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ON ELASTIC ELLIPSOIDAL SHELLS OF REVOLUTION

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ABSTRACT

By means of a more recent method of asymptotic integration due to Langer, a solution is obtained which is valid at the apex of the shell and involves Kelvin functions. This solution reduces in the limit to the known theory of shallow spherical shells. Specifically, the stress distribution is obtained for ellipsoidal shells under a uniform load distributed over a small area about the apex.

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INTRODUCTION

In a previous paper [1], it was shown that the resulting differential equations for small deformation of thin elastic shells of revolution, as given by Reissner [2], may be combined into a single second-order complex differential equation. This differential equation is valid for shells of revolution of uniform thickness and a large class of nonuniform thickness.

In the present paper, the solution of the complex differential equation mentioned is obtained by means of a more recent method of asymptotic integration due to Langer [3]. This solution is valid at the apex of the shell (where a second order pole is present in the differential equation) and involves Bessel functions of complex argument. Specifically, shells closed at the apex $\phi = 0$ and subjected to uniform distributed load over a small region about the apex is treated. Also included is the reduction of the solution to the known results for shallow spherical shells [4].

THE BASIC EQUATIONS

With the use of cylindrical coordinates r, Θ, z, the parametric equations of the middle surface of a shell of revolution, as shown in Figure 1, are

(1) $$r = r(\xi), \quad z = z(\xi)$$

and

(2) $$r' = \alpha \cos \phi, \quad z' = \alpha \sin \phi$$

where $\alpha = [(r')^2 + (z')^2]^{1/2}$,

ϕ is the inclination of the tangent to the meridian of the shell and the primes denote differentiation with respect to ξ.

To make the paper self-contained, we record here the relevant equations of the small deflection theory of elastic shells of revolution with axisymmetric loading, as given in [2].

$$
\begin{aligned}
rV &= -\int r\,\alpha\,p_V\,d\xi \\
\alpha N_\xi &= r'H + z'V; \quad \alpha Q = -z'H + r'V. \\
\alpha N_\Theta &= (rH)' + r\,\alpha\,p_H \\
rN_\xi &= (rH)\cos\phi + (rV)\sin\phi \\
rQ &= -(rH)\sin\phi + (rV)\cos\phi \qquad (3) \\
M_\xi &= \frac{D}{\alpha}\left[\beta' + \nu\frac{r'}{r}\beta\right] \\
M_\Theta &= \frac{D}{\alpha}\left[\frac{r'}{r}\beta + \nu\beta'\right] \\
u &= \frac{r}{Eh}(N_\Theta - \nu N_\xi) \\
w &= \int\left[\frac{z'}{Eh}(N_\xi - \nu N_\Theta) - r'\beta\right]d\xi,
\end{aligned}
$$

where N_ξ, N_Θ, and Q are the stress resultants; M_ξ and M_Θ are the stress couples (Figure 1); H and V denote the "horizontal" and "vertical" stress resultants related to Q and N_ξ; u and w are the displacements in the radial and axial directions, respectively; β is the negative change in ϕ due to deformation; p_H and p_V are the components of load intensity in the r and z directions; h is the thickness of the shell; $D = Eh^3/12(1-\nu^2)$ and E and ν are Young's modulus and Poisson's ratio, respectively.

The components of stress, due to the stress couples (bending) and due to stress resultants N_ξ and N_Θ (membrane), as well as the shearing stress τ, are defined in the usual manner by

$$
\begin{aligned}
&\bar{\sigma}_{\xi b} \equiv (\sigma_{\xi b})_{max} = \frac{6M_\xi}{h^2}, \quad \bar{\sigma}_{\Theta b} \equiv (\sigma_{\Theta b})_{max} = \frac{6M_\Theta}{h^2} \\
&\sigma_{\xi m} = \frac{N_\xi}{h}, \quad \sigma_{\Theta m} = \frac{N_\Theta}{h}, \quad \tau = \frac{3Q}{2h} \qquad (4)
\end{aligned}
$$

where the subscripts b and m refer to bending and membrane stresses, respectively

It was shown in [1] that the differential equations in β and (rH), resulting from (3) and the equations of equilibrium and compatibility, as derived by Reissner [2], may be combined in the normal form:

(5) $$W'' + [2\, i^3 \mu^2 \Psi^2 (\xi) + \Lambda(\xi)]\, W = \left[\frac{h}{h_o}\ \frac{r}{\alpha}\right]^{1/2} (F + ik\, G)\, f(\xi)$$

provided that k, given by[1]

(6) $$k = -\frac{i}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right) + \left\{1 - \left[\frac{1}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right)\right]^2\right\}^{1/2}$$

is constant.

