A graph-theoretic approach to the analysis
of international relations

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In their analysis of international relations, Bernstein and Weldon (1968) present their data in the form of four 35 × 35 matrices. They call these

A = Reciprocal Defense Relations Matrix
B = Organization Matrix
C = Trade Matrix
D = Embassy Matrix

In the usual manner of using a matrix to represent the adjacency relation in a graph, directed graph, or network (see Harary, Norman, and Cartwright, 1965), the i'th row and the i'th column stand for the i'th nation in their sample of 35 nations (as named in Table 1), and the i, j entry of the matrix (in the i'th row and the j'th column) tells the relationship of the i'th nation to the j'th. Thus, in matrix A (Bernstein and Weldon, p. 177), the i,j entry, denoted $a(i,j)$, is 1 if nation i has a "reciprocal defense relation" with nation j, $i \neq j$, and

is 0 otherwise. The diagonal entries are hyphens to indicate irrelevance.

Matrices C and D are also binary (with entries 0 and 1). The i,j entry is 1 in C if nation j is a "principal trader" of nation i and is 1 in D if nation i has an embassy in nation j. The i,j entry of matrix B is the percentage of possible times that nation i is in organizations with nation j.

Surprisingly, A is not a symmetric matrix. For example, the United States has a reciprocal defense relation with Ethiopia but Ethiopia does not have one with the United States. There are also other nonreciprocal pairs (i,j) of nations in which i has a "reciprocal defense relation" with j. The complete list of these pairs obtained from matrix A (with the corresponding i,j unit entry) is given in Table 2.

In light of Table 2, we contend that "reciprocal defense relation" is a misnomer and suggest that the name *defense commitment relation* would be more appropriate. We find it convenient to change A into a symmetric matrix in order to determine international cliques. Two different methods are used and both involve changing the nonsymmetric re-

TABLE 1

NATIONS IN THE SAMPLE*

1. United States	13. West Germany	25. U. of S. Africa
2. Bolivia	14. Ireland	26. Pakistan
3. Ecuador	15. Norway	27. Japan
4. Peru	16. Sweden	28. Burma
5. Colombia	17. Austria	29. Philippines
6. Dominican Rep.	18. Libya	30. Afghanistan
7. Honduras	19. Egypt	31. Thailand
8. Panama	20. Syria	32. Australia
9. Brazil	21. Iran	33. Hungary
10. Uruguay	22. Jordan	34. Yugoslavia
11. Denmark	24. Ghana	35. USSR

*From Bernstein and Weldon (1968, p. 161).

TABLE 2

THE NONRECIPROCAL "RECIPROCAL DEFENSE RELATIONSHIPS"

Nation i (is committed to defend)	Nation j	Matrix entry (i,j)
United States	Ethiopia	(1,23)
United States	Ghana	(1,24)
Iran	United Kingdom	(21,11)
Ghana	Australia	(24,32)
U. of S. Africa	Australia	(25,32)
Pakistan	Australia	(26,32)
Philippines	Australia	(27,32)
Thailand	Australia	(31,32)
USSR	Ghana	(35,24)
USSR	U. of S. Africa	(35,25)

lation given by matrix A into a symmetric relation.

As an illustration of the methods, Figure 1(a) represents the digraph[2] for the reciprocal defense relation with respect to the United States, Ghana, Australia, and Ethiopia. That is, a directed line is drawn from the point v_i, representing nation i to v_j if and only if the entry $a(i,j)$ is 1.

