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FOREWORD

This set of notes has been compiled with one primary objective in mind: to provide, in one volume, a handy reference for a large number of the commonly-used mathematical formulae, and to do so consistently with respect to notation, definition and normalization. Many of us keep these results available to us in an excessive number of references, in which notation or normalization varies, or formulae are so spread out that they are difficult to find, and their use is time-consuming.

Short explanations are included, with some examples, to serve two purposes: first, to recall to the user some of the ideas which may have slipped his mind since his detailed study of the material; second, for those who have never studied the material, to make its use at least plausible, and to help in his study of references.

No claim can be made that all the results anyone ever uses are here, but it is hoped that a sufficient quantity of material is included to make necessary only infrequent use of other references, except for integral tables, etc. for elementary work. Of course, the user may find it desirable to add some pages of his own.

Finally, it is recommended that those unfamiliar with the theory at any point not blindly apply the formulae herein, for this is risky business at best; a text should be studied (and, of course, understood) first.

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NOTES AND CONVERSION FACTORS

ORTHOGONAL FUNCTIONS

In general, orthogonal functions arise in the solution of certain boundary-value problems. The use of the properties of orthogonal functions may often greatly simplify and systematize the solution to such a problem, in addition to providing a natural way of making approximate solutions. Let us first make clear the concept of orthogonality. We begin at what may seem an improbable starting point.

Two vectors, $\underline{A}$ and $\underline{B}$ in a three-dimensional space are said to be "orthogonal" if the dot (inner) product $\underline{A}\cdot\underline{B}$ vanishes:

$$\underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i = 0.$$

This is easily generalized to a space with more than three dimensions. In an n-dimensional space the concept of orthogonality is unchanged, except that then the sum is over n terms A_iB_i, and we have for orthogonality

$$\underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + \dots\ A_nB_n = \sum_{i=1}^{n} A_iB_i = 0.$$

Now one may think of the components of a vector, A_1, A_2, A_3, as the values of a (real) function at three values of its argument; say $A_1 = f(r_1)$, $A_2 = f(r_2)$, $A_3 = f(r_3)$, or, in terms which will make our efforts here more clear, $A_i = f(r_i)$ $(r_i = r_1, r_2, r_3)$. That is, r has the values r_1, r_2, r_3, and to get A_i, put r_i in $f(r)$.

We may now think of r as having any number, say n, possible values in some range, so that f(r) evaluated at the various r's generates an n-dimensional vector. The step to considering a function as an infinitely-many-dimensional vector is now a natural one; we allow n to increase without bound, r taking all values in its range.

Let r have some range $a \leq r \leq b$ in which it takes on n values such that $r_j - r_{j-1} = \Delta r_j$, and suppose two such n-dimensional vectors $f(r_j)$ and

$g(r_j)$ are thus generated. The inner product of f with g is generalized as

$$\sum_{j=1}^{n} f(r_j)\, g(r_j) \Delta r_j.$$

Above, for $\underline{A} \cdot \underline{B}$, r_j takes on only the discrete values 1, 2, 3; so Δr is always unity in this simple case. Now let us consider the case as $n \to \infty$, r taking all values between a and b. The inner product, if it exists, is then

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(r_j)\, g(r_j) \Delta r_j.$$

With proper restrictions on Δr_j ($\max \Delta r_j \to 0$) and on the range of $r (a \leq r \leq b)$, this is just the limit occuring in the definition of the ordinary integral. Thus we say that the inner product of f(r) with g(r), which is often denoted (f,g), is

$$(f,g) = \int_a^b f(r)\, g(r)\, d\, r.$$

Of course, the range could be infinite.

This discussion constitutes the generalization of the dot, or inner, product, to functions.

Two functions are then by definition orthogonal over the range $a \leq r \leq b$ when

$$(f,g) = \int_a^b f(r)\, g(r)\, dr = 0 \quad .$$

As we shall see, this definition is subject to generalization by the inclusion of a "weight function" p(r), with which

$$(f,g) = \int_a^b p(r)\, f(r)\, g(r)\, dr.$$

Here f and g are said to be orthogonal "with respect to weight function p" over the range $a \leq r \leq b$. The function $p(r)$ may of course be unity.

Only the function $f(r) \equiv 0$, $a \leq r \leq b$ is orthogonal to itself. In general we denote the inner product of a function with itself as

$$(f,\ f) = \int_a^b f(r^2)\, dr = N^2_f,$$

and call it the norm of the function.

Orthogonal sets may or may not possess two other properties, normality and completeness. A set of orthogonal functions $\left\{ U_i \; (u) \right\}$ is said to be normal or orthonormal if $N_i^2 = 1$ for all i.

A set of functions $\left\{ U_i(u) \right\}$ orthogonal on the interval $u_1 \leq u \leq u_2$ is complete if there exists no other function orthogonal to all the U_i on the same interval, with respect to the same weight function, if one is involved.

Importance of Orthogonal Functions

The importance of orthogonal sets in mathematical physics may perhaps be indicated by further considerations of their analogs, orthogonal coordnate vectors. It is true that any N-dimensional vector may be defined in terms of its components along N coordinates, provided that no more than two of the reference coordinates are coplanar. But if the reference coordinates are orthogonal, e.g., Cartesian coordinates, the equations take a particularly simple form. The situation is somewhat similar when it is desired to expand a function in terms of a set of other functions -- it is much simpler if the set is orthogonal.

Completeness is another important property. It is apparent that no two reference axes will suffice for the definition of a vector in 3-dimensional space. The set of two references axes is not complete in ordinary space, since a third coordinate can be added which is orthogonal to both of them. Addition of this coordinate makes the set complete. The situation with orthogonal functions is exactly analogous. Some authors define a complete set as a set in terms of which any other function defined on the same interval can be expressed.

Some more common sets of orthogonal functions are the sines and cosines, Bessel functions, Legendre polynomials, associated Legendre functions, spherical harmonics, Laguerre and Hermite polynomials; operational properties of which are listed in these notes.

Generation of Orthogonal Functions

In the mathematical formulation of physical problems, one often encounters partial differential equations or integro-differential equations (which contain not only derivatives of functions but also integrals of functions), with which are associated a set of boundary conditions. If the equation and its boundary conditions are such as to be "separable" in one of the variables, one may attempt to apply the method of "separation of variables".

Suppose we (admittedly rather abstractly) represent our equation

$$\Phi \left\{ F(u, v, w, \ldots \right\} = 0$$

where $\Phi\{\}$ is an operator involving the variables u, v, w,..., applied to the function F(u,v,...). For example, $\Phi\{\}$ might be something like

$$\frac{\partial^2}{\partial u^2} + v + \int_{\omega_1}^{\omega_2} K(\omega, \omega') \quad d\omega' + \Delta^2$$

so that when is applied to a function F the equation appears

$$\Phi\{F\} = \frac{\partial^2 F}{\partial u^2} + vF + \int_{\omega_1}^{\omega_2} K(\omega,\omega') F(u,v,\omega') d\omega' + \Delta^2 F = 0$$

The process of separation proceeds as follows. One attempts to find a variable, say u, such that if it is assumed that F(u,v,w,...) may be written

$$F(u,v,w) = U(u)\, \phi(v, \omega, \text{----})$$

then the equation

$$\Phi\{F\} = \Phi\{U(u)\, \phi(v,\omega)\} = 0$$

can be written

$$\mathcal{U}\{U(u)\} = \Phi\{\phi(v,\omega)\}$$

Here, $\mathcal{U}$ is an operator involving only u, and Φ is an operator involving only v, w---. Returning to the example, assume

$$F(u, v, w) = U(u)\, \phi(v, \omega)$$

then

$$\oplus\{F\} = \oplus\{U\Phi\} =$$

$$= \frac{\partial^2[U\Phi]}{\partial u^2} + vU\Phi + \Delta^2 U\Phi + \int_{w_1}^{w_2} K(w,w')U\phi\, dw'$$

$$= \Phi\frac{d^2U}{du^2} + vU\phi + \Delta^2 U\Phi + U\int_{w_1}^{w_2} K(w,w')\Phi(v,w')dw' = 0$$

We can, in this case, if $U\Phi \neq 0$, divide by $U\Phi$ to get

$$\frac{1}{U}\frac{d^2U}{du^2} + v + \frac{1}{\Phi}\int_{w_1}^{w_2} K(w,w')\phi(v,w')dw' + \Delta^2 = 0$$

which can be rearranged

$$\frac{1}{U}\frac{d^2U}{du^2} + \Delta^2 = -v - \frac{1}{\Phi}\int_{w_1} K(w,w')\phi(v,w')dw'$$

as we wished. In this case the separation has been successful, for on the left are functions of u only, while on the right stand only functions of v and w. Now suppose we were to vary u, fixing v and w. Then the right side would not change since it does not involve u, and is therefore a constant. We therefore state this fact

$$\frac{1}{U}\frac{d^2U}{du^2} + \Delta^2 = \mu^2 = -v - \frac{1}{\Phi}\int_{w_1}^{w_2} K(w,w')\phi(v,w')dw'$$

where μ^2 is a constant, called the "separation constant". We may choose it at our discretion. We now have two equations, where only one existed before:

$$\frac{d^2U}{du^2} + (\Delta^2 - \mu^2)U = 0$$

$$(v+\mu^2)\Phi + \int_{w_1}^{w_2} K(w,w')\phi(v,w')dw' = 0$$

In order to effect a solution using this method, not only the equation, but also the boundary conditions must be separable. Back in the example, say the boundary conditions are

$$F(u_1, v, w) = 0$$

$$\frac{\partial F}{\partial u}(u_2, v, w) = F(u_2, v, w)$$

Introducing our assumption as to the form of F, that is,

$$F(u, v, w) = U(u)\Phi(v, w)$$

these become

$$U(u_1)\,\phi(v,w) = 0$$

$$\phi(v,w)\frac{dU}{du}(u_2) = \phi(v,w)\,U(u_2)$$

For $\phi \neq 0$, we can divide by ϕ to get

$$U(u_1) = 0$$

$$\frac{dU}{du}(u_2) = U(u_2)$$

which are separated; they do not involve v and w.