The various quantities occurring in (5) and (6) are defined by

(7)
$$W = \left(\frac{h}{h_o}\right)^{3/2}\left(\frac{r}{\alpha}\right)^{1/2}(\beta + ik\,\psi);\quad i = \sqrt{-1}$$

$$\psi = \frac{m}{Eh^2}(rH),\quad m = [12(1-\nu^2)]^{1/2},\ \frac{\alpha^2 m}{r_2 h_o} = 2\mu^2 f(\xi)$$

$$\lambda = \left[\frac{h_o}{h} f(\xi)\right]^{-1}\left\{\frac{(r'/\alpha)'}{(r/\alpha)} + 3\,\frac{r'}{r}\ \frac{h'}{h}\right\}$$

$$\delta = 2\left[\frac{h_o}{h} f(\xi)\right]^{-1}\left\{\frac{h''}{h} + 2\nu\,\frac{r'}{r}\,\frac{h'}{h} + \frac{(r/\alpha)'}{(r/\alpha)}\ \frac{h'}{h}\right\}$$

$$F = 2\mu^2\,\frac{m}{Eh^2}\,(rV)\cot\phi$$

$$G = \left[\frac{h}{h_o} f(\xi)\right]^{-1}\left\{\left[\frac{z'r'}{r^2} + \nu\,\frac{(z'/\alpha Eh)'}{(r/\alpha Eh)}\right]\Omega + \nu\,\frac{z'}{r}\,\Omega' - \left[\frac{(r/\alpha Eh)'}{(r/\alpha Eh)} + \nu\,\frac{r'}{r}\right]P_H - P_H'\right\}$$

and

$$\Psi^2 = \left(k + i\,\frac{\nu\lambda}{2\mu^2}\right)\left(\frac{h_o}{h}\right) f(\xi)$$

(8) $$\Lambda = -\frac{1}{2}\,\frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4}\left[\frac{(r/\alpha)'}{(r/\alpha)}\right]^2 - \left(\frac{r'}{r}\right)^2 - \frac{3}{2}\,\frac{(r/\alpha)'}{(r/\alpha)}\,\frac{h'}{h} - \frac{3}{2}\,\frac{h''}{h} - \frac{3}{4}\left(\frac{h'}{h}\right)^2$$

---

[1]This restriction, although obtained in an entirely different manner, is similar to that given previously by Meissner [5].

(8) $$\Omega = \frac{m}{Eh_o^2}(rV), \qquad P_H = \frac{m}{Eh_o^2} r \alpha p_H$$

where h_o is the value of h at some reference section (say $\xi = \xi_o$) and r_2, the length of the normal intercepted between the generating curve of the middle surface and the axis of rotation, is given by $r_2 = r/\sin \phi$. In what follows, the other principal radius of curvature, that is the radius of curvature of the generating curve, will be denoted by r_1.

For ellipsoidal shells of revolution, the equation of the middle surface in rectangular Cartesian coordinates is specified by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1 ,$$

a and c being the semi-major axes of the ellipsoid. Selecting the independent variable ξ as ϕ, it follows from the geometry of the middle surface and (2) that $\alpha = r_1$. Thus, the radii of curvature are

(9) $$r_1 = \alpha = \frac{c^2}{a[1 + \rho^2 \cos^2 \phi]^{3/2}}, \quad r_2 = \frac{a}{[1 + \rho^2 \cos^2 \phi]^{1/2}}$$

and

(10) $$r = \frac{a \sin \phi}{[1 + \rho^2 \cos^2 \phi]^{1/2}}$$

(11) $$\rho^2 = \frac{c^2 - a^2}{a^2}$$

From (7) and (9), we have

(12) $$2\mu^2 = m \left(\frac{c}{a}\right)^4 \left(\frac{a}{h_o}\right), \quad f(\xi) = (1 + \rho^2 \cos^2 \phi)^{-5/2}$$

and when the thickness h is uniform,

(13) $$\delta = 0 , \qquad \lambda = -\left(\frac{c}{a}\right)^2 (1 + \rho^2 \cos^2 \phi)^{3/2}$$

Since we are mainly concerned with shells of uniform thickness, in the remainder of this paper h_o will be replaced by h, unless otherwise stated.

As remarked previously [1], for ellipsoidal shells r_1 is not constant; it follows that when h is uniform, k is a function of ϕ. However, with a view towards approximating k by a constant so that the condition for the validity of equation (5) is fulfilled, we note that restriction of (c/a) to 0 (1) is consistent with $\nu\lambda \ll 2\mu^2$ and by (6),

$$\left[1 - \left(\frac{\nu\lambda}{2\mu^2}\right)^2\right]^{1/2} \simeq 1 \quad \text{or} \quad k \simeq 1.$$

