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# TECHNICAL REPORT NO. 1

# ON THE EQUATIONS OF MOTION OF CYLINDRICAL SHELLS

By

P. M. NAGHDI
Assistant Professor of Engineering Mechanics

J. G. BERRY
Instructor in Engineering Mechanics

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ABSTRACT

From the basic equations of thin cylindrical shells consistent with Love's first approximation, three uncoupled displacement equations of motion are deduced without any further approximation. Comparison is made with the work of other authors, who have used a variety of approximations in arriving at the equations of motion of cylindrical shells. Thus, in a qualitative manner, further insight is gained on the effectiveness and practicality of these approximations in the solution of problems of cylindrical shells.

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## INTRODUCTION

The formulation of the classical theory of thin elastic shells, and in particular cylindrical shells, has received repeated attention in the literature.* The basic assumptions used in the classical theory consistent with the conventional assumptions for displacements** are: (1) the thickness h of the shell is small compared with the least radius of curvature R of the middle surface; (2) the strains and displacements are sufficiently small so that quantities of the second and higher orders may be neglected in the components of strain; (3) the component of stress normal to the middle surface is small compared with other components of normal stress and may be neglected in the stress-strain relations; and (4) the normals to the undeformed middle surface remain normal to the deformed middle surface and suffer no extension. The last two assumptions imply neglect of the transverse normal stress and shear deformation respectively.

The classical theory of shells in the sense of Love's first approximation[2] is based on the four assumptions mentioned above, with a further stipulation that the ratio Z/R (see Fig. 1) is neglected in comparison with unity in the expressions of both stress-resultants and strain-displacement

*The complete bibliography of the subject is beyond the scope of this paper. For an exposition of the general theory of shells, see reference 1.

**These are U, V, and W along the x, s, and z directions, respectively (see Fig. 1); $U(x,s,z) = u(x,s) + zu'(x,s)$, $V(x,s,z) = v(x,s) + zv'(x,s)$, $W(x,s,z) = w(x,s)$ where $u' = -(\partial W/\partial x)$ and $v' = -(\partial W/\partial s + V/R)$ are consequences of assumption (4).

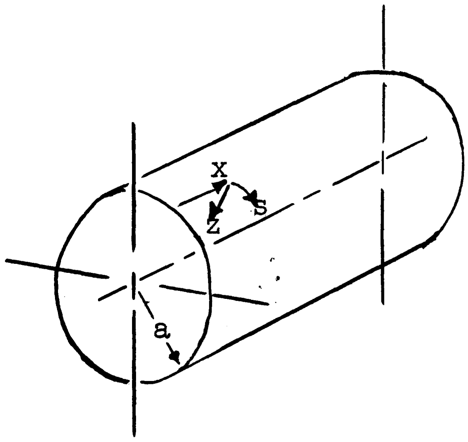


Fig. 1. The coordinate system for a circular cylindrical shell.

relations.* In applying the classical theory to special cases, some authors have introduced still further approximations besides Love's first approximation, while others have either abandoned assumption (3) or retained terms of the order $(Z/R)^2$ in the stress-resultants and strain-displacement relations. It should be mentioned that a general theory has recently been given[1]** which is based on assumptions (1) and (2) only, thus accounting for the effect of both transverse normal stress and shear deformation.

In the present paper, a set of uncoupled displacement equations of motion for circular cylindrical shells is obtained, consistent with Love's first approximation, without any further assumptions. Comparison is then made between the resulting characteristic equation and the works of other authors who have employed a variety of approximations in arriving at the coupled equations of motion.

## THE COORDINATE SYSTEM

The coordinate system for a circular cylindrical shell is shown in Fig. 1; the x-axis is directed along the generator of the cylinder, s is measured clockwise in the circumferential direction, and the z-axis is directed inward along the positive normal to the middle surface of the shell. The coördinate curves x and s, as lines of curvature, specify the position of a point on the middle surface, and the square of a linear element for the triply orthogonal coordinate system (x,s,z) may be shown to be

*See equations (4) and (6) of this report.
**Also see reference 3.

$$(ds)^2 = dx^2 + \left(1 - \frac{z}{a}\right)^2 ds^2 + dz^2 , \qquad (1)$$

where a is the radius of the circular cylinder.

## THE BASIC EQUATIONS OF THIN CIRCULAR CYLINDRICAL SHELLS

In order to make our discussion self-contained, we record here the basic equations for circular cylindrical shells in accordance with the classical theory, where the effects of transverse shear deformation and normal stress are neglected. From these, the basic equations commonly known as Love's first approximation will be deduced.

