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Page

3 Equation (2.2) should be numbered (2).

10 In lines 2 and 4, the equations should read

$$X(\lambda) = \int_{-\infty}^{\infty} x(t)\epsilon^{-j2\pi\lambda t}dt$$

and

$$X(t) = \int_{-\infty}^{\infty} X(\lambda)\epsilon^{j2\pi\lambda t}dt$$

15 In Eq. (27) the elements are 0 below diagonal.

17 Equation (33) should read:

$$R_X(f) = \begin{cases} r_i & \text{for } (i-1)\Delta f \leq f < i\Delta f, \quad i = 1, 2, \ldots, mN \\ 0 & \text{for } m \leq f < n \end{cases},$$

19 Line 4 should read: $C_Y(f)$ has been defined equal to zero for $m \leq f < n$ in order to emphasize the fact that the ···

In Line 8, a comma should be placed after "vectors".

Equation (39) should read:

$$\Lambda(f) = \begin{cases} \lambda_i & \text{for } (i-1)\Delta f \leq f < i\Delta f, \quad i = 1, 2, \ldots, mN \\ 0 & \text{for } m \leq f < n \text{ .} \end{cases}$$

26 Line 18 should read:

...," that is, the matrix in Eq. 3 is lower triangular.

29 Equation (50) should read:

$$\begin{bmatrix} y^1(0) \\ y^2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

35 In Line 4, replace

$$(\bar{\bar{x}}, \bar{\bar{x}}_1) \text{ with } (\bar{\bar{x}}, \bar{\bar{x}}_1)_{P_1}$$

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A TRANSFORM TECHNIQUE FOR MULTIVARIABLE, TIME-VARYING, DISCRETE-TIME, LINEAR SYSTEMS

by

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1. INTRODUCTION

In the last several years the use of matrices to characterize time-varying, discrete-time, linear systems has been growing. For example, Friedland (Ref. 1), has done much important work in this area. Cruz (Ref. 2) has shown how this idea can be employed in the design of control systems. Recently, the author showed (Ref. 3) that many of the frequency response concepts of time-invariant systems could be generalized so that they were meaningful for time-varying systems. So far attention has been centered on single-input, single-output systems. It is the purpose of this article to show that the previously developed methods can be applied, after certain changes and re-interpretations, to multivariable systems.

2. MATRIX REPRESENTATION FOR MULTIVARIABLE SYSTEMS

Consider the multivariable systems shown in Fig. 1. The input on the kth input channel, where k = 1, ..., m, is represented by the sequence $\{x^k(t_1), x^k(t_2), \ldots, x^k(t_N)\}$. Correspondingly, the output from the jth output channel, where j = 1, ..., n, is represented by the sequence $\{y^j(t_1), y^j(t_2), \ldots, y^j(t_N)\}$. Note that it has been assumed

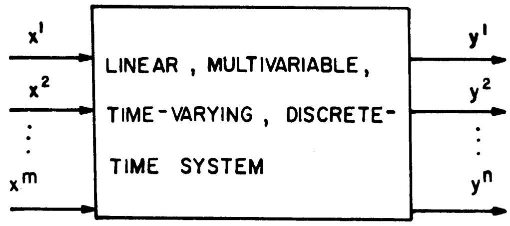


Fig. 1. Block diagram of multivariable system under consideration

that the sampling times are the same for all channels. Although this assumption is not required, it is employed here to simplify the equations which follow. The mathematical model assumed for this system is

$$y^j(t_\ell) = \sum_{k=1}^{m} \sum_{s=1}^{N} g_{jk}[t_\ell, t_s] x^k(t_s) \qquad \begin{pmatrix} j = 1, \ldots, n \\ \ell = 1, \ldots, N \end{pmatrix}, \tag{1}$$

where $g_{jk}[t_\ell, t_s]$ is the output of the jth channel at time t_ℓ caused by a unit input in the kth channel at time t_s. It is assumed here that the system is in its zero state before t_1, that is, all initial conditions are assumed to be zero.[1] Equation 1 can be recast in the following matrix formulation:

[1]In the case where this assumption is not possible, the initial state can be incorporated into the input vector and a development similar to the one presented here can be carried out.

$$
\begin{bmatrix} y^1(t_1) \\ y^2(t_1) \\ \cdot \\ y^n(t_1) \\ y^1(t_2) \\ y^2(t_2) \\ \cdot \\ y^n(t_2) \\ \cdots \\ y^1(t_N) \\ y^2(t_N) \\ \cdot \\ y^n(t_N) \end{bmatrix}
=
\left[\begin{array}{cccc|c|cccc}
g_{11}[t_1,t_1] & g_{12}[t_1,t_1] & \cdots & g_{1m}[t_1,t_1] & & g_{11}[t_1,t_N] & g_{12}[t_1,t_N] & \cdots & g_{1m}[t_1,t_N] \\
g_{21}[t_1,t_1] & g_{22}[t_1,t_1] & \cdots & g_{2m}[t_1,t_1] & \cdots & g_{21}[t_1,t_N] & g_{22}[t_1,t_N] & \cdots & g_{2m}[t_1,t_N] \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
g_{n1}[t_1,t_1] & g_{n2}[t_1,t_1] & \cdots & g_{nm}[t_1,t_1] & & g_{n1}[t_1,t_N] & g_{n2}[t_1,t_N] & \cdots & g_{nm}[t_1,t_N] \\
\hline
g_{11}[t_2,t_1] & g_{12}[t_2,t_1] & \cdots & g_{1m}[t_2,t_1] & & & & & \\
g_{21}[t_2,t_1] & g_{22}[t_2,t_1] & \cdots & g_{2m}[t_2,t_1] & \cdots & & \cdots & & \\
\cdot & \cdot & \cdot & \cdot & & & & & \\
g_{n1}[t_2,t_1] & g_{n2}[t_2,t_1] & \cdots & g_{nm}[t_2,t_1] & & & & & \\
\hline
 & \cdots & & & \cdots & & \cdots & & \\
\hline
g_{11}[t_N,t_1] & g_{12}[t_N,t_1] & \cdots & g_{1m}[t_N,t_1] & & g_{11}[t_N,t_N] & g_{12}[t_N,t_N] & \cdots & g_{1m}[t_N,t_N] \\
g_{21}[t_N,t_1] & g_{22}[t_N,t_1] & \cdots & g_{2m}[t_N,t_1] & \cdots & g_{21}[t_N,t_N] & g_{22}[t_N,t_N] & \cdots & g_{2m}[t_N,t_N] \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
g_{n1}[t_N,t_1] & g_{n2}[t_N,t_1] & \cdots & g_{nm}[t_N,t_1] & & g_{n1}[t_N,t_N] & g_{n2}[t_N,t_N] & \cdots & g_{nm}[t_N,t_N]
\end{array}\right]
\begin{bmatrix} x^1(t_1) \\ x^2(t_1) \\ \cdot \\ x^m(t_1) \\ \\ \\ \cdots \\ \\ \\ x^1(t_N) \\ x^2(t_N) \\ \cdot \\ x^m(t_N) \end{bmatrix}
\tag{2.2}
$$

Equation 2 can be more simply written

$$
\begin{bmatrix} \bar{y}(t_1) \\ \bar{y}(t_2) \\ \cdot \\ \bar{y}(t_N) \end{bmatrix}
=
\begin{bmatrix} G[t_1,t_1] & \cdots & G[t_1,t_N] \\ G[t_2,t_1] & \cdots & G[t_2,t_N] \\ \cdot & \cdot & \cdot \\ G[t_N,t_1] & \cdots & G[t_N,t_N] \end{bmatrix}
\begin{bmatrix} \bar{x}(t_1) \\ \bar{x}(t_2) \\ \cdot \\ \bar{x}(t_N) \end{bmatrix} , \tag{3}
$$

where

$$
\bar{y}(t_\ell) = \begin{bmatrix} y^1(t_\ell) \\ y^2(t_\ell) \\ \cdots \\ y^n(t_\ell) \end{bmatrix} \qquad (\ell = 1, \ldots, N) , \tag{4}
$$

$$
G[t_\ell,t_s] = \begin{bmatrix} g_{11}[t_\ell,t_s] & \cdots & g_{1m}[t_\ell,t_s] \\ & \cdots & \\ g_{n1}[t_\ell,t_s] & \cdots & g_{nm}[t_\ell,t_s] \end{bmatrix} \qquad (\ell, s = 1, \ldots, N) , \tag{5}
$$

$$\bar{x}(t_s) = \begin{bmatrix} x^1(t_s) \\ x^2(t_s) \\ \cdots \\ x^m(t_s) \end{bmatrix} \qquad (s = 1, \ldots, N) \ . \tag{6}$$

It is clear that $\bar{y}(t_\ell)$ characterizes the output on all channels at time t_ℓ; $\bar{x}(t_s)$ characterizes the input on all channels at time t_s. $G[t_\ell, t_s]$ characterizes the output on all channels at time t_ℓ caused by a unit input on all channels at time t_s and can be looked upon as the "multivariable unit response." In order to simplify the expressions which appear below, Eq. 2 is further condensed by writing it in the form

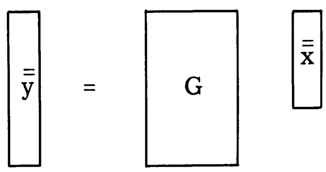
$$\bar{\bar{y}} = G\bar{\bar{x}} \ , \tag{7}$$

where

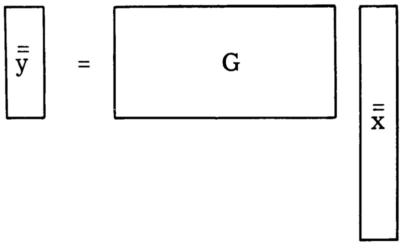
$$\bar{\bar{y}} = \begin{bmatrix} \bar{y}(t_1) \\ \bar{y}(t_2) \\ \cdots \\ \bar{y}(t_N) \end{bmatrix} , \quad G = \begin{bmatrix} G[t_1, t_1] & \cdots & G[t_1, t_N] \\ & & \\ G[t_N, t_1] & \cdots & G[t_N, t_N] \end{bmatrix} , \quad \bar{\bar{x}} = \begin{bmatrix} \bar{x}(t_1) \\ \bar{x}(t_2) \\ \cdots \\ \bar{x}(t_N) \end{bmatrix} , \tag{8}$$

and G is referred to as the system matrix.