With this approximation, equation (5) is valid and the coefficient functions Ψ^2 and Λ read:

$$\Psi^2 = (1 + \rho^2 \cos^2 \phi)^{-5/2}$$

(14)
$$\Lambda = -\frac{3}{4}\left(\frac{c}{a}\right)^4 \frac{\cot^2 \phi}{(1 + \rho^2 \cos^2 \phi)^2} + \Lambda_1$$

where

$$\Lambda_1 = \frac{3}{16}\left[\frac{\rho^2 \sin 2\phi}{1 + \rho^2 \cos^2\phi}\right]^2 + \frac{3}{2}\left[\frac{\rho^2 \cos 2\phi}{1 + \rho^2 \cos^2 \phi}\right] + \frac{1}{2}\left(\frac{c}{a}\right)^2\left[\frac{\sin^2 \phi + \rho^2 \cos^2 \phi}{(1 + \rho^2 \cos^2 \phi)^2}\right]$$

and the transformation for W in (7) is

(15)
$$W = \left(\frac{a}{c}\right)\left[\sin \phi \, (1 + \rho^2 \cos^2 \phi)\right]^{1/2} (\beta + i\,\psi)$$

SOLUTION OF THE DIFFERENTIAL EQUATION BY ASYMPTOTIC INTEGRATION

Inspection of the coefficient function Λ as given by (14) reveals the presence of a pole of second order (at $\phi = 0$) in the differential equation (5). Consequently, the solution of (5) by the classical method of asymptotic integration fails to yield a solution valid at the apex of the shell. In order to obtain a solution of (5) which is valid at $\phi = 0$, recourse will be made to a more recent technique of asymptotic integration, due to Langer [3].

According to Langer, it becomes necessary to modify (5) into a new normal form. For this purpose, we introduce a new independent variable t as follows:

(16)
$$t = \sin \frac{\phi}{2}, \quad d\,\phi = 2\,\frac{dt}{(1-t^2)^{1/2}}$$

Then (5) may be written as[2]

(17)
$$\frac{d^2W}{dt^2} - \frac{t}{1-t^2}\,\frac{dW}{dt} + 4\left[2i^3\mu^2\,\frac{\Psi^2(t)}{1-t^2} + \frac{\Lambda(t)}{1-t^2}\right]W = R(1-t^2)^{1/4}$$

---

[2]If it is desired to cover the entire region of the shell, i.e., $0 \leq \phi \leq \pi$, instead of transformation (16) $t = \sin \phi/2q$ should be used. This entails replacing the factor 4 by $4q^2$ in (17) where $q > 1$.

where

(18) $$R = \frac{4}{(1-t^2)^{5/4}} \left(\frac{r}{\alpha}\right)^{1/2} f(t)\ [F + i\ G]$$

Expanding the coefficient function $4\Lambda(t)/1-t^2$ in (17) by partial fractions, there results

(19) $$\frac{4\Lambda(t)}{(1-t^2)} = \frac{-3/4}{t^2} + \Lambda_2$$

where

(20) $$\Lambda_2 = \frac{4\Lambda_1}{1-t^2} + \left[\frac{-3/4\ (c/a)^4\ (1-2t^2)^2}{t^2\ (1-t^2)^2\ [1+\rho^2\ (1-2t^2)^2]^2} + \frac{3/4}{t^2}\right]$$

is bounded in $/t/ < 1$, i.e., in $0 \leq \phi < \pi$.

Now, by means of the transformation

(21) $$\bar{W} = (1-t^2)^{1/4}\ W$$

we obtain from (17) the new normal form

(22) $$\frac{d^2\bar{W}}{dt^2} + \left\{8i^3\ \mu^2 \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^2 + \frac{-3/4}{t^2} + \Lambda_3\right\}\bar{W} = R$$

where

(23) $$\Lambda_3 = \Lambda_2 + \frac{1}{2}\ (1-t^2)^{-1} + \frac{3}{4}\ t^2(1-t^2)^{-2}$$

is bounded in $/t/ < 1$.

Since the particular solution of (22) depends on the specific loading, we shall consider in this section only the solution of the homogeneous differential equation associated with (22).

As pointed out by Langer, the homogeneous solution of (22) may be written as[3]

(24) $$\left\{\begin{matrix}\bar{W}_1 \\ \bar{W}_2\end{matrix}\right\} = \left\{\left(\frac{\pi}{2}\right)^{1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Phi^{1/4}\ \eta^{1/4}\right\} \left\{\begin{matrix}\exp\frac{3\pi}{4}i\ \ H_1^{(1)}(\eta) \\ \exp\frac{-3\pi}{4}i\ \ H_1^{(2)}(\eta)\end{matrix}\right\} + \kappa$$

[3]The numerator of the second term in the coefficient of $\bar{W}$ in (22), when written as $1/4\ [1-(2)^2]$ corresponds to $1/4\ [1-A^2]$ in Langer's notation. The order of the Bessel functions in the solution of (22) is determined by $1/2$ A.

where

(25) $$\kappa = \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Phi^{1/4} \eta^{-3/4} \frac{\log\,[(2i^3 \mu^2)^{1/2}]}{(2i^3 \mu^2)^{1/2}} \; O(\epsilon)$$

and $O(\epsilon)$ denotes a bounded function.