In terms of its digraph, a relation is symmetric if and only if every pair of points in the relation is joined by a symmetric pair of

directed lines, or equivalently, in terms of the adjacency matrix, if and only if $a(i,j)$ and $a(j,i)$ have the same value (0 or 1). From Figure 1(a) we see that, although v_1 and v_{32} are joined by a symmetric pair, the other pairs of points in the relation are not. To obtain a symmetric relation, one of two operations can be performed: either join every pair of points in the relation by a symmetric pair of directed lines, or remove all directed lines which do not form part of a symmetric pair. The first operation yields the *symmetric closure relation* and the second the (maximal) *symmetric subrelation;* these are shown in Figures 1(b) and 1(c) respectively for the relation in Figure

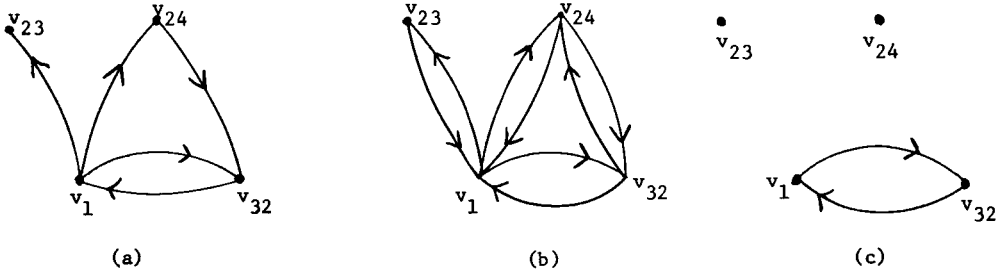[2]In every figure, the points are numbered according to Table 1.

FIG. 1. An illustration of the symmetric closure relation and symmetric subrelation for the reciprocal defense relations.

1(a). These two operations are equivalent to changing certain entries in the adjacency matrix from 0 to 1 or 1 to 0, respectively, so that the resultant matrix is symmetric in both cases. Consequently, matrix A was made symmetric by setting the ten i,j unit entries of the matrix cited in Table 2 to zero (for the symmetric subrelation or by setting the corresponding ten j,i zero entries to one (for the symmetric closure relation).

Our object is to analyze matrix A in terms of the corresponding graph G (compare Harary, 1961, where a similar study was made for the nations involved in the Middle East in 1959) by exploiting the subgraphs of G. In the *complete graph* K_p (see Harary, 1969, for more details concerning the definitions and notation in graph theory used here) with p points, every pair of distinct points are *adjacent*, that is, they are joined by an undirected line (see Figure 2). A (connected) *component* of a graph is illustrated by the graph of Figure 3, which has three compo-

nents. Note that a component is not necessarily complete. A *clique* of a graph G is defined as a maximal complete subgraph, that is, a subgraph which is no longer complete when any other points of G are included. For example, the graph in Figure 4 has five cliques formed by the following sets of points: {a,b,c,d,e}, {a,f,d}, {g,d,h}, {a,i}, {i,j,k,l}. A *cutpoint* of a graph is a point whose removal increases the number of components. For example, the graph in Figure 4 has three cutpoints: *d, a,* and *i*. A *bridge* is a line whose removal increases the number of components. The graph in Figure 4 has one bridge, namely the line joining points *a* and *i*.

The graph[3] representing the symmetric subrelation of the relation of matrix A is shown in Figure 5, and the cliques of the largest component of the graph are listed in Table 3.

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[3]For simplicity, the interior lines of the K_{10} clique have been omitted.
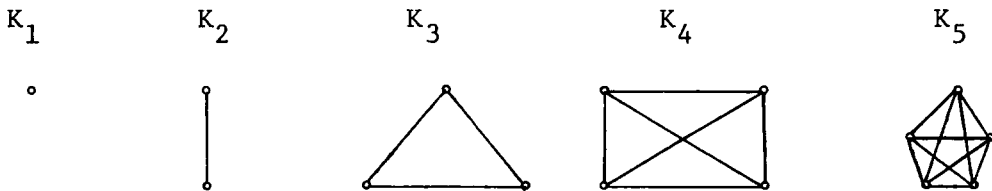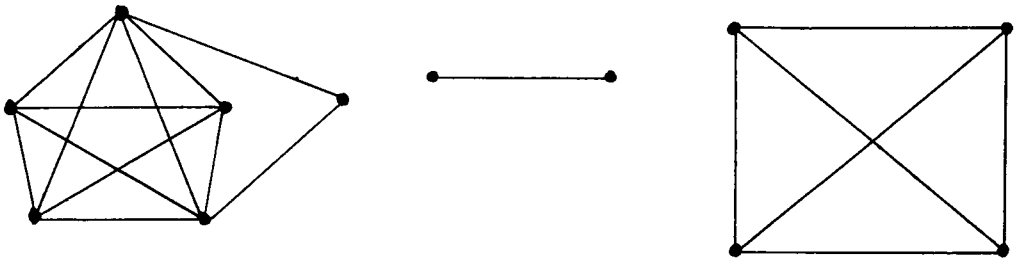