By the process of separation of variables we have, from the original equation involving u, v, w generated a new set of equations, some of which (those involving u) are a complete problem.

$$\begin{cases} \dfrac{d^2U}{du^2} + (\Delta^2 - \mu^2)U = 0 \\ U(u_1) = 0 \;, \quad \dfrac{dU}{du}(u_2) = U(u_2) \end{cases}$$

$$(v+\mu^2)\phi + \int_{w_1}^{w_2} K(w,w')\,\phi(v,w')\,dw' = 0$$

This was our objective in applying the method of separation of variables. The process may be repeated on the remaining equation or performed on another variable.

Now if, after separation, the u- equation can be put in the form

$$(1) \quad \frac{d}{du}\left[r(u)\frac{dU}{du}\right] - \left[q(u) + \mu\, p(u)\right] U = 0$$

and the boundary conditions in the form

$$(2) \quad \begin{cases} a_1U(u_1) + a_2U'(u_1) + \alpha_1U(u_2) + \alpha_2U'(u_2) = 0 \\ b_1U(u_1) + b_2U'(u_1) + \beta_1U(u_2) + \beta_2U'(u_2) = 0 \end{cases}$$

where a's, b's, α's, and β's are constants (some may be zero) then the system of differential equation and boundary conditions is called a "Sturm-Liouville* system". It will be noted that the Sturm-Liouville system is very general, and includes many important equations as special cases, for example the wave equation with those boundary conditions which are commonly applied.

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* pronounced LEE-oo-vil, NOT LOO-i-vil.

This system under quite general conditions generates a complete set of orthogonal functions, one for each of an infinite, discrete set of values of the parameter μ. One finds the values of μ, called "eigenvalues", for which solutions exist, and the solution functions corresponding to these eigenvalues, called "eigenfunctions". If the eigenvalues are $\mu_1, \mu_2, ,,$ and the corresponding eigenfunctions are $U_1(u)_1 U_2(u), \ldots,$ then in general the functions are orthogonal over the range u_1 to u_2, with respect to weight function $p(u)$, that is,

$$(3) \quad \int_{u_1}^{u_2} p(u)\, U_i(u)\, U_j(u) du = N_i^2\, \delta_{ij},$$

Here δ_{ij} is the Kronecker delta, and $p(u)$ is the same as in equation (1).

One must take care to find all possible eigenvalues (when the equation and boundary conditions are written exactly as in (1) and (2) they are all real). When all eigenvalues are found, the set of eigenfunctions is complete, and any function reasonably well behaved between u_1 and u_2 may be represented in terms of them. Say we seek an expansion of a function $f(u)$ in terms of our eigenfunctions,

$$f(u) = \sum_i f_i U_i(u)$$

where f_i are a set of constants. Multiply on the right and left by $U_j p(u)$ and integrate with respect to u in the range u_1 to u_2.

$$\int_{u_1}^{u_2} p(u)\, U_j(u)\, f(u)\, du = \int_{u_1}^{u_2} \sum_i p(u)\, U_j(u)\, f_i U_i(u)\, du$$

$$= \sum_i f_i \int_{u_1}^{u_2} p(u)\, U_j\, (u)\, U_i\, (u)\, du$$

As a consequence of the orthogonality of the $U_i(u)$'s, defined in equation (3) this becomes

$$= \sum_i f_i N_i^2 \delta_{ij} = f_j N_j^2$$

We solve then for the f_j's;

$$f_j = \frac{1}{N_j^2}\int_{u_1}^{u_2} p(u)\, U_j(u)\, f(u)\, du = \frac{1}{N_j^2}\,(U_j,f).$$

In order to justify the switching of the order of summation and integration here, and to guarantee the existence of (U_j,f), we usually require that $f(u)$ be absolutely integrable, i.e.,

$$\int_{u_1}^{u_2} f(u)\, du \text{ exists.}$$

It is to be noted that the outline of procedure above requires the solution of an ordinary differential equation, perhaps not an easy task, but one hopes not as difficult as the problem of solving the partial differential equation. Very often the eigenfunctions which fit a given problem are known, and so this process can be bypassed.

Different sets of eigenfunctions have different sets of operational properties, that is, sets of relationships between members, which may be found useful.

We note in conclusion that sets of orthogonal functions are generated by other means than by Sturm-Liouville systems; by sets of differential equations and boundary conditions which are not of Sturm-Liouville type, and by integral equations, to mention two.

Use of Orthogonal Functions

Let us consider a linear partial differential equation outlining the elimination of one variable from the equation. Under rather general conditions, we may expand F in an infinite series of orthagonal functions which we assume known;

$$F(u,v,w,\ldots) = \sum_{i=0}^{\infty} f_i(v,w,\ldots)\, U_i(u)$$

where

$$f_i(v,w,\ldots) = \frac{1}{N_i^2}\int_{\mathcal{U}_1}^{\mathcal{U}_2} p(\mathcal{U})\, F(u,v,w,\ldots)\, U_i(\mathcal{U})\, d\mathcal{U}.$$

The formula for the coefficient f_i follows immediately from multiplying the first equation by $p(\mathcal{U})\, U_i(\mathcal{U})$ and integrating. Let the original partial differential equation be represented abstractly.

$$\Phi F(u,v,w,\ldots) = S(u,v,w,\ldots)$$

where Φ is a linear differential operator and ΦF merely represents that part of the differential equation that involves F. Assume F to be expanded in a series in f_iU_i; multiply the equation by $p(\mathcal{U})U_j(\mathcal{U})$ and integrate over $\mathcal{U}$ from $\mathcal{U}_1$ to $\mathcal{U}_2$.

$$\Phi F = S$$

$$\Phi \sum_i f_iU_i = S$$

$$\int_{\mathcal{U}_1}^{\mathcal{U}_2} p(\mathcal{U})\, U_j(\mathcal{U})\, \Phi \sum_i f_iU_i\, d\mathcal{U} = \int_{\mathcal{U}_1}^{\mathcal{U}_2} Sp(\mathcal{U})U_j(\mathcal{U})\, d\mathcal{U} = N_j^2 S_j(v,w,\ldots)$$

where

$$S_j = \int_{\mathcal{U}_1}^{\mathcal{U}_2} S(u,v,w,\ldots)\, p(\mathcal{U})U_j(\mathcal{U})\, d\mathcal{U}$$

so that

$$S(u,v,w,\ldots) = \sum_j^{\infty} S_jU_j.$$

Now, by using the operational properties of the U_i's, one reduces the equations (an infinite set, one for each i) to a set in the f_i's and s_i's. The particular steps taken depend upon the exact nature of the operator Φ, and the set of equations may be coupled, i.e., f's with several indices may appear in the same equation (for example, f_{i-1}, f_i, f_{i+1}). These equations do not involve derivatives with respect to variable $\mathcal{U}$, and we have gained in this respect. But we have to contend with the infinite set of equations.

All is not lost at this point, for, as it turns out, the series for F, (called a generalized Fourier series because of the manner in which the coefficients of U_i's are chosen in the expansion for F), is the most rapidly convergent series possible in the U_i's. Thus solving for only the coefficients of the leading terms in the series may enable us to obtain a satisfactory approximation to F. Also, very often, we are interested only in one or two of the f_i's on physical or other grounds.

Operational Properties of Some Common Sets of Orthogonal Functions

There follow now a few pages on which are outlined, rather concisely, some basic operational properties of some commonly occuring sets of orthogonal functions. The familiar set of sines and cosines used in the construction of Fourier series is included as an example. These lists of properties contained in these notes are by no means complete, though they may suffice for the solution of many problems. The references listed with each section give detailed derivations, more extensive lists of properties, more discussions of the method and its limitations, or examples of the use of orthogonal functions. It is recommended that one unfamiliar with these functions read in some of these references, in order to avoid the pitfalls of using mathematics beyond the realm of its applicability. These notes have been assembled mainly for reference. Of special interest may be the tables in Margenau and Murphy, page 254, which lists twelve special cases of the Sturm-Liouville equation with the name of the orthogonal set which satisfies each one.

I. Fourier Series, Range $-\ell \leq x \leq \ell$:

A. Boundary Value Problem satisfied by sines and cosines:

$$\frac{d^2f}{dx^2} + k^2 f = 0 \quad (k \text{ real})$$

$$f(-\ell) = f(\ell), \quad f'(-\ell) = f'(\ell)$$

B. Orthogonal Set

$$\sin \frac{n\pi x}{\ell} \quad , \cos \frac{n\pi x}{\ell}$$

for $\ell \leq x \leq \ell$, $n = 0,1,2,\ldots$

C. Expansion in Fourier Series

$$F(x) = \frac{a_o}{z} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}\right)$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x) \cos \frac{n\pi x}{\ell}\, dx \qquad n=0,1,2\ldots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x) \sin \frac{n\pi x}{\ell}\, dx \qquad n=1,2,\ldots$$

D. Normalization Factors:

$$N_n^2 \cos = N_n^2 \sin = \ell \qquad n = 1,2,\ldots$$

$$N_o^2 \cos = 2\ell;$$

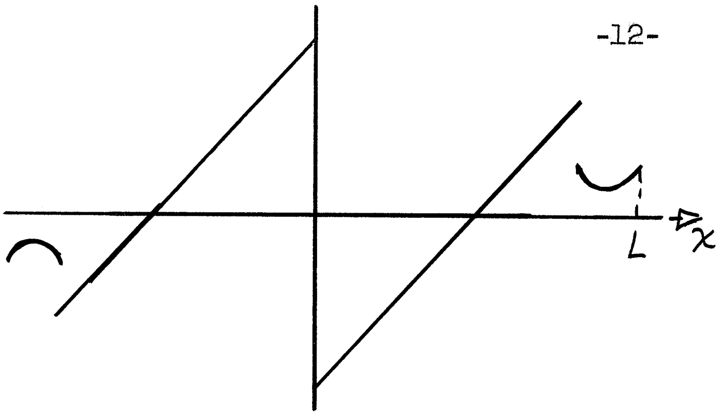
E. Orthonormal Set

$$\frac{1}{\sqrt{2\ell}},\ \frac{1}{\sqrt{\ell}} \cos \frac{n\pi x}{\ell},\ \frac{1}{\sqrt{\ell}} \sin \frac{n\pi x}{\ell}$$

