AN EXACT SOLUTION OF THE RAYLEIGH-BESANT EQUATION

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ABSTRACT: A power series solution of the Rayleigh-Besant equation governing the collapse of a spherical cavity in an inviscid fluid is presented. Numerical values for the non-dimensional solution $t = t(R)$ are obtained, where R is expressed in terms of the initial bubble radius R_o and time is expressed in terms of $R_o \sqrt{\rho / p_\infty}$

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1. Introduction.

In the earliest studies of the collapse or growth of a spherical cavity within a continuous liquid medium, Besant[1] in 1859, and later Rayleigh[2] in 1917, obtained by different methods a differential equation governing the motion of the bubble wall. This equation, based on the assumption of an inviscid and incompressible fluid, is written:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{p_\infty}{\rho}$$

where ρ is the density of the liquid and p_∞ the pressure at a very large distance from the collapsing or growing cavity, which is assumed to contain a vacuum.

By integration in t, the above equation reduces to:

$$\boxed{\dot{R}^2 = \frac{2}{3}\frac{p_\infty}{\rho}\left[\left(\frac{R_o}{R}\right)^3 - 1\right]} \qquad (1)$$

of which no solutions exist in the literature.

An exact solution of (1) would be of considerable interest in connection with bubble dynamic studies, since it could be compared to the results obtained from more sophisticated analyses, such as those of Gilmore[7] or Flynn[8], in which viscosity and compressibility are taken into account.

The function R = R(t) determines the radius of a collapsing bubble of initial radius R_o as a function of time, and will be therefore a single-valued function. In consequence, a solution of the form t = t(R) will be equally useful and easier to obtain. In fact, Rayleigh[1] gave in his original paper a solution for the "time of complete collapse" required for the bubble to reduce its diameter to zero. For all other times, as pointed out by Lamb[3], the solution is not so easily found.

2. Power Series Solution.

To express equation (1) in a more tractable form, let:

$$K = R_o\sqrt{\frac{3}{2}}\sqrt{\frac{\rho}{p_\infty}} \qquad (2)$$

and define a new dependent variable β as:

$$\beta = \frac{R}{R_o} \qquad (3)$$

Thus:

$$\frac{d\beta}{dt} = \frac{1}{R_o}\frac{dR}{dt}$$

and equation (1) becomes:

$$\frac{d\beta}{dt} = \pm\frac{1}{K}\sqrt{\frac{1-\beta^3}{\beta^3}}$$

But for the collapsing bubble, $dR/dt < 0$, and hence, also $d\beta/dt < 0$. Since K is a positive number, only the negative sign has physical meaning in the above expression, which is then written:

$$dt = -K\frac{\beta^{3/2}}{(1-\beta^3)^{1/2}}$$

The time required for the bubble to collapse from an initial radius R_o (i.e., $\beta = 1$) to an arbitrary radius R is thus:

$$t = -K\int_1^\beta \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} = K\int_\beta^1 \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} \qquad (4)$$

For $\beta = 0$, the time of complete collapse τ is obtained:

$$\tau = K\int_0^1 \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} \qquad (5)$$

As shown by Rayleigh[2], this integral is easily solved in terms of the gamma function. Let:

$$\beta^3 = z \qquad \text{and then} \qquad 3\beta^2 d\beta = dz$$

The integral (5) becomes then:

$$\tau = K\,\frac{1}{3}\int_0^1 z^{-1/6}\,(1-z)^{-1/2}\,dz$$

Recalling now the well known formula*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

we have for this case, for m = 5/6 and n = 1/2:

$$\tau = \frac{K}{3}\;\frac{\Gamma(5/6)\,\Gamma(1/2)}{\Gamma(4/3)}$$

After calculation of the gamma functions and replacing K by its value, (2), one finally obtains:

$$\boxed{\tau = .91468\; R_o\sqrt{\frac{\rho}{p_\infty}}} \qquad (6)$$

as given by Rayleigh.

But we are interested in calculating t when the final value of β is different from zero. For that case, equation (4) is rewritten as follows:

$$t = K\int_0^1 \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} - K\int_0^{\beta} \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}}$$

or also, considering equation (5):

$$t = \tau - K\int_0^{\beta} \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} \qquad (7)$$

The integral

$$I = \int_0^{\beta} \frac{\beta^{3/2}\,d\beta}{(1-\beta^3)^{1/2}} \qquad (8)$$

will exist for all $0 \leq \beta < 1$, since the integrand is defined within this interval. This integrand is then expressed in terms of the

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(*) See Reference 4, page 383, for example.

powers of β, starting from the formula[(*)]

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots.$$

valid for any real number m and for $|x|<1$.

