Since unambiguous ranking of income distributions according to their degree of inequality is not always possible, choice of inequality measure must rest on the appropriateness of particular measures for particular substantive problems. This article provides a complete account of one measure of inequality, δ, defined, for *$x > 0$, as the ratio of the geometric mean to the arithmetic mean—a measure that is closely linked to the sense of distributive justice. Its properties are summarized, and formulas reported for the effects of transfers and of location changes. Analytic expressions for δ for three classical probability distributions—the Pareto, Lognormal, and Rectangular families—are provided, and δ's behavior in within-family comparisons discussed. The measure δ's behavior in between-family comparisons is explored using a new procedure for bounding the zones of ambiguity in inequality comparisons. Finally, a newly obtained decomposition formula for δ is reported.*

Measuring Inequality

Using the Geometric Mean/Arithmetic Mean Ratio

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Unambiguous ranking of income distributions according to their degree of inequality is known to be possible only for certain sets of distributions, notably, for observed distributions whose Lorenz curves do not intersect (Atkinson, 1970, 1975; Rothschild and Stiglitz, 1973; Sen, 1973; Champernowne, 1974; Cowell, 1977; Allison, 1978; Fields and Fei, 1978) and for a priori, mathematically specified distributions drawn from the same distributional family (Cowell, 1977; Jasso, 1980). In all other cases, use of different measures of inequality yields different orderings, and, hence, different answers to the question: Which distribution is more (or less) unequal?

Thus, if one is comparing the relative inequality of five Pareto distributions or of five observed distributions possessing non-intersecting Lorenz curves, then choice of inequality measure

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does not affect the result: All measures yield the same rank ordering. However, if one is comparing the relative inequality of two Paretos and three Lognormals, or of five observed distributions possessing intersecting Lorenz curves, then choice of inequality measure *determines* the resulting rank ordering.

The fact that different measures of inequality can produce different rank orderings suggests that they vary in their sensitivities to the amounts of dispersion and concentration at varying points along the income continuum—suggests, that is, that they highlight different aspects of the distribution's inequality. The question of which inequality measure to use thus reduces to a question about which aspect of the distribution's inequality the investigator wishes to tap. This choice is crucial because: (1) correct specification of the substantive phenomenon depends on it; and (2) empirical samples of social aggregates (e.g., nation-states) are likely to include some whose income distributions have intersecting Lorenz curves. In this view, a measure of inequality is never regarded as in general superior to another, but only as more appropriate for a particular substantive problem.

Informed choice of inequality measure involves fitting the measure's properties to the substantive relation under investigation. It follows that the social scientist's inventory of tools should contain many measures of inequality and thorough knowledge of each measure's properties. Note that some measures of inequality have been originally obtained by solving a substantive problem. For example, Gini's Index of Concentration is a scale-invariant version of a measure obtained by solving for the mean of all the pairwise absolute differences, i.e., obtained as an answer to the substantive question, "What is the expected absolute difference between two randomly selected individuals?" (See Jencks, 1972, for an interesting use of this measure.)

This article reports a detailed description of a measure of dispersion δ and of its derivation from a substantive problem in the study of distributive justice. This measure is defined as the ratio of the geometric mean to the arithmetic mean of a distribution of a positive magnitude. In the observed case,

$$\delta = \left[\prod_{i=1}^{N} x_i \right]^{1/N} \div (1/N) \sum_{i=1}^{N} x_i \qquad [1]$$

In the a priori case,

$$\delta = \frac{e^{\int y\, g(y)\, dy}}{\int x\, f(x)\, dx} \qquad [2]$$

where $y = \ell n\ x$.

Since in all unequal distributions of a positive magnitude, the geometric mean is smaller than the arithmetic mean, and in the equal distribution the two are equal, δ can take values from near-zero to one. Inequality increases as δ grows smaller. Although, as will be seen, this has a natural interpretation, the usual convention of denoting increasing inequality by an increasing positive number can be achieved by defining the transformation

$$\Delta = 1 - \delta \qquad [3]$$

This article describes the substantive foundations of the measure δ, summarizes its properties, presents formulas for δ in three classical probability distributions, examines δ's behavior in cross-family comparisons, and reports decomposition formulas.

