the case of the elastic medium. With the notation

$$\sigma_{r,b} = \frac{6M_r}{h^2}, \quad \sigma_{\theta,b} = \frac{6M_\theta}{h^2}$$
$$\sigma_{r,m} = \frac{N_r}{h}, \quad \sigma_{\theta,m} = \frac{N_\theta}{h} \tag{32}$$

where subscripts b and m refer to the bending and membrane stresses, respectively, the plots of amplitudes of axial displacement, maximum bending stresses, and membrane stresses versus (r/ℓ) for all three media and for two values of $(R\omega)/(E/\rho)^{1/2}$ (i.e. 0.8717 and 1.1136) are displayed in Figs. 1 to 5. We note that the amplitudes of membrane stresses in Figs. 3(b) and 4(b), although small at the origin, do not vanish at $r = 0$, and that for both values of $(R\omega)/(E/\rho)^{1/2}$ the amplitude of $(\sigma_{\theta,m})$ is 180° out of phase with that of $\sigma_{r,m}$ for both Maxwell and Kelvin media. In comparing the amplitudes of the stresses shown in Figs. 3 to 5, it may be of interest to note that the stresses of the elastostatic solution of the corresponding problem, shown in Fig. 6, have the same character as those of Fig. 5(a); other features of the results in Figs. 1 to 5 are self-explanatory.

Two interesting characteristics of the steady state solution (24) exemplified in Figs. 1 to 4, should be emphasized. The first is the difference between the displacement and the difference between the stresses for the the two frequency ranges in the case of the Maxwell material. The second is that a corresponding difference is not to be found for the Kelvin material. As discussed below both of these characteristics are dependent upon the nature of the Kelvin functions of complex argument in solution (24).

The argument of the Kelvin functions involved in the solution may be specialized to the Maxwell and Kelvin materials according to (28) and (30) respectively. Thus, it is seen that in the case of the Maxwell material the phase of the complex argument is nearly zero for the "low" frequency range $[\omega^2 < (E/\rho)/R^2]$, and nearly $\pi/4$ for the "high" frequency range $[\omega^2 > (E/\rho)/R^2]$, while in the case of the Kelvin material the phase of the arguments is nearly $\pi/8$ for both frequency ranges. Turning to the relations between the Bessel and Kelvin functions of complex argument, such as (15)

$$\operatorname{ber} z = \frac{1}{2}\left[J_0\left(e^{-i\frac{\pi}{4}}z\right) + J_0\left(e^{i\pi/4}z\right)\right]$$

which occur in the solutions, and again considering the two materials separately, it is seen that for the Maxwell material excited in the "low" frequency range, the Bessel functions must be evaluated along a path which approaches the lines $\pi/4$ or $-\pi/4$ asymptotically in the complex plane. When the Maxwell material is excited in the "high" frequency range, on the other hand, the Bessel functions must be evaluated either along a line which approaches the real axis asymptotically for increasing z or approaches the imaginary axis asymptotically for increasing z. The distinctly different behavior of the Bessel functions along the imaginary axis as compared to that along the real axis accounts for the different stress and displacement distributions in the two frequency ranges.

Since the solution for the Kelvin material does not exhibit a change of phase in the argument as the exciting frequency moves from one frequency range to another, the stresses and displacements are not qualitatively distinct for the two ranges. The dissimilarity of displacement and stresses for the Kelvin material as compared to those of Maxwell for either frequency range is again due to the phase angle of the argument. From the properties of the Bessel functions occurring in (21), one may conclude, in fact, that for any linear viscoelastic material the qualitative nature of the steady state stresses and displacement is determined exclusively by the behavior of the phase angle in the argument of the Kelvin functions.

REFERENCES

1. T. Alfrey, "Non-homogeneous Stresses in Viscoelastic Media," Quart. Appl. Math., 2, 113-119 (1944).

2. E. H. Lee, "Stress Analysis in Visco-elastic Bodies," Quart. Appl. Math., 13, 183-190 (1955).

3. E. Sternberg, "On Transient Thermal Stress in Linear Viscoelasticity," Proc. 3rd U. S. Natl. Congr. Appl. Mech., pp. 673-683 (1958).

4. E. H. Lee, and I. Kanter, "Wave Propagation in Finite Rods of Viscoelastic Material," J. Appl. Phys., 24, 1115-1122 (1953).

5. E. H. Lee, and J. A. Morrison, "A Comparison of the Propagation of Longitudinal Waves in Rods of Viscoelastic Materials," J. Polymer Sci., 19, 93-110 (1956).

6. E. H. Lee, "Stress Analysis in Viscoelastic Materials," J. Appl. Phys., 27, 665-672 (1956).

7. D. S. Berry, and S. C. Hunter, "The Propagation of Dynamic Stresses in Viscoelastic Rods," J. Mech. Phys. Solids, 4, 72-95 (1956).

8. P. M. Naghdi, and W. C. Orthwein, "Response of Shallow Viscoelastic Spherical Shells to Time-Dependent Axisymmetric Load," to be published.

9. E. Reissner, "On Transverse Vibrations of Thin Shallow Elastic Shells," Quart. Appl. Math., 13, 169-176 (1955).

10. K. Marguerre, "Zur Theorie der Gekrümmten Platte grosser Formänderung," Proc. 5th Intern. Congr. Appl. Mech., 93-101 (1938).