It should be noted at this point that G is an nN x mN matrix (i. e., with nN rows and mN columns). Therefore the matrix G is not necessarily square, which leads to an interesting and obvious consequence. The matrix G can be rectangular either in the form (for $n > m$)



or in the form (for $m > n$)



In the first case, it follows from the rectangularity of G that not all output vectors in nN-dimensional space are realizable (i. e., for some $\bar{\bar{y}}$ there does not exist an $\bar{\bar{x}}$ such that $\bar{\bar{y}} = G\bar{\bar{x}}$). In fact, all realizable $\bar{\bar{y}}$'s must be in a subspace of dimension less than or equal to mN (the equality holds if G is of maximal rank). In the second case, it follows from the rectangularity of G that, regardless of the explicit form of G, $\bar{\bar{x}}$-vectors in a subspace of dimension (m-n)N cause zero output [i. e., the null space of G has a dimension of at least (m-n)N]. Using the vocabulary of filter theory, one could say that such a system would have a nontrivial stop band.

3. BACKGROUND TO TRANSFORM METHODS

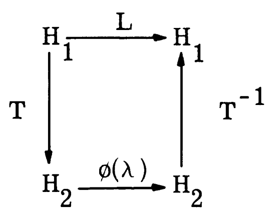
At this point it is worthwhile to reconsider the basic goal behind transform techniques. Simply stated, the goal is to transform, where possible, a given operation into the operation of multiplication by a function (e.g., the transfer function). Let L denote some given linear operation such that

$$y = Lx \quad , \tag{9}$$

where x is an element of the input space[2] or domain and y is an element of the output space or range. L is transformed to a multiplication in the desired sense if an invertible transformation T and a function $\phi(\lambda)$ can be found such that

$$L = T^{-1}\phi(\lambda)T \ . \tag{10}$$

Thus, if L is a linear transformation of a space H_1 into itself, T is a linear transformation of H_1 onto a space H_2, and multiplication by $\phi(\lambda)$ is a linear transformation of H_2 into itself, we have the following situation



A classic example of such a transformation to a multiplication is offered by the well-known use of the Laplace transform with linear, time-invariant systems. There T is the direct Laplace transform; $\phi(\lambda)$ is the transfer function with the complex numbers, λ, taken along the Wagner-Bromwich contour; and T^{-1} is the inverse Laplace transform.

---

[2] The element x might be a function or a sequence, for example; the corresponding spaces would be function or sequence spaces. In the case of the systems considered in this article the element x is a sequence and the space is a sequence space.

Another well-known example in the same spirit is the diagonalization of a square matrix by means of a similarity transformation. There L is the matrix to be diagonalized; T is a nonsingular (i.e., invertible) matrix; and $\phi(\lambda)$ is a diagonal matrix. This diagonal matrix can be viewed as equivalent to a function defined on the integers from 1 through N (L being an N x N matrix). Thus, $\phi(1) = \lambda_1$, the first entry on the diagonal; $\phi(2) = \lambda_2$, the second entry; and so on through $\phi(N) = \lambda_N$. If, in a similar manner, an arbitrary vector upon which the diagonal matrix operates is viewed as a function--say $x(\lambda)$, defined on the integers 1 to N--then operation with the diagonal matrix can be viewed as a multiplication of the function $x(\lambda)$ by the function $\phi(\lambda)$. This latter example leads one to refer to the general process as a "diagonalization of L" whether L is a matrix or not. A detailed discussion of the philosophy behind such diagonalization applied to continuous-time systems is given by Zadeh (Ref. 4). Finally, it must be emphasized that such a diagonalization is not always possible.

In addition to the obvious fact that such a "diagonalization" simplifies the representation of the operator L (assuming L is "diagonalizable"), several specific aspects should be noted. Perhaps the most important of these relates to the tandem operation of two operators, say L_1 and L_2, which can be diagonalized by the same transform T. In this case,

$$L_1 = T^{-1}\phi_1(\lambda)T$$

and

$$L_2 = T^{-1}\phi_2(\lambda)T \quad . \tag{11}$$

It immediately follows that

$$\begin{aligned} L_1L_2 &= T^{-1}\phi_1(\lambda)TT^{-1}\phi_2(\lambda)T \\ &= T^{-1}\phi_1(\lambda)\phi_2(\lambda)T \\ &= T^{-1}\phi_2(\lambda)\phi_1(\lambda)T \\ &= L_2L_1 \quad . \end{aligned} \tag{12}$$

Thus, T also diagonalizes the operators L_1L_2 and L_2L_1 as well as L_1 and L_2. Carrying Eq. 12 further, if L_1, L_2, ..., L_n form a set of operators which can be diagonalized by

T, then any polynomial function[3] of these operators and their inverses where they exist can be diagonalized by T. (For example, consider the familiar case of polynomials of the derivative operator, d/dt, and the Laplace transform as T.) Finally, it should be noted in Eq. 11 and Eq. 12 that a necessary condition for L_1 and L_2 to be diagonalized by the same T is $L_1L_2 = L_2L_1$. Since not all linear operators commute, one cannot expect to find a T which will diagonalize all operators. Thus, an all-purpose transform, in the sense of Eq. 10, is not possible. On the other hand, it is possible to find T's which diagonalize all members of large classes of linear operators. One such class is made up of time-invariant linear differential operators.

An important result of the above statement about polynomials is that the diagonalized identity operator is the identity operator in the transform domain regardless of the transform used. This fact is easily appreciated, since $I = L^0$ or

$$T^{-1}IT = I \quad , \tag{13}$$

where I is used to denote the identity operator before as well as after transformation. The importance of this property arises in the diagonalization of polynomials of the form (I + L), for if T diagonalizes L, it also diagonalizes (I + L). For example, if L_1 and L_2 can both be diagonalized by T, then both sides of

$$(I + L_1)y = L_2x \tag{14}$$

can also be diagonalized by T; and the importance of equations in the form of Eq. 14 for feedback systems is well-known.

It is convenient for the magnitude $|\phi(\lambda)|$ to have some significance. In particular, Parseval's (Plancherel's) theorem or its analog is desirable. Returning to Eqs. 9 and 10

$$\begin{aligned} y &= Lx \\ &= T^{-1}\phi(\lambda)Tx \quad . \end{aligned}$$

Assume that L is a bounded linear transformation from a Hilbert space H_1 into itself and

[3]This statement is valid for a larger class of functions than polynomials.

T is an invertible linear transformation of H_1 onto H_2. Let $Tx = X(\lambda)$ and $Ty = Y(\lambda)$ and denote the inner products on H_1 and H_2 by $(y, x)_{H_1}$ and $(Y, X)_{H_2}$, respectively. In the case of square-integrable functions these become

$$(y, x)_{H_1} = \int y(t)\bar{x}(t)dt$$

and

$$(Y, X)_{H_2} = \int Y(\lambda)\bar{X}(\lambda)d\lambda \ , \qquad (15)$$

where the bar denotes the complex conjugate. The goal is to relate $(y, y)_{H_1}$ to $\phi(\lambda)$ and $X(\lambda)$. It follows from Eqs. 9 and 10 that

$$(y, y)_{H_1} = (Lx, Lx)_{H_1}$$
$$= [T^{-1}\phi(\lambda)Tx, T^{-1}\phi(\lambda)Tx] \ . \qquad (16)$$

If the adjoint[4] of T^{-1} is designated by $(T^{-1})^*$ and $(T^{-1})^*T^{-1}$ is replaced by Q, then Eq. 16 is equivalent to

$$(y, y)_{H_1} = [\phi(\lambda)X(\lambda), Q\phi(\lambda)X(\lambda)]_{H_2} \qquad (17)$$

The usefulness of the above expression depends on the nature of the transformation Q.

The desirable situation is for $Q\phi(\lambda)X(\lambda)$ to be easily expressible in terms of $\phi(\lambda)X(\lambda)$. For example, if T is a unitary transformation,[5] then $T^{-1} = T^*$; therefore, $Q = I$, the identity transformation, and Eq. 17 becomes

$$(y, y)_{H_1} = [\phi(\lambda)X(\lambda), \phi(\lambda)X(\lambda)]_{H_2} \qquad (18)$$

---

[4]Recall that the adjoint of an operator A which maps H_1 into H_2 is that operator A^*, mapping H_2 into H_1, for which

$$(Au, v)_{H_2} = (u, A^*v)_{H_1}$$

for all u in H_1 and all v in H_2.

[5]Recall that L can be "diagonalized" by a unitary transformation if and only if it is normal, i.e., commutes with its adjoint, $L^*L = LL^*$. It is not true that all "diagonalizable" operators are normal.

