APPLYING THE QUANTILE OPTIMIZATION METHODS TO WATER-SUPPLY SYSTEM DESIGN

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Abstract

In this paper we consider an optimimization problem in water supply systems. This problem can be formulated as a minimization with probabilistic constraints. We present a nontraditional approach for the solution of this problem using quantile optimization algorithms. It may be practically useful, because the confidence level of probabilistic constraints is closed to unity, and it is difficult to use other gradient methods [1],[5],[8]. We present a method to obtain a guaranteeing solution of this

problem based on the Generalized Minimax Approach [4] and a sequence of linear program solutions. We show how to use stochastic quasi-gradient algorithms for quantile function optimization [2],[3] in this case. Numerical results are also discussed.

Keywords: optimization problem, probabilistic constraints, quantile optimization algorithm, guaranteeing solution, stochastic quasi-gradient algorithm.

1 Introduction.

Optimization in water-supply systems often arises in the design of distant solar powered plant for fresh water production when fresh water sources are absent. The model considers stochastic deviation of the solar plant power and the possibility of monthly external fresh water supplies. The objective function of this problem is the sum of the capital cost of solar powered plant constraction and the cost of external water supplies. We consider the probability of complete satisfaction of the demands to be restricted.

Similar problems were considered in [1],[6],[7],[8]. In general, previous methods for such problems are based on efficient probability function calculation and different kind of gradient methods using. However these methods are inefficient for solution of the problem under consideration, because their computational reguirements depend on the probabilistic constraints confidence level α approximatly as $\sim \frac{1}{(1-\alpha)^2}$ [4]. We

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consider the problem, where α is interpreted as the reliability of the system, and so α is close to unity. Our approach is based on transforming the initial problem to an equivalent unconditional quantile optimization problem. It is a specific convolution of a nonrandom objective function and probabilistic constraints. Our method for the equivalent problem uses known numerical algorithms [2],[3], which are suited to deal with high confidence level quantile functions. We also consider a guaranteeing solution method, i.e. a method for obtaining an upper bound on the optimal value of the objective function. In fact that the upper bound tends to an exact optimal value of objective function as α increases [4]. So guaranteeing solution is a good initial condition for the stochastic quasi-gradient method [3]. A combination of these methods can be effectively used for the original problem solution.

2   Problem statesment.

Let $h = (h_1, ..., h_n)$ be a nonrandom vector of monthly water demends over n months and $w = (w_1, ..., w_n)$ be a nonrandom vector of monthly external supply plans. The internal water-supply system consists of a solar powered plant for water production with area $S(m^2)$ and cistern of volume $V(m^3)$ for keeping monthly water overflow. The solar powered plant's productivity depends on monthly solar activity and can be expressed as a random vector $t = (t_1, ..., t_n)$. The probability distribution of random vector t is known with support $\Re^n_+$. The total cost of the water-supply system is given

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by:

$$\Phi = \Phi(S, V, w) = aS + bV + q\sum_{j=1}^{n} w_j \qquad (2.1)$$

where a is the cost per m^2 of solar powered plant;

b is cost per m^3 of cistern;

q is cost per m^3 of externally delivered water.

The surplus water at the end of ith month is expressed by:

$$\begin{cases} x_j = \bar{x}_{j-1} + St_j + w_j - h_j \\ x_0 = 0 \qquad\qquad\qquad j = \overline{1,n} \\ \bar{x}_{j-1} = \min(x_{j-1}, V) \end{cases} \qquad (2.2)$$

where $\bar{x}_{j-1}$ is water remainder from previous month, which can not exceed the volume of the cistern.

The system works normally if the following condition is satisfied:

$$\min_{j=1,n} x_j \geq 0 \qquad (2.3)$$

Since equation (2.2) includes the random vector t, then holding (2.3) is random event.

The optimization problem is to find nonnegative system parameters S, V and nonnegative external water supply parameter w, which minimizes the objective function (2.1). The optimal parameters are:

$$(S^*, V^*, w^*) = \arg\min_{S,V,w} \Phi(S, V, w) \qquad (2.4)$$

by given values of a, b, q, h and distribution t, subject to probabilistic constraint:

$$P(S, V, w) = P\{t : \min_{\overline{1,n}} x_j(t) \geq 0\} \geq \alpha \qquad (2.5)$$

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Here $\alpha \in (0,1)$ is the given system reliability.

Using (2.2) and (2.3) we can transform (2.5) to the probabilistic constraint of simultaneously holding of $N = n * (n+1)/2$ bilinear inequalities, because every ith month we must add i inequalities to the previous inequality system.