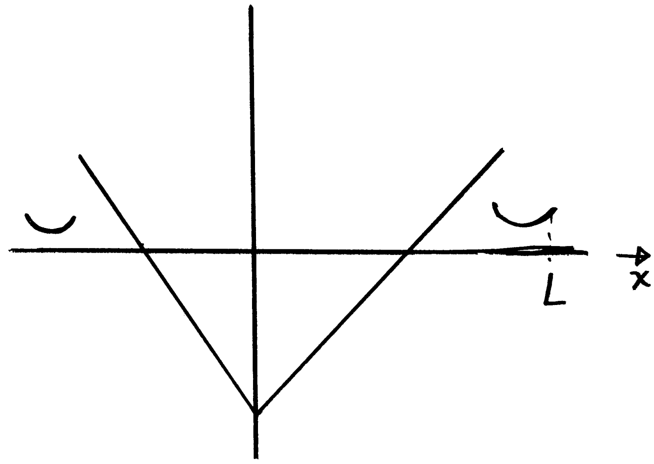
$$-\ell \leq x \leq \ell, \quad n=1,2,\ldots$$

F. Fourier Series, range $0 \leq x \leq L$.

One may wish to expand a function defined on an interval $0 \leq x \leq L$, in a Fourier series. One may choose at his discretion either of two ways, expanding either in a series of sines or one of cosines. The sine- series expansion yields an odd function, the cosine- series an even function, when the series is considered as a continuation of the function outside the range $0 \leq x \leq L$. An "odd function" is one such that $F(x) = -F(-x)$, an "even function" such that $F(x) = F(-x)$, as below.



Odd Function



Even Function

Of course, either series represents the function on the interval $0 \leq x \leq L$. Note that on this range either $\left\{\sin \frac{n\pi x}{L}\right\}$ or $\left\{1, \cos \frac{n\pi x}{L}\right\}$ $n = 1,2,\ldots,$ is a complete set, thus the two possible expansions.

G. Expansion in "Half-Range" Series

$$F(x) = \frac{a_0}{z} \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

or

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_0^L F(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 0,1,2,\ldots$$

$$b_n = \frac{2}{L}\int_0^L F(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1,2,\ldots$$

H. Comment on Expansions of "Odd" and "Even" Functions in Full-Range Series

Consider the coefficients in a full-range expansion for an even function, i.e., $F(x)$ such that $F(x) = F(-x)$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(x) \cos \frac{n\pi x}{\ell}\, dx =$$

$$\frac{1}{\ell}\left[\int_{-\ell}^{0} F(x) \cos \frac{n\pi x}{\ell}\, dx + \int_0^{\ell} F(x) \cos \frac{n\pi x}{\ell}\, dx\right]$$

putting $-x$ for x in the first integral,

$$= \frac{1}{\ell}\left[\int_{\ell}^{0} F(-x) \cos \frac{-n\pi x}{\ell}\,(-dx) + \int_0^{\ell} F(x) \cos \frac{n\pi x}{\ell}\, dx\right]$$

use $F(-x) = F(x)$, $\cos\left(-\frac{n\pi x}{\ell}\right) = \cos\frac{n\pi x}{\ell}$, and switch order of integration

$$= \frac{1}{\ell}\left[\int_0^\ell F(x)\cos\frac{n\pi x}{\ell}\,dx + \int_0^\ell F(x)\cos\frac{n\pi x}{\ell}\,dx\right]$$

$$a_n = \frac{2}{\ell}\left[\int_0^\ell F(x)\cos\frac{n\pi x}{\ell}\,dx\right] \neq 0 \text{ in general.}$$

$$b_n = \frac{1}{\ell}\left[\int_{-\ell}^0 F(x)\sin\frac{n\pi x}{\ell}\,dx\right.$$

$$= \frac{1}{\ell}\left[\int_{-\ell}^0 F(x)\sin\frac{n\pi x}{\ell}\,dx + \int_0^\ell F(x)\sin\frac{n\pi x}{\ell}\,dx\right]$$

put $-x$ for x, in first integral,

$$= \frac{1}{\ell}\left[\int_\ell^0 F(-x)\sin\frac{-n\pi x}{\ell}\,(-dx) + \int_0^\ell F(x)\sin\frac{n\pi x}{\ell}\,dx\right]$$

use $F(x) = F(-x)$, $\sin\left(\frac{-n\pi x}{\ell}\right) = -\sin\frac{n\pi x}{\ell}$, and switch order of integrations.

$$= \frac{1}{\ell}\left[-\int_0^\ell F(x)\sin\frac{n\pi x}{\ell}\,dx + \int_0^\ell F(x)\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$= 0 \qquad \text{all } n = 1,2,\ldots$$

Thus, the coefficients of the sine terms in the full range expansion of an even function are all zero; the cosine coefficients do not, of course, vanish for all n.

The situation is much the same with respect to coefficients in the full range expansion of odd functions, except that in this case it is the coefficients of the cosine terms which vanish, for $n = 0,1,2\ldots$

II. Legendre Polynomials (Range $-1 \leq x \leq 1$)

A. Generating Function

$$H(x,y) = \frac{1}{(1 - 2xy + y^2)^{1/2}} = \sum_{=0}^{\infty} P_\ell(x)y^\ell$$

Then

$$P_\ell(x) = \frac{1}{\ell!}\frac{\partial^\ell H}{\partial y^\ell}\bigg|\; y = 0$$

B. Recurrence Relations

1. $(\ell + 1) P_{\ell+1}(x) - (2\ell + 1) x P_\ell(x) + \ell P_{\ell-1}(x) = 0$

2. $\ell P_{\ell-1}(x) - P'_\ell(x) + xP'_{\ell-1}(x) = 0$

C. Differential Equation Satisfied by $P_\ell(x)$

1. $(1 - x^2) P_\ell''(x) - 2 x P_\ell'(x) + \ell(\ell + 1) P_\ell(x) = 0$ (ℓ an integer)

Very often $x = \cos\theta$.

2. Normal Form

$$\frac{\partial^2 P_\ell}{\partial x^2} + \frac{\ell(\ell+1)(1 - x^2) + 1}{(1 - x^2)^2} P_\ell = 0$$

D. Rodrigues's Formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell$$

E. Normalizing Factor

Norm of $P_\ell(x)$ is

$$N_\ell^2 = \int_{-1}^{1} (P_\ell(x))^2 \, dx = \frac{2}{2\ell + 1}$$

F. Orthogonality

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x) \, dx = \delta_{\ell\ell'} N_\ell^2$$

where $\delta_{\ell\ell'}$ is the Kronecker delta, defined $\delta_{\ell\ell'} = \begin{cases} 0, \ell \neq \ell' \\ 1, \ell = \ell' \end{cases}$

G. Expansion in Legendre Polynomials

Any function $f(x)$ which is defined over the range $-1 \leq x \leq 1$, and which is absolutely integrable over this range may be expanded in an infinite series of Legendre Polynomials;

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(x)$$

where

$$f_\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} f(x) P_\ell(x) \, dx$$

$$= \frac{1}{N_\ell^2} \int_{-1}^{1} f(x) P_\ell(x) \, dx$$

H. Normalized Legendre Polynomials

$$p_{\ell}(x) = \frac{P_{\ell}(x)}{N_{\ell}}$$

I. Expansion in Normalized Polynomials

$$g(x) = \sum_{\ell=0}^{\infty} g_{\ell} p_{\ell}(x)$$

where

$$g_{\ell} \doteq \int_{-1}^{1} g(x)\, p_{\ell}(x)\, dx.$$

J. A Few Low-Degree Legendre Polynomials and Respective Norms

$P_0(x) = 1$	$N_0^2 = 2$
$P_1(x) = x$	$N_1^2 = 2/3$
$P_2(x) = 1/2(3x^2 - 1)$	$N_2^2 = 2/5$
$P_3(x) = \cancel{1/2(5x^2 - 3x)} = \frac{1}{2}(5x^3 - 3x)$	$N_3^2 = 2/7$
$P_4(x) = 1/8(35x^4 - 30x^2 + 3)$	$N_4^2 = 2/9$
$P_5(x) = 1/8(63x^5 - 70x^3 + 15x)$	$N_5^2 = 2/11$
$P_6(x) = 1/16(231x^6 - 315x^4 + 105x^2 - 5)$	$N_6^2 = 2/13$

K. Integral Representation of $P_{\ell}(x)$

$$P_{\ell}(x) = \frac{1}{\pi}\int_0^{\pi} \left[x + (x^2 - 1)^{1/2} \cos\varphi\right]^{\ell} d\varphi$$

L. Bounds on $P_{\ell}(x)$

For $-1 \leq x \leq 1$,

$|P_{\ell}(x)| \leq 1$, all ℓ.

References on Legendre Polynomials

1. D. Jackson; "Fourier Series and Orthogonal Polynomials", Carus Math. Monographs, Number 6, 1941, pp. 45-68.

2. H. Margenau and G. M. Murphy; "The Mathematics of Physics and Chemistry", D. Van Nostrand, First Edition, 1943, pp. 94-109.

3. A. G. Webster; "Partial Differential Equations of Math. Phys.", G.E. Stechert and Company, 1927, pp. 302-320.

4. R.V. Churchill; "Fourier Series and Boundary Value Problems", McGraw Hill, 1941, pp.175-201. (for discussion of concept of orthogonality, see Chap. III).