FIG. 2.   The smallest complete graphs.
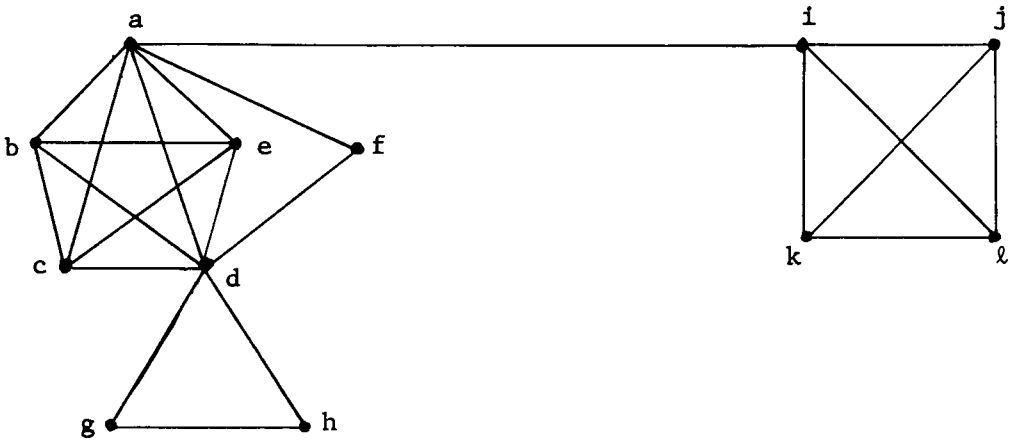
FIG. 3. A graph with three components.

FIG. 4. A graph with five cliques.

Similarly, the graph[4] for the symmetric clo-sure relation appears in Figure 6 and the cliques in Table 4.

The largest component of the graph of Figure 5 has two cutpoints, the United States and the USSR. It also has two bridges, one between the United States and Japan (a bridge across the Pacific) and the other between the USSR and Hungary. The only nation which is adjacent to both the United States and the USSR is the Union of South Africa! The

[4]In order to afford a clearer view of the overall structure of the graph, all but two of the interior lines of the two K_6 cliques have been omitted (the two lines being retained since they also form part of a K_4 clique) and the interior lines of the K_{10} clique.