Problem (2.4),(2.5) is a stochastic program with a nonrandom objective function and probabilistic constraints.

3   Reduction to a quantile optimization problem.

Let us reduce the problem (2.4),(2.5) to a quantile optimization problem. This means that we must optimize a confidence level α quantile function for some auxiliary random variable distribution. This allows us to use effective numerical algorithms for high confidence level quantile function minimization. Let us express variable w_1 using (S, V, Φ) and $(w_j, \quad j = \overline{2,n})$ from (2.1). Define a new variable $z = V - \Phi/2$. After changing variables, the probability constraint (2.5) can be reduced to the following form:

$$P\{t : F_i(S, z, w_2, ..., w_n, t) \leq \Phi \quad i = \overline{1, N}\} \geq \alpha \tag{3.1}$$

where functions F_i have the structure:

$$F_i(\cdot) = S f_i(t) + \hat{f}_i(z, w_2, ..., w_n). \tag{3.2}$$

Here, $\hat{f}_i(\cdot)$ are linear functions, and $f_i(t)$ have the form:

$$f_i(t) = \sum_{m=l_i}^{k_i} t_m \quad l_i \leq k_i \leq n \quad i = \overline{1, N} \tag{3.3}$$

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Define value $\Phi_\alpha(S, z, w_2, ..., w_n)$ as the minimal value of Φ when constraints (3.1) hold. $\Phi_\alpha(\cdot)$ is the confidence level α quantile function of random variable F's distribution, where $F = F(S, z, w_2, ..., w_n, t) = \max_i\{F_i(\cdot)\}$. Define $u = (S, z, w_2, ..., w_n)$. Then the initial problem can be reduced to the following form:

$$u_\alpha = \arg\min_u \Phi_\alpha(u) \qquad (3.4)$$

subject to:

$$u \in U = \{u : S \geq 0; V \geq 0; w_j \geq 0 \quad j = \overline{1, n}\}$$

4 Guaranteeing solution algorithm.

The quantile optimization problem (3.4) is computationally difficult, because the quantile function $\Phi_\alpha(u)$ can not be expressed analytically. First we consider an algorithm for obtaining a guaranteeing solution of this problem. We use the Generalized Minimax Approach [4] to rewrite problem (3.4) as:

$$(u_\alpha, E_\alpha) = \arg\min_{u \in U, E_\alpha \in \mathbf{E}_\alpha} \max_{t \in E_\alpha} F(u, t) \qquad (4.1)$$

where $\mathbf{E}_\alpha$ is the class of confidence sets whith probability measure α. Denote $\mathbf{B}$ as the Borel σ- field on the disturbance space. Then $\mathbf{E}_\alpha \in \mathbf{B}$. Choosing an optimum set $E_\alpha \in \mathbf{E}_\alpha$ is a difficult operation. However if we want to find a guaranteeing solution, we can replace E_α by some initial approximation $E_0 \in \mathbf{E}_\alpha$. For standard Gaussian random disturbances it is convenient to select E_0 as a sphere C_α [4]. The center of

C_α is the mathematical expectation point of the random vector t and $P(C_\alpha) = \alpha$. In this case then the radius of C_α can be obtained analytically. Then the function:

$$\hat{\Phi}_\alpha(u) = \max_{t \in C_\alpha} F(u, t) \qquad (4.2)$$

will be a majorant of the quantile function $\Phi_\alpha(u)$ (3.4), i.e. :

$$\Phi_\alpha(u) \leq \hat{\Phi}_\alpha(u) \quad \forall u \in U \qquad (4.3)$$

Given $u \in U$ and (4.2), $\hat{\Phi}_\alpha(u)$ is the minimal value Φ, that satisfies all of the inequalities:

$$\max_{t \in C_\alpha} F_i(u, t) \leq \Phi \quad i = \overline{1, N} \qquad (4.4)$$

For every $i = \overline{1, N}$, the maximum in (4.4) is attained at the point $t_i^* \in \Re^n$, which is a tangency point of the hyperplane $f_i(t) = const$ and sphere C_α and can be found analytically. Hence, for calculating the guaranteeing solution, it is sufficient to solve the following linear programming problem:

$$\hat{u}_\alpha = \arg\min_{u \in U} \Phi \qquad (4.5)$$

subject to linear constraints:

$$F_i(u, t_i^*) \leq \Phi \quad i = \overline{1, N} \qquad (4.6)$$

This problem can be solved by the standard simplex algorithm.