5. E. Jahnke and F. Emde; "Tables of Functions", Dover Publications, 1945, pp. 107-125. (Lists some properties and tabulates functions).

III. Associated Legendre Functions

A. Definition of the Associated Legendre Functions (Associated Legendre Polynomials

1. $$P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^{|m|}}{dx^{|m|}} P_\ell(x) \quad (0 \leq m \leq \ell)$$

2. $$P_\ell^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^\ell \ell!} \frac{d^{\ell+m}}{dx^{\ell-m}} (x^2-1)^\ell$$

B. Recurrence Relations

1. $$P_\ell^{m+1}(x) - 2m \frac{x}{\sqrt{1-x^2}} P_\ell^m(x) + \left[\ell(\ell+1) - m(m-1)\right] P_\ell^{m-1}(x) = 0$$

2. $$xP_\ell^m(x) = \frac{(\ell+m) P_{\ell-1}^m(x) + (\ell-m+1) P_{\ell+1}^{m+1}(x)}{2\ell+1}$$

3. $$(1-x^2)^{1/2} P_\ell^m(x) = \frac{P_{\ell+1}^{m+1}(x) - P_{\ell-1}^{m+1}(x)}{2\ell+1}$$

4. $$P^{m+1} = \frac{2m}{\sqrt{1-x^2}\,(2\ell+1)} \left[(\ell+m) P_{\ell-1}^m(x) + (\ell-m+1) P_{\ell+1}^m(x)\right] - \left[\ell(\ell+1) - m(m-1)\right] P_\ell^{m-1}(x)$$

C. Differential Equation Satisfied by $P_\ell^m(x)$

1. $$(1-x^2)\frac{d^2P_\ell^m}{dx^2} - 2x\frac{dP_\ell^m}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] P_\ell^m = 0$$

2. Since $P_\ell^m(x)$ is defined on the interval $-1 \leq x \leq 1$, in physical applications $P_\ell^m(x)$ is often associated with an angle θ through the relation $x = \cos\theta$. Then the equation satisfied by $P_\ell^m(x)$ may be found in the form

$$\frac{d^2P_\ell^m(x)}{d\theta^2} + \cot\theta \frac{dP_\ell^m(x)}{d\theta} + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P_\ell^m(x) = 0$$

or

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\right] P_\ell^m(x) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P_\ell^m(x) = 0$$

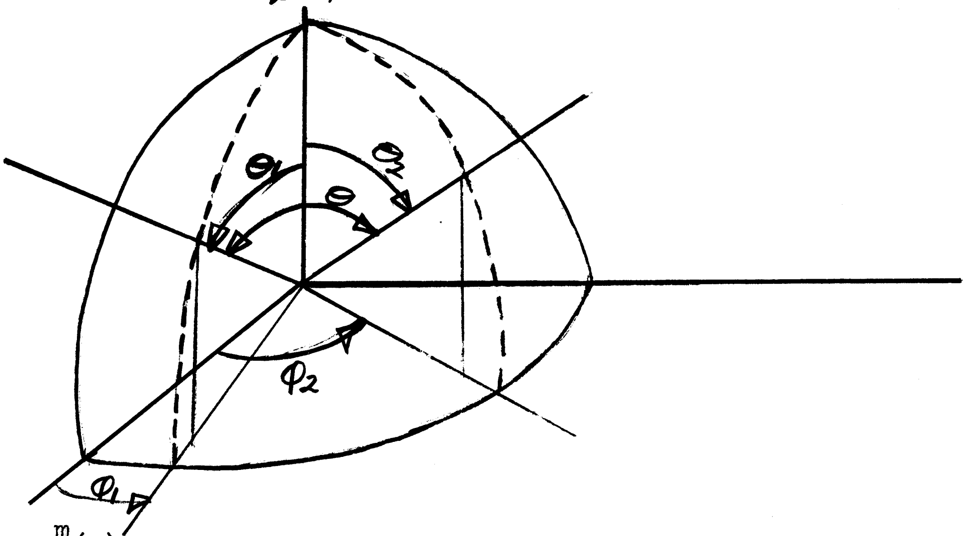
D. Expression of $P_\ell(\cos\theta)$ in terms of P_ℓ^m (The Addition Theorem)

1. Expression Θ_1, Φ_1 and Θ_2, Φ_2 denote, respectively, the polar and azimuthal angles of two lines passing through the origin, then θ, the angle between these two lines, is given by

$\cos\theta = \cos\Theta_1 \cos\Theta_2 + \sin\Theta_1 \sin\Theta_2 \cos(\Phi_1 - \Phi_2)$, see figure.

With these definitions, $P(\cos\theta)$ may be expressed

$$P_\ell(\cos\theta) = P_\ell(\cos\Theta_1)\,P_\ell(\cos\Theta_2) + 2\sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\Theta_1)\,P_\ell^m(\cos\Theta_2)\cos m(\Phi_1 - \Phi_2).$$



2. If $P_\ell^m(x)$ is defined in a slightly different manner, allowing negative values for m,

$$P_\ell^m(x) = (1 - x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} P_\ell(x) \qquad (|m| \leq \ell)$$

then the expansion may be written

$$P_\ell(\cos\theta) = \sum_{m=-\ell}^{\ell} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^m(\cos\Theta_1)\,P^m(\cos\Theta_2)\cos m(\Phi_1 - \Phi_2)$$

E. Normalizing Factor

$$(N_\ell^m)^2 = \int_{-1}^{1}\left[P_\ell^m(x)\right]^2 dx = \frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}$$

F. References

1. H. Margenan and G. M. Murphy; "The Mathematics of Physics and Chemistry, D. Van Nostrand, 1st edition (1934).

2. A. G. Webster; "Partial Differential Equations of Math. Physics", G. E. Stechert and Company (1927).

3. D. K. Holmes and R. V. Meghreblian, "Notes on Reactor Analysis, Part II, Theory", U.S.A.E.C. Document CF- 4-7-88 (Part II), August 1955, pp. 164-165.

4. E. Jahnke and F. Emde; "Tables of Functions", Dover Publications (1945).

IV. Spherical Harmonics

A. Definition of $Y^m(\underline{\Omega})$

The spherical harmonics are a complete, orthonormal set of complex functions of two variables, defined on the unit sphere. Below, the vector symbol $\underline{\Omega}$ will be used to denote a pair of variables, θ, φ here taken to be, respectively, the polar and azimuthal angles specifying a point on the unit sphere with reference to a coordinate system at its center. With these conventions, the functions $Y_\ell^m(\underline{\Omega})$ are defined

$$Y_\ell^m(\underline{\Omega}) = \left\{ \frac{2\ell+1}{4\pi} \; \frac{(\ell-|m|)!}{(\ell+|m|)!} \right\}^{1/2} P_\ell^m(\cos\theta) e^{im\varphi}$$

also, it defines $\mu = \cos\theta$,

$$Y_\ell^m(\underline{\Omega}) = \left\{ \frac{2\ell+1}{4} \; \frac{(\ell-|m|)!}{(\ell+|m|)!} \right\}^{1/2} \frac{(1-\mu^2)^{\frac{|m|}{2}}}{2\ell!} \frac{d^{\ell+|m|}}{d\mu^{\ell+|m|}} (\mu^2-1)^\ell e^{im\varphi}$$

Note: $Y_\ell^m = (-1)^m Y_\ell^{-m*}$ where * denotes complex conjugate)

B. Expression of $P_\ell(\cos\theta)$ in Terms of the Spherical Harmonics

Define θ_1, φ_1, i.e. $\underline{\Omega}_1$ and θ_2, φ_2, i.e., $\underline{\Omega}_2$ as the polar and azimuthal angles specifying two points on the unit sphere, with respect to a coordinate system at its center. Denote by θ the angle between the lines drawn from each point to the origin of the coordinates. Then:

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{m=\ell} Y_\ell^m(\underline{\Omega}_1)\, Y_\ell^{m*}(\underline{\Omega}_2)$$

C. Orthonormality of $Y^m(\underline{\Omega})$

$$\int_{\underline{\Omega}} Y_j^k(\underline{\Omega})\, Y_\ell^{m*}(\underline{\Omega})\, d\underline{\Omega} = \delta_{j\ell}\, \delta_{km}$$

where the integral over vector $\underline{\Omega}$ indicates a double integration over the full ranges of θ, φ; $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq \varphi < 2\pi$.

D. Expansion in Spherical Harmonics

Any functions, perhaps complex, of the variables θ and φ, i.e., $\underline{\Omega}$, absolutely integrable over θ and φ, can be expanded in terms of the functions $Y_\ell^m(\underline{\Omega})$:

$$F(\underline{\Omega}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} F_\ell^m \, Y_\ell^m(\underline{\Omega})$$

where

$$F_\ell^m = \int_{\Omega} F(\underline{\Omega}) \, Y_\ell^{m*}(\underline{\Omega}) \, d\Omega$$

E. Differential Equation Satisfied by $Y_\ell^m(\underline{\Omega})$:

$$\frac{\partial}{\partial\theta}\left(\sin\frac{\partial y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \ell(\ell+1)(\sin\theta)\, Y = 0 \qquad (n \quad \text{integer})$$

Assume $Y = \Phi(\varphi)\Theta(\cos\theta)$ and say

$\dfrac{\partial^2 Y}{\partial\varphi^2} = -m^2 Y$; impose the conditions Y bounded at $\cos\theta = \pm 1$

(makes ℓ an integer) Y single-valued in φ (makes m an integer).