TABLE 3

THE CLIQUES OF THE LARGEST COMPONENT OF THE GRAPH OF FIGURE 5

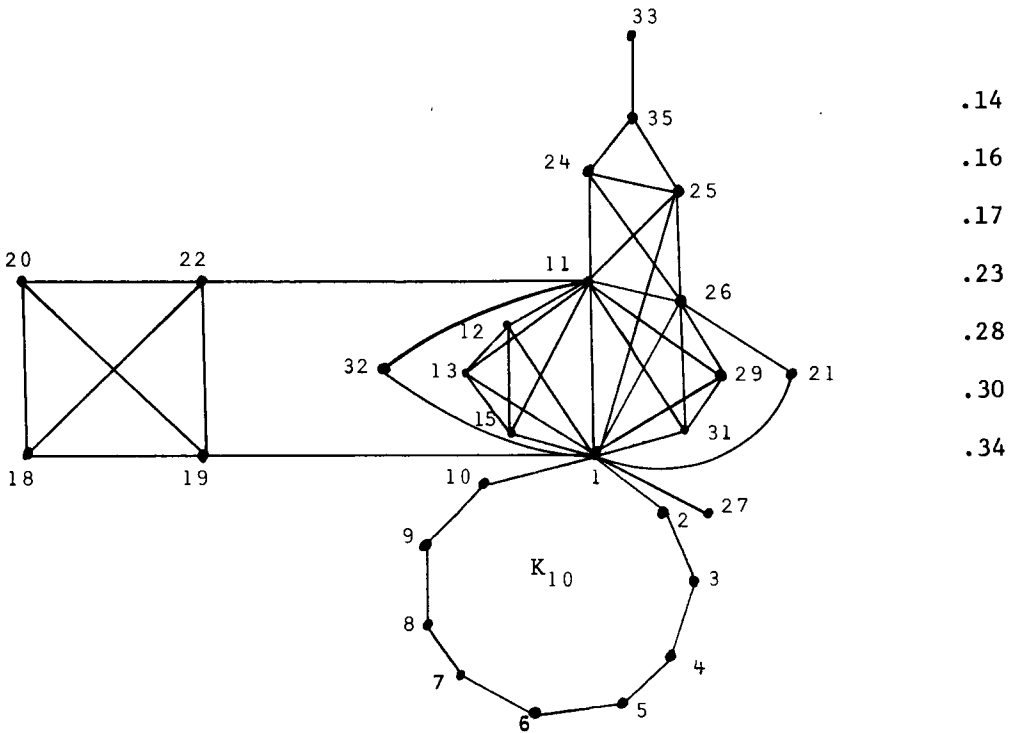| Description	Nations in the clique
K_{10}	1,2,3,4,5,6,7,8,9,10
K_5	1,11,12,13,15
K_5	1,11,26,29,31
K_4	1,11,25,26
K_4	11,24,25,26
K_4	18,19,20,22
K_3	1,11,32
K_3	1,11,21
K_3	24,25,35
K_2	1,19
K_2	1,27
K_2	11,22
K_2	33,35

FIG. 5. The graph representing the symmetric subrelation of the reciprocal defense relation.

United States is a member of the largest number of cliques (eight out of 13), followed by

TABLE 4

THE CLIQUES OF THE LARGEST COMPONENT
OF THE GRAPH OF FIGURE 6

Description	Nations in the clique
K_{10}	1,2,3,4,5,6,7,8,9,10
K_6	1,11,26,29,31,32
K_6	1,11,24,25,26,32
K_5	1,11,12,13,15
K_4	18,19,29,22
K_4	1,11,21,26
K_3	24,25,35
K_2	1,19
K_2	1,23
K_2	1,27
K_2	11,22
K_2	33,35

the United Kingdom (seven out of 13). These two nations also have the greatest number of relationships (21 and 11, respectively). The largest component of the graph of Figure 6 has the same two cutpoints and bridges as well as a bridge between the United States and Ethiopia. Only Ghana and the Union of South Africa are adjacent to both the United States and the USSR! As in the other graph, the United States is a member of the largest number of cliques (eight out of 12) and has the greatest number of relationships (23) followed by the United Kingdom (five out of 12 cliques and 11 relationships).