SUBSTANTIVE FOUNDATIONS

The measure δ has been independently derived in two substantively meaningful contexts.

ATKINSON'S MEASURE OF INEQUALITY

The ratio of the geometric mean to the arithmetic mean first emerged as a measure of inequality when it was noticed that it arises as a special case of Atkinson's Index of Inequality (1970, 1975),

$$A = 1 - \left[(1/N) \sum_{i=1}^{N} (x_i/\mu)^{1/\epsilon} \right]^{1/(1-\epsilon)} \qquad \epsilon > 0 \qquad [4]$$

where ϵ is an explicit inequality aversiveness parameter. That special case occurs when ϵ approaches one (Champernowne, 1974; Bartels and Nijkamp, 1976; Cowell, 1977; Allison, 1978). The proof utilizes a proof that

$$\lim_{\epsilon \to 1} \left[(1/N) \sum_{i=1}^{N} (x_i)^{1/\epsilon} \right]^{1/(1-\epsilon)} = \text{geometric mean} \qquad [5]$$

based on L'Hôpital's Rule (see, for example, Kendall and Stuart [1977: 37]). Thus,

$$\lim_{\epsilon \to 1} A = 1 - \delta \qquad [6]$$

Substantively, a value of one on the inequality aversiveness parameter means that the desire to reduce inequality is of the following strength: In administering a transfer of size h from a relatively richer person R (of income x_R) to a relatively poorer person P (of income x_P), society (or the person making the judgment) is willing to lose up to a $(1 - x_P/x_R)$ fraction of h (see Atkinson, 1970, 1975; Cowell, 1977). For example, one would be willing to take \$10,000 from a person of income \$50,000 in order to give \$2,000 to a person of income \$10,000, thereby losing \$8,000 in the transfer process (and still not achieving equality).

Of course, choice of a particular value of ϵ implies imputing to society's members a certain inequality aversiveness—implies, moreover, imputing to all society's members the same degree of inequality aversiveness. No research has been undertaken to discover an empirically based ϵ.

THE DISTRIBUTIVE JUSTICE FOUNDATIONS OF δ

The foundation of δ as a measure of inequality has been recently strengthened by new developments in the field of distributive justice. Recent empirical and theoretical work (Jasso, 1978, 1980) suggests that the magnitude of the sentiment of justice or injustice experienced by an individual about his/her amount of a social resource (such as income) varies with the logarithm of the ratio of the amount to the arithmetic mean of the relevant population:

$$\text{justice evaluation} = \ln \frac{x_i}{\mu} \qquad [7]$$

In this formulation, the sense of distributive justice is represented by the full real-number line, with zero the point of perfect justice, the negative segment representing the sense of unjust underreward, and the positive segment representing the sense of unjust overreward.

It is suggested further that a social aggregate may be usefully represented and characterized by the distribution of the sentiment of justice among its members. That is to say, the justice distribution is as pertinent a description of a collectivity as is the income distribution that gives rise to it. The location, dispersion, and other characteristics of the justice distribution, too, are considered as useful as the characteristics of the underlying income distribution. Moreover, it is hypothesized that each of the attributes of the justice distribution predicts different social phenomena, that is, each exercises hegemony over a different area of the social life.

Of course, the income distribution and the justice distribution are closely linked. The latter can be obtained from the former by a change of variable. Their characteristics, too, are closely linked. For example, the quantity "proportion that feels unjustly underrewarded" is equal to the quantity "proportion with incomes below the mean."

The measure δ was obtained by Jasso (1980) as part of the solution to the question, "What is the arithmetic mean of the justice distribution?" Equation 15 in Jasso (1980) reports the definitional formula for the arithmetic mean of the justice distribution,

$$\text{justice mean} = \frac{\sum_{i=1}^{N} (\ell n\, x_i/\mu)}{N} \qquad [8]$$

where the x_i are the individuals' amounts of the resource (such as income) and μ is the arithmetic mean of the resource distribution. Equations 16 and 17 in Jasso (1980) show that the definitional formula for the justice mean is equivalent to

$$\ell n \frac{\left[\prod_{i=1}^{N} x_i\right]^{1/N}}{\mu} \qquad [9]$$

or the logarithm of the ratio of the geometric mean of the income distribution to the arithmetic mean. That is,

$$\text{justice mean} = \ell n\, \delta \qquad [10]$$

This link between the income distribution and the justice distribution may be expressed as

$$\text{justice mean} = \ell n \text{ income inequality} \qquad [11]$$

where income inequality is measured by δ. The mean of the justice distribution has a range from negative infinity to zero, the point of perfect justice, which it realizes only when income is equally distributed. Thus, as δ grows toward unity (i.e., as inequality decreases), the justice mean increases toward zero. Stated differently, δ realizes a magnitude of one if and only if income is equally distributed.