11. E. Reissner, "On Axi-symmetrical Vibrations of Shallow Spherical Shells," Quart. Appl. Math., 13, 279-290 (1955).

12. R. V. Churchill, Operational Mathematics, 2nd ed., McGraw-Hill Book Co., 1958.

13. J. P. Dalton, Symbolic Operators, Witwatersrand University Press (Johannesburg), 1954.

14. E. Reissner, "Stress and Small Displacements of Shallow Spherical Shells," J. Math. and Phys., 25, 80-85, 279-300 (1945); 27, 240 (1948).

15. W. Magnus, and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea Publishing Co., 1949.

For small values of the argument (λa), the Bessel functions and their derivatives become

$$J_0(\lambda a) \simeq 1 \quad , \quad I_0(\lambda a) \simeq 1$$

$$J_0'(\lambda a) \simeq -\frac{a\lambda^2}{2} \quad , \quad I_0'(\lambda a) \simeq \frac{a\lambda^2}{2}$$

$$Y_0(\lambda a) \simeq \frac{2}{\pi}\left[\bar{\gamma} + \ln\frac{a\lambda}{2}\right] \quad , \quad Y_0'(\lambda a) \simeq \frac{2}{\pi a}$$

$$K_0(\lambda a) \simeq -\left[\bar{\gamma} + \ln\frac{a\lambda}{2}\right] \quad , \quad K_0'(\lambda a) \simeq -\frac{1}{a}$$

$$\frac{J_0(\lambda a)}{J_0'(\lambda a)} \simeq -\frac{2}{a\lambda^2} \quad , \quad \frac{I_0(\lambda a)}{I_0'(\lambda a)} \simeq \frac{2}{a\lambda^2}$$

$$\frac{Y_0(\lambda a)}{Y_0'(\lambda a)} \simeq a\left[\bar{\gamma} + \ln\frac{a\lambda}{2}\right]$$

$$\frac{K_0(\lambda a)}{K_0'(\lambda a)} \simeq a\left[\bar{\gamma} + \ln\frac{a\lambda}{2}\right]$$

where $\bar{\gamma}$ is Euler's constant.

Likewise for small values of the argument (λa), the Kelvin functions and their derivatives may be written as

$$\mathrm{ber}(\lambda_0 a) = 1 - \frac{\lambda_0^4 a^4}{2^6} + \cdots$$

$$\mathrm{bei}(\lambda_0 a) = \frac{\lambda_0^2 a^2}{2^2} - \frac{\lambda_0^6 a^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$\mathrm{ker}(\lambda_0 a) = -\left[\bar{\gamma} + \ln\frac{\lambda a}{2}\right] + \frac{\pi \lambda_0^2 a^2}{2^4} + \cdots$$

$$\mathrm{kei}(\lambda_0 a) = -\frac{\pi}{4} + \left[1 - \bar{\gamma} - \ln\frac{\lambda_0 a}{2}\right]\frac{\lambda_0^2 a}{4} + \cdots$$

$$\mathrm{ber}'(\lambda_0 a) = -\frac{\lambda_0^4 a^3}{2^4} + \frac{\lambda_0^8 a^7}{2^9 \cdot 6^2} - \cdots$$

$$\mathrm{bei}'(\lambda_0 a) = \frac{\lambda_0^2 a}{2} - \frac{\lambda_0^6 a^5}{2^6 \cdot 6} + \cdots$$

$$\mathrm{ker}'(\lambda_0 a) = -\frac{1}{a} + \frac{\pi \lambda_0^2 a}{8} + \cdots$$

$$\mathrm{kei}'(\lambda_0 a) = \left[\frac{1}{2} - \bar{\gamma} - \ln\frac{\lambda_0 a}{2}\right]\frac{\lambda_0^2 a}{2} + \frac{\pi \lambda_0^4 a^3}{2^6} + \cdots$$