In (24), $H_1^{(1)}(\eta)$ and $H_1^{(2)}(\eta)$ are Hankel functions[4]

(26a) $$\eta = (8i^3 \mu^2)^{1/2} \Phi$$

and

(26b) $$\Phi = \int_0^t \frac{\Psi}{(1-t^2)^{1/2}} \, dt \, .$$

In view of (24) and (25), it follows that the homogeneous solution of (22) may be represented asymptotically by

(27) $$\overline{W}_H = \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Phi^{1/4} \eta^{1/4} \left\{ \overline{A}\, H_1^{(1)}(\eta) + \overline{B}\, H_1^{(2)}(\eta) \right\}$$

which is valid in $/t/ < 1$ and where the constants $\overline{A}$ and $\overline{B}$ are complex. It is advantageous to express solution (27) in terms of Kelvin functions, as follows:

(28) $$\overline{W}_H = \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Phi^{1/2} \Big\{ \overline{A}_1 \, [\mathrm{ber}_1 (s) + i \, \mathrm{bei}_1 (s)] + \overline{B}_1 \, [\mathrm{ker}_1 (s) + i \, \mathrm{kei}_1 \, s] \Big\}$$

where

(29a) $$s = (8\mu^2)^{1/2} \Phi$$

and

(29b) $$\overline{A}_1 = A_o + i A_1 \, , \quad \overline{B}_1 = B_o + i B_1 \, .$$

By means of (15), (21), and solution (28), we record the expressions for β, ψ, β', and ψ', since they are required in the solution of specific problems.

(30a) $$\beta = \left(\frac{\Phi}{2t}\right)^{1/2} \left\{(1-t^2)[1+\rho^2(1-2t^2)^2]\right\}^{-1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Big\{ A_o \, \mathrm{ber}_1 (s) - A_1 \, \mathrm{bei}_1 (s) + B_o \, \mathrm{ker}_1 (s) - B_1 \, \mathrm{kei}_1 (s) \Big\}$$

---

[4]The notation used is that of Watson. See reference[6].

(30a)

$$\psi = \left(\frac{\Phi}{2t}\right)^{1/2} \left\{(1-t^2)[1+\rho^2(1-2t^2)^2]\right\}^{-1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \Big\{A_0 \operatorname{bei}_1(s) + A_1 \operatorname{ber}_1(s) + B_0 \operatorname{kei}_1(s) + B_1 \operatorname{ker}_1(s)\Big\}$$

(30b)

$$\beta' = \beta \mathcal{L} + (2\mu^2)^{1/2} \left(\frac{\Phi}{2t}\right)^{1/2} [1+\rho^2(1-2t^2)^2]^{-1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{1/2} \mathcal{M}$$

$$\psi' = \psi \mathcal{L} + (2\mu^2)^{1/2} \left(\frac{\Phi}{2t}\right)^{1/2} [1+\rho^2(1-2t^2)^2]^{-1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{1/2} \mathcal{N}$$

where again primes denote differentiation with respect to ϕ, $t = \sin \phi/2$, and

(30c)

$$\mathcal{L} = \frac{1}{4} [1+\rho^2(1-2t^2)^2]^{-5/4} \Phi^{-1} \left\{1 - \Phi \frac{t}{(1-t^2)^{1/2}} [1+\rho^2(1-2t^2)^2]^{1/4} \times [1+\rho^2(1-2t^2)(11-12t^2)]\right\} - \frac{1}{4}\left\{\frac{1-3t^2}{t(1-t^2)^{1/2}} - \frac{8\rho^2 t(1-t^2)^{1/2}(1-2t^2)}{[1+\rho^2(1-2t^2)^2]}\right\}$$

$$\mathcal{M} = \frac{d}{ds} [A_0 \operatorname{ber}_1(s) - A_1 \operatorname{bei}_1(s) + B_0 \operatorname{ker}_1(s) - B_1 \operatorname{kei}_1(s)]$$

$$\mathcal{N} = \frac{d}{ds} [A_0 \operatorname{bei}_1(s) + A_1 \operatorname{ber}_1(s) + B_0 \operatorname{kei}_1(s) + B_1 \operatorname{ker}_1(s)]$$

REDUCTION TO THE THEORY OF SHALLOW SPHERICAL SHELLS

In this section, we consider the transition from the solution (30) to the known results for shallow spherical shells, due to E. Reissner [4]. For this purpose, we first examine solution (30) for the case of a spherical shell and then proceed to show its reduction and correspondence to the limiting case of shallow shells.