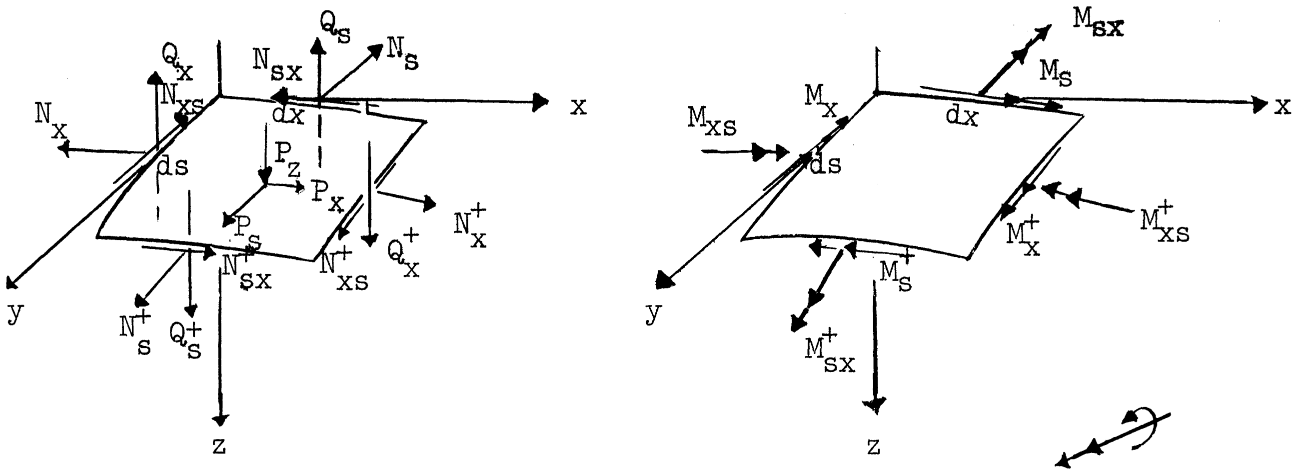


Fig. 2. An element of cylindrical shell showing the stress-resultants and the stress couples; $N_x^+ \equiv N_x + (\partial N_x/\partial x)\, dx$, $N_s^+ \equiv N_s + (\partial N_s/\partial s)\, ds$, etc.

The stress-resultant and the stress-couple differential equations of equilibrium, which may be obtained from Fig. 2, are:

$$\left.\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} + P_x &= 0 \\
\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} - \frac{Q_s}{a} + P_s &= 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_s}{\partial s} + N_s + P_z &= 0
\end{aligned}\right\} \qquad (2)$$

and

$$\left.\begin{aligned} \frac{\partial M_x}{\partial x} \quad \frac{\partial M_{sx}}{\partial s} - Q_x &= 0 \\ \frac{\partial M_{xs}}{\partial x} \quad \frac{\partial M_s}{\partial s} - Q_s &= 0 \\ N_{xs} - N_{sx} &= \frac{M_{sx}}{a} \end{aligned}\right\} \qquad (3)$$

where the stress-resultants and couples are defined by

$$\left.\begin{aligned} N_x &= \int_{-(h/z)}^{+(h/z)} \sigma_x \left(1 - \frac{z}{a}\right) dz \; ; \; N_s = \int_{-(h/z)}^{+(h/z)} \sigma_s \, dz \\ N_{xs} &= \int_{-(h/z)}^{+(h/z)} \sigma_{xs} \left(1 - \frac{z}{a}\right) dz \; ; \; N_{sx} = \int_{-(h/z)}^{+(h/z)} \sigma_{xs} \, dz \end{aligned}\right\} \qquad (4a)$$

$$\left.\begin{aligned} Q_x &= \int_{-(h/z)}^{+(h/z)} \sigma_{xz} \left(1 - \frac{z}{a}\right) dz \; ; \; Q_s = \int_{-(h/z)}^{+(h/z)} \sigma_{sz} \, dz \end{aligned}\right\} \qquad (4b)$$

$$\left.\begin{aligned} M_x &= \int_{-(h/z)}^{+(h/z)} \sigma_x \left(1 - \frac{z}{a}\right) zdz \; ; \; M_s = \int_{-(h/z)}^{+(h/z)} \sigma_s \, zdz \\ M_{xs} &= \int_{-(h/z)}^{+(h/z)} \sigma_{xs} \left(1 - \frac{z}{a}\right) zdz \; ; \; M_{sx} = \int_{-(h/z)}^{+(h/z)} \sigma_{xs} \, zdz \end{aligned}\right\} \qquad (4c)$$

and $P_x$, $P_s$, and $P_z$ represent the effective external as well as body forces per unit area of the middle surface of the shell. These may be written as

$$\left.\begin{aligned} P_x &= -\rho h \frac{\partial^2 u}{\partial t^2} + \overline{P}_x \\ P_s &= -\rho h \frac{\partial^2 v}{\partial t^2} + \overline{P}_s \\ P_z &= -\rho h \frac{\partial^2 w}{\partial t^2} + \overline{P}_z \end{aligned}\right\} \quad (5)$$

where $\overline{P}_x$, $\overline{P}_s$, and $\overline{P}_z$ do not involve the displacements u, v, and w and $\rho$ denotes the density.

It may be mentioned that the last equilibrium equation in (3) is merely an identity, and that in the first two of equations (3) the effect of rotary inertia has been neglected.

The pertinent strain-displacement relations are

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \\ \epsilon_s &= \frac{1}{(1 - \frac{z}{a})} \left[\left(\frac{\partial v}{\partial s} - \frac{w}{a}\right) - z\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{a}\frac{\partial v}{\partial s}\right)\right] \\ \epsilon_{xs} &= \frac{1}{(1 - \frac{z}{a})} \left\{\left(1 - \frac{z}{a}\right)\left[\frac{\partial v}{\partial x} - z\left(\frac{\partial^2 w}{\partial x \partial s} \quad \frac{1}{a}\frac{\partial v}{\partial x}\right)\right] + \left[\frac{\partial u}{\partial s} - z \frac{\partial^2 w}{\partial x \partial s}\right]\right\} \end{aligned}\right\} \quad (6)$$

and the corresponding stress-strain relations are given by

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_s) \\ \sigma_s &= \frac{E}{1 - \nu^2} (\epsilon_s + \nu \epsilon_x) \\ \sigma_{xs} &= \frac{E}{2(1 + \nu)} \epsilon_{xs} \end{aligned}\right\} \quad (7)$$

where E is Young's Modulus and $\nu$ is Poisson's ratio.

The basic equations of the theory of thin shells in the sense of Love's first approximation may now be deduced if the ratio z/a is neglected in comparison with unity in equations (1), (4), and (6). Thus, this approximation affects the stress resultants and stress couples, as well as the components of strain given by equation (6).