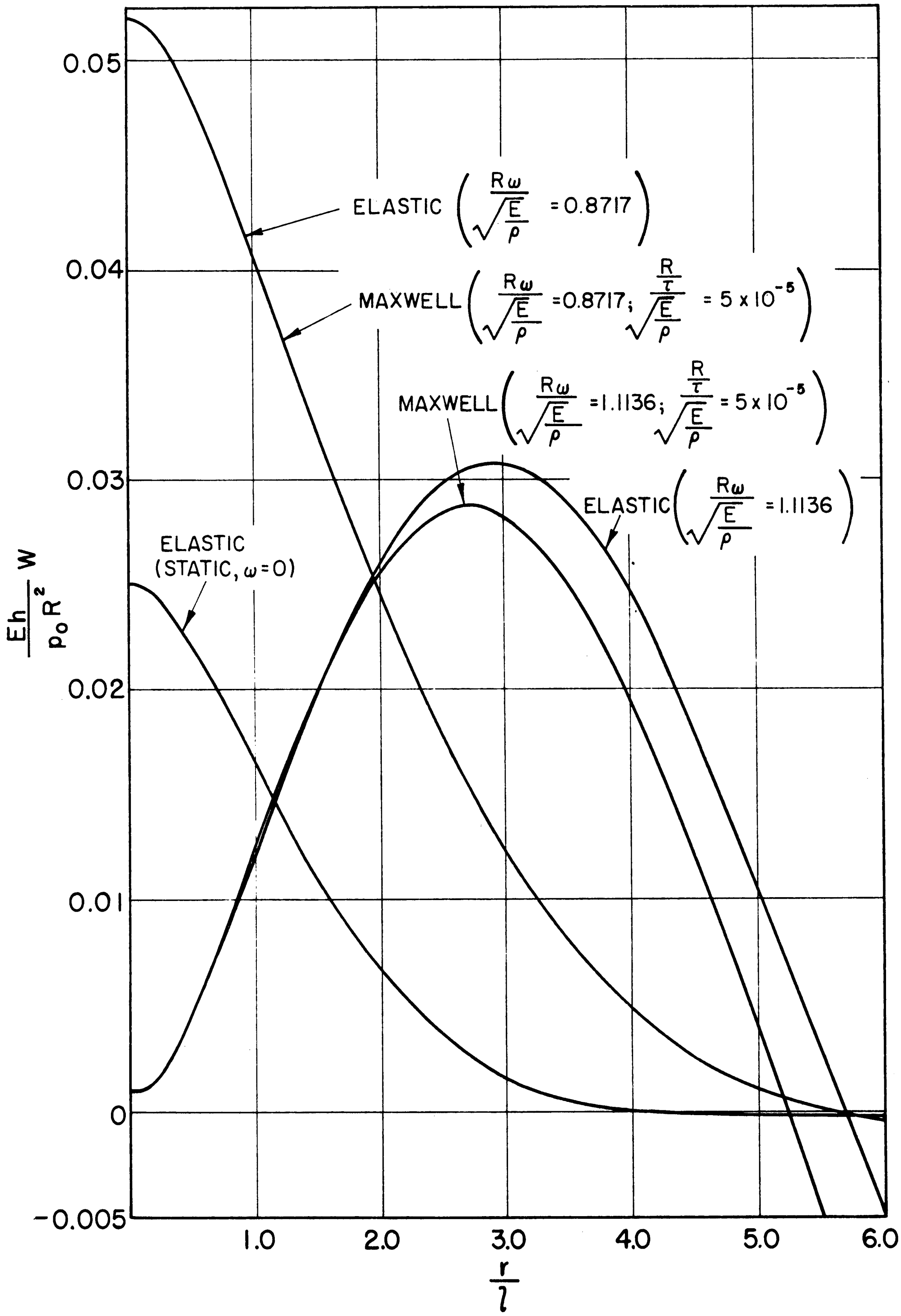

Fig. 1. Axial displacement for Maxwell and elastic materials.

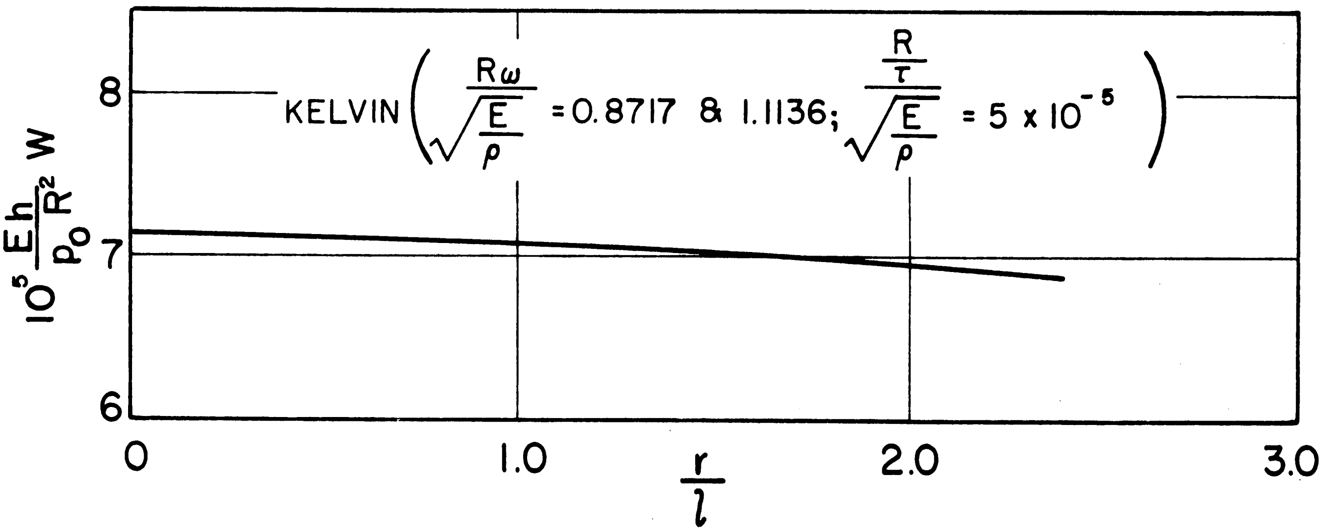

Fig. 2. Axial displacement for Kelvin material.