Since for spherical shells $c/a = 1$, then $\alpha = r_1 = r_2$, $\rho^2 = 0$, and by (7), $f = 1$. Hence $2\mu^2 = (a/h)m$, and the quantities Ψ and Φ in (14) and (26) become

(31)

$$\Psi = 1 , \quad \Phi = \phi/2.$$

The solution (30) is now considerably simplified and reads as follows:

(32a)

$$\beta = \left[\frac{\phi}{2 \sin \phi}\right]^{1/2} \Big\{A_0 \operatorname{ber}_1(s) - A_1 \operatorname{bei}_1(s) + B_0 \operatorname{ker}_1(s) - B_1 \operatorname{kei}_1(s)\Big\}$$

(32a)

$$\psi = \left[\frac{\phi}{2\sin\phi}\right]^{1/2}\left\{A_0\,\mathrm{bei}_1\,(s) + A_1\,\mathrm{ber}_1\,(s) + B_0\,\mathrm{kei}_1\,(s) + B_1\,\mathrm{ker}_1\,(s)\right\}$$

$$\beta' = \beta\mathcal{L} + \left(\frac{a}{h}m\right)^{1/2}\left[\frac{\phi}{2\sin\phi}\right]^{1/2}\frac{d}{ds}\left\{A_0\,\mathrm{ber}_1\,(s) - A_1\,\mathrm{bei}_1\,(s)\right.$$

(32b)

$$\left. + B_0\,\mathrm{ker}_1\,(s) - B_1\,\mathrm{kei}_1\,(s)\right\}$$

$$\psi' = \psi\mathcal{L} + \left(\frac{a}{h}m\right)^{1/2}\left[\frac{\phi}{2\sin\phi}\right]^{1/2}\frac{d}{ds}\left\{A_0\,\mathrm{bei}_1\,(s) + A_1\,\mathrm{ber}_1\,(s) + B_0\,\mathrm{kei}_1(s)\right.$$

$$\left. + B_1\,\mathrm{ker}_1\,(s)\right\}$$

where the argument "s" is reduced to

(32c) $$s = \left(\frac{a}{h}m\right)^{1/2}\phi$$

and

(32d) $$\mathcal{L} = \frac{1}{2}\left(\frac{1}{\phi} - \cot\phi\right) .$$

According to Reissner [4] a spherical shell is defined as shallow when r/a is small compared to unity. This assumption implies that the theory of shallow shells is valid only for small values of ϕ; in particular, since $r = a\sin\phi$, then for small values of ϕ, $r/a \simeq \phi$. While the argument "s" given by (32a) is valid for $0 \leq \phi < \pi$, for small values of ϕ it may be written as $s \simeq (a/h\ m)^{1/2}\ r/a$ which is identical with the argument of Kelvin functions employed in [4]. In view of this discussion, it is clear that for shallow shells, $\sin\phi \simeq \phi$ and $\mathcal{L} \simeq 0$ in solution (32). Thus for shallow spherical shells, using well-known recurrence relations for Kelvin functions, we have

$$\beta = \frac{d}{ds}\left\{\frac{A_0-A_1}{2}\,\mathrm{ber}\,(s) - \frac{A_0+A_1}{2}\,\mathrm{bei}\,(s) + \frac{B_0-B_1}{2}\,\mathrm{ker}\,(s) - \frac{B_0+B_1}{2}\,\mathrm{kei}\,(s)\right\}$$

(33)

$$\psi = \frac{d}{ds}\left\{\frac{A_0-A_1}{2}\,\mathrm{bei}(s) + \frac{A_0+A_1}{2}\,\mathrm{ber}\,(s) + \frac{B_0-B_1}{2}\,\mathrm{kei}\,(s) + \frac{B_0+B_1}{2}\,\mathrm{ker}\,(s)\right\}.$$

To show the correspondence of solution (33) with that of Reissner [4], we recall that the basic dependent variables of his work are a stress function F and the displacement normal to the middle surface of the shell, which for small ϕ is the same as the deflection w of the present paper. These dependent variables are related to our β and ψ by

(34) $$\beta = \frac{dw}{dr}, \qquad \psi = \frac{m}{Eh^2}\ \frac{dF}{dr} .$$

If (33) and the solution given in [4] are substituted in (34), it is seen that a one to one correspondence exists between the two solutions, provided the coefficient of 1/s in dF/dr is set equal to zero; the function 1/s arises from the solution of the differential equation $\nabla^2 F = 0$. An equivalent differential equation does not arise in the formulation of shells of revolution employed here, and this is also borne out by Reissner's subsequent work on shallow shells [2, p. 243].

ELLIPSOIDAL SHELL UNDER A UNIFORM NORMAL LOAD DISTRIBUTED OVER A SMALL REGION ABOUT THE APEX

We consider an ellipsoidal shell clamped at the edge $\phi = \pi/2$ subjected to a uniform normal load p_n distributed symmetrically in the region $0 \leq \phi \leq \bar{\phi}$, $\bar{\phi}$ being small.