The relationship between inequality as measured by δ and the sense of distributive justice provide δ with a firm, empirically based, substantive foundation. The measure δ may appropriately be used for any substantive problem in which the social scientist specifies a social variable as dependent on a linear, additive combination of the magnitudes of justice or injustice experienced by all the collectivity's members, with each person's sentiment of justice or injustice receiving equal weight.

PROPERTIES OF δ

The principal properties of a measure of inequality are its limits and its sensitivities to the location, scale, and shape parameters of the distribution and to the size N of the sample or population. (For a fuller discussion of these properties, see Dalton, 1920; Yntema, 1933; Champernowne, 1974; Allison, 1978; Cowell, 1977: 62-73, who provides a convenient tabular summary of some properties of many common measures of inequality.)

LIMITS

The value of δ has fixed limits of zero and one, being open at 0 and closed at one. Since in any distribution of a positive quantity, except an equal one, the geometric mean is less than the arithmetic mean, the value of δ is a proper fraction for all unequal distributions. As noted above, in an equal distribution δ equals unity. Since both the geometric mean and the arithmetic mean of a set of positive magnitudes are themselves positive magnitudes, δ never reaches zero. The measure δ thus satisfies Yntema's (1933: 423) criterion of definite limits, "preferably zero and one."

It should be noted, however, that fixed limits are not unambiguously desirable, since, as Cowell (1977: 69) argues, there might be good substantive reason to specify inequality as partly dependent on the size N of the population. In any case, fixed limits are no rare property, for "there are many ways of transforming the measure such that it lies in the zero-to-one range" (Cowell, 1977: 69).

SENSITIVITY TO LOCATION

Consider an income distribution of any specified location and shape. If a constant amount c is added (or subtracted) to each income, then the graph of the distribution (either the frequency distribution in the observed case, or the probability density function in the a priori case) will move sideways along the x-axis. The visual shape, as well as the size of the graph, remains intact, although, as will be seen below, the shape parameter of an a priori distribution may change.

What is the effect of such a change on measured inequality? Stated differently, do distributions whose graphs are horizontal translates of each other differ in their degree of inequality? The answer will reflect an inequality measure's sensitivity to the cardinal properties of the order statistics, that is, to the absolute amounts of income, as well as to changes in the individuals' relative shares.

To derive δ's sensitivity to location, first define the new quantity

$$\hat{\delta} = \left[\prod_{i=1}^{N} x_i + c \right]^{1/N} \div (\mu + c) \qquad [12]$$

By restating the formulas for δ and $\hat{\delta}$:

$$\delta = \exp \left[\sum_{i=1}^{N} \ell n \, (x_i/\mu)^{1/N} \right] \qquad [13]$$

and

$$\hat{\delta} = \exp \left[\sum_{i=1}^{N} \ell n \, ((x_i + c)/(\mu + c))^{1/N} \right] \qquad [14]$$

(from Equation 8 above), it can be seen that the elementary units of the formulas are of the form x_i/μ and $(x_i + c)/(\mu + c)$.

Since the function $1/N$ is in this case a positive constant, and since the logarithmic and exponential functions are monotonic, the effect on δ of adding or subtracting a constant may be observed by constructing inequalities for the two formulas' elementary units.

Addition of a positive constant yields the inequalities,

$$\frac{x + c}{\mu + c} > \frac{x}{\mu} \qquad x < \mu$$

$$\frac{x + c}{\mu + c} < \frac{x}{\mu} \qquad x > \mu \qquad [15]$$

and subtraction of a positive constant yields the opposite, namely,

$$\frac{x-c}{\mu-c} < \frac{x}{\mu} \qquad x<\mu$$
$$\frac{x-c}{\mu-c} > \frac{x}{\mu} \qquad x>\mu \qquad [16]$$

For x less than μ, the ratio is a proper fraction, and its logarithm is negative; for x greater than μ, the ratio exceeds unity and its logarithm is a positive quantity. A basic property of logarithms is that only reciprocals have equal absolute magnitudes. Hence, adding a constant decreases the absolute value of the negative logged quantities more than it decreases the magnitude of the positive logged quantities; subtracting a constant increases the absolute value of the negative logged quantities more than it increases the magnitudes of the positive logged quantities.