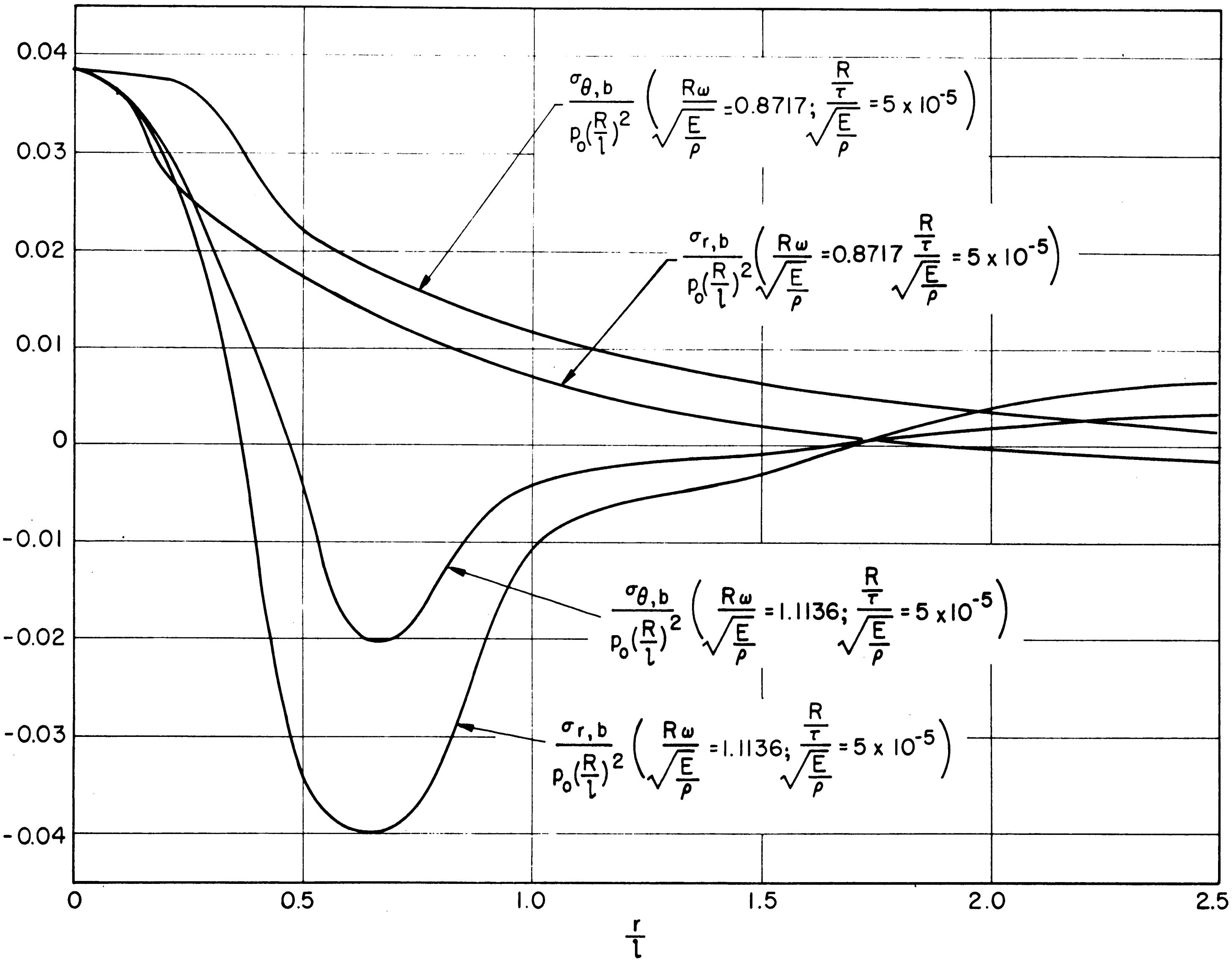

Fig. 3(a). Maximum bending stresses for Maxwell material.