If we write the components of strain in the form

$$\epsilon_x = e_x + z\chi_x; \qquad \epsilon_s = e_s + z\chi_s; \qquad \epsilon_{xs} = e_{xs} + z\chi_{xs} \tag{8}$$

then from equation (6), the membrane strains or components of strain at the middle surface (z = 0) are

$$e_x = \frac{\partial u}{\partial x}; \qquad e_s = \frac{\partial v}{\partial s} - \frac{w}{a}; \qquad e_{xs} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \tag{9}$$

and the expressions for the change of curvatures are given by*

$$\chi_x = -\frac{\partial^2 w}{\partial x^2}; \qquad \chi_s = -\frac{1}{a}\frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial s^2}; \qquad \chi_{xs} = -\frac{1}{a}\frac{\partial v}{\partial x} - z\frac{\partial^2 w}{\partial x \partial s} \tag{10}$$

Introducing relations (7) into the stress-resultants (4a) and the stress-couples (4c) with the appropriate neglects of z/a terms, performing the required integration, and making use of relations (8), the following stress-strain relations result:

$$\left.\begin{aligned} N_x &= \frac{Eh}{1-\nu^2}(e_x + \nu e_s) \\ N_s &= \frac{Eh}{1-\nu^2}(e_s + \nu e_x) \\ N_{xs} = N_{sx} &= \frac{Eh}{2(1+\nu)} e_{xs} \end{aligned}\right\} \tag{11a}$$

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*It appears that there is no uniformity in the literature for the expression of twist $\chi_{xs}$. Love,[2] Timoshenko,[4] and others write $[-(2/a)(\partial v/\partial x) - 2(\partial^2 w/\partial x\partial s)]$ for the twist (when conformed to our notation), while $\chi_{xs}$ of this paper is in accord with that of reference 1. In the spirit of Love's first approximation, if "the neglect of z/a in comparison with unity" is introduced in both equations (6) and (4) before deriving equations (11), then $\chi_{xs}$ is given by equation (10). However, if this approximation is introduced following the substitution of (6) into equation (4), then $\chi_{xs}$ would have the form given by Love and Timoshenko; in this connection, see reference 5.

$$\left.\begin{aligned} M_x &= D(\chi_x + \nu\chi_s) \\ M_s &= D(\chi_s + \nu\chi_x) \\ M_{xs} &= \frac{D(1 - \nu)}{2}\chi_{xs} \end{aligned}\right\} \qquad (11b)$$

where $D = Eh^3/12(1 - \nu^2)$.

By equations (8), (7), and (11), the pertinent components of the stress tensor are given by

$$\left.\begin{aligned} \sigma_x &= \frac{N_x}{h} + \frac{12z}{h^3} M_x \\ \sigma_s &= \frac{N_s}{h} + \frac{12z}{h} M_s \\ \sigma_{xs} &= \frac{N_{xs}}{h} + \frac{12z}{h} M_{xs} \end{aligned}\right\} \qquad (12)$$

It should be noted that $Q_x$ and $Q_s$ cannot be obtained from the expressions (4b) since, by virtue of assumption (4), $\sigma_{xz}$ and $\sigma_{sz}$ have been neglected. However, if equation (10) is substituted into the first two of equations (3), the expressions for $Q_x$ and $Q_s$ will result.

The five independent differential equations of equilibrium, namely equations (2) and the first two of (3), remain unaltered in the present approximation. But it is noted that in view of the last of equations (11a), the identity of the last of moment equilibrium equations (3) is not satisfied. This discrepancy is due to the approximation introduced and it is difficult to see how this can appreciably affect the solution of a specific problem.

Equations (9), (10), (11), (12), and the first two of (3) constitute seventeen equations in the seventeen unknowns $M_x$, $M_s$, $M_{xs}$; $N_x$, $N_s$, $N_{xs}$; $Q_x$, $Q_s$; $e_x$, $e_s$, $e_{xs}$; $\chi_x$, $\chi_s$, $\chi_{xs}$; and u, v, and w. These equations describe completely the state of stress and deformation of a thin circular cylindrical shell.

## REDUCTION TO THREE PARTIAL DIFFERENTIAL EQUATIONS

We now reduce the seventeen equations of the preceding section to three displacement equations of motion, as follows. Eliminate $Q_x$ and $Q_s$ from (2) by (3), and using relations (11), (9) and (10), we arrive at three coupled independent equations involving u, v, and w.

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1 - \nu)}{2}\frac{\partial^2 u}{\partial s^2} + \frac{(1 + \nu)}{2}\frac{\partial^2 v}{\partial x \partial s} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{(1 - \nu^2)}{Eh} P_x = 0 \qquad (13a)$$

$$\left.\begin{aligned} &\frac{(1 + \nu)}{2}\frac{\partial^2 u}{\partial x \partial s} + \frac{(1 - \nu)}{2}(1 + k)\frac{\partial^2 v}{\partial x^2} + (1 + k)\frac{\partial^2 v}{\partial s^2} - \frac{1}{a}\frac{\partial w}{\partial s} \\ &+ ka\frac{\partial^3 w}{\partial x^2 \partial s} + ka\frac{\partial^3 w}{\partial s^3} + \frac{(1 - \nu^2)}{Eh} P_s = 0 \end{aligned}\right\} \qquad (13b)$$

$$\left.\begin{aligned} &\frac{\nu}{a}\frac{\partial u}{\partial x} - ka\frac{\partial^3 v}{\partial x^2 \partial s} - ka\frac{\partial^3 v}{\partial s^3} + \frac{1}{a}\frac{\partial v}{\partial s} - ka^2\frac{\partial^4 w}{\partial x^4} \\ &-2ka^2\frac{\partial^4 w}{\partial x^2 \partial s^2} - ka^2\frac{\partial^4 w}{\partial s^4} - \frac{w}{a^2} + \frac{(1 - \nu^2)}{Eh} P_z = 0 \end{aligned}\right\} \qquad (13c)$$

where $k = h^2/12a^2$.

We shall now uncouple equations (13) and obtain three equations, one involving u and w, one in v and w, and a single differential equation involving w alone. This may be carried out as follows:

1. Apply the operators $\partial^2/\partial x^2$ and $\partial^2/\partial s^2$ to (13a) independently of one another, and in each case solve for terms containing v.