For our case: $m = -1/2$

$x = -\beta^3$

and therefore:

$$\frac{\beta^{1/2}}{(1-\beta^3)^{1/2}} = \beta^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\beta^{3n}\right] \qquad (9)$$

valid for i.e., within the interval of interest $0 \leq \beta < 1$, and where:

$$(2n-1)!! = 1.3.5\ldots\ldots(2n-3)(2n-1)$$

and: $$(2n)!! = 2.4.6\ldots\ldots(2n-4)(2n-2)2n$$

The radius of convergence r^* of the power series (9) can be found by using the formula[(++)]:

$$r^* = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

For this case:

$$c_n = \frac{(2n-1)!!}{(2n)!!}$$

$$c_{n+1} = \frac{(2n+1)!!}{(2n+2)!!}$$

and therefore:

$$r^* = \lim_{n\to\infty} \frac{(2n-1)!!\,(2n+2)!!}{(2n)!!\;(2n+1)!!} = \lim_{n\to\infty}\frac{2n+2}{2n+1} = 1$$

The power series is then absolutely convergent in the interval

$$0 \leq \beta < 1$$

(*) Reference 4, page 359, for example
(++) Reference 4, page 350.

and converges uniformly for:

$$0 \leqslant \beta \leqslant r_1 < 1$$

Since a power series can be integrated term by term within the interval of convergence*, by substitution of (9) into (8) we obtain:

$$I = \int_0^\beta \frac{\beta^{3/2}\, d\beta}{(1-\beta^3)^{1/2}} = \int_0^\beta \left[\beta^{3/2} + \beta^{3/2} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \beta^{3n}\right] d\beta$$

$$= \int_0^\beta \beta^{3/2} d\beta + \int_0^\beta \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \beta^{3n+\frac{3}{2}}\, d\beta$$

or:

$$I = \frac{2}{5}\beta^{5/2} + \beta^{5/2} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{\beta^{3n}}{3n + \frac{5}{2}} \tag{10}$$

For $\beta = 1$, the integration term by term is not valid, since the power series (9) diverges at that point. For our purposes, this is not important, since for $\beta = 1$ the value of I is, from equation (5):

$$I = \frac{\tau}{K} \qquad \text{and in (7): } t = 0$$

By substitution of (10) into (7), then:

$$t = \tau - K\left\{\frac{2}{5}\beta^{5/2} + \beta^{5/2} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{\beta^{3n}}{3n + \frac{5}{2}}\right\} \tag{11}$$

Replacing τ and K by their respective values (6) and (2), the desired solution of the Rayleigh-Besant equation is expressed:

$$t = R_o \sqrt{\frac{\rho}{P_\infty}} \left\{0.91468 - \sqrt{\frac{3}{2}}\, \beta^{5/2}\left(\frac{2}{5} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{\beta^{3n}}{3n + \frac{5}{2}}\right)\right\} \tag{12}$$

for $0 < \beta < 1$, and:

$$t = 0$$

for $\beta = 1$. This is also evident from physical considerations.

(*) Reference 4, page 352, Theorem 37.

3. Numerical Solution.

Practical applications of equation (12) require the calculation of numerical values that are independent of the particular problem under consideration. Clearly, then, the first step is to express (12) in a dimensionless form. For that, one observes that the quantity:

$$R_o \sqrt{\frac{\rho}{p_\infty}}$$

has the dimensions of a time. Hence, if t is expressed in terms of this quantity, equation (12) becomes:

$$\boxed{t = 0.91468 - \sqrt{\frac{3}{2}}\,\beta^{5/2}\left[\frac{2}{5} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\frac{\beta^{3n}}{3n + \frac{5}{2}}\right]} \qquad (14)$$

valid for $0 \leq \beta < 1$, where $\beta = R/R_o$. This non-dimensional equation can also be written, in a more compact form:

$$t = 0.91468 - S(\beta) \qquad (15)$$

where $S(\beta)$ is defined as:

$$S(\beta) = 1.224745\left[0.4 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\frac{\beta^{3n}}{3n+2.5}\right]\beta^{5/2} \qquad (16)$$

The next problem, then, is to determine how many terms one must calculate to obtain $S(\beta)$ with an error less than a given value ϵ.