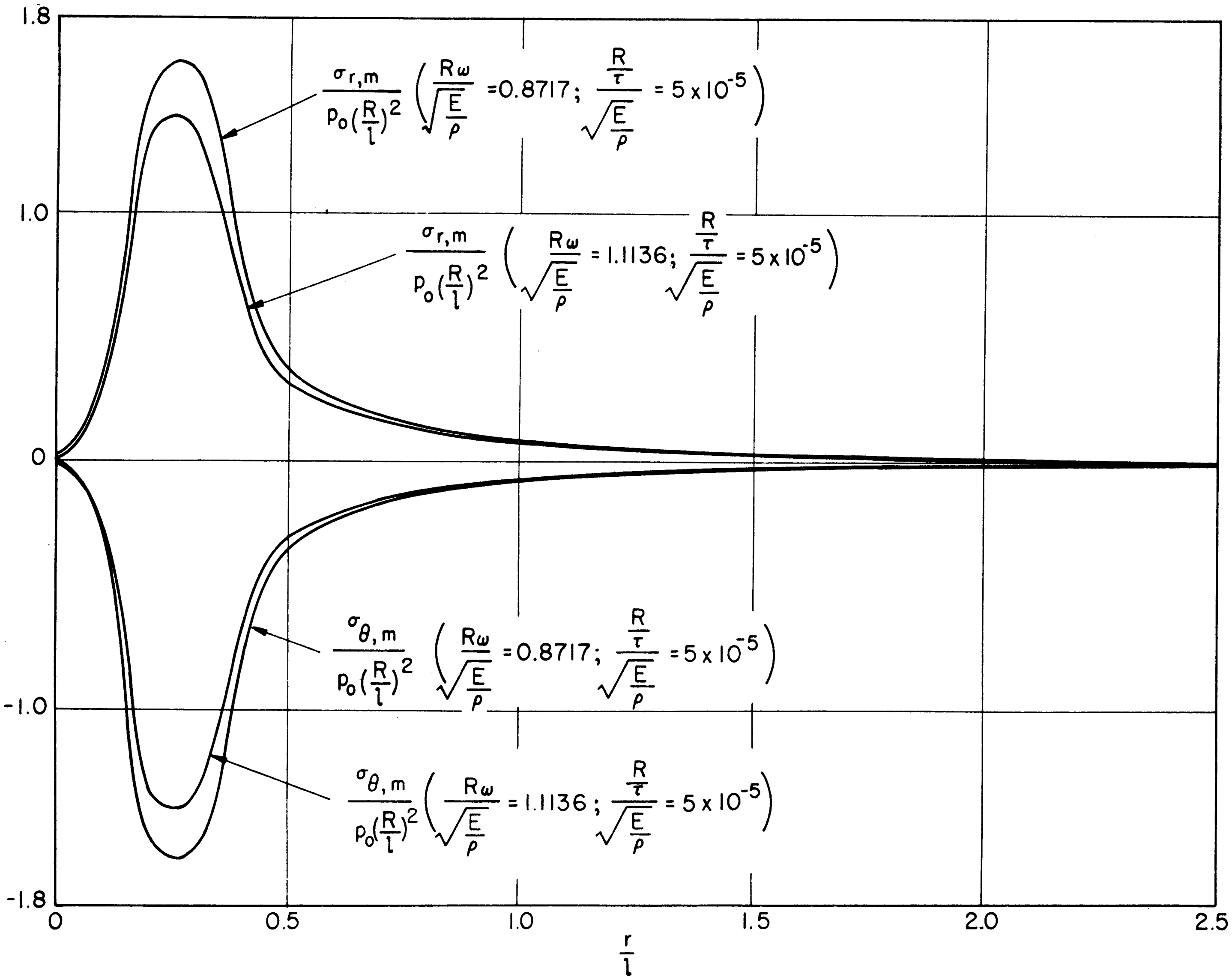

Fig. 3(b). Membrane stresses for Maxwell material.

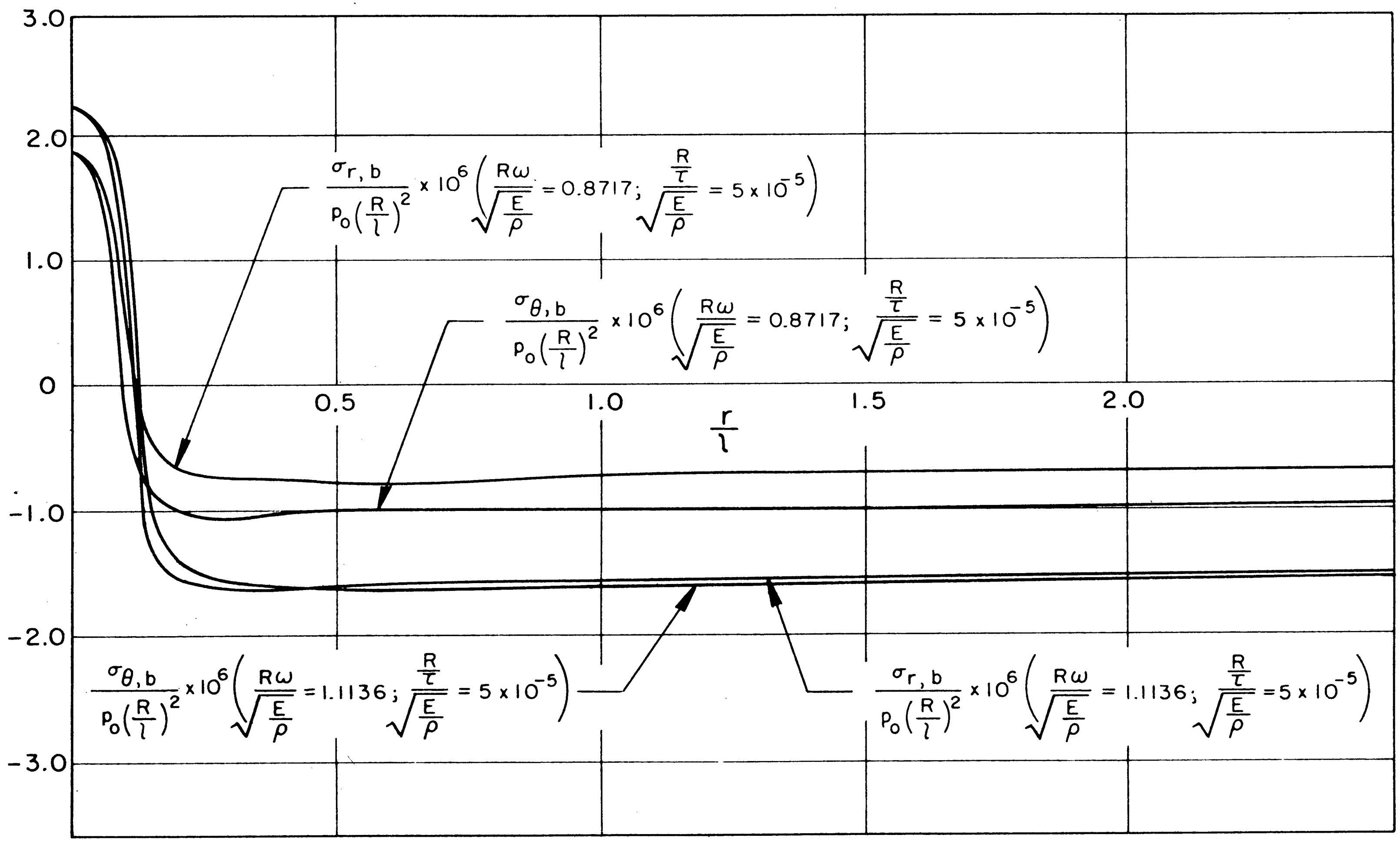

Fig. 4(a). Maximum bending stresses for Kelvin material.