In view of the presence of a distributed load, it becomes necessary to obtain a particular solution of (22). Since by the first of (3)

$$(35) \qquad rV = -\frac{p_n}{2} a^2 \sin^2 \phi \, \frac{1+\rho^2 \cos^2 \bar{\phi}}{(1+\rho^2 \cos^2 \phi)^2} \quad , \; 0 \leq \phi \leq \bar{\phi}$$

$$= -\frac{p_n}{2} a^2 \sin^2 \bar{\phi} \, \frac{1}{(1+\rho^2 \cos^2 \bar{\phi})} \quad , \; \bar{\phi} \leq \phi \leq \pi/2$$

then, as in [4], a suitable particular solution may be obtained approximately by the membrane theory of shells. Thus, for small values of ϕ, the particular solution is

$$(36) \qquad \beta_P = 0$$

$$\psi_P = -\frac{m}{E}\left(\frac{a}{h}\right)^2 \frac{p_n}{2} \, \frac{\tan \phi}{(1+\rho^2 \cos^2 \phi)}$$

In the following, the loaded and unloaded regions will be distinguished by subscripts I and II, respectively. The requirement that the quantities M_ϕ, M_θ, N_ϕ, N_θ, Q, w, and w' remain finite at the apex $\phi = 0$ demands that

$$(37a) \qquad \phi = 0; \quad \beta_I \, , \, \frac{1}{r}\beta_I \, , \, \beta_I' \, , \, \frac{1}{r}\psi_I \, , \, \psi_I'$$

remain finite where the quantities in the above include both the homogeneous and particular solutions, i.e.,

$$\beta_I = (\beta_H)_I + (\beta_P)_I \quad \text{etc.}$$

The boundary conditions at the clamped edge $\phi = \pi/2$ are

(37b) $$\phi = \pi/2; \quad \beta_{II} = 0, \quad \psi'_{II} = \frac{\nu m}{Eh^2} (rV)$$

where $\beta_{II} = (\beta_H)_{II}$ etc., since no particular solution is needed for the unloaded region. Also, the continuity conditions at $\phi = \bar{\phi}$ are

(37c) $$\phi = \bar{\phi}; \quad \beta_I = \beta_{II}, \quad \psi_I = \psi_{II}, \quad \psi'_I = \psi'_{II}, \quad \beta'_I = \beta'_{II} .$$

Examination of (30) reveals that condition (37a) is satisfied provided that the constant coefficients of the functions $\ker_1(s)$ and $\mathrm{kei}_1(s)$ are set equal to zero at the outset.

Hence, for region I, the solution is given by

(38) $$\beta_I = \left(\frac{\Phi}{2t}\right)^{1/2} \left\{(1-t^2)[1+\rho^2(1-2t^2)^2]\right\}^{-1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \left\{C_0 \,\mathrm{ber}_1(s) - C_1 \mathrm{bei}_1(s)\right\}$$

$$\psi_I = \left(\frac{\Phi}{2t}\right)^{1/2} \left\{(1-t^2)[1+\rho^2(1-2t^2)^2]\right\}^{1/2} \left[\frac{\Psi}{(1-t^2)^{1/2}}\right]^{-1/2} \left\{C_0 \,\mathrm{bei}_1\, s + C_1 \,\mathrm{ber}_1 s\right\}$$

$$- \frac{m}{E} \left(\frac{a}{h}\right)^2 \frac{p_n}{2} \frac{\tan \phi}{(1+\rho^2 \cos^2 \phi)} .$$

The solution for region II, on the other hand, involves (30) only with all four functions retained. The six constants of integration are determined from the conditions (37b,c) which result in the following six simultaneous equations:

$$C_0 \,\mathrm{ber}_1(\bar{s}) - C_1 \,\mathrm{bei}_1(\bar{s}) - A_0 \,\mathrm{ber}_1(\bar{s}) + A_1 \,\mathrm{bei}_1(\bar{s}) - B_0 \ker_1(\bar{s}) + B_1 \,\mathrm{kei}_1(\bar{s}) = 0.$$

$$C_0 \,\mathrm{ber}'_1(\bar{s}) - C_1 \,\mathrm{bei}'_1(\bar{s}) - A_0 \,\mathrm{ber}'_1(\bar{s}) + A_1 \,\mathrm{bei}'_1(\bar{s}) - B_0 \ker'_1(\bar{s}) + B_1 \,\mathrm{kei}'_1(\bar{s}) = 0.$$

$$C_0 \,\mathrm{bei}_1(\bar{s}) + C_1 \,\mathrm{ber}_1(\bar{s}) - A_0 \,\mathrm{bei}_1(\bar{s}) - A_1 \,\mathrm{ber}_1(\bar{s}) - B_0 \,\mathrm{kei}_1(\bar{s}) - B_1 \ker_1(\bar{s})$$
$$= \Omega_0 \frac{\tan \bar{\phi}}{(1+\rho^2 \cos^2 \bar{\phi})} \frac{(1+\rho^2)}{2} .$$

$$C_0 \,\mathrm{bei}'_1(\bar{s}) + C_1 \,\mathrm{ber}'_1(\bar{s}) - A_0 \,\mathrm{bei}'_1(\bar{s}) - A_1 \,\mathrm{ber}'_1(\bar{s}) - B_0 \,\mathrm{kei}'_1(\bar{s}) - B_1 \ker'_1(\bar{s})$$
$$= \Omega_0 \frac{(1+\rho^2)^{5/4}}{(2\mu^2)^{1/2}} \left\{\frac{1}{\cos^2 \bar{\phi}\,(1+\rho^2 \cos^2 \bar{\phi})} + \frac{2\rho^2 \sin^2 \bar{\phi}}{(1+\rho^2 \cos^2 \bar{\phi})^2} - \frac{\tan \bar{\phi}\, \mathcal{L}(\bar{\phi})}{(1+\rho^2 \cos^2 \bar{\phi})}\right\}$$