Therefore, adding a constant produces a larger value of δ, reducing inequality; subtracting a constant produces a smaller value of δ, increasing inequality. This means that when inequality is measured by δ, an across-the-board bonus of fixed absolute size reduces inequality, while an across-the-board tax of fixed absolute size increases it.

SENSITIVITY TO SCALE

Sensitivity to scale is an important property of measures of inequality. Changes in scale do not alter the shape of the graph of a frequency distribution or of a probability density function, but they do alter the size of the graph. There are two main arguments advanced in favor of scale-invariance. The first, as Cowell (1977: 63) puts it, is that "measured inequality should not depend on the size of the cake. If everyone's income changes by the same proportion then it can be argued that there has been no essential alteration in the income distribution, and thus that the value of the inequality measure should remain the same." The second is that measured inequality should be "independent of the units in which income or wealth is measured" (Yntema, 1933: 423), that is, as Allison (1978: 866) puts it, measured inequality "should not depend on whether income is measured in dollars or yen."

The measure δ can be quickly shown to be scale-invariant by using a version of the argument used in discussing sensitivity to location, where

$$\text{for all } x \qquad \frac{cx}{c\mu} = \frac{x}{\mu} \qquad [17]$$

Thus, inequality as measured by δ varies only with changes in the ratios of own income to the mean income. This means that only proportional taxation leaves inequality intact; progressive taxation can be seen to be an instrument of reducing inequality.

SENSITIVITY TO SHAPE

Sensitivity to shape is examined separately for observed and for mathematically specified distributions.

Observed Distributions

An appealing way to explore sensitivity to shape in an observed distribution is by considering the effect of transfers from one to another person in the distribution. Given the value of the pretransfer δ, and expressing the value of the posttransfer measure as δ^*, the ratio of δ^* to δ, for the case in which one transfer is made, equals:

$$\frac{\delta^*}{\delta} = \frac{\left[(x_j - c)(x_i + c)\right]^{1/N}}{(x_i x_j)^{1/N}} \qquad [18]$$

Since the product of two quantities whose sum is fixed varies inversely with the absolute difference between them, it follows that transfers from scores with lower order statistics to scores with higher order statistics (i.e., from relatively poorer to relatively richer) produce a δ^* smaller than δ, and that transfers from scores with higher order statistics to scores with lower order statistics (i.e., from relatively richer to relatively poorer) produce a δ^* greater than δ. This means that transfers from poor to rich persons reduce the value of δ, thereby increasing inequality, and vice versa.

How sensitive is δ to the size of the transfer? The magnitude of the ratio of δ^* to δ varies curvilinearly with the size of the transfer. For a transfer from x_j to x_i given that $x_j > x_i$,

$$\frac{\delta^*}{\delta} = \left[\frac{x_i x_j + cx_j - cx_i - c^2}{x_i x_j}\right]^{1/N} \qquad [19]$$

which

$$= \left[\frac{x_i x_j + c(x_j - x_i) - c^2}{x_i x_j}\right]^{1/N} \qquad [20]$$

This expression shows that there is no change in δ (i.e., $\delta^* = \delta$) when the distance between the two pretransfer scores equals the size of the transfer and that the value of the ratio of δ^* to δ peaks when the size of the transfer equals one-half the distance between the two pretransfer scores. This property illustrates δ's strong sensitivity to pairwise equality.

Classical Mathematically Specified Distributions

In the study of classical probability distributions, variation in shape appears in two ways. First, there is a sense in which shape is coterminous with distributional family or distributional form. For example, the Normal family is identified by a *distinc-*

tive shape, as are the Lognormal, Gamma, Pareto, and all other families of curves. Second, variation in shape also occurs *within* distributional family and is usually determined by a "shape parameter" (Hastings and Peacock, 1974: 20), which, like the other basic parameters, "appear[s] explicitly in the specification of the distribution" (Kendall and Stuart, 1977: 32).