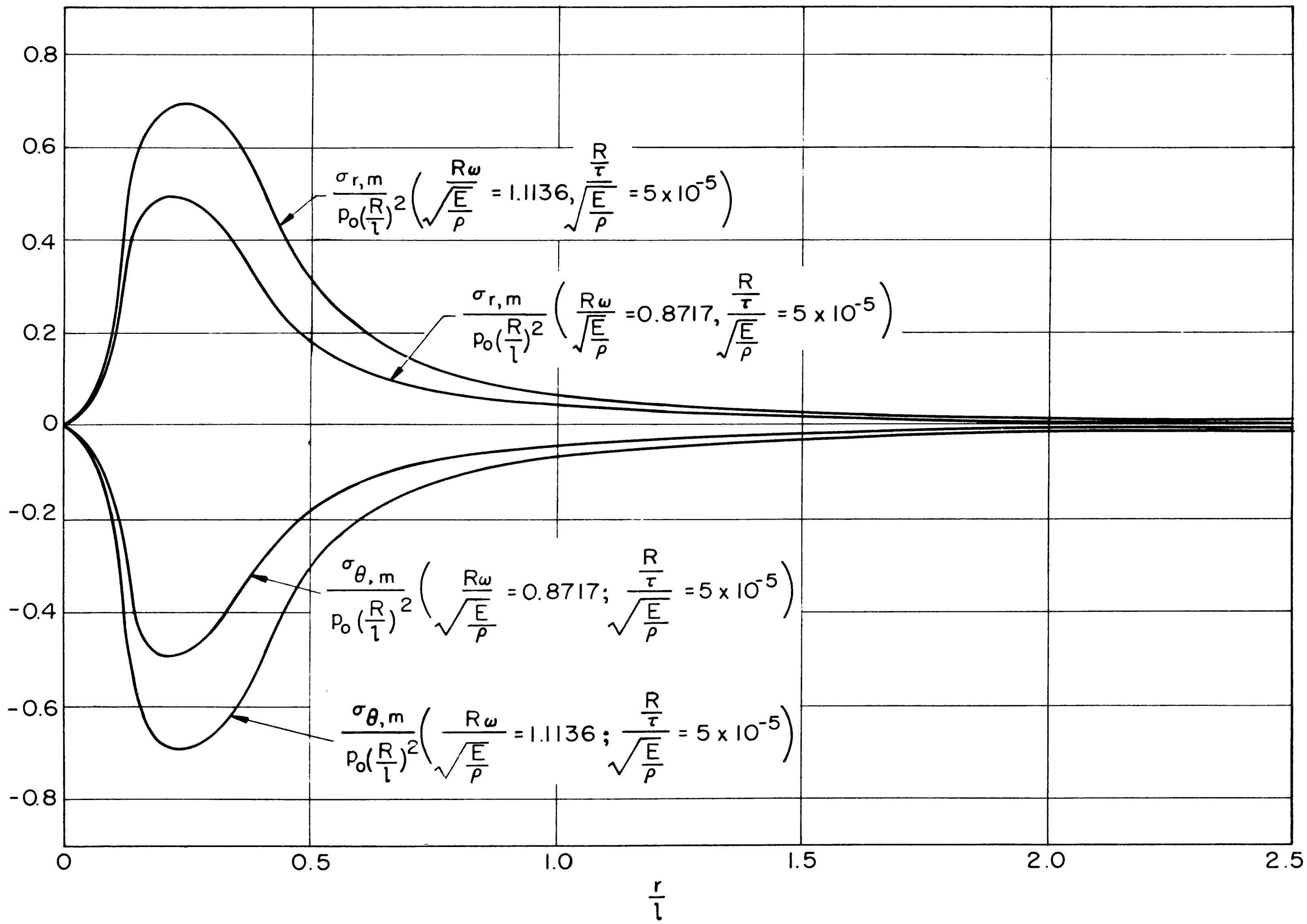

Fig. 4(b). Membrane stresses for Kelvin material.