Suppose that the guaranteeing solution $(\hat{u}_\alpha, \hat{\Phi}_\alpha)$ is discovered. Then the set of all vectors t satisfying following inequalities:

$$F_i(\hat{u}_\alpha, t) \leq \hat{\Phi}_\alpha \quad i = \overline{1, N} \qquad (4.7)$$

is a polyhedral set:

$$\hat{C} = \{t : f_i(t) \leq c_i(\hat{\Phi}_\alpha, \hat{u}_\alpha) \quad i = \overline{1, N}\} \tag{4.8}$$

where

$$c_i(\hat{\Phi}_\alpha, \hat{u}_\alpha) = \frac{\hat{\Phi}_\alpha - \hat{f}(\hat{u}_\alpha^\circ)}{\hat{S}} \quad i = \overline{1, N} \tag{4.9}$$

and $\hat{u}_\alpha^\circ = (\hat{z}, \hat{w}_2, ..., \hat{w}_n)$.

By construction of t_i^*, the polyhedron

$$f_i(t) \leq f_i(t_i^*) \quad i = \overline{1, N} \tag{4.10}$$

is circumscribed around sphere C_α.

By (4.10),(4.6) and that $\hat{S} \geq 0$, all t which satisfy (4.10) belong to $\hat{C}$. Hence $C_\alpha \in \hat{C}$, and $P(\hat{C}) > \alpha$, because polyhedron (4.10) circumscribed the sphere C_α.

Let us consider a new confidence sphere $C_{\alpha 1}$ with a radius $R_{\alpha 1} < R_\alpha$, where R_α is the radius of C_α. It is obvious that $P(C_{\alpha 1}) < \alpha$. Choose at the sphere $C_{\alpha 1}$ points t_i^{**} by analogy with the selection of t_i^*. Now, consider problem (4.5) subject to:

$$F_i(u, t_i^{**}) \leq \hat{\Phi} \quad i = \overline{1, N} \tag{4.11}$$

Let $(\hat{u}_\alpha^*, \hat{\Phi}_\alpha^*)$ be a solution to this problem. Denote:

$$\hat{C}^* = \{t : f_i(t) \leq c_i(\hat{\Phi}_\alpha^*, \hat{u}_\alpha^*) \quad i = \overline{1, N}\} \tag{4.12}$$

where the structure of $c_i(\cdot)$ is described in (4.9).

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Lemma 4.1

If $P(\hat{C}^*) \geq \alpha$, then $\hat{\Phi}_\alpha^*$ is an upper bound of the quantile function optimal value of problem (3.4) and the following inequality holds: $\hat{\Phi}_\alpha^* \leq \hat{\Phi}_\alpha$.

Proof of the Lemma 4.1 see in **Appendix A**.

This Lemma allows us to present the following algorithm for obtaining an improved upper bound of the $\Phi_\alpha(\cdot)$:

Step 1. We find a solution $(\hat{u}_\alpha, \hat{\Phi}_\alpha)$ of the linear programming problem (4.5),(4.6), where t_i^* $i = \overline{1, N}$ are selected on confidence sphere C_α with probability measure α, and the t_i^* maximize $F_i(\cdot)$ (3.2).

Step 2. Using a numerical simulation algorithm , we check inequality $P(\hat{C}) \geq \alpha$, where $\hat{C}$ is from (4.8).

Step 3. If $P(\hat{C}) > \alpha$, we decrease the confidence sphere radius with the given increment h_α and continue the cycle from **Step 1**.

Step 4. In the opposite case, we select the previous step's linear program solution, $\hat{\Phi}_\alpha(\cdot)$, as an upper bound on the optimum value of $\Phi_\alpha(\cdot)$ in problem (3.4) and set $u_\alpha = \hat{u}_\alpha$.

5   Stochastic quasi-gradient algorithm.

As mentioned above, the quantile function can not be analyticaly expressed in this problem. This complicates obtaining an exact solution. We propose algorithms, which are based on the idea of using stochastic quasi-gradient vector of $\Phi_\alpha(\cdot)$. Let us

consider possible ways of getting quantile function statistical estimates. Using these estimates we can calculate a stochastic quasi-gradient vector.

Denote an independent sample of random vector t values as $\{t_k\}_{k=1}^\tau$ and $\{F_k\}_{k=1}^\tau$ to be an ordered collection of function $F(u, t_k)$ values: $F_1 \leq F_2 \leq ... \leq F_\tau$, i.e. a variational sequence. Then the confidence level α quantile function $\Phi_\alpha(u)$ of random variable $F(u, t)$ distribution can be estimated by the following statistic:

$$\Phi_\alpha^\tau(u) = F_{[\tau\alpha]}, \quad \tau = [\frac{1}{(1-\alpha)}] \tag{5.1}$$

where $[\tau\alpha]$ is the integer path of the $\tau\alpha$ value. Estimate (5.1) is $\sqrt{\tau}$- consistent and $\Phi_\alpha^\tau(u)$ is also an asymptotically effective estimate.