Thus, in the case of the Fourier transform pair[6]

$$X(\lambda) = \int_{-\infty}^{\infty} x(t)\epsilon^{-j2\pi\lambda} dt$$

and

$$x(t) = \int_{-\infty}^{\infty} X(\lambda)\epsilon^{j2\pi\lambda} d\lambda ,$$

where the implied T is unitary and, thus, Q = I, Eq. 18 becomes

$$\int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty} \phi(\lambda)\overline{\phi(\lambda)}X(\lambda)\overline{X(\lambda)}d\lambda . \tag{19}$$

Thus, $|\phi(\lambda)|^2$ indicates the energy transfer capabilities of the system. On the other hand the transformation Q in Eq. 17 may not lead to a simple interpretation of the magnitude of $\phi(\lambda)$. In fact, there is no reason to expect a simple correlation between the $|\phi(\lambda)|$ and the energy transfer capabilities of the system. Thus, much of the insight and many of the analytic techniques associated with the use of transfer functions based on the Fourier (or Laplace) transforms may not carry over to the general case. Clearly, it is unfortunate when they do not.

In summary, then, the diagonalization discussed above has certain advantages and certain disadvantages. The advantages are as follows:

A_1. If the operators $L_1, \ldots, L_n$ can be diagonalized by a transformation T, then any polynomial function of these operators and their inverses (where they exist) can be diagonalized by T. Among other things this property allows an operational calculus based on the multiplication and addition of transfer functions to be developed. In particular, if the transform T diagonalizes L, then it also diagonalizes I + L, where I is the identity operator.

[6] Here x(t) is restricted to the intersection of square-integrable and absolutely integrable functions. In order to consider all square-integrable functions the Fourier-Plancherel transform must be used.

A_2. In certain cases, for example when L is normal, L can be diagonalized in a way that leads to a meaningful generalization of Parseval's (Plancherel's) theorem.

The disadvantages are as follows:

D_1. Not all linear operators can be diagonalized in the above way. For example, not all matrices can be so diagonalized.

D_2. Given any transformation T, only a relatively small class of linear operators will be diagonalizable with it. In other words, there does not exist one transformation T which will diagonalize all or even a relatively large segment of the set of all linear operators.

D_3. Parseval's (Plancherel's) theorem can be meaningfully generalized only in special cases.

In the next section, a transform technique is introduced which overcomes some of the above difficulties at the cost of sacrificing advantages. Moreover, it adds an advantage which not even the Laplace transform as usually applied to time-invariant systems has: for a multivariable system, it yields <u>one transfer function</u> instead of a matrix of transfer functions.

4. TRANSFORM TECHNIQUE

Lanzcos (Ref. 5) has shown that all system matrices, G, can be decomposed as follows:

$$G = (Y\Delta f)\Lambda X^T , \qquad (20)$$

where

(i) Y is a matrix whose columns are pairwise orthogonal to one another and each is of norm $\sqrt{N}$. At this point the norm employed is the familiar Euclidean norm. This decomposition is generalized subsequently so that norms based on arbitrary inner products can be employed. If n, the number of output channels, is greater than or equal to m, the number of input channels, then Y is an (nN x mN)-matrix. If $n < m$, then Y is an (nN x nN)-matrix.

(ii) $\Delta f = 1/N$ and is referred to as an increment of generalized frequency.

(iii) Λ is a diagonal matrix with all non-negative entries. If $n \geq m$, Λ is an (mN x mN)-matrix. If $n < m$, Λ is an (nN x nN)-matrix.

(iv) X is a matrix (X^T is the transpose of X) whose columns are pairwise orthogonal to one another and each is of norm $\sqrt{N}$. If $n \geq m$, X^T is an (mN x mN)-matrix. If $n < m$, X^T is an (nN x mN)-matrix.

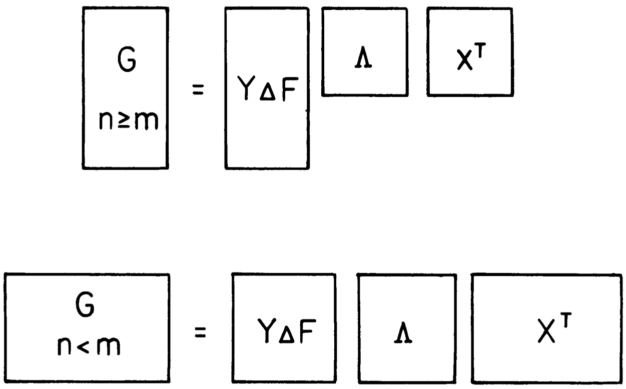


Fig. 2. Form of the decomposition

The relation between the sizes of the above matrices is illustrated in Fig. 2. This decomposition is essentially the same as the one used with single-input, single-output systems (Ref. 3). The fact that it is valid for nonsquare matrices allows the present extension to multivariable systems. As in the case of single-input, single-output systems, it is shown below that X^T acts as a direct transform, Λ acts as a transfer function, and $(Y\Delta f)$ acts as an inverse transform.

Given a system matrix G the decomposition indicated in Eq. 20 can be carried out in the following manner. Since

$$G^T = (X\Delta f)\Lambda Y^T$$

and

$$Y^T Y = I \frac{1}{\Delta f}$$

even for rectangular Y's, it follows that

$$G^T G = (X\Delta f)\Lambda^2 X^T$$

Thus, the columns of X are eigenvectors of $G^T G$ and the main diagonal entries in Λ are the positive square roots of the eigenvalues of $G^T G$. Note that Λ is uniquely determined but X is not. If the entries in Λ are distinct this lack of uniqueness for X arises from the fact that (-1) times an eigenvector is also an eigenvector. If the entries in Λ are not distinct the lack of uniqueness is evidenced in a less trivial manner. For example, if $G^T G = I$ or 0, any X-matrix satisfying (iv) is suitable. Given X and Λ, the Y-matrix is determined from the relation

$$Y\Lambda = GX$$

This relation uniquely determines Y if Λ is nonsingular; otherwise, Y is only uniquely determined on the range of G.

Case I: $n > m$

Assume for the moment that there are more output than input channels, that is, $n > m$. The matrices in the decomposition are then of the following structure:

$$Y = \begin{bmatrix} y_1^1(t_1) & y_2^1(t_1) & \cdots & y_{mN}^1(t_1) \\ y_1^2(t_1) & y_2^2(t_1) & \cdots & y_{mN}^2(t_1) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^n(t_1) & y_2^n(t_1) & \cdots & y_{mN}^n(t_1) \\ y_1^1(t_2) & y_2^1(t_2) & \cdots & y_{mN}^1(t_2) \\ y_1^2(t_2) & y_2^2(t_2) & \cdots & y_{mN}^2(t_2) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^n(t_2) & y_2^n(t_2) & \cdots & y_{mN}^n(t_2) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^1(t_N) & y_2^1(t_N) & \cdots & y_{mN}^1(t_N) \\ y_1^2(t_N) & y_2^2(t_N) & \cdots & y_{mN}^1(t_N) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^n(t_N) & y_2^n(t_N) & \cdots & y_{mN}^n(t_N) \end{bmatrix} \qquad (21)$$

An equivalent, simplified form of this matrix is

$$Y = \begin{bmatrix} \bar{y}_1(t_1) & \bar{y}_2(t_1) & \cdots & \bar{y}_{mN}(t_1) \\ \bar{y}_1(t_2) & \bar{y}_2(t_2) & \cdots & \bar{y}_{mN}(t_2) \\ \cdots & \cdots & \cdots & \cdots \\ \bar{y}_1(t_N) & \bar{y}_2(t_N) & \cdots & \bar{y}_{mN}(t_N) \end{bmatrix} , \tag{22}$$

where

$$\bar{y}_j(t_k) = \begin{bmatrix} y_j^1(t_k) \\ y_j^2(t_k) \\ \cdots \\ y_j^n(t_k) \end{bmatrix} \qquad \left\{ \begin{matrix} j = 1, \ldots, mN \\ k = 1, \ldots, N \end{matrix} \right\} . \tag{23}$$

An even more concise notation is

$$Y = [\bar{\bar{y}}_1, \bar{\bar{y}}_2, \ldots, \bar{\bar{y}}_{mN}] , \tag{24}$$

where

$$\bar{\bar{y}}_j = \begin{bmatrix} \bar{y}_j(t_1) \\ \bar{y}_j(t_2) \\ \cdots \\ \bar{y}_j(t_N) \end{bmatrix} \tag{25}$$

Thus, the vectors $\bar{\bar{y}}_j$, $j = 1, 2, \ldots, mN$, are mutually orthogonal and of norm $\sqrt{N}$, that is, $(\bar{\bar{y}}_j, \bar{\bar{y}}_k) = \delta_{jk} \frac{1}{\Delta f} = \delta_{jk} N$, where δ_{jk} = Kronecker delta symbol. Here

$$(\bar{\bar{y}}_j, \bar{\bar{y}}_k) = y_j^1(t_1)\, y_k^1(t_1) + y_j^2(t_1)\, y_k^2(t_1) + \ldots + y_j^n(t_N)\, y_k^n(t_N) . \tag{26}$$

The matrix Λ is an (mN x mN) diagonal matrix with nonnegative main diagonal entries, that is,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \lambda_3 & & \\ & & & \cdots & \\ & & & & \lambda_{mN} \end{bmatrix} \tag{27}$$

where $\lambda_i \geq 0$, $i = 1, 2, \ldots, mN$.