$$A_0 \,\mathrm{ber}_1(\bar{\bar{s}}) - A_1 \,\mathrm{bei}_1(\bar{\bar{s}}) + B_0 \ker_1(\bar{\bar{s}}) - B_1 \,\mathrm{kei}_1(\bar{\bar{s}}) = 0$$

$$A_0[\mathrm{bei}_1(\bar{\bar{s}})\mathcal{L}(\pi/2) + (2\mu^2)^{1/2}\,\mathrm{bei}_1'(\bar{\bar{s}})] + A_1[\mathrm{ber}_1(\bar{\bar{s}})\mathcal{L}(\pi/2) + (2\mu^2)^{1/2}\mathrm{ber}_1'(\bar{\bar{s}})]$$

$$+B_0[\mathrm{kei}_1(\bar{\bar{s}})\mathcal{L}(\pi/2) + (2\mu^2)^{1/2}\,\mathrm{kei}_1'(\bar{\bar{s}})] + B_1[\mathrm{ker}_1(\bar{\bar{s}})\mathcal{L}(\pi/2) + (2\mu^2)^{1/2}\mathrm{ker}_1'(\bar{\bar{s}})]$$

$$= -\,\Omega_0\,\nu\,\frac{\sin^2\bar{\phi}}{(1+\rho^2\cos^2\bar{\phi})}\left(\frac{c}{a}\right)^2 [2\,\Phi\,(\pi/2)]^{-1/2}$$

where $\Omega_0 = \frac{m}{E}\left(\frac{a}{h}\right)^2 p_n$, $\bar{s} = s(\bar{\phi})$, $\bar{\bar{s}} = s(\pi/2)$

and the primes in these equations denote differentiation with respect to 's'.

Taking $a/h = 20$, $c/a = \sqrt{2}(\rho^2 = 1)$, $\bar{\phi} = 10°$, and $\nu = 0.3$,

then the constants of integration are determined as follows:

(40)

$$C_0 = 0.065738\,\Omega_0$$

$$C_1 = -0.126213\,\Omega_0$$

$$B_0 = -0.016310\,\Omega_0$$

$$B_1 = 0.010468\,\Omega_0$$

$$A_0 = -5.5574 \times 10^{-8}\,\Omega_0$$

$$A_1 = -2.5676 \times 10^{-8}\,\Omega_0 \quad .$$

Using (4), the stress distribution for the example treated is as shown in Figure 2. It is noteworthy that, although the ratio of c/a in the present example is $\sqrt{2}$, the resulting bending stresses in the loaded region are in very good agreement with those of the corresponding example of shallow spherical shells [4].