2. Apply the operator $\partial^2/\partial x \partial s$ to (13b) and using the results of the previous step after some combination, the resulting equation becomes

$$\left.\begin{aligned}&(1+k)\nabla^4 u + \frac{(1+\nu)^2}{2(1-\nu)} k \frac{\partial^4 u}{\partial x^2 \partial s^2} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1}{a}\frac{\partial^3 w}{\partial x \partial s^2}\\ &- \frac{k}{a}\left[v \frac{\partial^3 w}{\partial x^3} + \frac{2v}{(1-\nu)} \frac{\partial^3 w}{\partial x \partial s^2}\right] - \frac{(1+\nu)}{(1-\nu)} ka \left[\frac{\partial^5 w}{\partial x^3 \partial s^2} + \frac{\partial^5 w}{\partial x \partial s^4}\right]\\ &+ \frac{(1+\nu)}{Eh}(1+k)\left[2\frac{\partial^2 P_x}{\partial s^2} + (1-\nu)\frac{\partial^2 P_x}{\partial x^2}\right] - \frac{(1+\nu)^2}{Eh}\frac{\partial^2 P_s}{\partial x \partial s} = 0\end{aligned}\right\} \quad (14a)$$

where $\nabla^4 \equiv \partial^4/\partial x^4 + 2(\partial^4)/\partial x^2 \partial s^2 + \partial^4/\partial s^4$ .

In an analogous manner, application of the same operator as before to (13b) and (13a) yields

$$\left.\begin{aligned}&(1+k)\nabla^4 v + \frac{(1+\nu)^2}{2(1-\nu)} k \frac{\partial^4 v}{\partial x^2 \partial s^2} - \frac{(2+\nu)}{a}\frac{\partial^3 w}{\partial x^2 \partial s} - \frac{1}{a}\frac{\partial^3 w}{\partial s^3} + ka\\ &\cdot\left[\frac{2}{(1-\nu)}\frac{\partial^5 w}{\partial x^4 \partial s} + \frac{(3-\nu)}{(1-\nu)}\frac{\partial^5 w}{\partial x^2 \partial s^3} + \frac{\partial^5 w}{\partial s^5}\right] + \frac{(1+\nu)}{Eh}\\ &\cdot\left[2\frac{\partial^2 P_s}{\partial x} + (1-\nu)\frac{\partial^2 P_s}{\partial s}\right] - \frac{(1+\nu)^2}{Eh}\frac{\partial^2 P_x}{\partial x \partial s} = 0\end{aligned}\right\} \quad (14b)$$

Similarly, by operating on (14a) with $\partial/\partial x$ and on (14b) with $1/\partial s$ and combining these results with that obtained from the application of $\nabla^4$ to (13c), we obtain

$$
\begin{aligned}
&\nabla^8 w + \frac{1}{ka^4}\nabla^4 w - \frac{1}{a^4k(1+k)}\Bigg\{\nabla^4 w - (1-u^2)\frac{\partial^4 w}{\partial x^4} \\
&+\nu^2 k\left[\frac{\partial^4 w}{\partial x^4} + \frac{2}{(1-\nu)}\frac{\partial^4 w}{\partial x^2\partial s^2}\right] - ka^2\Bigg[(2+\nu)\frac{\partial^6 w}{\partial x^4\partial s^2} \\
&+ (3+\nu)\frac{\partial^6 w}{\partial k^2\partial s^4} + \frac{\partial^6 w}{\partial s^6}\Bigg] - \frac{(1+\nu)^2}{2(1-\nu)}ka\left[\nu\frac{\partial^5 u}{\partial x^3\partial s^2} + \frac{\partial^5 v}{\partial x^2\partial s^3}\right]\Bigg\} \\
&+ \frac{1}{a}\nabla^4\left[\frac{\partial^3 v}{\partial x^2\partial s} + \frac{\partial^3 v}{\partial s^3}\right] + \frac{v}{(1-\nu)}\frac{1}{aD}\left[2\frac{\partial^3 P_x}{\partial x\partial s^2} + (1-\nu)\frac{\partial^3 P_x}{\partial x^3}\right] \\
&- \frac{(1+\nu)}{(1-\nu)}\frac{\nu}{aD}\frac{1}{(1+k)}\frac{\partial^3 P_s}{\partial x^2\partial s} + \frac{1}{(1-\nu)}\frac{2}{aD}\frac{1}{(1+k)} \\
&\left[\frac{\partial^3 P_s}{\partial x^2\partial s} + \frac{(1-\nu)}{2}\frac{\partial^3 P_s}{\partial s^3}\right] - \frac{1}{aD}\frac{1}{(1+k)}\frac{(1+\nu)}{(1-\nu)}\frac{\partial^3 P_x}{\partial x\partial s^2} - \frac{1}{D}\nabla^4 P_z = 0
\end{aligned}
\qquad (14c)
$$

If we now introduce equations (5) into equations (14), we observe that the latter are not as yet uncoupled. The terms containing v and u in equations (14a) and (14b) respectively may be eliminated as follows:

1. Operate on both equations (13a) and (13b) with $2(1 + \nu)\ (\rho/E)(\partial^2/\partial t^2)$.

2. Subtract the results in each case from (14a) and (14b) respectively, to arrive at equations (15a) and (15b). In equation (14c), the terms involving u and v are

$$\frac{(1+\nu)^2}{2(1-\nu)} \frac{1}{a^3(1+k)} \frac{\partial^4}{\partial x^2 \partial s^2}\left[\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s}\right] + \frac{1}{a}\frac{\partial}{\partial s}(\nabla^6 v) - \frac{\nu}{1-\nu}\frac{\rho h}{aD}\frac{\partial^2}{\partial z^2}$$

$$\cdot\left[2\frac{\partial^3 u}{\partial x \partial s^2} + (1-\nu)\frac{\partial^3 u}{\partial x^3}\right] + \frac{(1+\nu)}{(1-\nu)}\frac{\nu\rho h}{aD}\frac{1}{1+k}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^3 v}{\partial x^2 \partial s}\right)$$

$$-\frac{1}{1-\nu}\frac{\rho h}{aD}\frac{1}{1+k}\frac{\partial^2}{\partial t^2}\left[2\frac{\partial^3 v}{\partial x^2 \partial s} + (1-\nu)\frac{\partial^3 v}{\partial s^3}\right] + \frac{1+\nu}{1-\nu}\frac{\rho h}{aD}\frac{1}{1+k}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^3 u}{\partial x \partial s^2}\right).$$

and the elimination of these from (14c) is accomplished in the following manner:

1. Apply the operator $\nu/[k(1+k)a^3][\partial/\partial x]$ to (14a).

2. Apply the operator $1/[k(1+k)a^3][\partial/\partial s]$ to (14b).

3. Add the results of steps 1 and 2.

4. Apply the operator $1/(ka^4)\nabla^4$ to (13c) and subtract the result from that of step 3.

5. Subtract the result of step 4 from (14c) to obtain equation (15c).

The differential equations (15) are the uncoupled equations of motion of circular cylindrical shells in the unknowns u, v and w.*

It is clear from the differential equations (15) that the order of the primitive equations (13) has been raised from an eighth-order to a sixteenth-order set of partial differential equations. But, since the homogeneous differential equations associated with (15a,b) do not yield independent solutions, the result is mathematically consistent with the existence of four boundary conditions at each edge of the cylindrical shell.

*Since $k = h^2/12a^2 \ll 1$, it is reasonable to dimplify these equations by replacing the quantities $(1 + k)$ by unity.

$$
\left.
\begin{aligned}
&(1+k)\nabla^4 u + \frac{(1+\nu)^2}{2(1-\nu)} k \frac{\partial^4 u}{\partial x^2 \partial s^2} - \frac{(1+\nu)}{E}\rho \frac{\partial^2}{\partial t^2} \\
&\left\{(1+k)\left[2\frac{\partial^2 u}{\partial s^2} + (1-\nu)\frac{\partial^2 u}{\partial x^2}\right] + 2\frac{\partial^2 u}{\partial x^2} + (1-\nu)\frac{\partial^2 u}{\partial s^2} - 2\frac{(1-\nu^2)}{E}\rho\frac{\partial^2 u}{\partial t^2}\right\} \\
&- \frac{\nu}{a}(1+k)\frac{\partial^3 w}{\partial x^3} + \frac{1}{a}\left(1 - \frac{2\nu}{1-\nu}k\right)\frac{\partial^3 w}{\partial x \partial s^2} - \frac{(1+\nu)}{(1-\nu)} ka \\
&\left[\frac{\partial^5 w}{\partial x^3 \partial s^2} + \frac{\partial^5 w}{\partial x \partial s^4}\right] + \frac{(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left[\frac{2\nu}{a}\frac{\partial w}{\partial x}\right] + \frac{(1+\nu)}{Eh}(1+k) \\
&\left[2\frac{\partial^2 \overline{P_x}}{\partial s^2} + (1-\nu)\frac{\partial^2 \overline{P_x}}{\partial x^2}\right] - \frac{(1+\nu)^2}{Eh}\frac{\partial^2 \overline{P_s}}{\partial x \partial s} - \frac{2(1+\nu)(1-\nu^2)}{E^2 h}\rho\frac{\partial^2 \overline{P_x}}{\partial t^2} = 0
\end{aligned}
\right\} \quad (15a)
$$

$$
\left.
\begin{aligned}
&(1+k)\nabla^4 v + \frac{(1+\nu)^2}{2(1-\nu)} k \frac{\partial^4 v}{\partial x^2 \partial s^2} - \frac{(1+\nu)}{E}\rho \frac{\partial^2}{\partial t^2} \\
&\left\{(1+k)\left[2\frac{\partial^2 v}{\partial s^2} + (1-\nu)\frac{\partial^2 v}{\partial x^2}\right] + 2\frac{\partial^2 v}{\partial x^2} + (1-\nu)\frac{\partial^2 v}{\partial s^2} - 2\frac{(1-\nu^2)}{E}\rho\frac{\partial^2 v}{\partial t^2}\right\} \\
&- \frac{1}{a}\frac{\partial^2 w}{\partial s^3} - \frac{2+\nu}{a}\frac{\partial^3 w}{\partial x^2 \partial s} + ka\left[\frac{2}{1-\nu}\frac{\partial^5 w}{\partial x^4 \partial s} + \frac{3-\nu}{1-\nu}\frac{\partial^5 w}{\partial x^2 \partial s^3} + \frac{\partial^5 w}{\partial s^5}\right] \\
&- \frac{(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left[-\frac{2}{a}\frac{\partial w}{\partial s} + 2ka\frac{\partial^3 w}{\partial x^2 \partial s} + 2k\frac{\partial^3 w}{\partial s^3}\right] + \frac{1+\nu}{Eh} \\
&\left[2\frac{\partial^2 \overline{P_s}}{\partial x^2} + (1-\nu)\frac{\partial^2 \overline{P_s}}{\partial s^2} - (1+\nu)\frac{\partial^2 \overline{P_x}}{\partial x \partial s}\right] - \frac{2(1+\nu)(1-\nu^2)}{E^2 h}\rho\frac{\partial^2 \overline{P_s}}{\partial t^2} = 0
\end{aligned}
\right\} \quad (15b)
$$