F. Some Low-Order Spherical Harmonics

$$Y_0^0(\underline{\Omega}) = \frac{1}{(4\pi)^{1/2}}$$

$$Y_1^{-1}(\underline{\Omega}) = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\varphi}$$

$$Y_1^0(\underline{\Omega}) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$Y_1^1(\underline{\Omega}) = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\varphi}$$

G. A Useful Relationship

If the vector $\underline{\Omega}$ is considered to represent a point on the unit sphere, its components can be represented

$$\Omega_x = \sin\theta\cos\varphi$$

$$\Omega_y = \sin\theta\sin\varphi$$

$$\Omega_z = \cos\theta$$

If a new set of components be constructed,

$$\Omega_{-1} = \frac{1}{\sqrt{2}} (\Omega_x - i\Omega y) = \frac{1}{\sqrt{2}} \sin\theta(\cos\phi\ i \sin\phi)$$

$$\Omega_0 = \Omega_z = \cos\theta$$

$$\Omega_1 = \frac{1}{\sqrt{2}} (\Omega_x + i\Omega y) = \frac{1}{\sqrt{2}} \sin\theta(\cos\phi + i \sin\phi)$$

then these are readily seen to be expressible as the $\ell = 1$ spherical harmonics (see E),

$$\Omega_{-1} = \left(\frac{4\pi}{3}\right)^{1/2} Y_1^{-1}$$

$$\Omega_0 = \left(\frac{4\pi}{3}\right)^{1/2} Y_1$$

$$\Omega_1 = \left(\frac{4\pi}{3}\right)^{1/2} Y_1^{1}$$

H. References

1. H. Margenan and G. M. Murphy: "The Mathematics of Chemistry and Physics", D. Van Nostrand, 1st edition, (1934).

2. A. G. Webster; "Partial Differential Equations of Mathematical Physics", G. E. Stechert, (1927).

3. E. Jahnke and F. Emde; "Tables of Functions", Dover Publications, 1945, pp. 107-125 (Lists some properties and tabulates functions).

4. D. K. Holmes and R. V. Meghreblian; "Notes on Reactor Analysis, Part II, Theory", U.S.A.E.C. Document CF-54-7-88 (Part II), August 1955.

5. L. I. Schiff, "Quantum Mechanics", 2nd Edition, McGraw Hill, 1955, p. 73.

6. Whittaker and Watson; "Modern Analysis"; 4th edition, Cambridge University Press (1927), pp. 391-396.

V. Laguerre Polynomials

A. Derivative Definition

$$L_n^{(\alpha)}(x) = (-1)^n\ x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{\alpha+n} e^{-x})$$

B. Generating Function

$$H(x,t) = (1 - t)^{-(\alpha+1)} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L_n^{(\alpha)}(x)\, t^n$$

Thus $$L_n^{(\alpha)}(x) = \left[\frac{d^n}{dt^n} (1-t)^{-(\alpha+1)} e^{\frac{-xt}{1-t}} \right]_{t=0}$$

C. Differential Equation Satisfied by $L_n^{(\alpha)}(x)$:

$$x \frac{d^2y}{dx^2} + (\alpha - x + 1) \frac{dy}{dx} + n\, y_n = 0 \qquad n = 0,1,2,\ldots$$

D. Orthogonality, range $0 \leq x \leq \infty$

$$\int_0^{\infty} x^{\alpha} e^{-x} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, dx = N_n^2\, \delta mn$$

where

$$N_n^2 = (-1)^n\, n!\, \Gamma(n + \alpha + 1)$$

E. Expansion in Laguerre Polynomials

$$F(x) = \sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(x)$$

where

$$f_n = \frac{1}{N_n^2} \int_0^{\infty} x^{\alpha} e^{-x} F(x)\, L_n^{(\alpha)}(x)\, dx$$

F. Expansion of x^m in Laguerre polynomials.

$$x^m = \sum_{n=0}^{m} \frac{(-1)^n\, m!\, \Gamma(\alpha + m + 1)}{n!\, \Gamma(n + \alpha + 1)} L_n^{(\alpha)}(x)$$

G. Recurrence Relations

a. $$(x - 2n - \alpha - 1)\, L_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + n(n + \alpha)\, L_{n-1}^{(\alpha)}(x)$$

b. $$L'^{(\alpha)}_{n+1}(x) = (n + 1) \left[L'^{(\alpha)}_n(x) - L_n^{(\alpha)}(x) \right]$$

VI. Bessel Functions

A. Differential Equation Satisfied by Bessel Functions.

1. $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$

or 2. $$\frac{1}{x}\frac{d}{dx}\left(x \frac{dy}{dx}\right) + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$

(An extensive listing of other equations satisfied by Bessel functions is given in Reference 2.)

B. General Solution of Above Equations.

$y = A J_\nu(x) + BJ_{-\nu}(x)$ (ν non-integral)

$y = A J_n(x) + BN_n(x)$ (n integral)

($N_n(x)$ is frequently represented by the symbol $Y_n(x)$.

where A and B are arbitrary co^n^stants.

$J_n(x)$ = Bessel Functions of the first kind of n^{th} order.

$N_n(x)$ = *Neumann Functions, or Bessel Functions of the second kind.

For non-integers,

$$N_\nu(x) = \frac{J\ (x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu \pi}$$

For $\nu = n$, an integer, the above expression reduces to**

$$N_n(x) = \frac{2}{\pi} J_n(x) \log \frac{1}{2} x - \frac{1}{\pi}\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2} x\right)^{n+2r}}{r!\ (n+r)!} F(r) + F\ (n+r)$$

$$- \frac{1}{\pi}\sum_{r=0}^{n-1} \frac{(n-r-1)!}{r!} \left(\frac{1}{2} x\right)^{-n+2r}$$

where $F(r) = \sum_{s=1}^{r} \frac{1}{s}$

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* In reference 4, these are called Weber functions

** This is shown in reference 8, pg. 577.

A third function which sometimes finds use is the Hankel function, or Bessel function of the third kind. There are two such functions, defined by

$$H_\nu^{(1)}(x) = J_\nu(x) + i\,N_\nu(x) \qquad (\nu \text{ unrestricted})$$
$$H_\nu^{(2)}(x) = J_\nu(x) - i\,N_\nu(x)$$

Then, for integers, we see that a solution to Bessel's equation will be

$$y = A_1 H_n^{(1)}(x) + A_2 H_n^{(2)}(x)$$

where A_1 and A_2 are arbitrary constants and may be complex.

These functions bear the same relation to the Bessel function $J_\nu(x)$ and $N_\nu(x)$ as the functions exp $(\pm \nu x)$ bear to cos νx and sin νx. They satisfy the same differential equation and recursion relations as $J_\nu(x)$. Their importance results from the fact that they alone vanish for an infinite complex argument, vis. $H^{(1)}$ if the imaginary part of the argument is positive, $H^{(2)}$ if it is negative, i.e, $\lim_{r\to\infty} H^{(1)}(re^{i\theta}) = 0$, $\lim_{r\to\infty} H^{(2)}(re^{-i\theta})$, $0 \le \theta \le \lambda$.

From the above equations, we can also write

$$J_\nu(x) = \frac{1}{2}\left[H_\nu^{(2)}(x) + H_\nu^{(1)}(x)\right] \qquad (\nu \text{ unrestricted})$$
$$N_\nu(x) = \frac{i}{2}\left[H_\nu^{(2)}(x) - H_\nu^{(1)}(x)\right]$$

C. Series Representation for Bessel Functions of the first kind of Order n.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x)^{2r+n}}{2^{2r+n}\, r!\, (1+n+r)}$$

D. Properties of Bessel Functions

1. $J_n(x) = (-1)^n J_{-n}(x)$
2. $J_n(x) = (-1)^n J_n(-x)$
3. Bounds on $J_n(x)$
   a. $|J_n(x)| \le 1 \qquad (n = 0,1,2\ldots) \quad x \ge 0$
   b. $\left|\frac{d^k}{dx^k} J_n(x)\right| \le 1 \quad (n = 0,1,2..;\ k = 1,2..) \quad x \ge 0$

4. Limits for $J_n(x)$ where n equal zero or positive integer.

a. $\lim_{n\to\infty} J_n(x) = 0$

b. $\lim_{x\to\infty} J_n(x) = 0$

c. $\lim_{x\to\infty} J_n(x) = \frac{x^n}{2^n n!}$

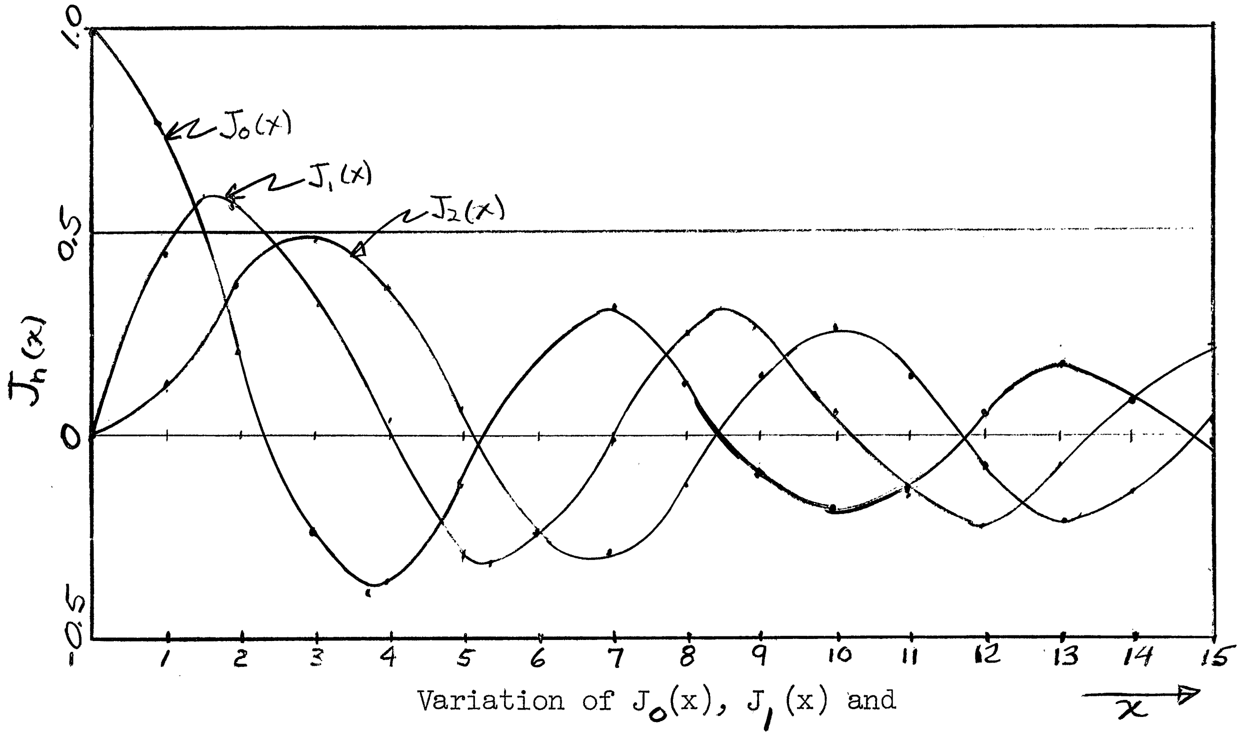
5. Limits for $N_n(x)$ for n = 0 or positive integer and x real.