The matrix X^T is an (mN x mN)-matrix with mutually orthogonal rows of the form

$$X^T = \begin{bmatrix} x_1^1(t_1), x_1^2(t_1), \ldots, x_1^m(t_1), x_1^1(t_2), x_1^2(t_2), \ldots, x_1^m(t_2), \ldots, x_1^1(t_N), \ldots, x_1^m(t_N) \\ x_2^1(t_1), x_2^2(t_1), \ldots, x_2^m(t_1), x_2^1(t_2), x_2^2(t_2), \ldots, x_2^m(t_2), \ldots, x_2^1(t_N), \ldots, x_2^m(t_N) \\ \cdots \\ x_{mN}^1(t_1), x_{mN}^2(t_1), \ldots, x_{mN}^m(t_1), x_{mN}^1(t_2), \ldots, x_{mN}^m(t_2), \ldots, x_{mN}^1(t_N), \ldots, x_{mN}^m(t_N) \end{bmatrix} \quad (28)$$

This matrix can be simplified to [7]

$$X = \begin{bmatrix} \bar{x}_1(t_1) & \bar{x}_2(t_1) & \cdots & \bar{x}_{mN}(t_1) \\ \bar{x}_1(t_2) & \bar{x}_2(t_2) & \cdots & \bar{x}_{mN}(t_2) \\ \cdots & \cdots & \cdots & \cdots \\ \bar{x}_1(t_N) & \bar{x}_2(t_N) & \cdots & \bar{x}_{mN}(t_N) \end{bmatrix} \quad (29)$$

or

$$X = [\bar{\bar{x}}_1, \bar{\bar{x}}_2, \ldots, \bar{\bar{x}}_{mN}] , \quad (30)$$

where substitutions analogous to those used with the Y-matrix are employed. Moreover, the vectors $\bar{\bar{x}}_j$, $j = 1, 2, \ldots, mN$, are mutually orthogonal and $(\bar{\bar{x}}_j, \bar{\bar{x}}_k) = \delta_{jk}/\Delta f = \delta_{jk}N$. It is important to note that $G\bar{\bar{x}}_i = \lambda_i \bar{\bar{y}}_i$, $i = 1, 2, \ldots, mN$. This relation and the orthogonality of the $\bar{\bar{x}}_i$-vectors and the $\bar{\bar{y}}_i$-vectors is the key to what follows.

It can be seen from Eq. 20 that the first step in the operation of a decomposed G on an arbitrary input, $\bar{\bar{x}}$, is $X^T\bar{\bar{x}}$. It will now be shown that this first step can be interpreted as taking the direct transform of $\bar{\bar{x}}$ to obtain its generalized frequency domain representation. First note that the orthogonal set of vectors, $\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_{mN}$, spans the linear space of all possible inputs. Thus, an arbitrary input, $\bar{\bar{x}}$, can be uniquely represented in the form

---

[7] Note that X and not X^T is written.

$$\bar{\bar{x}} = \sum_{j=1}^{mN} r_j \bar{\bar{x}}_j \Delta f \qquad (31)$$

where the r_j's are constants. Taking the inner product of both sides of Eq. 31 with $\bar{\bar{x}}_i$ yields

$$r_i = (\bar{\bar{x}}, \bar{\bar{x}}_i) \qquad i = 1, 2, \ldots, mN \qquad (32)$$

for the determination of the r_i's. In a manner similar to that employed in the single-input, single-output case (Ref. 3), the sequence made up of the r_i's may be considered to be a "frequency domain representation" or transform of $\bar{\bar{x}}$. Moreover, a piecewise constant function, $R_X(f)$, of generalized frequency can be introduced as the frequency domain representation of $\bar{\bar{x}}$. This function is obtained by associating an increment Δf of generalized frequency with each $\bar{\bar{x}}_i$-vector. Thus, $R_X(f)$ is defined as follows

$$R_X(f) = \begin{cases} r_i & \text{for } (i-1)\Delta f \le f < i\Delta f, \quad i = 1, 2, \ldots, mN \\ 0 & \text{for } m \le f < n \end{cases} , \qquad (33)$$

where, again, $\Delta f = 1/N$. An illustration is shown in Fig. 3. The frequency domain is defined to include the interval $m \le f < n$ and $R_X(f)$ is defined to be equal to zero on this interval so that the input and output frequency domain representations can be compatible. In general, the generalized frequency domain is defined to be $0 \le f < \max[m, n]$. Since it can be seen from Eq. 30 and Eq. 32 that

$$X^T\bar{\bar{x}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdots \\ r_{mN} \end{bmatrix} , \qquad (34)$$

it follows that $X^T\bar{\bar{x}}$ can indeed be viewed as the direct transform, that is, $X^T\bar{\bar{x}}$ yields $R_X(f)$.

Finally, it is easily shown that this generalized frequency domain representation leads to a meaningful generalization of Parseval's (Plancherel's) theorem. In fact, trivial calculations show that

$$(\bar{\bar{x}}, \bar{\bar{x}}) = \sum_{j=1}^{mN} r_j^2 \Delta f = \int_o^n R_X^2(f) df , \qquad (35)$$

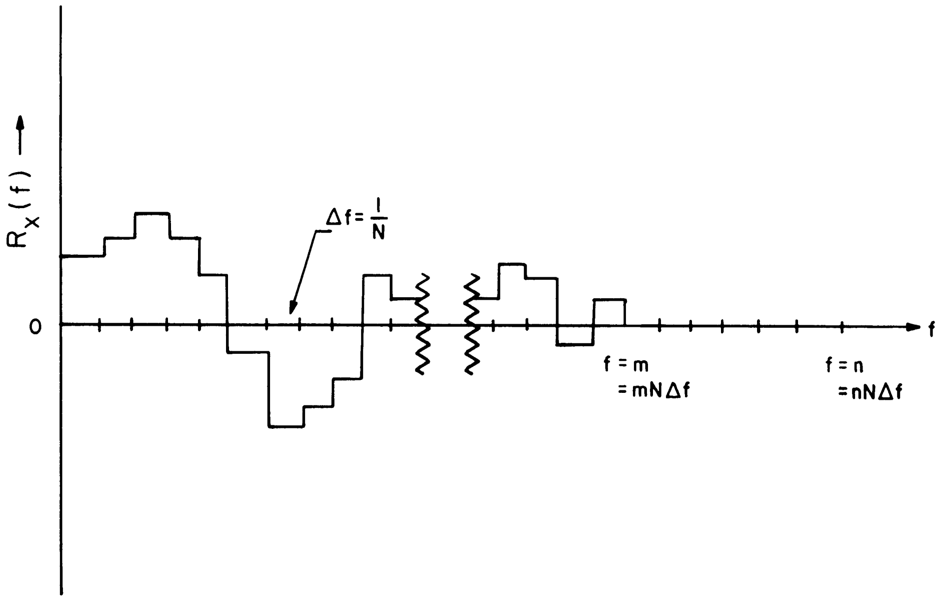


Fig. 3. Frequency domain representation of an input vector.

that is, $R_X{}^2(f)$ can be viewed as "energy" per unit bandwidth.

In a similar manner, the generalized frequency domain representation of an arbitrary output, $\bar{\bar{y}}$, is given by

$$\bar{\bar{y}} = \sum_{j=1}^{mN} c_j \bar{\bar{y}}_j \Delta f \ , \tag{36}$$

and it follows that a function $C_Y(f)$, analogous to $R_X(f)$, can be defined in the generalized frequency domain, $0 \leq f < n$, by

$$C_Y(f) = \begin{cases} c_i & \text{for } (i-1)\Delta f \leq f < i\Delta f, \quad i = 1, 2, \ldots, mN \\ 0 & \text{for } m \leq f < n \end{cases} \tag{37}$$

so that

$$(\bar{\bar{y}}, \bar{\bar{y}}) = \int_0^n C_Y{}^2(f) df \tag{38}$$

and $C_Y(f)$ can be viewed as the transform of $\bar{\bar{y}}$. It should be noted that arbitrary vectors in the nN-dimensional vector space which contains the output vectors, $\bar{\bar{y}}$, can not be represented as a linear combination of the $\bar{\bar{y}}_j$-vectors as in Eq. 36 because these vectors span only the mN-dimensional subspace which is the range of the G-matrix under consideration. Therefore, this transform is not, as presented here, applicable to the whole nN-dimensional vector space. $C_Y(f)$ has been defined equal to zero in order to emphasize the fact that the range of G is a proper subspace (recall that here we are considering the case $m < n$).

The functions $R_X(f)$ and $C_Y(f)$, then, are the frequency domain representations of the input and output respectively. The subscripts X and Y indicate that the transforms are with respect to the $\bar{\bar{x}}_i$- and $\bar{\bar{y}}_i$-vectors respectively. It is now easy to introduce a generalized transfer function relating $R_X(f)$ and $C_Y(f)$. It is clear from the decomposition (Eq. 20) that the output corresponding to an input $\bar{\bar{x}}_i$ is $\lambda_i \bar{\bar{y}}_i$. This follows from the fact that Λ is a diagonal matrix. Thus, in the spirit of the definitions of $R_X(f)$ and $C_Y(f)$, the transfer function for the system can be defined by

$$\Lambda(f) = \begin{cases} \lambda_i & \text{for } (i-1)\Delta f \leq f < i\Delta f, \quad i = 1, 2, \ldots, mN \\ 0 & \text{for } m \leq f < n \, . \end{cases} \tag{39}$$

It follows that the generalized frequency domain representation for the system operation is given by

$$C_Y(f) = \Lambda(f)\, R_X(f) \qquad 0 \leq f < n \tag{40}$$

and from Eq. 38 that

$$(\bar{\bar{y}}, \bar{\bar{y}}) = \int_0^n \Lambda^2(f)\, R_X{}^2(f) df \, .$$

Moreover, it follows from Eq. 24 and Eq. 36 that the matrix multiplication of $\Lambda X^T \bar{\bar{x}}$ by $(Y\Delta f)$ is equivalent to transforming $C_Y(f)$ to the time domain by the "inverse transform" $(Y\Delta f)$.