However, when we deal with a high confidence level quantile function, it is difficult to use this estimate, because in this case the sample size must be more then $T_\alpha = [\frac{1}{(1-\alpha)}]$, and process of the variational sequence tracing becomes very complicated. In this case we prefer to use another estimation procedure, based on an asymptotic propety of extremal order statistics $F_\tau, F_{\tau-1}$, i.e. extreme right terms of the variational sequence:

$$\hat{\Phi}_\alpha^\tau(u) = F_\tau - (F_\tau - F_{\tau-1})(\mu + \ln(\tau) + \ln(1 - \tau)) \tag{5.2}$$

where $\tau \leq [\frac{1}{(1-\alpha)}]$. Estimate (5.2) allows us to estimate high confidence level quantile function using small size samples. Moreover, we need only extreme members of the sample. Statistical propeties of estimates (5.1),(5.2) were analyzed in [2],[3], where the fitness of these estimates for using in stochastic quasi-gradient algorithms for quantile function minimization was shown.

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Let us define a quantile function stochastic quasi-gradient as a random vector:

$$\phi^s(u) = \frac{1}{2\beta_s} \sum_{i=1}^{r} [\hat{\Phi}_{\alpha}^{\tau}(\tilde{u}_1, ..., u_i + \beta_s, ..., \tilde{u}_r) - \hat{\Phi}_{\alpha}^{\tau}(\tilde{u}_1, ..., u_r - \beta_s, ..., \tilde{u}_r]e_i \qquad (5.3)$$

where β_s is a nonrandom decreasing sequence term; r is the dimention of vector u; $\tilde{u}_i \quad i = \overline{1,r}$ are random variables, which have uniform distribution on $[u_i - \beta_s, u_i + \beta_s]$; $\hat{\Phi}_{\alpha}^s(u)$ are independent quantile function estimates (5.1),(5.2); $e_i \quad i = \overline{1,r}$ are basis vectors.

The stochastic quasi-gradient procedure for quantile function minimization is :

$$u^{s+1} = \rho_U(u^s - \phi^s(u)p_s) \quad u^0 \in U \qquad (5.4)$$

where $\rho_U(u)$ is projector to the set U; p_s is a nonrandom sequence of step multipliers; s is the number of the procedure step. Conditions of strong convergence of u^s to u_α are considered in [3].

As an initial condition u^0 for procedure (5.4) we take $\hat{u}_\alpha$, the guaranteeing solution obtained in Section 4.

6 Numerical example.

The considered problem is a system with $n = 6$ months. Two reliability levels, $\alpha = 0.99$ and $\alpha_1 = 0.999$ are considered. Initial data for this problem are : $a = 3.75[unit/m^2]$; $b = 10[unit/m^3]$; $q = 25[unit/m^3]$.

The object water demand per month:

$h = [29.6; 0.0001; 23.9; 36.2; 82.1; 173.4].$

We suppose, that the random vector t has normal distribution with parameters:

$$E_t = [0.00837; 0.00828; 0.0185; 0.0631; 0.123; 0.137]$$

$$\sigma_t = [0.000582; 0.000552; 0.00123; 0.00421; 0.00818; 0.00916].$$

Random variables t_j $j = \overline{1,6}$ are independent.

This problem is model one. We choose short system life for obtaining results convenient for analysis. The results essentially depend on the a,q,E_t,σ_t parameters relations. We omit the description of computational aspects of this problem. It is more important that this example reflects correlations between guaranteeing and stochastic quasi-gradient approaches.

Results of the guaranteeing and asymptotically exact solutions are given for $\alpha = 0.99$ and $\alpha_1 = 0.999$ in Table 1. Results are given with accuracy two significant figure after point.

Table 1.: Numerical Example Solutions

	Guaranteeing solution GS		Asymptotic exact solution AES		abs(GS-AES)	
var.	$\alpha = 0.99$	$\alpha_1 = 0.999$	$\alpha = 0.99$	$\alpha_1 = 0.999$	$\alpha = 0.99$	$\alpha_1 = 0.999$
Φ	4905.96	4995.02	4879.05	4973.78	26.91	21.24
S	1000.19	1017.03	1001.29	1017.22	1.1	0.19
V	61.33	63.21	59.14	61.11	2.19	2.1
w_1	22.48	22.61	22.06	22.76	0.42	0.15
w_2	0.	0.	0.	0.	0.	0.
w_3	0.	0.	0.	0.	0.	0.
w_4	0.	0.	0.	0.	0.	0.
w_5	0.	0.	0.	0.	0.	0.
w_6	0.	0.	0.	0.	0.	0.