a. $\lim_{x\to\infty} N_n(x) = 0$

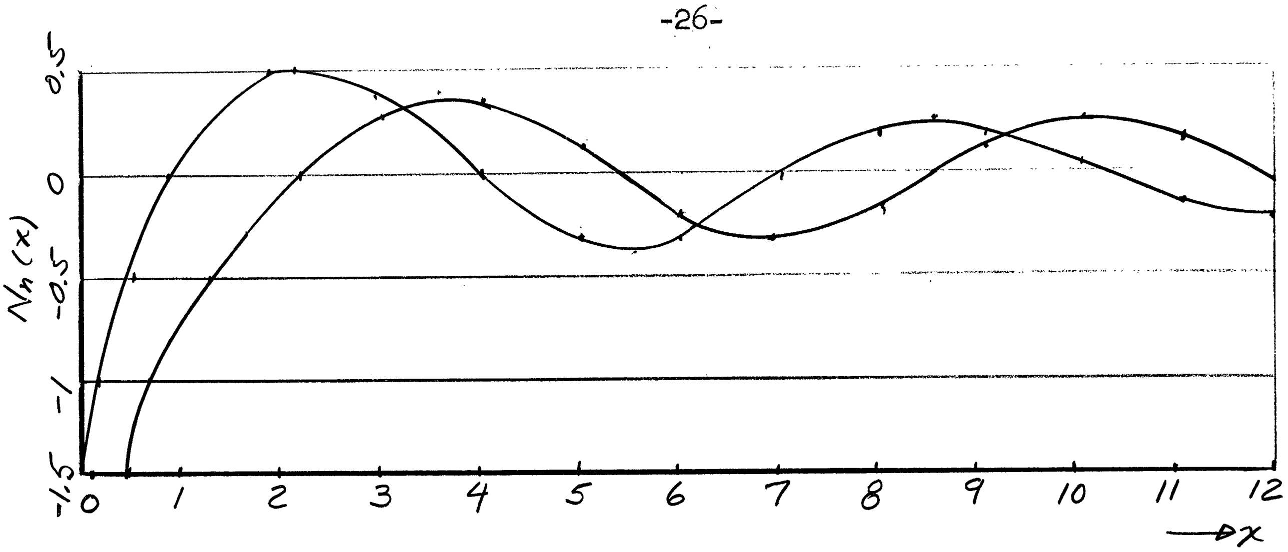
b. $\lim_{x\to 0} N_n(x) = -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n \quad n \geq 1.$

c. $\lim_{x\to 0} N_0(x) = -\frac{2}{\pi}\ln\frac{2}{\gamma x} \qquad (c\gamma = 1.781)$

6. Graphs of $J_n(x)$ and $N_n(x)$



Variation of $J_0(x)$, $J_1(x)$ and $J_2(x)$ with x.



Variations of $N_0(x)$ and $N_1(x)$ with x.

E. Generating Function

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{h=-\infty}^{\infty} J_n(x)\, t^n \qquad \text{(n integral)}$$

F. Recursion Formulae

a. $2J'n(x) = J_{n-1}(x) - J_{n+1}(x)$

b. $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

c. $xJn'(x) = x J_{n-1}(x) - n J_n(x)$

$= x J_n(x) - xJ_{n+1}(x)$

G. Differential Formulae:

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

H. Orthogonality, range $0 \leq x \leq C$

1. $\int_0^c x J_n(\lambda_{nj}x) J_n(\lambda_{nk}x)\, dx = \delta_{jk} N_{nj}^2$

where λ_{nj} are positive roots of the equation $J_n(\lambda c) = 0$

and $N_{nj}^2 = \frac{C^2}{2}\left[J_{n+1}(\lambda_{nj}C)\right]^2$

2. $\int_0^c x\, J_n(\lambda_{n\ell} x)\, J_n(\lambda_{nm} x)\, dx = \delta_{\ell m} M_{n\ell}^2$

where $\lambda_{n\ell}$ are the positive roots of the equation $(\lambda c) J'n(\lambda c) =$ $-hJ_n(\lambda C)$ or its equivalent

$(n + h) J_n(\lambda c) - \lambda c J_{n+1}(\lambda c) = 0$

where h is some constant > 0

$n = 0,1,2...$

and $M_{n\ell}^2 = \left[\dfrac{\lambda^2_{n\ell} c^2 + h^2 - n^2}{2\lambda^2_{n\ell}}\right]\left[J_n(\lambda_{n\ell} c)\right]^2$

I. Expansion in Bessel Functions

$$f(x) = \sum_{=0}^{\infty} A_j J_n(\lambda_{nj} x)$$

where A_j will be represented either by

a. $A_j = \dfrac{2}{c^2\left[J_{n+1}(\lambda_{nj} c)\right]^2} \displaystyle\int_0^c x\, f(x)\, J_n(\lambda_{nj} x)\, dx$

when $J_n(\lambda_{nj} c) = 0$

b. or $A_j = \dfrac{2\lambda_{nj}^2}{\lambda^2_{nj} c^2 + h^2 - n^2)\left[J_n(\lambda_{nj} c)\right]^2} \displaystyle\int_0^c x\, f(x)\, J_n(\lambda_{nj} x)\, dx$

when $\lambda c J'_n(\lambda c) = -h\, J_n(\lambda c)$

J. Bessel Integral Form

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\,\theta - x \sin\theta)\, d\theta \quad (n = 0,1,2...)$$

VII. Modified Bessel Functions

A. Differential Equation satisfied by Modified Bessel Functions.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) y = 0$$

B. General Solution of Above Equation.

$y = AI_{\nu}(x) + BI_{-\nu}(x)$ (ν non-integral)

$y = AI_n(x) + BK_n(x)$ (n integral)

where

A and B are arbitrary constants.

$I_n(x)$ = Modified Bessel Function of the first kind of n^{th} order.

$K_n(x)$ = Modified Bessel Function of the second kind of n^{th} order.

For non-integers,

$$K_\nu(x) = \frac{\pi}{2}\left[I_{-\nu}(x) - I_\nu(x)\right]\frac{1}{\sin \nu\pi}$$

$$= \frac{\pi}{2}\, i^{\nu+1}\, H_\nu^{(1)}(ix) = \frac{\pi}{2}\, i^{-\nu-1}\, H_\nu^{(2)}(-ix)$$

For integers,

$$K_n(x) = (-1)^{n+1}\,\frac{2}{\pi}\, I_n(x)\log(1/2\,x) + \frac{1}{\pi}\sum_{r=0}^{n-1}(-1)^r\,\frac{r(n-r-1)!}{r!}\,(1/2\,x)^{-n+2r}$$

$$+ (-1)^n\,\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(1/2\,x)^{n+2r}}{r!\,(n+r)!}\left\{F(r) + F(n+r)\right\}$$

where

$$F(r) = \sum_{s=1}^{r}\frac{1}{s}$$

C. Relation of Modified Bessel Function of First Kind to Bessel Function of First Kind.

For ν unrestricted,

$$I_\nu(x) = i^{-\nu}\, J_\nu(ix)$$

$$= \frac{i}{2}\left[H_\nu^{(2)}(x) - H_\nu^{(1)}(x)\right]$$

$$= \frac{x^\nu}{2^\nu\,\Gamma(\nu+1)}\sum_{r=0}^{\infty}\frac{(1/4\,x^2)^r}{r!\,(\nu+1)\,r}$$