To do that, let:

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\frac{\beta^{3n}}{3n + \frac{5}{2}} = \sum_{n=1}^{\infty} u_n$$

and calculate:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+2)}\,\frac{\beta^{3n+3}}{3n+\frac{11}{2}}\quad\frac{3n + \frac{5}{2}}{\beta^{3n}}\right| = \beta^3$$

Then, the remainder of the series is given by the expression*:

$$|R_n| \leq \frac{u_{n+1}}{1-\beta^3} = \frac{(2n+1)!!\ \beta^{3n+3}}{(2n+2)!!(3n+\frac{11}{2})(1-\beta^3)} \tag{17}$$

This formula gives only an upper bound of the error. One notices that for $\beta \to 1, (1-\beta^3) \to 0$, and therefore the values of R_n will increase as β approaches 1. If β_1 is the maximum value of β in the interval of interest, the actual error for β will be less than the value given by (17) for $\beta = \beta_1$.

In consequence, expression (17) can be used to obtain an estimate of the error that will result from computing only n terms of the sum, for different values of β. The following values were obtained:

β_1	n	$R_n \leq$
0.90	10	0.00054
0.99	10	0.1145
	20	0.0334
0.999	10	1.5285
	20	0.5852

Using a 7090 IBM digital computer, values of t as given by (14) were calculated for different values of n and β. A preliminary calculation showed that the actual error, for given n and β, is as follows:

	n	$\epsilon \leq$
$0 < \beta \leq 0.7$	10	10^{-6}
$0.7 \leq \beta \leq 0.9$	30	10^{-6}
$0.9 \leq \beta \leq 0.96$	80	10^{-6}
$0.96 \leq \beta \leq 0.99$	100	10^{-3}

(*) See Reference 4, page 328.

confirming that the values of ϵ are much less than the values R_n predicted by using expression (17). Using then these values of n a final machine calculation was performed, incrementing β by 0.01 between 0 and 0.99. The results are listed in Table I, while the corrsponding plot of t vs. β is presented in Figure 1. One must keep in mind that β represents a non-dimensional radius, defined as:

$$\beta = \frac{R}{R_o}$$

and that t is a non-dimensional time, related to the actual time t' by the expression:

$$t = \frac{t'}{R_o \sqrt{\rho / p_\infty}}$$

It is interesting to note from the above that the usual considerations of dynamic similarity apply to this case in that times of collapse for different fluids would be equal as long as the available "head drop", ie., p_∞ / ρ, were the same.

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β	t	β	t	β	t
0.99	0.016145	0.64	0.733436	0.29	0.892245
0.98	0.079522	0.63	0.741436	0.28	0.894153
0.97	0.130400	0.62	0.749154	0.27	0.895956
0.96	0.174063	0.61	0.756599	0.26	0.897658
0.95	0.212764	0.60	0.763782	0.25	0.899262
0.94	0.247733	0.59	0.770712	0.24	0.900769
0.93	0.279736	0.58	0.777398	0.23	0.902182
0.92	0.309297	0.57	0.783847	0.22	0.903505
0.91	0.336793	0.56	0.790068	0.21	0.904738
0.90	0.362507	0.55	0.796068	0.20	0.905885
0.89	0.386662	0.54	0.801854	0.19	0.906947
0.88	0.409433	0.53	0.807433	0.18	0.907928
0.87	0.430965	0.52	0.812810	0.17	0.908829
0.86	0.451377	0.51	0.817993	0.16	0.909654
0.85	0.470770	0.50	0.822988	0.15	0.910404
0.84	0.489229	0.49	0.827798	0.14	0.911083
0.83	0.506830	0.48	0.832431	0.13	0.911692
0.82	0.523635	0.47	0.836890	0.12	0.912234
0.81	0.539701	0.46	0.841181	0.11	0.912713
0.80	0.555078	0.45	0.845308	0.10	0.913130
0.79	0.569810	0.44	0.849277	0.09	0.913489
0.78	0.583937	0.43	0.853090	0.08	0.913793
0.77	0.597495	0.42	0.856752	0.07	0.914045
0.76	0.610515	0.41	0.860268	0.06	0.914248
0.75	0.623027	0.40	0.863640	0.05	0.914406
0.74	0.635059	0.39	0.866872	0.04	0.914523
0.73	0.646633	0.38	0.869969	0.03	0.914604
0.72	0.657773	0.37	0.872933	0.02	0.914652
0.71	0.668498	0.36	0.875768	0.01	0.914675
0.70	0.678830	0.35	0.878477	0.00	0.91468
0.69	0.688784	0.34	0.881062		
0.68	0.698377	0.33	0.883528		
0.67	0.707625	0.32	0.885876		
0.66	0.716542	0.31	0.888110		
0.65	0.725142	0.30	0.890232		