Case II: $m \geq n$

So far it has been assumed that $n > m$; if $m \geq n$ it is merely necessary to interchange the roles of m and n. The frequency domain becomes $0 \leq f < m$. The matrix Y becomes an (nN x nN)- instead of an (nN x mN)-matrix, Λ becomes an (nN x nN)- instead of an (mN x mN)-matrix, and X^T becomes an (nN x mN)- instead of an (mN x mN)-matrix.

Thus the original operation has been transformed to a multiplication by a function $\Lambda(f)$ defined on the interval $0 \leq f < n$ (or $0 \leq f < m$ depending on Case I or II). However, this has not been accomplished by means of the similarity transformation discussed in Section 3, for the inverse transform implied by $(Y\Delta f)$ is not necessarily the inverse of the direct transform implied by X^T. The question is how do the advantages and ıdvantages of this transform method relate to those listed at the end of Section 3.

First, what about tandem operation of the two systems G_1 and G_2? Given

$$G_1 = (Y_1\Delta f)\Lambda_1 X_1^T$$

and

$$G_2 = (Y_2\Delta f)\Lambda_2 X_2^T$$

the desirable situation is to have[8]

$$G_1 G_2 = (Y_1\Delta f)\Lambda_1\Lambda_2 X_2^T \ .$$

A sufficient condition for this reduction to take place is $Y_2 = X_1$, for then $X_1^T(Y_2\Delta f) = I$. When G_1 and G_2 are invertible this condition is necessary for then

$$(Y_1\Delta f)\Lambda_1\Lambda_2 X_2^T = (Y_1\Delta f)\Lambda_1 X_1^T(Y_2\Delta f)\Lambda_2 X_2^T$$

which yields

$$\Lambda_1\Lambda_2 = \Lambda_1 X_1^T(Y_2\Delta f)\Lambda_2$$

and

$$X_1^T(Y_2\Delta f) = I$$

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[8]Here it is assumed that the indicated matrix multiplication makes sense.

which implies that $Y_2 = X_1$. In case G_1 or G_2 or both are not invertible, it again follows that

$$\Lambda_1\Lambda_2 = \Lambda_1 X_1^T(Y_2\Delta f)\Lambda_2 .$$

When Λ_1 and Λ_2 are both invertible (this can happen in spite of G_i not being invertible), $Y_2 = X_1$ is still a necessary and sufficient condition. If either Λ_1 or Λ_2 or both are not invertible, that is, have zero entries on the main diagonal, then $Y_2 = X_1$ is no longer a necessary condition, for some of the rows in X_1^T and some of the columns of Y_2 may be multiplied by the possible zero entries in Λ_1 and Λ_2, respectively. It follows that the necessary condition becomes that $Y_2 = X_1$ except for the columns multiplied by zero. However, since these latter columns are not uniquely determined and can be chosen so that the corresponding ones in Y_2 and X_1 are equal to one another, it can be said that the desired reduction takes place if and only if Y_2 and X_1 can be selected so that they equal one another. Finally, note that this is a general statement that applies in all cases where the decomposition of G_1 or G_2 is not unique.

It follows from the foregoing remarks that given $G_1 = (Y_1\Delta f)\Lambda_1 X_1^T$, one representation for the set of all G_2's for which the $G_1G_2 = (Y_1\Delta f)\Lambda_1\Lambda_2 X_2^T$ is

$$G_2 = (X_1\Delta f)\Lambda_2 X_2^T ,$$

where Λ_2 and X_2 are arbitrary or constrained by the requirements of physical realizability. In any event, this class of G_2's is not empty; therefore, the concept of the product of two transfer functions representing tandem operation does carry over in a certain sense. On the other hand, it is true, for example, that the transfer function of G^2 is not necessarily $\Lambda^2(f)$; and this would be true if a similarity transformation were used.

In summary, when the decomposition presented in this article is employed a multiplication of transfer functions results for certain classes of tandemly connected systems--just as multiplication of transfer functions results for certain, presumably other, classes of tandemly connected systems when a decomposition based on a similarity transformation is used.

Next consider the question of the decomposition of I + G. If the decomposition of G is given by[9]

$$G = (Y\Delta f)\Lambda X^T$$

and $Y \neq X$, it follows that $(Y\Delta f)IX^T \neq I$. Then

$$I + G \neq (Y\Delta f)\,(I+\Lambda)X^T ,$$

which means that except for the special case Y = X, transformation of I + G and G cannot be carried out by the same Y and X transformations. This is not the case, of course, when similarity transformations are used.

In regard to a generalization of Parseval's (Plancherel's) theorem, it is clear from the foregoing discussion that a meaningful generalization is always possible. This is an extremely important property and one which is not present when similarity transformations are employed. Another advantageous property of the present decomposition, and one not present with a similarity transformation, is that it can be applied to an arbitrary system matrix, G. So much for advantages and disadvantages.

One last point: The nature of the transform implied by the matrix X^T and its constituent $\bar{\bar{x}}_i$-vectors should be carefully appreciated. Since arbitrary inputs, $\bar{\bar{x}}$, are represented as linear combinations of the $\bar{\bar{x}}_i$-vectors, the $\bar{\bar{x}}_i$'s can be viewed as basic or fundamental inputs, and the response to these fundamental inputs completely characterizes the system. The key point is that these fundamental inputs can and probably do involve simultaneous inputs on more than one channel, which is in contrast, for example, to the approach to time-invariant multivariable systems implied when the final system characterization is a matrix made up of transfer functions. There each column of the transfer function matrix is the Laplace transform of the output when a unit impulse is applied to one input channel <u>while the other input channels have zero input</u>. Thus, the present approach might be characterized

[9] Here G is assumed to be square.

as treating all input channels simultaneously rather than one at a time. Related remarks can be made regarding the $(Y\Delta f)$-matrix and its constituent $\bar{\bar{y}}_i$-vectors.

5. EXTENSION OF SINGLE-VARIABLE RESULTS TO MULTIVARIABLE SYSTEMS

It has been shown in Ref. 3 that many important results follow from the matrix decomposition discussed in the foregoing section when it is applied to matrices representing two-port systems (single-input, single-output channel systems). Since the same decomposition has been applied here to matrices representing multivariable systems, it is not too surprising that many of the results pertaining to two-port systems also pertain to multivariable systems. Some of these extensions are outlined below:

Gain. The entry λ_i on the main diagonal of the Λ-matrix is referred to as the gain over the frequency interval $(i-1)\Delta f \leq f < i\Delta f$.

Gain-Squared Bandwidth Product. The gain-squared bandwidth product, $\Phi(G)$, is defined by

$$\Phi(G) = \int_0^{\max[m,n]} \Lambda^2(f)df = \sum_{i=1}^{\min[m,n]} \lambda_i^2 \Delta f \, . \tag{41}$$

Moreover, it should be noted that

$$\Phi(G) = \sum_{i,j,k,\ell} g_{ij}^2[t_k,t_\ell]\Delta f$$

where the $g_{ij}^2[t_k,t_\ell]$'s are the elements of G.

Norm of G. The norm of G, $||G||$, is defined by

$$||G|| = \max_{||\bar{\bar{x}}||=1} ||G\bar{\bar{x}}|| \, .$$

It can be shown that

$$||G|| = \max_i \lambda_i \tag{42}$$

Assuming for the moment that the λ_i's are distinct and that $\lambda_1 > \lambda_2 > \dots > \lambda_{nN}$ (or λ_{mN}), it follows that for all $\bar{\bar{x}}$-vectors of a fixed norm, say $\sqrt{N}$, the one causing an output vector with a maximum norm is $\bar{\bar{x}}_1$ and the corresponding output vector is $\lambda_1\bar{\bar{y}}_1$. Considering all input vectors of norm $\sqrt{N}$ which are orthogonal to $\bar{\bar{x}}_1$, the one which causes an output vector with a maximum norm is $\bar{\bar{x}}_2$ and the corresponding output vector is $\lambda_2\bar{\bar{y}}_2$. This pattern continues through $\bar{\bar{x}}_{nN}$ (or $\bar{\bar{x}}_{mN}$) and $\lambda_{nN}\bar{\bar{y}}_{nN}$ (or $\lambda_{mN}\bar{\bar{y}}_{mN}$). This is an important property of the decomposition presented and shows that the $\bar{\bar{x}}_i$ and $\bar{\bar{y}}_i$-vectors characterize the "extremal" inputs and outputs of the system. In case the λ_i's are not all distinct, as assumed above, any fixed norm linear combination (say norm equal $\sqrt{N}$) of $\bar{\bar{x}}_i$-vectors associated with equal λ_i's yields an output vector whose norm is both independent of the linear combination used and the maximum output norm possible over the appropriate subspace of inputs. For example, if $\lambda_1 > \lambda_2 > \dots > \lambda_j = \lambda_{j+1} > \lambda_{j+2} > \dots > \lambda_{nN}$ (or λ_{mN}) and the input vectors of norm $\sqrt{N}$ which are orthogonal to $\bar{\bar{x}}_1, \bar{\bar{x}}_2, \dots, \bar{\bar{x}}_{j-1}$ are considered, the $\bar{\bar{x}}$'s associated with the maximum output are all linear combinations of the form $a\bar{\bar{x}}_j + b\bar{\bar{x}}_{j+1}$, where $a^2 + b^2 = 1$, and the corresponding outputs are $a\lambda_j\bar{\bar{y}}_j + b\lambda_{j+1}\bar{\bar{y}}_{j+1}$.