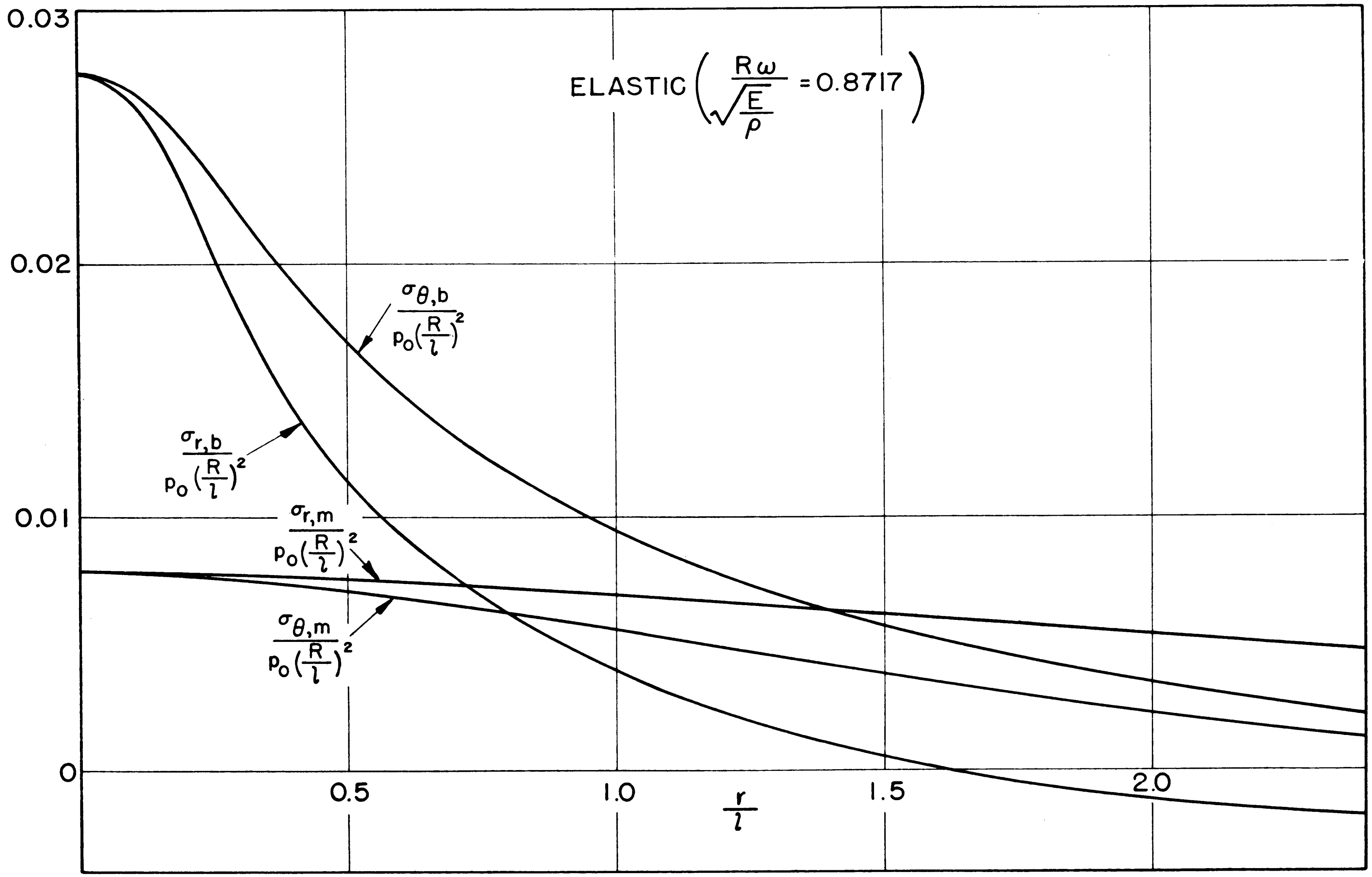

Fig. 5(a). Maximum bending and membrane stresses for elastic material.