$$
\left.
\begin{aligned}
&\nabla^8 w + \frac{1}{a^4}\nabla^4 w + \frac{12(1-\nu^2)}{a^2h^2}\frac{\partial^4 w}{\partial x^4} - \frac{\nu^2}{a^4}\left[\frac{\partial^4 w}{\partial x^4} + \frac{2}{(1-\nu)}\frac{\partial^4 w}{\partial x^2 \partial s^2}\right] \\
&+ \frac{1}{a^2}\left[(2+\nu)\frac{\partial^6 w}{\partial x^4 \partial s^2} + (3+\nu)\frac{\partial^6 w}{\partial x^2 \partial s^4} + \frac{\partial^6 w}{\partial s^6}\right] + \frac{(2+\nu)}{a^2}\frac{\partial^4}{\partial x^2 \partial s^2} \\
&\cdot (\nabla^2 w) + \frac{1}{a^2}\frac{\partial^4}{\partial s^4}(\nabla^2 w) + \frac{1}{2a^4}\frac{(1+\nu)^2}{(1-\nu)}\frac{\partial^4 w}{\partial x^2 \partial s^2} - k \\
&\cdot\left[\frac{2}{(1-\nu)}\frac{\partial^6}{\partial x^4 \partial s^2}(\nabla^2 w) + \frac{(3-\nu)}{(1-\nu)}\frac{\partial^6}{\partial x^2 \partial s^4}(\nabla^2 w) + \frac{\partial^6}{\partial s^6}(\nabla^2 w) - \frac{(1+\nu)^2}{2(1-\nu)}\frac{\partial^4}{\partial x^2 \partial s^2}(\nabla^4 w)\right] \\
&+ \frac{\rho h}{aD}\frac{\partial^2}{\partial t^2}\Bigg\{a\nabla^4 w + \frac{ka}{2}\frac{(1+\nu)^2}{(1-\nu)}\frac{\partial^4 w}{\partial x^2 \partial s^2} + k^2a^3\frac{(1+\nu)^2}{(1-\nu)}\frac{\partial^2}{\partial x^2}(\nabla^4 w) + \frac{k}{a} \\
&\cdot\frac{(1+\nu)}{(1-\nu)}\frac{\partial^2 w}{\partial x^2} - \frac{4ka}{(1-\nu)}\frac{\partial^2}{\partial s^2}(\nabla^2 w) + \frac{2k^2a^3}{(1-\nu)}\frac{\partial^2}{\partial s^2}(\nabla^4 w) - \frac{(3-\nu+2k)}{a(1-\nu)}\nabla^2 w \\
&- \frac{ka^3(3-\nu+2k)}{(1-\nu)}\nabla^6 w + \frac{2\nu^2}{a(1-\nu)}\frac{\partial^2 w}{\partial x^2} + \frac{2}{a(1-\nu)}\frac{\partial^2 w}{\partial s^2} + ka(1+\nu)^2\frac{\rho}{E}\frac{\partial^2}{\partial x^2} \\
&\cdot\left(\frac{\partial^2 w}{\partial t^2}\right) - a(3-\nu+2k)(1+\nu)\frac{\rho}{E}\nabla^2\left(\frac{\partial^2 w}{\partial t^2}\right) + 2ka^3(1+\nu)\frac{\rho}{E}\nabla^4\left(\frac{\partial^2 w}{\partial t^2}\right) \\
&+ \frac{2(1+\nu)}{a}\frac{\rho}{E}\frac{\partial^2 w}{\partial t^2} + 2a(1+\nu)(1-\nu^2)\left(\frac{\rho}{E}\right)^2\frac{\partial^4 w}{\partial t^4} - \frac{ka(1+\nu)^2}{Eh}\frac{\partial^2 \bar{P}_z}{\partial x^2} \\
&+ \frac{2ka^2(1+\nu)}{Eh}\frac{\partial}{\partial s}(\nabla^2 \bar{P}_s) + \frac{a(3-\nu+2k)(1+\nu)}{Eh}\nabla^2 \bar{P}_z - \frac{2a(1+\nu)(1-\nu^2)}{Eh} \\
&\cdot\frac{\rho}{E}\frac{\partial^2 \bar{P}_z}{\partial t^2} - \frac{2(1+\nu)}{Eh}\left(\nu\frac{\partial \bar{P}_x}{\partial x} + \frac{\partial \bar{P}_s}{\partial s}\right)\Bigg\} + \frac{1}{aD}\Bigg\{-\frac{\partial^3 \bar{P}_x}{\partial x \partial s^2} + \nu\frac{\partial^3 \bar{P}_x}{\partial x^3} + (2-\nu)\frac{\partial^3 \bar{P}_s}{\partial x^2 \partial s} \\
&+ \frac{\partial^3 \bar{P}_s}{\partial s^3} - a\nabla^4 \bar{P}_z\Bigg\} + \frac{1}{aEh}\Bigg\{-2(1+\nu)\frac{\partial^3}{\partial x^2 \partial s}(\nabla^2 \bar{P}_s) + (1-\nu^2)\frac{\partial^3}{\partial s^3}(\nabla^2 \bar{P}_s) - \frac{2(1+\nu)^3}{a} \\
&\frac{\partial^4 \bar{P}_z}{\partial x^2 \partial s^2}\Bigg\} = 0
\end{aligned}
\right\} \quad (15c)
$$

## DISCUSSION AND COMPARISON WITH THE WORKS OF OTHER AUTHORS

In order to verify at least partially the results of equations (15), let the displacements u, v, and w be of the form

$$\left.\begin{aligned} u &= \sum_{n=0}^{\infty} A_n\, e^{\lambda x/a} \cos \frac{ns}{a}\, e^{ipt} \\ v &= \sum_{n=1}^{\infty} B_n\, e^{\lambda x/a} \sin \frac{ns}{a}\, e^{ipt} \\ w &= \sum_{n=0}^{\infty} C_n\, e^{\lambda x/a} \cos \frac{ns}{a}\, e^{ipt} \end{aligned}\right\} \qquad (16)$$