where $(\alpha)_r = (\alpha)\,(\alpha+1)\,\dots\dots\dots(\alpha+r-1)$

D. Properties of I_n (where n is integral).

1. $I_{-n}(x) = In(x)$

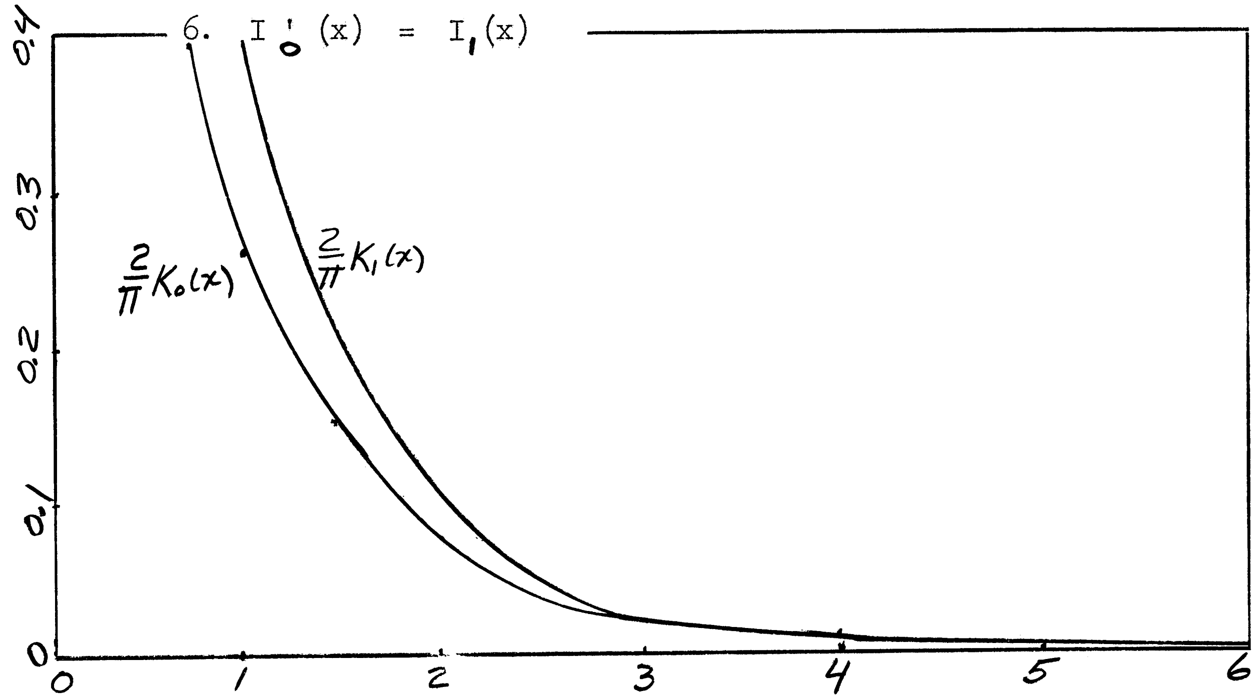
2. $2I'_n(x) = I_{n-1}(x) + I_{n+1}(x)$

3. $\frac{2n}{x}\, In(x) = I_{n-1}(x) - I_{n+1}(x)$

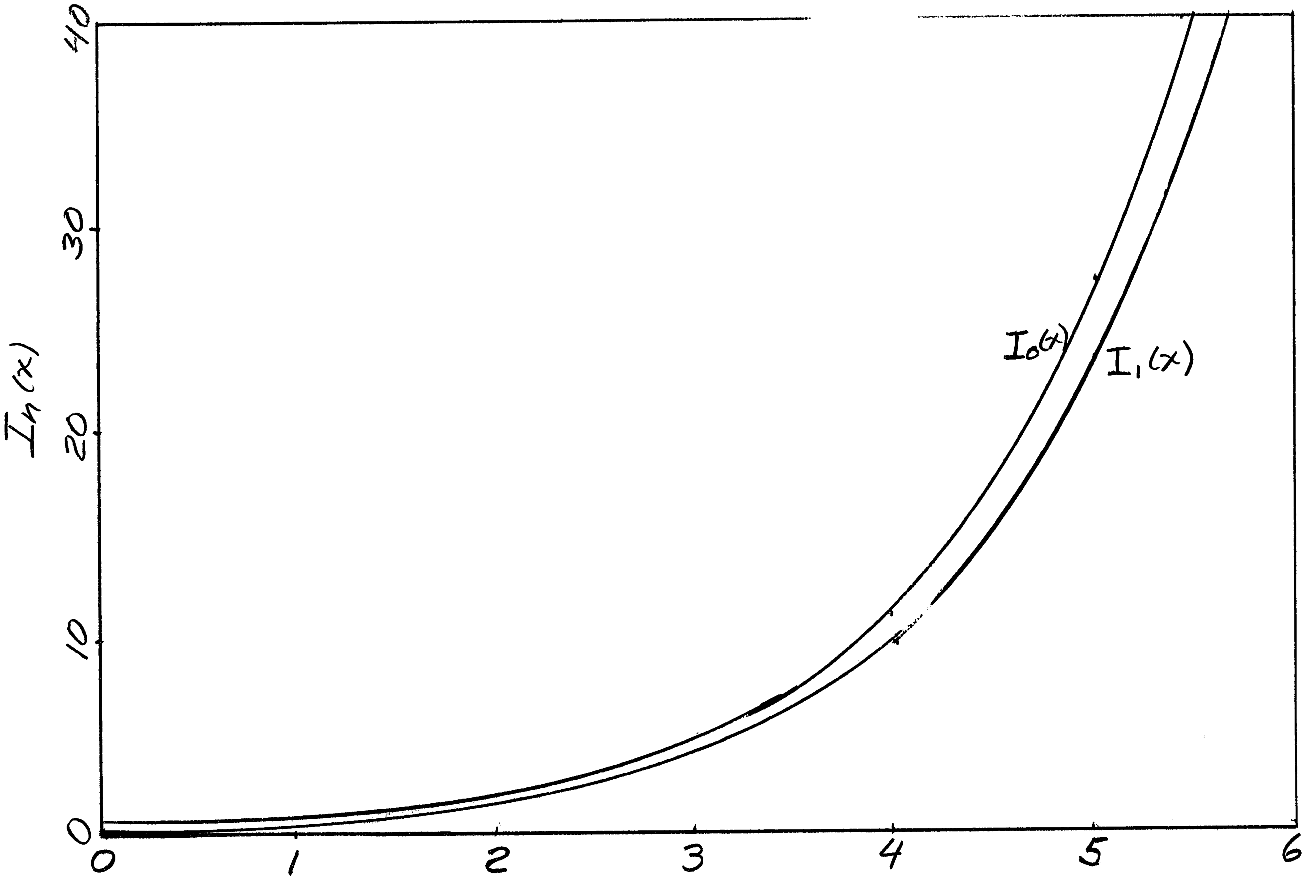
4. $x\, I_n'(x) = x\, I_{n-1}(x) - n I_n(x)$

5. $x\, I_n'(x) = n\, I_n(x) + x\, I_{n+1}(x)$

6. $I_0'(x) = I_1(x)$



Variation of $K_0(x)$ and $K_1(x)$ with x



Variation of $I_0(x)$ and $I_1(x)$ with x

*Comment on notation of Jahnke and Emde.

1. A general cylindrical function $Z_p(x)$ is defined on page 144 by

$$Z_p(x) = C_1 J_p(x) + c_2 N_p(x) \qquad \text{(p integer or arbitrary positive)}$$

where c_1, c_2 denote arbitrary (real or complex) constants. Thus $Z_p(x)$ can apply to $J_p(x)$ by letting $c_2 = 0$, to $N_p(x)$ for $c_1 = 0$, and to $H_p(x)$ by other constants.

All formulae on pages following use this definition of $Z_p(x)$.

2. The function $I_p(x)$ is not listed as such, but is found as $i^{p} J_p\ (ix)$ on pages 224-229.

3. The function $K_p(x)$ is not listed, but

$$\frac{2}{\pi} K_n(x) = i^{n+1} H_n^{(1)}(ix)$$
$$= -i^{-n+1} H_n^{(2)}(ix)$$

The functions $iH_o^{(1)}(ix) = -iH_o^{(2)}(-ix)$ and $-H_1^{(1)}(ix) = -H_1^{(2)}(-ix)$ are tabulated on pages 236-243.

4. This reference is full of extremely interesting, beautiful and helpful pictures of many functions, almost suitable for hanging in the living room.

References.

1. G. N. Watson; "A Treatise on the Theory of Bessel Functions", 2nd edition, Cambridge University Press, 1944, (exhaustive treatment)

2. E. Jahnke and F. Emde; "Tables of Functions" (extensive tabulation of equations leading to Bessel Functions and of the related cylindrical functions. Also has further properties, see "Comment on notation of Jahnke and Emde).

3. R. V. Churchill; "Fourier Series and Boundary Value Problems", McGraw Hill, 1941 (good elementary discussion)

4. I. N. Sneddon; "Special Functions of Mathematical Physics and Chemistry" University Math. Texts, 1956 (thorough discussion for practical use. His use of subscript is not always clear or general)

5. D. Jackson; "Fourier Series and Orthogonal Polynomials", Carus Math. Monographs, No. 6, 1941 (good brief discussion)

6. Whittaker and Watson; "Modern Analysis" 4th edition, Cambridge University Press, 1958 (more rigorous development)

7. N. W. McLachlan; "Bessel Functions for Engineers", 2nd edition, Oxford, Clarendon Press, 1955.

8. H. and B. S. Jeffreys; "Mathematical Physics", 3rd edition, Cambridge University Press, 1955, (compact, but rigorous presentation. They use a different notation, but it is clearly defined.)

THE LAPLACE TRANSFORMATION

I. Introduction

A. Description

The Laplace transformation permits many relatively complicated operations upon a function, such as differentiation and integration for instance, to be replaced by simpler algebraic operations, such as multiplication or division, upon the transform. It is analogous to the way in which such operations as multiplication and division are replaced by simpler processes of addition and subtraction when we work not with the numbers themselves but with their logarithms.

B. Definition

The Laplace transformation applied to a function f(t) associates a function of a new variable with f(t). This function of s is denoted by $\mathcal{L}\, f(t)$ or where no confusion will result, simply by $\mathcal{L}(f)$ or F(s); and the transform is defined by:

$$\mathcal{L}(f) = \int_0^\infty f(t)\, e^{-st}\, dt$$

C. Existance Conditions

For a Laplace transformation of f(t) to exist and for f(t) to be recoverable from its transform it is sufficient that f(t) be of <u>exponential order</u>, i.e. that there should exist a constant, a, such that the product;

$$e^{-at} \left| f(t) \right|$$

is bounded for all values of t greater than some finite number T; and that f(t) should be at least piecewise continuous over every finite interval $0 \leq t \leq T$, T any finite number. These conditions are usually met by functions occuring in physical problems. The number a is called the exponential order of f(t). If a number a exists such that $e^{-at} \left| f(t) \right|$ is bounded, f is said to be of exponential order.