Within distributional family, the measure δ varies with the shape parameter c.[1] A discussion of the functions that describe this variation for three distributional families follows. Across distributional form, however, variation in δ is irregular and cannot be completely predicted a priori. The final section of the article attacks this problem through an examination of two situations: one, in which very different distributions have identical values of δ; and another, in which δ and the Gini Index of Concentration produce different rank orderings of distributions as to their degree of inequality.

SENSITIVITY TO THE SIZE N OF THE SAMPLE OR POPULATION

The measure δ is insensitive to the size N of the sample or population. Two ways which render visible this insensitivity to N are: First, if k identical discrete distributions are pooled, the value of δ remains unchanged; and second, adding persons to a distribution of rank-order statistics leaves δ unchanged (see following discussion of rectangular distributions). This property satisfies another of Yntema's (1933: 423) criteria, namely, that the measure of inequality be "independent of the number of persons in a distribution."

Note, however, that the appropriateness of a measure with this property depends on the substantive problem under investigation. As Cowell (1977: 63-64) observes, a two-person world in which one person has all the income has rather different implications for the social life than a four-person world in which two persons have all the income.

FORMULAS FOR δ IN SOME CLASSICAL PROBABILITY DISTRIBUTIONS

When the distributional family is known a priori, the formula for δ may be expressed in terms of the analytic formulas for the geometric and arithmetic means, which in turn are expressed in terms of the distribution's location, scale, and shape parameters. The formulas for δ are given below for three distributional families, the Pareto, the Lognormal, and the Rectangular.

PARETO DISTRIBUTIONAL FAMILY

The Pareto is a two-parameter family of positively skewed curves. Its probability density function, with the location parameter fixed at the arithmetic mean μ, is given by

$$f(x) = \mu^{c}(c - 1)^{c} c^{-c+1} x^{-c-1} \qquad c > 1 \tag{21}$$

where c the shape parameter, known as Pareto's constant, is restricted to values greater than one (in order that the mean be defined). The graph of the Pareto's probability density function has a single mode at the lower limit, which is a quantity greater than zero, and is everywhere decreasing at an increasing rate. For a fuller treatment of the Pareto family, see Hastings and Peacock (1974: 102-105) and Johnson and Kotz (1970a: 233-249).

The formula for δ is obtained by evaluating the expression in Equation 2. The first step, obtaining the expected value of the logarithmic function of the Pareto density (see Hoel, 1971: 72 for this procedure), yields

$$\int \ln x \, f(x)\, dx = \ln \frac{\mu(c - 1)}{c} + \frac{1}{c} \tag{22}$$

Taking the exponential of that expression and dividing by μ produces

$$\delta = \frac{(c-1)e^{1/c}}{c} \qquad c > 1 \tag{23}$$

The measure δ varies with c, approaching zero as c approaches one and approaching one as c goes to infinity. Pareto's constant is regarded as a crude measure of inequality (see Cramer, 1971: 51-58). In fact, in all within-family comparisons, it operates as a general measure of inequality, controlling all the other measures of inequality.[2]

LOGNORMAL DISTRIBUTIONAL FAMILY

The Lognormal is a two-parameter family of positively skewed curves. Its probability density function, with location parameter fixed equal to the arithmetic mean, is given by

$$f(x) = \frac{\mu^{1/2}\, e^{-c^2/8}\, e^{-(\ell n(x/\mu))^2/2c^2}}{cx(2x\pi)^{1/2}} \qquad c > 0 \tag{24}$$

where the shape factor c is the standard deviation of the logged Lognormal variate. The graph of the Lognormal's probability density function has a lower limit approaching zero, a mode at a subsequent value, and two points of inflection, one on either side of the mode. For a fuller treatment of the Lognormal family, see Hastings and Peacock (1974: 84-89) and Johnson and Kotz (1970a: 40-111).

The formula for δ is very easily obtained, since the Lognormal's

$$\text{geometric mean} = \mu/e^{c^2/2} \tag{25}$$

Hence,

$$\delta = e^{-c^2/2} \tag{26}$$

The measure δ varies with the Lognormal's shape parameter c, decreasing to zero (i.e., increasing inequality) as c goes to infinity and increasing to one (perfect equality) as c diminishes to zero.