Substitution of (16) into the primitive differential equations (13), in the absence of quantities denoted by $\overline{P}_x$, $\overline{P}_s$, and $\overline{P}_z$, yields three equations in the coefficients $A_n$, $B_n$ and $C_n$. The characteristic equation obtained by setting the determinant of these coefficients equal to zero is identical with the characteristic equation which may be obtained directly from equation (15c). If, in the characteristic equation, the quantities (1 + k) are replaced by unity, then there results

$$\lambda^8 - \left[4n^2 - \frac{(3-\nu)}{(1-\nu)}\gamma p^2\right]\lambda^6 + \left[6n^4 - 2(2+\nu)\,n^2 + \frac{1-\nu^2}{k} - \gamma p^2 \cdot \left(\frac{3(3-\nu)}{(1-\nu)}n^2 + \frac{1}{k}\right) + \frac{2}{1-\nu}\gamma^2 p^4\right]\lambda^4 - \left[4n^6 - 2(3+\nu)\,n^4 + \frac{(5+3\nu)}{2}n^2 - \gamma p^2\left(\frac{3(3-\nu)}{(1-\nu)}n^4 + \frac{2n^2}{k} + \frac{(3+2\nu)}{k}\right) + \gamma^2 p^4\left(\frac{(3-\nu)}{(1-\nu)}\frac{1}{k} + \frac{4}{(1-\nu)}n^2\right)\right]\lambda^2 + \left[n^8 - 2n^6 + n^4 - \gamma p^2\left(\frac{3-\nu}{1-\nu}n^6 + \frac{n^4}{k} + \frac{n^2}{k}\right) + \gamma^2 p^4 \cdot \left(\frac{3-\nu}{1-\nu}\frac{n^2}{k} + \frac{2}{1-\nu}n^4 + \frac{2}{1-\nu}\frac{1}{k}\right) - \frac{2}{1-\nu}\frac{1}{k}\gamma^3 p^6\right] = 0\,, \qquad (17)$$

where $\gamma = [(1 - \nu^2)\rho a^2]/E$.

The characteristic equation associated with the differential equations (15) when the displacements are independent of time may be obtained from equation (17) by putting the terms involving $p^2$ equal to zero. Thus,

$$\left.\begin{aligned} &\lambda^8 - 4n^2\lambda^6 + \left[\frac{1 - \nu^2}{k} + 6n^4 - 2(2 + \nu)\, n^2\right] \lambda^4 \\ &+ \left[-4n^6 + 2(3 + \nu)\, n^4 - \frac{(5 + 3\nu)}{2}\, n^2\right] \lambda^2 \\ &+ [n^8 - 2n^6 + n^4] = 0 \,. \end{aligned}\right\} \tag{18}$$

Since the characteristic equation is relatively easy to obtain, it furnishes a convenient basis for comparison with the results (equations of motion) given by other authors. To be consistent, in the comparison that follow, quantities of the type $(1 + k)$ will be replaced by unity. Also, the displacements will be assumed to have the form of equations (16).

Flügge[6] and Byrne[7] retain terms of the order $(z/a)^2$ as compared to one in the stress-resultants (4) and strain-displacement relation (6), which results in the following characteristic equation when the displacements are independent of time.*

$$\left.\begin{aligned} &\lambda^8 - 2(2n^2 - \nu)\, \lambda^6 + \left[\frac{1 - \nu^2}{k} + 6n^4 - 6n^2\right] \lambda^4 \\ &+ [-4n^6 + 2(4 - \nu)\, n^4 - 2\,(2 - \nu)\, n^2]\, \lambda^2 \\ &+ [n^8 - 2n^6 + n^4] = 0 \,. \end{aligned}\right\} \tag{19}$$

Comparison of equations (18) and (19) reveals that retaining terms of the order $(z/a)^2$ in equations (4) and (6) may result in only a small effect** in the equations of motion.

If, in the second of equations (2), the term $Q_s/a$ is neglected as compared with the other terms, and at the same time the expressions for the change of curvatures are simplified to read

$$\chi_x = \frac{\partial^2 w}{\partial x^2} \,; \qquad \chi_s = -\frac{\partial^2 w}{\partial s^2} \,; \qquad \chi_{xs} = -2\frac{\partial^2 w}{\partial x \partial s}, \tag{20}$$

*See equation (78), page 125 of reference 6.

**One should recall that replacing $(1 + k)$ by unity does not nullify the effect of retaining the $(z/a)^2$ terms; compare equations (11), page 118 of reference 6, with equations (13) of this report.

then the resulting characteristic equation associated with the equation of motion is

$$\begin{aligned}
&\lambda^8 + \left[-4n^2 + \frac{(3-\nu)}{(1-\nu)}\gamma p^2\right]\lambda^6 + \left[6n^4 + \frac{(1-\nu^2)}{k} - 8p^2\left(\frac{3(3-\nu)}{1-\nu}n^2 + \frac{1}{k}\right)\right. \\
&\quad \left. + \frac{2}{1-\nu}\gamma^2 p^4\right]\lambda^4 + \left[-4n^6 + \gamma p^2\left(\frac{3+2\nu}{k} + \frac{2n^2}{k} + \frac{3(3-\nu)}{1-\nu}n^4\right) - \gamma^2 p^4\right. \\
&\quad \left. \cdot\left(\frac{4n^2}{1-\nu} + \frac{(3-\nu)}{(1-\nu)}\frac{1}{k}\right)\right]\lambda^2 + \left[n^8 - \gamma p^2\left(\frac{3-\nu}{1-\nu}n^6 + \frac{n^4}{k} + \frac{n^2}{k}\right) + \gamma^2 p^4\right. \\
&\quad \left. \cdot\left(\frac{2n^4}{1-\nu} + \frac{2}{1-\nu}\frac{1}{k} + \frac{(3-\nu)}{1-\nu}\frac{n^2}{k}\right) - \frac{2}{1-\nu}\frac{1}{k}\gamma^3 p^6\right] = 0 .
\end{aligned} \tag{21}$$

Although this type of approximation* leading to the characteristic equation (21) has been justly used in special cases,[9] its validity in general is questionable. In fact, comparison of equations (17) and (21) indicates that the present approximation has left unaltered the terms involving time explicitly. It is the complicated form of these terms which gives rise to most of the practical difficulty encountered in the solution of vibration problems of shells.