From the table, we see that for the given conditions the optimal water supply

system satisfies demand by using fresh water produced by the solar powered plant. Only in the first month, when plant productivity is small, does the optimal solution uses an external water supply. This fact can be explained also by the small capital cost of solar powered plant construction.

With increasing α, the guaranteeing solution tends to the asymtotically exact solution. This fact allows us effectively to use the guaranteeing solution algorithm itself, and also to accelerate convergence of stochstic quasi-gradient methods using guaranteeing solution as a good start point.

7   Conclusion.

Choosing optimal water supply system structure is a typical optimization problem with probabilistic constraints. We present a nontraditional algorithm for this problem solution, reducing it to a quantile optimization problem. This approach allows us to avoid the main lack of solution procedures for optimization problems with probabilistic constraints. Algorithms traditionally used suffer increasing computational difficulties and decreasing precision when the confidence level α increases. We considered the guaranteeing solution algorithm for obtaining successively improving upper bounds on the objective function value. This algorithm can be reduced to the sequential solution of linear programming problems. The computational requirements depend of the confidence level α only by checking the condition $P(\hat{C}) \geq \alpha$ under each iteration of the algorithm, where $\hat{C}$ is polyhedral set of random vector t. However,

we can regulate the number of iterations by choosing the negative increment of the confidence sphere radius. On the other hand, under increasing of α, the guaranteeing solution tends to the exact one. Thus we can obtain a good approximation of the exact solution and find a good starting condition for the stochastic quasi-gradient method.

Moreover, the computational complecity of the considered stochastic quasi-gradient method $(o(\frac{1}{1-\alpha}))$ increases more slowly than previous methods $(o(\frac{1}{(1-\alpha)^2}))$

So the combination of the two presented methods is an effective algorithm for solving the cosidered problems.

Appendix A

Proof. of **Lemma 4.1**

1.   Consider set $\hat{C}^*$ (4.12) as a confidence set instad of C_α. Then inequalities (4.6) are:

$$\frac{\hat{\Phi}^*_\alpha - \hat{f}_i(\hat{u}^{*\circ}_\alpha)}{\hat{S}^*} S + \hat{f}_i(u^\circ) \leq \Phi \quad i = \overline{1,N} \tag{7.1}$$

where $u^\circ = (z, w_2, ..., w_n)$.

It is obvious, that (7.1) holds for the solution $(\hat{u}^*_\alpha, \hat{\Phi}^*_\alpha)$. Hence $(\hat{u}^*_\alpha, \hat{\Phi}^*_\alpha)$ is a feasiable solution for problem (4.5),(4.6), where $t^*_i \quad i = \overline{1,N}$ are selected on the confidence set $\hat{C}^*$. It follows from the condition of the lemma, that $P(\hat{C}^*) \geq \alpha$. Therefore, $\hat{\Phi}^*_\alpha$ is an upper bound on the objective function optimal value in the problem (3.4) [4].

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2. Functions $f_i(\cdot)$ (3.3) are linear combinations of t_j $j = \overline{1,n}$ with non-negative coefficients. By $R_\alpha > R_{\alpha 1}$, and the fact that vector $\mathbf{E_t}$ consists of positive elements, the following inequalities hold:

$$f_i(t_i^{**}) < f_i(t_i^*) \quad i = \overline{1,N} \tag{7.2}$$

In fact, $(\hat{u}_\alpha, \hat{\Phi}_\alpha)$ is the solution of the problem (4.5),(4.6). So by (3.2),(7.2) and $S \geq 0$, inequalities

$$F_i(\hat{u}_\alpha, t_i^{**}) \leq \hat{\Phi}_\alpha \quad i = \overline{1,N} \tag{7.3}$$

also hold. Hence, $(\hat{u}_\alpha, \hat{\Phi}_\alpha)$ is a feasible solution for problem (4.5),(4.11), where the t_i^{**} are selected on the cofidence set $C_{\alpha 1}$. However $(\hat{u}_\alpha^*, \hat{\Phi}_\alpha^*)$ is the optimal solution of this problem. Therefore, $\hat{\Phi}_\alpha^* \leq \hat{\Phi}_\alpha$. This completes the proof of this Lemma.

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