RECTANGULAR DISTRIBUTIONS

In some applications the rank-order statistics are themselves of interest, either instead of or in addition to the order statistics. If the rank-order statistics are employed directly, the resulting distribution is a discrete rectangular, described in Hastings and Peacock (1974: 52-53) and Johnson and Kotz (1969: 238-240). If the probability integral transformation is used (for example, to model infinitely large populations), the resulting distribution is a continuous rectangular, described in Hastings and Peacock (1974: 116-119) and Johnson and Kotz (1970b: 57-74). The formula for δ is easily obtained in both cases, its value approaching a constant as N approaches infinity.

First, consider a distribution of rank-order statistics from one to N. Since the geometric mean of the set of integers from one to N can be expressed as the N^{th} root of N-factorial, and since the arithmetic mean is half the quantity (N + 1),

$$\delta = 2(N!)^{1/N}/(N + 1) \qquad [27]$$

The limit of that expression, as N goes to infinity, is the quantity

$$2/e \cong .7358 \qquad [28]$$

where e is the base of the natural logarithms.

Second, consider a unit rectangular, that is, defined on the zero-to-one range. Using Equation 2, obtaining the expected value of the logarithmic function of the unit rectangular yields the quantity −1. Its exponential, divided by the arithmetic mean of one-half, produces the quantity 2/e, as in Equation 28.

δ IN CROSS-FAMILY COMPARISONS

As noted previously, unambiguous ranking of income distributions according to their degree of inequality is generally not possible across families of distributions. That is to say, two measures of inequality may yield different rank orderings. But is a more precise statement possible? As Fields and Fei (1978: 315) wonder with respect to the Lorenz dominance criterion, might it be possible "to reduce further the zones of ambiguity?"

This section reports the preliminary results of work in progress aimed at placing bounds on the zones of ambiguity in cross-family comparisons. The principal task is to establish analytically, for any given distribution of specified parameters, which distributions from other families it can unambiguously be ranked with, and which distributions from other families it cannot unambiguously be ranked with.

Consider a Pareto distribution of given shape parameter c. Suppose that its inequality is to be ranked relative to that of all possible Lognormal distributions. Suppose further that two measures of inequality are to be used, δ and the Gini Index of Concentration (GIC). How can the zone of ambiguity be bounded?

A simple approach is proposed here: First, calculate the values of δ and GIC for the given Pareto distribution. Second, find the Lognormal curve whose δ equals that Pareto's δ. Third, find the Lognormal whose GIC equals that Pareto's GIC. The zone of ambiguity for comparisons between the given Pareto and all Lognormals, therefore, lies between these two Lognormals.

Table 1 reports the results of this procedure for three cases. In each case, the reference distribution is a member of the Pareto

TABLE 1
Bounding the Zones of Ambiguity for Inequality Rankings of Cross-Family Distributions: Three Members of the Pareto Family Each Compared to All Lognormals

		δ	Δ	GIC
A.	Pareto:1.5	.6492	.3508	.5
	Lognormal:0.9295	.6492	.3508	.4890
	Lognormal:0.9539	.6345	.3655	.5
		Zone of Ambiguity:	Lognormal:c|0.9295 < c < 0.9539	
B.	Pareto:2.0	.8244	.1756	.3333
	Lognormal:0.6215	.8244	.1756	.3397
	Lognormal:0.6091	.8307	.1693	.3333
		Zone of Ambiguity:	Lognormal:c|0.6091 < c < 0.6215	
C.	Pareto:2.5	.8951	.1049	.25
	Lognormal:0.4708	.8951	.1049	.2608
	Lognormal:0.4506	.9035	.0965	.25
		Zone of Ambiguity:	Lognormal:c|0.4506 < c < 0.4708	

NOTES: δ is the ratio of the geometric means to the arithmetic mean.
$\Delta = 1 - \delta$
GIC is the Gini Index of Concentration.
See text for formulas and procedures.

family, and the "other" distributions are Lognormal. Equations 23 and 26 are used to solve for δ, as well as to solve "backwards" for the Lognormals. The formulas for the Gini Index of Concentration are:

$$\text{in the Pareto,} \quad 1/(2c - 1) \qquad [29]$$

$$\text{in the Lognormal,} \quad 2F_N(c/\sqrt{2}) - 1 \qquad [30]$$

where F_N is the cumulative distribution of the Normal family (see Cowell, 1977: 153). To simplify comparisons, $(\Delta = 1 - \delta)$ is tabulated along with δ. Thus, greater values of both Δ and GIC indicate greater inequality. Following Hastings and Peacock (1974), distributions are identified by family name and numerical value of the shape parameter (the location parameter being irrelevant for these measures of relative inequality).