Error less than 10^{-6} for $0 \leq \beta \leq 0.96$

TABLE I

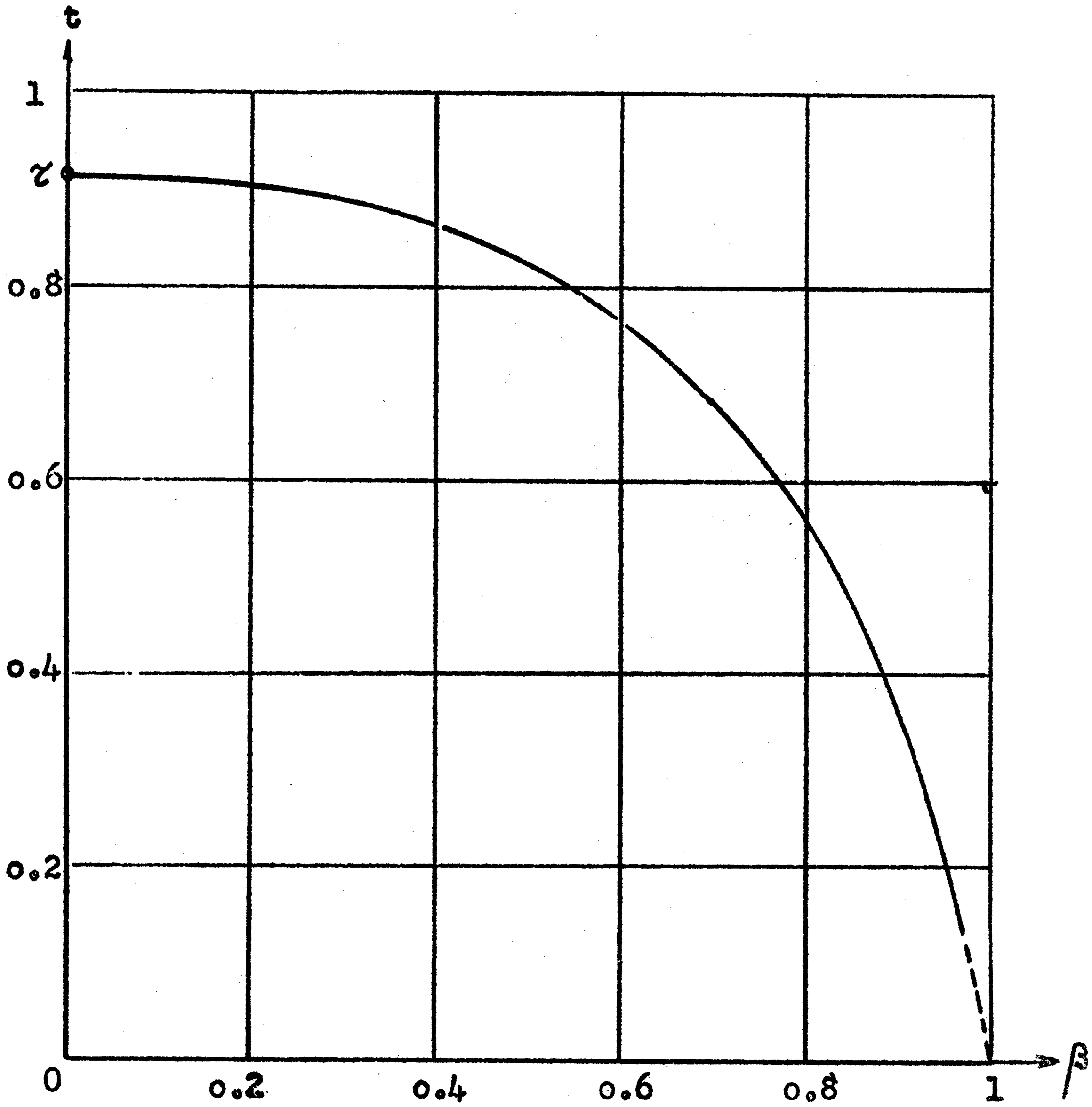

Fig.1. Non-dimensional Time vs. Non-dimensional Radius for Collapsing Spherical Void.

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