Bandwidth. As in the case of two-port systems, a meaningful generalization of the concept of bandwidth is given by

$$Bw = \frac{\Phi(G)}{\|G\|^2} . \tag{43}$$

Physical Realizability. Let the columns of the matrix X^T be designated by $\bar{\bar{u}}_j(t_k)$, where

$$\bar{\bar{u}}_j(t_k) = \begin{bmatrix} x_1^j(t_k) \\ x_2^j(t_k) \\ x_3^j(t_k) \\ \dots \\ x_{mN}^j(t_k) \end{bmatrix} \quad \begin{cases} k = 1, 2, \dots, N \\ j = 1, \dots, m \end{cases} \tag{44}$$

The $\bar{\bar{u}}_j(t_k)$'s are referred to here as the input ensemble vectors. Let the rows of the matrix $(Y\Delta f)\Lambda$ be designated by $\bar{\bar{v}}_j(t_k)$:

$$\bar{\bar{v}}_j(t_k) = \begin{bmatrix} \lambda_1 y_1^j(t_k) \\ \lambda_2 y_2^j(t_k) \\ \cdots \\ \lambda_{mN} y_{mN}^j(t_k) \end{bmatrix} \quad \begin{cases} k = 1,2,\ldots,N \\ j = 1,2,\ldots,n \end{cases} \tag{45}$$

The $\bar{\bar{v}}_j(t_k)$'s are referred to as the output ensemble vectors. Referring to Eq. 3 it can be appreciated that the matrix G corresponds to a physically realizable system if

$$G[t_k, t_r] = 0 \quad \text{for } r > k \ . \tag{46}$$

This is easily shown to be the case if and only if

$$[\bar{\bar{v}}_j(t_k),\ \bar{\bar{u}}_\ell(t_r)] = 0 \quad \text{for } r > k \ . \tag{47}$$

That is, the output ensemble vector at time t_k must be orthogonal to all input ensemble vectors occurring later in time. Roughly speaking, the output at any time is "independent" of all future inputs.

In the case of single-input, single-output systems the G-matrix for a physically realizable system is lower triangular. For multivariable systems this is not the case; however, it can be seen from Eq. 2, Eq. 46 and Eq. 47 that matrices representing physically realizable systems are what might be called "lower staircase," that is, Eq. 3 is lower triangular.

Pseudo-Inverse of G. The system matrix G is not necessarily invertible; in fact, if G is rectangular it will not be invertible regardless of its structure. On the other hand, given a desired output $\bar{\bar{y}}$ it is often necessary to find an input $\bar{\bar{x}}$, if one exists, which causes $\bar{\bar{y}}$. That is, given G and $\bar{\bar{y}}$, an $\bar{\bar{x}}$ must be found such that

$$G\bar{\bar{x}} = \bar{\bar{y}} \ . \tag{48}$$

Assuming that

$$G = (Y\Delta f)\Lambda X^T ,$$

it follows that Eq. 48 has a solution if $\bar{\bar{y}}$ is an element of the space spanned by the columns of Y (i. e., $\bar{\bar{y}}_i$-vectors) associated with nonzero λ_i's. Simply stated, $\bar{\bar{y}}$ must be in the range of G. If $\bar{\bar{y}}$ is in $\mathcal{R}(G)$, then it follows that a solution to Eq. 48 is given by

$$\bar{\bar{x}}_o = (X\Delta f)\Lambda^+ Y^T \bar{\bar{y}} , \tag{49}$$

where Λ^+ is a diagonal matrix for which $\lambda_i^+ = 1/\lambda_i$ if $\lambda_i \neq 0$ and $\lambda_i^+ = 0$ if $\lambda_i = 0$. If G has a nontrivial null space, then the general solution to Eq. 48 is

$$\bar{\bar{x}} = \bar{\bar{x}}_o + \bar{\bar{x}}_N ,$$

where $\bar{\bar{x}}_N$ is any element in the null space of G and, owing to the selection of $\bar{\bar{x}}_o$, $\bar{\bar{x}}_N$ is orthogonal to $\bar{\bar{x}}_o$. The validity of this latter statement follows from the fact that the null space of G is orthogonal to the subspace spanned by the columns in X associated with nonzero λ_i's. Thus, $\|\bar{\bar{x}}_o\| \leq \|\bar{\bar{x}}_o + \bar{\bar{x}}_N\|$ for all $\bar{\bar{x}}_N$, that is, if a solution exists $\bar{\bar{x}}_o$ is the "smallest" one.

Now assume that $\bar{\bar{y}}$ is not in the range of G, that is, $G\bar{\bar{x}}_o = \bar{\bar{y}}_o \neq \bar{\bar{y}}$, where $\bar{\bar{x}}_o$ is determined from Eq. 49. Since

$$\begin{aligned} \bar{\bar{y}}_o = G\bar{\bar{x}}_o &= (Y\Delta f)\Lambda X^T (X\Delta f)\Lambda^+ Y^T \bar{\bar{y}} \\ &= (Y\Delta f)\Lambda\Lambda^+ Y^T \bar{\bar{y}} , \end{aligned}$$

it follows that $\bar{\bar{y}}_o$ is the orthogonal projection of $\bar{\bar{y}}$ onto the range of G. Thus, $\bar{\bar{y}}_o$ is the vector in the range of G "closest" to $\bar{\bar{y}}$, that is, $\|\bar{\bar{y}} - G\bar{\bar{x}}\|$ is minimized by $\bar{\bar{x}}_o$. Obviously,

$$\bar{\bar{y}}_o = G\bar{\bar{x}}_o = G(\bar{\bar{x}}_o + \bar{\bar{x}}_N) ,$$

thus, again, $\|\bar{\bar{x}}_o\| \leq \|\bar{\bar{x}}_o + \bar{\bar{x}}_N\|$. Note that when G^{-1} exists it equals $(X\Delta f)\Lambda^+ Y^T$. Whether G^{-1} exists or not, $(X\Delta f)\Lambda^+ Y^T$ is referred to as the pseudo-inverse (Refs. 5, 6 and 7) and is designated by G^+.

Finally, the physical realizability of G does not necessarily imply that G^+ is physically realizable. For example, consider the following matrix representation of a physically realizable single-input, single-output system. If

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

its pseudo-inverse is

$$G^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which does not correspond to a physically realizable system.

6. EXAMPLE

Consider the two-input, two-output channel system characterized by the following difference equation:

$$\begin{bmatrix} y^1(k) \\ y^2(k) \end{bmatrix} = \begin{bmatrix} \frac{k-1}{N} & 1 - \frac{k-1}{N} \\ 1 - \frac{k-1}{N} & \frac{k-1}{N} \end{bmatrix} \begin{bmatrix} y^1(k-1) \\ y^2(k-1) \end{bmatrix} + \begin{bmatrix} x^1(k) \\ x^2(k) \end{bmatrix} \qquad (k = 1,2,\ldots,N)$$

$$\begin{bmatrix} y^1(0) \\ y^2(0) \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \tag{50}$$

It follows from Eq. 50 that the inverse, G^{-1}, of the system matrix for N = 10 is given by

$$G^{-1} = \left[\begin{array}{cccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/10 & -9/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9/10 & -1/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2/10 & -8/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8/10 & -2/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3/10 & -7/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -7/10 & -3/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4/10 & -6/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6/10 & -4/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5/10 & -5/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5/10 & -5/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6/10 & -4/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4/10 & -6/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7/10 & -3/10 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/10 & -7/10 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8/10 & -2/10 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2/10 & -8/10 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9/10 & -1/10 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/10 & -9/10 & 0 & 1
\end{array}\right] \tag{51}$$

Note that the nonsingularity of this matrix follows from its lower triangularity and from the fact that all its main diagonal elements are nonzero. Since the decomposition of G is given by

$$G = (Y\Delta f)\Lambda X^T ,$$

it follows that

$$G^{-1} = (X\Delta f)\Lambda^{-1}Y^T .$$

Therefore, the decomposition of G^{-1} as given in Eq. 51 is equivalent to the decomposition of G itself. It was found with the aid of a digital computer that the parts of this decomposition are as follows:

Y =

0.145	0.423	0.003	-0.824	-0.664	-0.046	-0.745	0.845	-0.716	0.001	0.951	1.020	0.243	0.973	0.007	1.467	0.908	-0.763	-0.550	0.288
0.145	0.423	-0.003	0.824	-0.664	0.046	0.745	0.845	0.716	-0.001	0.951	-1.020	-0.243	0.973	-0.007	-1.467	0.908	-0.763	-0.550	0.288
0.288	0.763	-0.005	1.377	-0.973	0.049	0.795	0.845	0.485	-0.001	0.423	-0.102	-0.024	-0.145	0.008	1.491	-0.664	0.951	0.908	-0.550
0.288	0.763	0.005	-1.377	-0.973	-0.049	-0.795	0.845	-0.485	0.001	0.423	0.102	0.024	-0.145	-0.008	-1.491	-0.664	0.951	0.908	-0.550
0.423	0.951	0.004	-1.328	-0.763	0.014	0.234	0	0.742	-0.002	-0.763	-1.394	-0.332	-0.951	0.004	0.760	-0.423	-0.423	-0.951	0.763
0.423	0.951	-0.004	1.328	-0.763	-0.014	-0.234	0	-0.742	0.002	-0.763	1.394	0.332	-0.951	-0.004	-0.760	-0.423	-0.423	-0.951	0.763
0.550	0.951	-0.003	0.781	-0.145	-0.086	-1.411	-0.845	-0.844	0.002	-0.763	-1.242	-0.296	0.288	0.001	0.219	0.973	-0.423	0.663	-0.908
0.550	0.951	0.003	-0.781	-0.145	0.086	1.411	-0.845	0.844	-0.002	-0.763	1.242	0.296	0.288	-0.001	-0.219	0.973	-0.423	0.663	-0.908
0.664	0.763	0.001	-0.224	0.550	0.081	1.322	-0.845	-1.728	0.004	0.423	-0.443	-0.106	0.908	0	0.003	-0.288	0.951	-0.145	0.973
0.664	0.763	-0.001	0.224	0.550	-0.081	-1.322	-0.845	1.728	-0.004	0.423	0.443	0.106	0.908	0	-0.003	-0.288	0.951	-0.145	0.973
0.763	0.423	0.123	0	0.951	1.172	-0.072	0	0.004	1.811	0.951	0.132	-0.554	-0.423	0.047	0	-0.763	-0.763	-0.423	-0.951
0.763	0.423	-0.123	0	0.951	-1.172	0.072	0	-0.004	-1.811	0.951	-0.132	0.554	-0.423	-0.047	0	-0.763	-0.763	-0.423	-0.951
0.845	0	0.454	0.002	0.845	1.486	-0.091	0.845	-0.001	-0.522	0	-0.343	1.441	-0.845	-0.334	0.002	0.845	0	0.845	0.845
0.845	0	-0.454	-0.002	0.845	-1.486	0.091	0.845	0.001	0.522	0	0.343	-1.441	-0.845	0.334	-0.002	0.845	0	0.845	0.845
0.908	-0.423	0.913	0.003	0.288	0.802	-0.049	0.845	-0.002	-0.987	-0.951	0.277	-1.165	0.550	1.056	-0.005	0.145	0.763	-0.973	-0.664
0.908	-0.423	-0.913	-0.003	0.288	-0.802	0.049	0.845	0.002	0.987	-0.951	-0.277	1.165	0.550	-1.056	0.005	0.145	0.763	-0.973	-0.664
0.951	-0.763	1.305	0.004	-0.423	-0.224	0.014	0	0	-0.083	-0.423	0.136	-0.572	0.763	-1.701	0.009	-0.951	-0.951	0.763	0.423
0.951	-0.763	-1.305	-0.004	-0.423	0.224	-0.014	0	0	0.083	-0.423	-0.136	0.572	0.763	1.701	-0.009	-0.951	-0.951	0.763	0.423
0.973	-0.951	1.497	0.005	-0.908	-0.841	0.051	-0.845	0.001	0.684	0.763	-0.194	0.816	-0.664	0.937	-0.005	0.550	0.423	-0.288	-0.145
0.973	-0.951	-1.497	-0.005	-0.908	0.841	-0.051	-0.845	-0.001	-0.684	0.763	0.194	-0.816	-0.664	-0.937	0.005	0.550	0.423	-0.288	-0.145
$\bar{\bar{y}}_1$	$\bar{\bar{y}}_2$	$\bar{\bar{y}}_3$	$\bar{\bar{y}}_4$	$\bar{\bar{y}}_5$	$\bar{\bar{y}}_6$	$\bar{\bar{y}}_7$	$\bar{\bar{y}}_8$	$\bar{\bar{y}}_9$	$\bar{\bar{y}}_{10}$	$\bar{\bar{y}}_{11}$	$\bar{\bar{y}}_{12}$	$\bar{\bar{y}}_{13}$	$\bar{\bar{y}}_{14}$	$\bar{\bar{y}}_{15}$	$\bar{\bar{y}}_{16}$	$\bar{\bar{y}}_{17}$	$\bar{\bar{y}}_{18}$	$\bar{\bar{y}}_{19}$	$\bar{\bar{y}}_{20}$

(52)

$$\Lambda = \begin{bmatrix}
6.691 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.247 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.818 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.818 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.369 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.954 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.954 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.802 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.801 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.801 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.682 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.638 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.638 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.605 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.555 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.523 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.506
\end{bmatrix} \tag{53}$$

$$X^T = \begin{bmatrix}
0.973 & 0.973 & 0.951 & 0.951 & 0.908 & 0.908 & 0.845 & 0.845 & 0.763 & 0.763 & 0.664 & 0.664 & 0.550 & 0.550 & 0.423 & 0.423 & 0.288 & 0.288 & 0.145 & 0.145 \\
0.951 & 0.951 & 0.763 & 0.763 & 0.423 & 0.423 & 0 & 0 & -0.423 & -0.423 & -0.763 & -0.763 & -0.951 & -0.951 & -0.951 & -0.951 & -0.763 & -0.763 & -0.423 & -0.423 \\
0.005 & -0.005 & -0.004 & 0.004 & 0.003 & -0.003 & -0.002 & 0.002 & 0 & 0 & 0.224 & -0.224 & 0.781 & -0.781 & 1.328 & -1.328 & 1.377 & -1.377 & 0.824 & -0.824 \\
-1.497 & 1.497 & 1.305 & -1.305 & -0.913 & 0.913 & 0.455 & -0.455 & -0.123 & 0.123 & 0.001 & -0.001 & 0.003 & -0.003 & 0.004 & -0.004 & 0.005 & -0.005 & 0.003 & -0.003 \\
-0.908 & -0.908 & -0.423 & -0.423 & 0.288 & 0.288 & 0.845 & 0.845 & 0.951 & 0.951 & 0.550 & 0.550 & -0.145 & -0.145 & -0.763 & -0.763 & -0.973 & -0.973 & -0.664 & -0.664 \\
-0.051 & 0.051 & 0.137 & -0.137 & 0.049 & -0.049 & -0.091 & 0.091 & 0.072 & -0.072 & 1.322 & -1.322 & 1.411 & -1.411 & 0.234 & -0.234 & -0.795 & 0.795 & -0.745 & 0.745 \\
-0.841 & 0.841 & 0.224 & -0.224 & 0.802 & -0.802 & -1.486 & 1.486 & 1.172 & -1.172 & -0.081 & 0.081 & -0.086 & 0.086 & -0.014 & 0.014 & 0.049 & -0.049 & 0.046 & -0.046 \\
0.845 & 0.845 & 0 & 0 & -0.845 & -0.845 & -0.845 & -0.845 & 0 & 0 & 0.845 & 0.845 & 0.845 & 0.845 & 0 & 0 & -0.845 & -0.845 & -0.845 & -0.845 \\
-0.684 & 0.684 & -0.083 & 0.083 & 0.987 & -0.987 & -0.522 & 0.522 & -1.811 & 1.811 & 0.004 & -0.004 & -0.002 & 0.002 & -0.002 & 0.002 & 0 & 0 & 0.001 & -0.001 \\
0.001 & -0.001 & 0 & 0 & -0.002 & 0.002 & 0.001 & -0.001 & 0.004 & -0.004 & 1.728 & -1.728 & -0.844 & 0.844 & -0.743 & 0.743 & 0.485 & -0.485 & 0.716 & -0.716 \\
0.763 & 0.763 & -0.423 & -0.423 & -0.951 & -0.951 & 0 & 0 & 0.951 & 0.951 & 0.423 & 0.423 & -0.763 & -0.763 & -0.763 & -0.763 & 0.423 & 0.423 & 0.951 & 0.951 \\
0.816 & -0.816 & 0.572 & -0.572 & -1.165 & 1.165 & -1.441 & 1.441 & -0.554 & 0.554 & 0.106 & -0.106 & -0.296 & 0.296 & 0.332 & -0.332 & -0.242 & 0.242 & -0.242 & 0.242 \\
0.194 & -0.194 & 0.136 & -0.136 & -0.277 & 0.277 & -0.343 & 0.343 & -0.132 & 0.132 & -0.443 & 0.443 & 1.242 & -1.242 & -1.393 & 1.393 & 0.102 & -0.102 & 1.020 & -1.202 \\
0.664 & 0.664 & -0.763 & -0.763 & -0.550 & -0.550 & 0.845 & 0.845 & 0.423 & 0.423 & -0.908 & -0.908 & -0.288 & -0.288 & 0.951 & 0.951 & 0.145 & 0.145 & -0.973 & -0.973 \\
0.005 & -0.005 & 0.009 & -0.009 & 0.005 & -0.005 & 0.002 & -0.002 & 0 & 0 & 0.030 & -0.030 & -0.219 & 0.219 & 0.760 & -0.760 & -1.491 & 1.491 & 1.467 & -1.467 \\
0.937 & -0.937 & 1.701 & -1.701 & 1.056 & -1.056 & 0.334 & -0.334 & 0.047 & -0.047 & 0 & 0 & 0.001 & -0.001 & -0.004 & 0.004 & 0.008 & -0.008 & -0.008 & 0.008 \\
0.550 & 0.550 & -0.951 & -0.951 & 0.145 & 0.145 & 0.845 & 0.845 & -0.763 & -0.763 & -0.288 & -0.288 & 0.973 & 0.973 & -0.423 & -0.423 & -0.664 & -0.664 & 0.908 & 0.908 \\
-0.423 & -0.423 & 0.951 & 0.951 & -0.763 & -0.763 & 0 & 0 & 0.763 & 0.763 & -0.951 & -0.951 & 0.423 & 0.423 & 0.423 & 0.423 & -0.951 & -0.951 & 0.763 & 0.763 \\
-0.288 & -0.288 & 0.763 & 0.763 & -0.973 & -0.973 & 0.845 & 0.845 & -0.423 & -0.423 & -0.145 & -0.145 & 0.664 & 0.664 & -0.951 & -0.951 & 0.908 & 0.908 & -0.550 & -0.550 \\
0.145 & 0.145 & -0.423 & -0.423 & 0.664 & 0.664 & -0.845 & -0.845 & 0.951 & 0.951 & -0.973 & -0.973 & 0.908 & 0.908 & -0.763 & -0.763 & 0.550 & 0.550 & -0.288 & -0.288
\end{bmatrix}
\begin{matrix}
\bar{\bar{x}}_1 \\ \bar{\bar{x}}_2 \\ \bar{\bar{x}}_3 \\ \bar{\bar{x}}_4 \\ \bar{\bar{x}}_5 \\ \bar{\bar{x}}_6 \\ \bar{\bar{x}}_7 \\ \bar{\bar{x}}_8 \\ \bar{\bar{x}}_9 \\ \bar{\bar{x}}_{10} \\ \bar{\bar{x}}_{11} \\ \bar{\bar{x}}_{12} \\ \bar{\bar{x}}_{13} \\ \bar{\bar{x}}_{14} \\ \bar{\bar{x}}_{15} \\ \bar{\bar{x}}_{16} \\ \bar{\bar{x}}_{17} \\ \bar{\bar{x}}_{18} \\ \bar{\bar{x}}_{19} \\ \bar{\bar{x}}_{20}
\end{matrix} \tag{54}$$