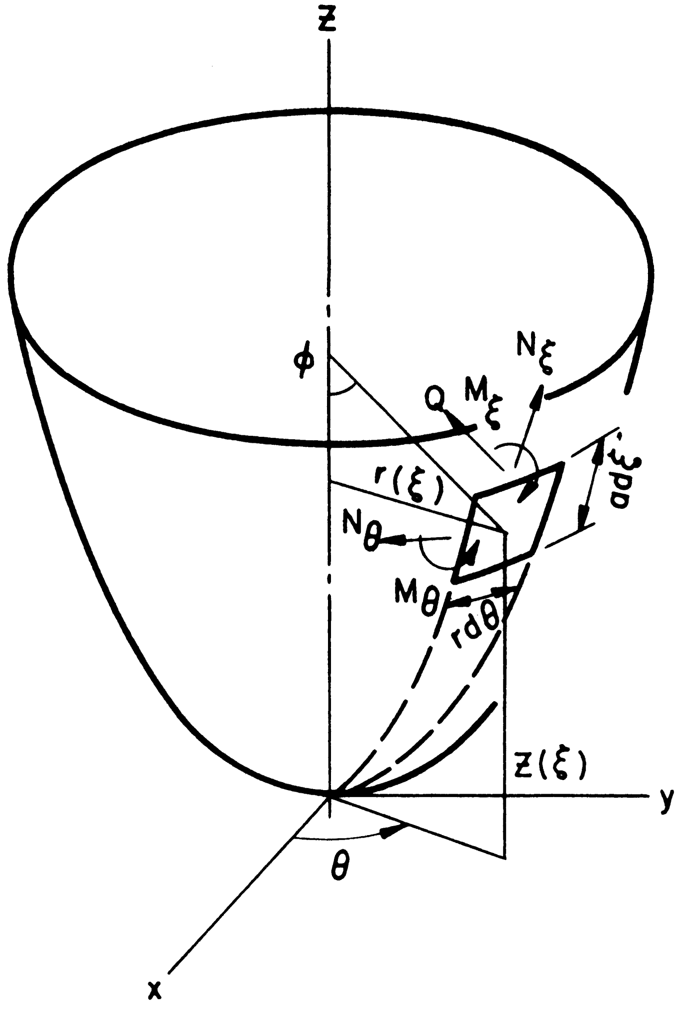
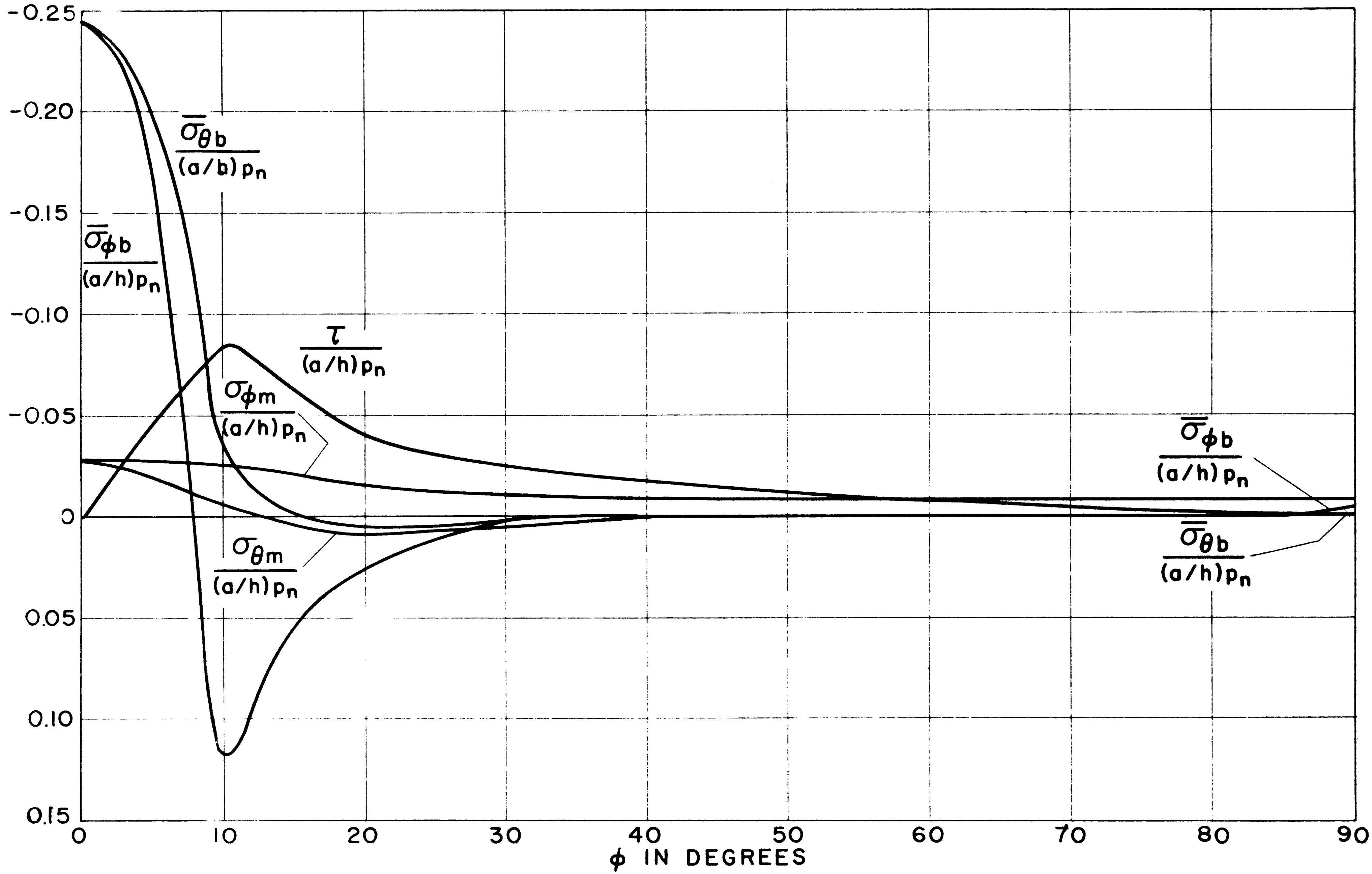


Fig. 1



-0.25
-0.20
-0.15
-0.10
-0.05
0
0.05
0.10
0.15
0
10
20
30
40
50
60
70
80
90
ϕ IN DEGREES
$\frac{\bar{\sigma}_{\theta b}}{(a/b)p_n}$
$\frac{\bar{\sigma}_{\phi b}}{(a/h)p_n}$
$\frac{\tau}{(a/h)p_n}$
$\frac{\sigma_{\phi m}}{(a/h)p_n}$
$\frac{\sigma_{\theta m}}{(a/h)p_n}$
$\frac{\bar{\sigma}_{\phi b}}{(a/h)p_n}$
$\frac{\bar{\sigma}_{\theta b}}{(a/h)p_n}$

Fig. 2

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