Only a few of the many properties of a graph have been described to demonstrate its applicability to the study of international relations. Nevertheless, it should be clear from these illustrative observations that the
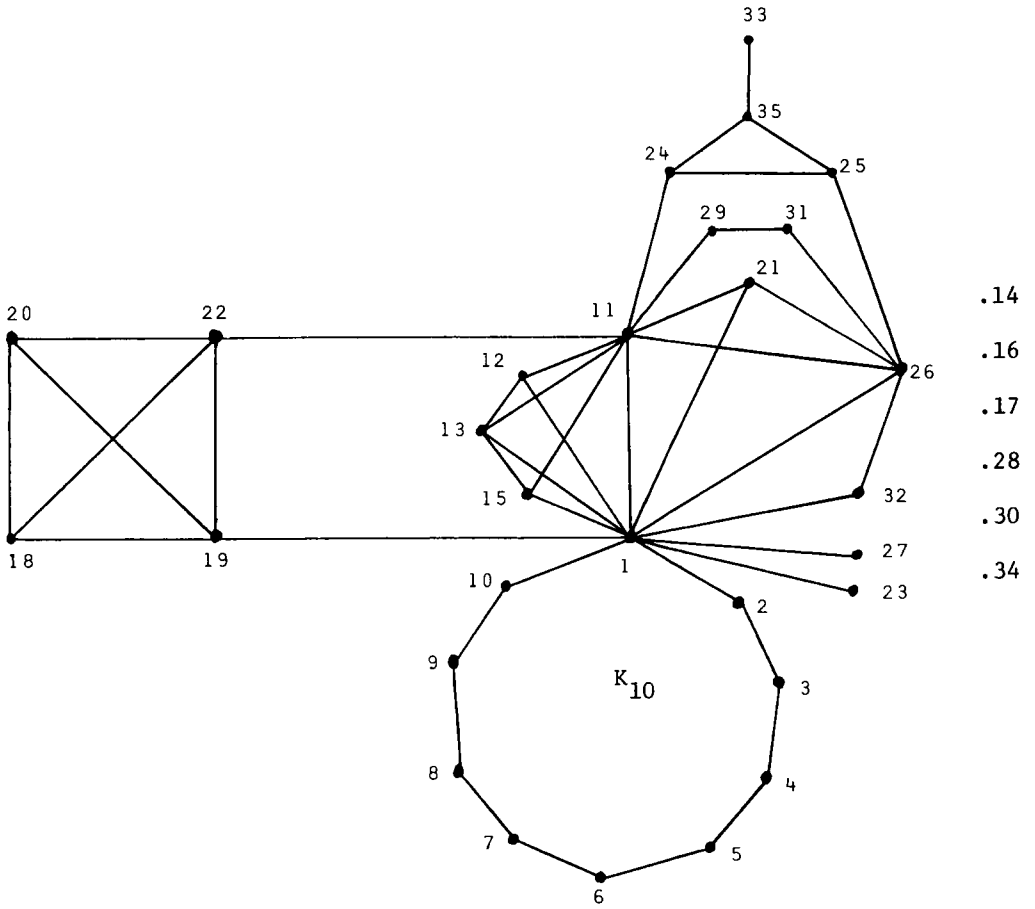
FIG. 6.   The graph representing the symmetric closure relation of the reciprocal defense relation.
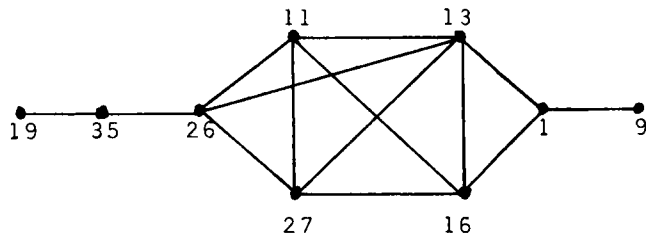

FIG. 7.   Graph redrawn from Bernstein and Weldon (1968).

analytical power provided by a structural approach using concepts from graph theory is far more useful than the mere listing of relationships in matrix form. In fact, Bernstein and Weldon (1968, p. 172) themselves say that "the patterns may be more easily visualized in schematic form," and proceed to construct a "schematic diagram," commonly known as a graph. Nation *i* and nation *j* are in the relation (i.e., are adjacent points in the graph) if there is "an overlap in [their] spheres of influence which cannot be attributed to chance more than five times in a hundred." The graph (see Figure 7) has been redrawn from Bernstein and Weldon (1968, p. 173) in order to more clearly show the cliques. The reader will see that there are six

cliques formed by the following sets of points: {11,13,16,27}, {11,13,26,27}, {1,13,16}, {19,35}, {26,35}, {1,9}.

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