To illustrate use of Table 1, suppose that the reference distribution of interest is the Pareto: 2.0. Panel B shows that it has magnitudes of 0.1756 and 0.3333 for Δ and GIC, respectively. Solving for a Lognormal of equal Δ, the result is Lognormal: 0.6215; solving for a Lognormal of equal GIC, the result is Lognormal: 0.6091. These results indicate that: (1) when inequality is measured by Δ, the Pareto: 2.0 is more unequal than any Lognormal whose shape parameter is less than 0.6215; and (2) when inequality is measured by the GIC, the Pareto: 2.0 is less unequal than any Lognormal whose shape parameter is greater than 0.6091. Hence, the zone of ambiguity for comparisons of the Pareto: 2.0 and any Lognormal comprises the set of all Lognormals whose shape parameter c lies between 0.6091 and 0.6215. This means that when inequality is measured by Δ and GIC, the Pareto: 2.0 can be unambiguously ranked relative to all *other* Lognormals, that is, to all except the ones in the zone of ambiguity.[3]

This procedure can be applied to as many pairs of inequality measures as are desired. For example, if four measures are to be used, this procedure would be repeated six times, each time yielding a pairwise-specific zone of ambiguity. The final zone of ambiguity would comprise all the pairwise-specific zones of ambiguity.

The major limitation of this procedure is that it requires a specific reference distribution (e.g., a Pareto of specified shape parameter). Obviously, a theorem expressing the relation between any Pareto and any Lognormal would be preferred.

DECOMPOSITON OF δ

By algebraic manipulation, the measure δ can be nicely decomposed into a between-groups component and a within-groups component.[4] The decomposition formula, where there are J groups in the population, μ the population mean, and, for each group j = 1, . . . , J, $\bar{X}_j$ the group arithmetic mean and p_j the proportion of the population in group j, is given by

$$\delta = \left[\prod_{j=1}^{J} (\delta_j)^{p_j}\right] \left[\left[\prod_{j=1}^{J} (\bar{X}_j)^{p_j}\right] \Big/ \mu\right] \quad [31]$$

That is, the population δ is the product of two factors. The first factor to the right of the equal sign measures the within-groups component, that portion of δ attributable to dispersion among group members. The second factor measures the between-groups component, that portion of δ attributable to differences in means across groups. As with δ generally, the bounds on the components are zero and one (open at zero, closed at one). Thus, as the value of a component approaches unity, that component has a smaller effect.

If the means of all groups are equal, then the population δ is equal to the within-groups component. That is to say, in this case, the population δ is equal to the product of the groups' δ's, where each of the δ_j's is raised to the p_j power.

If there is no variation within groups, that is, if the population consists of J groups, each with an equal distribution, then the population δ equals the between-groups component.

To assess the relative contributions of the two components to the population δ, the rule is straightforward: The component with the smaller value is providing more inequality.

Sometimes the decomposition—as well as the assessment of the relative contributions—may be regarded as simpler if the components are additive. This can be done easily for δ by taking logarithms. Thus,

$$\ell n\, \delta = \sum_{j=1}^{J} (p_j\, \ell n\, \delta_j) + \sum_{j=1}^{J} (p_j\, \ell n\, \bar{X}_j) - \ell n\, \mu \quad [32]$$

or the equivalent expression

$$\ell n\, \delta = \sum_{j=1}^{J} (p_j\, \ell n\, \delta_j) + \sum_{j=1}^{J} \ell n \left[((\bar{X}_j)^{p_j}) \div \mu^{1/J}\right] \quad [33]$$

Recalling that $\ell n\ \delta$ has range $(-\infty, 0)$ and that the greater the absolute magnitude the greater the inequality, it follows that the greater the absolute magnitude of the logarithm of a component, the greater its contribution to inequality.