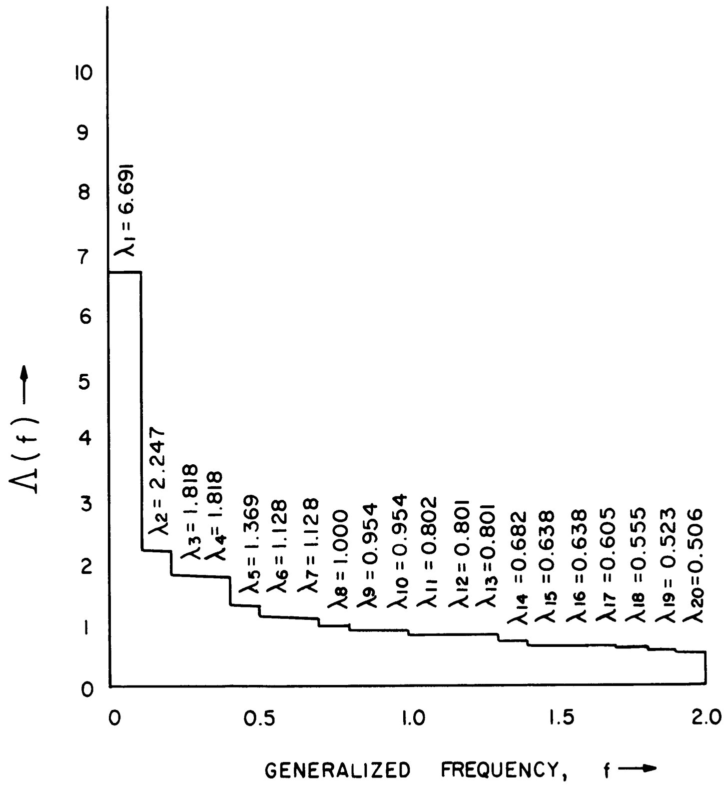

Fig. 4. Generalized frequency response for example system

A plot of the generalized frequency response, $\Lambda(f)$, is shown in Fig. 4. Note in Fig. 4 that λ_1 is considerably larger than the remaining λ_i's. If the number of sampling times, N, were increased from 10, it would be found that λ_1 approaches infinity as $N \rightarrow \infty$. This is because one of the eigenvalues of the matrix in Eq. 50 is 1 and its associated eigenvector is [1, 1] independently of k.

Finally, the insight into system behavior given by Eq. 53 or Fig. 4 is direct and obvious. On the other hand, the mass of numbers present in the Y and X-matrices, even for this relatively simple example, can be overwhelming. Clearly, there is a need for some limited, perhaps not unique, characterization of the $\bar{x}_i$ and $\bar{y}_i$-vectors which make up X and Y. For example, the number of sign changes or sign changes per channel might be used. Many other possibilities suggest themselves. As more experience is obtained in the use of these decompositions it should be possible to discover which characterizations are the more suitable.

7. ALTERNATE INPUT AND OUTPUT NORMS

So far the decomposition of the system matrix, G, has been based on the following inner product:

$$(\bar{\bar{x}}, \bar{\bar{z}}) = y^1(t_1)\, z^1(t_1) + \ldots + y^n(t_1)\, z^n(t_1) + \ldots$$

$$+ y^1(t_N)\, z^1(t_N) + \ldots + y^n(t_N)\, z^n(t_N) \quad . \tag{55}$$

Moreover, this inner product is used to characterize the range as well as the domain of G. There are many situations in which this characterization is not desirable. For example, it may not be realistic to weight the inputs and outputs on all the channels exactly the same. This uniform weighting is implied by Eq. 55 and it may not be desirable to weight all the sampling times exactly the same, which is also implied by Eq. 55.

Therefore, the decomposition of G is generalized here so that an arbitrary input and an arbitrary output inner product can be employed. An aribtrary inner product on a finite dimensional vector space can be represented by

$$(\bar{\bar{x}}_1, \bar{\bar{x}}_2)_{P^2} = \bar{\bar{x}}_2^{\,T} P^2 \bar{\bar{x}}_1 \ ,$$

where[9] P is a symmetric, positive definite matrix. Let the desired input and output inner products be characterized by P_i and P_o respectively. Given

$$\bar{\bar{y}} = G\bar{\bar{x}} \ , \tag{56}$$

let

$$\bar{\bar{x}} = P_i^{-1}\bar{\bar{u}}$$

---

[9] The matrix P^2 is used instead of P in order to avoid the notation $\sqrt{P}$. Since every positive definite symmetric matrix has a positive definite symmetric square root, this procedure results in no loss of generality.

$$\bar{\bar{y}} = P_o^{-1}\bar{\bar{v}} .$$

Then

$$\bar{\bar{v}} = P_o G P_i^{-1}\bar{\bar{u}}$$

$$= H\bar{\bar{u}} .$$

But the norm in the $\bar{\bar{v}}$- and $\bar{\bar{u}}$-spaces is again Eq. 55; therefore H can be decomposed as before. Thus

$$H = (Y\Delta f)\Lambda X^T .$$

Then

$$G = (P_o^{-1}Y\Delta f)\Lambda X^T P_i$$

$$= (P_o^{-1}Y\Delta f)\Lambda(P_i^{-1}X)^T P_i^2$$

$$= (Y_{P_o}\Delta f)\Lambda_{P_oP_i} X_{P_i}^T P_i^2 , \tag{57}$$

where $Y_{P_o} = P_o^{-1}Y$, $\Lambda_{P_oP_i} = \Lambda$, and $X_{P_i} = P_i^{-1}X$. Thus, the decomposition of G takes on a slightly altered form. Nevertheless, the analogies discussed previously carry over. It is easily seen that operation on an arbitrary input vector, $\bar{\bar{x}}$, with $X_{P_i}^T P_i^2$ is equivalent to taking the direct transform. Let the rows of $X_{P_i}^T$ be denoted $\bar{\bar{x}}_i$. These vectors are pairwise orthogonal and $||\bar{\bar{x}}_i|| = \sqrt{N} = 1/\sqrt{\Delta f}$ relative to the input norm determined by P_i, for

$$X_{P_i}^T P_i^2 X_{P_i} = X^T P_i^{-1} P_i^2 P_i^{-1} X$$

$$= X^T X$$

$$= NI .$$

If $\bar{\bar{x}}$ is expanded in terms of the $\bar{\bar{x}}_i$-vectors,

$$\bar{\bar{x}} = \sum_{k=1}^{mN} r_k \bar{\bar{x}}_k \Delta f$$

and

$$r_k = (\bar{\bar{x}}, \bar{\bar{x}}_k)_{P_i} = \bar{\bar{x}}_k{}^T P_i{}^2 \bar{\bar{x}} \ ,$$

but

$$X_{P_i}^T P_1{}^2 \bar{\bar{x}} = \begin{bmatrix} (\bar{\bar{x}}, \bar{\bar{x}}_1) \\ \cdots \\ (\bar{\bar{x}}, \bar{\bar{x}}_{mN})_{P_1} \end{bmatrix} .$$

Therefore the analogy does indeed carry over. Clearly, $\Lambda_{P_o P_i}$ in Eq. 57 acts as a generalized frequency response, and $(Y_{P_o} \Delta f)$ acts as the inverse transform. It should be noted that the columns of Y_{P_o} are pairwise orthogonal and of norm $\sqrt{N}$ relative to the output norm determined by $P_o{}^2$.

A discussion of the properties of the decomposition in Eq. 57 would parallel the discussion given for the Euclidean norm, Eq. 55 which is now a special case of Eq. 57. However, it is of interest to consider the method whereby Eq. 57 is generated. Since

$$P_i^{-2} G^T P_o{}^2 G = X_{P_i} \Lambda^2_{P_o P_i} X^T_{P_i} P_i{}^2 \ ,$$

it follows that the $\bar{\bar{x}}_i$-vectors are the eigenvectors of $P_i^{-2} G^T P_o{}^2 G$ and that the $\lambda_i{}^2$'s are its eigenvalues. The $\bar{\bar{y}}_i$-vectors can be determined from the relation

$$G\bar{\bar{x}}_i = \lambda_i \bar{\bar{y}}_i$$

under the assumption that $\lambda_i \neq 0$. If $\lambda_i = 0$, then, as before, the $\bar{\bar{y}}_i$-vectors are also the $\bar{\bar{x}}_i$-vectors and need not be uniquely determined.

8. CONCLUSIONS

It has been shown that multivariable, time-varying, discrete-time, linear systems can be handled in a manner equivalent to single-input, single-output systems. A transform technique for the latter systems is extended to multivariable systems, and frequency response concepts are shown to carry over in a straightforward way to multivariable systems. The key point of the present development has been a de-emphasis of the channelized character of the input or output and the treatment of an arbitrary input or output as a single vector in a linear vector space. Thus, much of the insight associated with single-input, single-output systems has validity for multivariable systems.

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