In formulating a theory of shells, Vlasov[10] at first partially abandons assumptions (3) and (4) and retains terms of the order $(z/a)^2$ as compared to one in the stress-resultants and the strain-displacement relations. Later, his work is simplified and the resulting equations of motion fall within the assumptions of the classical theory, and the characteristic equation for cylindrical shells becomes**

$$\begin{aligned}
&\lambda^8 + \lambda^6\left[-4n^2 + 2\nu + \frac{(3-\nu)}{1-\nu}\gamma p^2\right] + \lambda^4\left[6n^4 - (4+\nu)n^2 + \frac{(1-\nu^2)}{k} + 1 + \gamma p^2\right. \\
&\quad \left.\cdot\left(-\frac{1}{k} - \frac{3(3-\nu)}{1-\nu}\frac{n^2}{k}\right) + \frac{2}{1-\nu}\gamma^2 p^4\right] + \lambda^2\left[-4n^6 + (8-\nu)n^4 - 2n^2 + \gamma p^2\right. \\
&\quad \left.\cdot\left(\frac{3(3-\nu)}{1-\nu}n^4 - 2(3+\nu)n^2 + \frac{2n^2}{k} + \frac{(3+2\nu)}{k}\right) + \gamma^2 p^4\left(-\frac{(3-\nu)}{1-\nu}\frac{1}{k} - \frac{4}{1-\nu}n^2\right)\right] \\
&\quad + \left[n^8 - 2n^6 + n^4 + \gamma p^2\left(-\frac{(3-\nu)}{1-\nu}n^6 + \frac{(3-\nu)}{1-\nu}n^2 - \frac{n^4}{k} - \frac{n^2}{k}\right) + \gamma^2 p^4\right. \\
&\quad \left.\cdot\left(\frac{2}{1-\nu}n^4 - \frac{4}{1-\nu}n^2 + \frac{(3-\nu)}{1-\nu}\frac{n^2}{k} + \frac{2}{1-\nu}\frac{1}{k} + \frac{2}{1-\nu}\right) - \frac{2}{1-\nu}\frac{\gamma^3 p^6}{k}\right] = 0.
\end{aligned} \tag{22}$$

*Yuan's equation (6a), reference 8, may be obtained directly with this approximation; his characteristic equation is identical with equation (21) above when $p^2 = 0$.

**See Table 2 of reference 10.

It appears that there are only minor differences between equations (17) and (22) and these are essentially in the terms involving $p^2$ in the coefficients of $\lambda^2$ and the last bracket.

Finally, we compare the characteristic equation (17) with that of Kennard's recent paper[11] which is based on the work of Epstein.[12] Kennard's simplified equations of motion* yield the following characteristic equation:

$$
\begin{aligned}
&\lambda^8 + \lambda^6 \left[-4n^2 + \frac{(3-\nu)}{1-\nu}\gamma p^2\right] + \lambda^4 \left[6n^4 - \frac{(4-\nu)}{2(1-\nu)} n^2 + \frac{(2+\nu)}{2(1-\nu)}\right. \\
&\quad \left. + \frac{(1-\nu^2)}{k} + \gamma p^2 \left(-\frac{3(3-\nu)}{1-\nu} n^2 - \frac{1}{k}\right) + \frac{2}{1-\nu}\gamma^2 p^4\right] + \lambda^2 \\
&\quad \cdot \left[-4n^6 + \frac{(8 - 8\nu - 3\nu^2)}{2(1-\nu)} n^4 - \frac{(2+\nu)(2-3\nu)}{2(1-\nu)} n^2 + \gamma p^2\right. \\
&\quad \cdot \left(\frac{3(3-\nu)}{1-\nu} n^4 - \frac{(4-\nu)(3-\nu)}{2(1-\nu)^2} n^2 + \frac{2n^2}{k} + \frac{3+2\nu}{k} + \frac{(2+\nu)(3-\nu)}{2(1-\nu)^2}\right) \\
&\quad \left. + \gamma^2 p^4 \left(-\frac{(3-\nu)}{(1-\nu)k} - \frac{4n^2}{1-\nu}\right)\right] + \left[n^8 - 2n^6 + n^4 + \gamma p^2\right. \\
&\quad \cdot \left(-\frac{(3-\nu)}{1-\nu} n^6 - \frac{(4+\nu)(3-\nu)}{2(1-\nu)^2} n^4 - \frac{(3-\nu)(2-\nu)}{2(1-\nu)^2} n^2 - \frac{n^4}{k} - \frac{n^2}{k}\right) \\
&\quad + \gamma^2 p^4 \left(-\frac{(4-\nu)}{(1-\nu)^2} n^2 + \frac{(2+\nu)}{(1-\nu)^2} + \frac{2}{1-\nu} n^4 + \frac{(3-\nu)}{1-\nu}\frac{n^2}{k} + \frac{2}{1-\nu}\frac{1}{k}\right) \\
&\quad \left. - \frac{2}{1-\nu^2}\gamma^3 p^6\right] = 0
\end{aligned}
\qquad (23)
$$

As in the previous case, the minor differences between equations (17) and (23) are in the terms involving $p^2$ in the coefficients of $\lambda^2$ and the last bracket. It is noteworthy that there is a close agreement between Vlasov's equation (22) and equation (23).

*Equations (22) of reference 11.

## CONCLUSION

We have not succeeded in arriving at definite conclusions concerning the relative merits of the different approximations used by various authors in the classical theory of shells. However, on the basis of comparisons made, at least for cylindrical shells, there appears to be little advantage in going beyond Love's first approximation. Indeed, any true improvement of the theory beyond Love's first approximation should take into account the effect of shear deformation and transverse normal stress.[1]

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