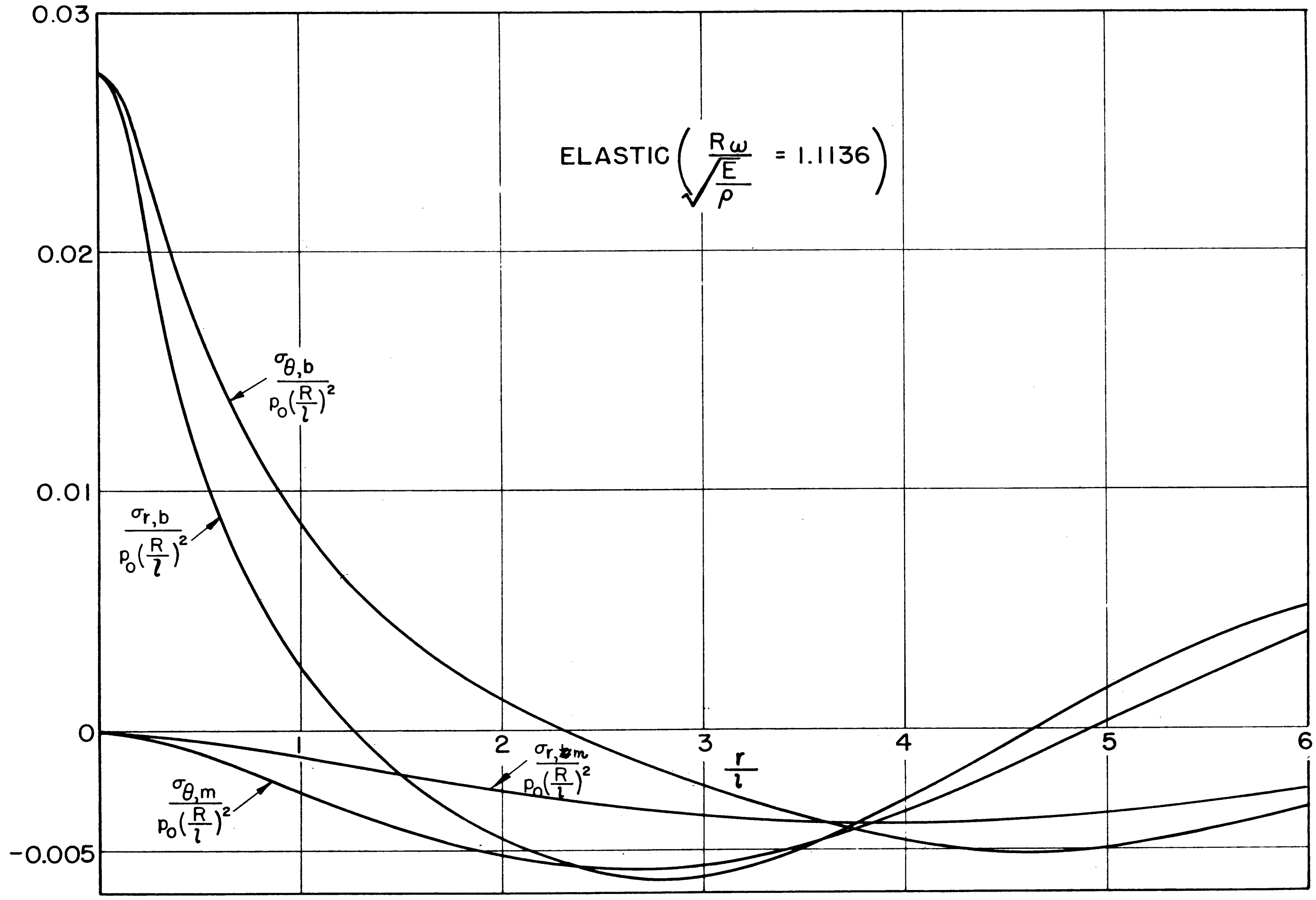

Fig. 5(b). Maximum bending and membrane stresses for elastic material.

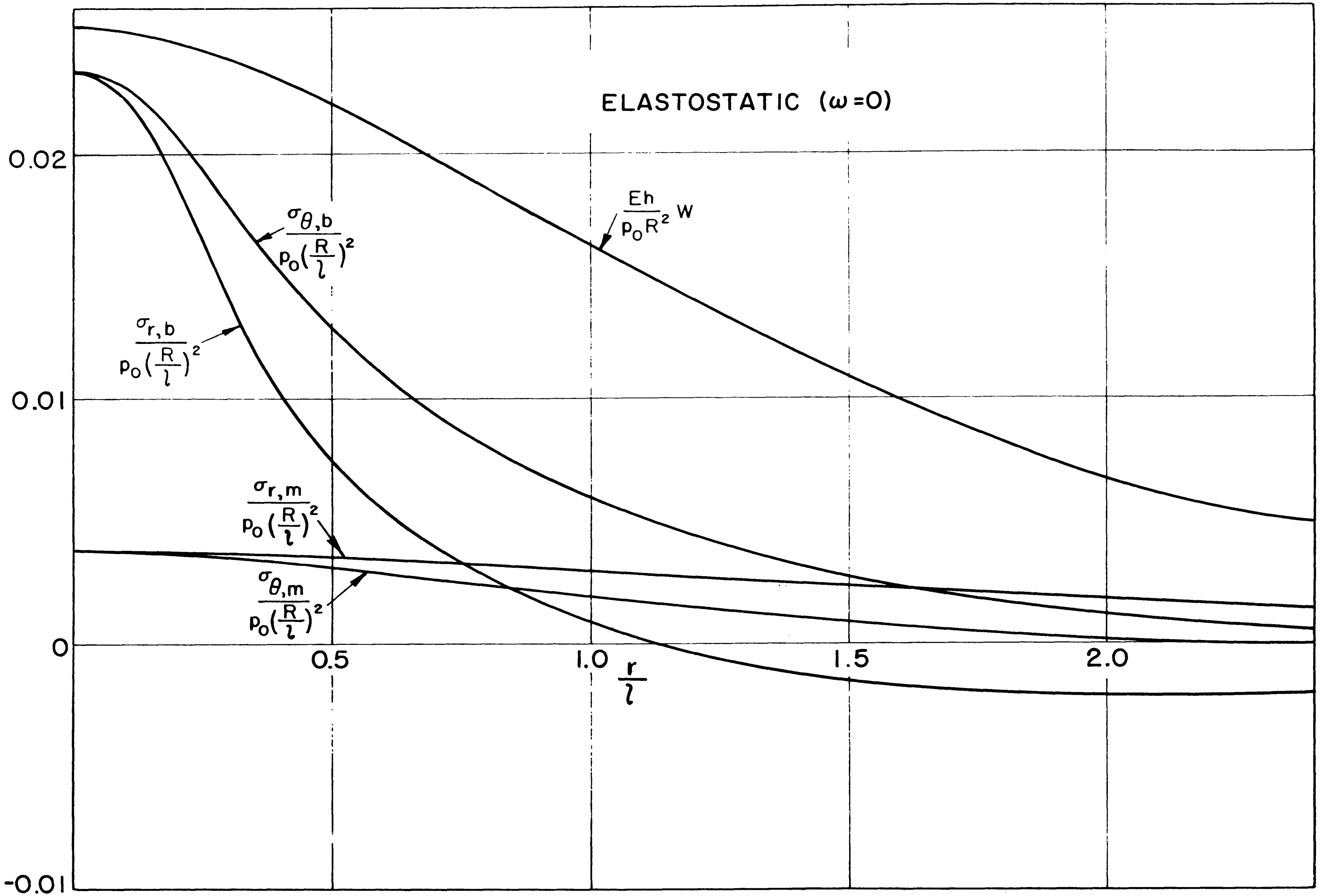

Fig. 6. **Maximum bending and membrane** stresses of the elastostatic solution ($\omega = 0$).