Assessment of the relative contributions can be carried one step further. The logarithm of each component may be divided by $\ell n\ \delta$, thereby providing the proportion of δ attributable to each component.

SUMMARY

This article has presented a detailed account of the measure of inequality δ, defined as the ratio of the geometric mean to the arithmetic mean. The appropriateness of this measure to a particular substantive problem can be judged by noting its properties and its relation to the sense of distributive justice. This measure appears suitable for any substantive problem in which the social scientist specifies a social variable as dependent on a linear, additive combination of the magnitudes of justice or injustice experienced by all the members of a social aggregate, with each person's sentiment of justice or injustice receiving equal weight.

NOTES

1. Distributional families whose usual definition includes two shape factors can be divided into two subfamilies, such that in each subfamily one shape factor is held constant and the other is allowed to vary. In such case, the varying shape factor controls variation in δ.

2. Since the lower extreme value of the Pareto variate occurs at a magnitude greater than zero, the Pareto variate provides a good illustration of the effects on the shape parameter—and on inequality—of adding or subtracting a constant to every income. The Pareto's lower extreme value (LEV) is a joint function of the arithmetic mean and the shape parameter:

$$\text{LEV} = \frac{\mu(c-1)}{c} \qquad [a]$$

Adding or subtracting a constant k alters both the lower extreme value and the arithmetic mean, thereby producing a new shape parameter c* which satisfies the equation

$$c^* = c + \frac{k}{\mu_1} \qquad [b]$$

where μ_1 is the original mean. This expression shows that addition of a constant increases the magnitude of the shape parameter, thereby reducing inequality, while subtraction of a constant reduces the magnitude of the shape parameter, thereby increasing inequality.

3. The bounds on the zone of ambiguity signal those special cases in which two different distributions have identical values of a measure of inequality. Jasso (forthcoming) describes a different approach to compare different income distributions that have identical magnitudes of the Gini Index of Concentration.

4. Derivation of the decomposition formula is reported in the Appendix. To our knowledge, a decomposition formula for δ has not previously been obtained.

APPENDIX

DERIVATION OF DECOMPOSITION FORMULA FOR δ

Let δ denote the population ratio of the geometric mean to the arithmetic mean, μ the population mean, and N the population size. Let J denote the number of groups in the population, such that j = 1 to J, δ_j each group's ratio of its geometric mean to its arithmetic mean, $\bar{X}_j$ each group's arithmetic mean, n_j the group size, and $p_j = (n_j/N)$ the proportion of the population in each group. Restating Equation 1 in the text,

$$\delta = \frac{\left[\prod_{i=1}^{N} x_i\right]^{1/N}}{\mu} \qquad [i]$$

Since N is equal to the sum of the n_j, each member of the population being a member of one and only one group j, the numerator can be restated to produce

$$\delta = \frac{\prod_{j=1}^{J}\left[\prod_{i=1}^{n_j} x_i\right]^{1/N}}{\mu} \qquad [ii]$$

Applying the laws of multiplication and exponentiation to the denominator yields

$$\delta = \prod_{j=1}^{J} \frac{\left[\prod_{i=1}^{n_j} x_i\right]^{1/N}}{\mu^{1/J}} \qquad [iii]$$

Utilizing the relations between n_j and N to obtain a new numerator, and reexpressing the denominator in terms of the group means and the population mean,

$$\delta = \prod_{j=1}^{J} \frac{\left[\left[\prod_{i=1}^{n_j} x_i\right]^{1/n_j}\right]^{n_j/N}}{\bar{X}_j^{n_j/N} \cdot \mu^{1/J} / \bar{X}_j^{n_j/N}} \qquad [iv]$$

Each group's δ_j can now be constructed from the group's geometric mean in the numerator and the group's arithmetic mean in the denominator, producing

$$\delta = \prod_{j=1}^{J} \delta_j^{p_j} \bar{X}_j^{p_j} \mu^{-1/J} \qquad [v]$$

which can be reexpressed as

$$\delta = \frac{1}{\mu} \prod_{j=1}^{J} \delta_j^{p_j} \bar{X}_j^{p_j} \qquad [vi]$$

This expression can be seen to equal the two-component form reported in Equation 31 in the text.

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