D. Analyticity of $F(s)$.

If $f(t)$ is piecewise continuous and of exponential order a, the transform of $f(t)$, i.e., $F(s)$, is an analytic function of s for $Re(s) > a$. Also, it is true that for $Re(s) > a$, $\lim_{x \to \infty} F(s) = 0$ and $\lim_{y \to \infty} F(s) = 0$ when $s = x + iy$.

E. Theorems.

Theorem I (Linearity)

The Laplace transform of a sum of functions is the sum of the transforms of the individual functions.

$$\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$$

Theorem II (Linearity)

The Laplace transform of a constant times a function is the constant times the transform of the function

$$\mathcal{L}(cf) = c\,\mathcal{L}(f)$$

Theorem III (Basic Operational Property)

If $f(t)$ is a function of exponential order which is continuous and whose derivative is at least piecewise continuous over every finite interval $0 \leq t \leq t_2$, and if $f(t)$ approaches the value $f(0^+)$ as t approaches zero from the right, then the Laplace transform of the derivative of $f(t)$ is given by

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0^+)$$

and

$$\mathcal{L}(f'') = s^2\,\mathcal{L}(f) - sf(0^+) - f'(0^+)$$

the latter, of course, requires an extension of the continuity of $f(t)$ and its derivatives to include $f''(t)$, and may be formally shown by partial integration. More generally, if $f(t)$ and its first n-1 derivatives are continuous and $\frac{d^n f}{dt^n}$ is piecewise continuous, then

$$\mathcal{L}\left[\frac{d^n f}{dt}\right] = s^n\,\mathcal{L}(f) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) \text{-----} - f^{(n-1)}(0^+)$$

Theorem IV (Transforms of Integrals)

If $f(t)$ is of exponential order and at least piecewise continuous, the transform of $\int_a^t f(t)\,dt$ is given by

$$\mathcal{L}\left[\int_a^t f(t)\,dt\right] = \frac{1}{s}\mathcal{L}(f) + \frac{1}{s}\int_a^0 f(t)\,dt$$

F. Further Properties

Below, let us assume all functions of the variable t are piecewise continuous, $0 \leq t \leq T$, and of exponential order as $t \to \infty$. Then

Theorem V

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s - a).$$

Theorem VI

If

$$f_b(t) \equiv \begin{cases} f(t-b), & t \geq b \\ 0 \quad , & t < b \end{cases}, \text{ then}$$

$$\mathcal{L}\left[f_b(t)\right] = e^{-bs}F(s)$$

Theorem VII (Convolution)

$$\mathcal{L}\left[\int_o^t f(t - \tau)\, g(\tau)\, d\tau\right] = F(s)*G(s)$$

$$= \mathcal{L}\left[\int_o^t g(t - \tau)\, f(\tau)\, d\tau\right].$$

Theorem VIII (Derivatives of Transforms)

$$\mathcal{L}\left[tf(t)\right] = \frac{-dF(s)}{ds}$$

and, in general,

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^nF}{ds^n}$$

II. Examples.

A. Solving Simultaneous Equations

Solve for y from the simultaneous equations

$$y' + y + 3\int_o^t z\,dt = \cos t + 3 \sin t$$

$$2y' + 3z' + 6z = 0$$

$$y_o = -3, \; z_o = 2$$

The transform of each equation is:

$$\mathcal{L}(y') + \mathcal{L}(y) + 3\mathcal{L}\int_0^t z\,dt = \mathcal{L}(\cos t) + 3\mathcal{L}(\sin t)$$

$$2\mathcal{L}(y') + 3\mathcal{L}(z') + 6\mathcal{L}(z) = 0$$

or;

$$\left[s\mathcal{L}(y) + 3\right] + \mathcal{L}(y) + \frac{3}{s}\mathcal{L}(z) = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

$$2\left[s\mathcal{L}(y) + 3\right] + 3\left[s\mathcal{L}(z) - 2\right] + 6\mathcal{L}(z) = 0$$

collecting terms and transposing

$$(s+1)\mathcal{L}(y) + \frac{3}{s}\mathcal{L}(z) = \frac{s+3}{s^2+1} - 3$$

$$2s\mathcal{L}(y) + 3(s+2)\mathcal{L}(z) = 0$$

The two original integro-differential equations are now reduced to two linear algebraic equations in $\mathcal{L}(y)$ and $\mathcal{L}(z)$. Applying Cramer's rule and solving for $\mathcal{L}(y)$ since it is y which we want;

$$\mathcal{L}(y) = \frac{\begin{vmatrix} \left(\frac{s+3}{s^2+2}\right) - 3 & \frac{3}{s} \\ 0 & 3(s+2) \end{vmatrix}}{\begin{vmatrix} (s+1) & \frac{3}{s} \\ 2s & 3(s+2 \end{vmatrix}} = \frac{3(s+2)\left(\frac{s+3}{s^2+1} - 3\right)}{3s\ (s+3)}$$

$$\text{or, } \mathcal{L}(y) = \frac{s+2}{s(s^2+1)} - 3\left[\frac{s+2}{s(s+3)}\right]$$

Applying the method of partial fractions;

$$\mathcal{L}(y) = \left(\frac{2}{s} + \frac{-2s+1}{s^2+1}\right) - \left(\frac{1}{s+3} + \frac{2}{s}\right)$$

$$= -2\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1} - \frac{1}{s+3}$$

And, finding the inverses in the table of transforms, which are tables relating functions of s to the corresponding functions of t, and will be found in section IV of this paper,

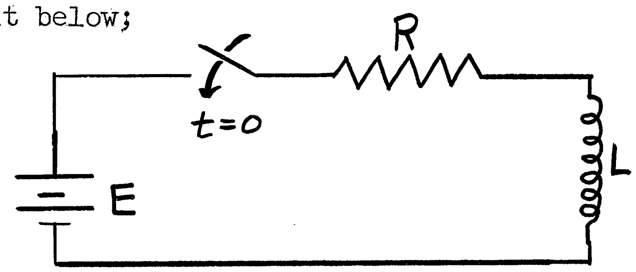
$$y = -2\cos t + \sin t - e^{-3t} \quad (\text{for } t > 0)$$

It should be noted that one of the inherent characteristics of solving differential equations by the use of Laplace transforms is that the initial conditions are included in the solution.

B. Electric Circuit Example

Since Laplace transforms are widely used in the determination of the transient response of electric circuits, a simple circuit example is given below.

Given, circuit below;



Find, equation of current flow after switch is closed,

a. Circuit equation; $E = iR + L\frac{di}{dt}$

b. TRANSFORMING; $\frac{E}{s} = I(s)R + LsI(s) - Lf(0^+)$

at $t = 0$, $i = 0$ so; $f(0^+) = 0$

c. solving for I(s)

$$I(s) = \frac{E}{s(R + Ls)}$$

d. TAKING INVERSE

$$i(t) = \frac{E}{R}(1 - e^{-Rt/L})$$

C. Transfer Functions

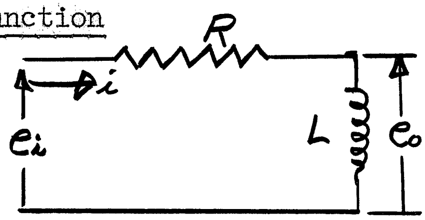
For certain control functions, and for representing the dynamic behavior of various devices such as reactors, heat exchangers, etc., it is advantageous to use a "transfer function" because of the convenience in manipulation which obtains. The transfer functions of many elements of a system, when strung together in a block diagram, represent a convenient way of writing complicated system equations. The transfer function of a system may be defined as the ratio of the output to the input of the system in transform (s) space.

<u>Conditions for using transfer functions</u>

1. Initial condition operator = 0.
2. No loading between transfer functions.
3. Transfer function satisfies existence conditions for Laplace transformations.
4. Linear system.

<u>Example of Transfer Function</u>

Find $\frac{Eo}{Ei}$ (s) of



a. Equations

$$e_i = R_i + L\frac{di}{dt}$$

$$e_o = L\frac{di}{dt}$$

b. Transforms

$$E_i(s) = RI(s) + sLI(s)$$

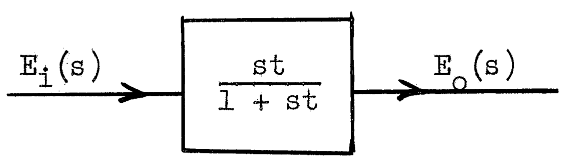
$$E_o(s) = sLI(s)$$

c. Solving

$$\frac{E_o}{E_i}(s) = \frac{sLI(s)}{(R+sL)\ I(s)} = \frac{sL}{R + sL} = \frac{st}{1 + st}$$

where $t = L/R$ = circuit time constant.

d. Block diagram



III. Inverse Transformations

A. Heaviside Methods.

When solving equations by the Laplace transform technique, it is frequently the most difficult part of the procedure to invert the transformed solution for F(s) into the desired function f(t). A simple way of making this inversion, but unfortunately a method only applicable to special cases, is to reduce the answer to a number of simple expressions by using partial fractions, and then apply the Heaviside theorems as outlined below:

Theorem I

If $y(t) = \mathcal{L}^{-1}\left[\frac{p(s)}{q(s)}\right]$, where p(s) and q(s) are polynomials, and the order of q(s) is greater than the order of p(s), then the term in y(t) corresponding to an unrepeated linear factor (s-a) of q(s) is

$$\frac{p(a)}{q'(a)} e^{at}, \text{ or } \frac{p(a)}{Q(a)} e^{at},$$

where Q(s) is the product of all factors of q(s) except (s-a)

Example: If $\mathcal{L}(f\ (t)) = \frac{s^2 + 2}{s(s+1)(s+2)}$, what is f(t)?

a. Roots of denominator are $s = 0$, $s = -1$, $s = -2$

b. $p(s) = s^2 + 2$

c. $q(s) = s^3 + 3s^2 + 2s$; $q'(s) = 3s^2 + 6s + 2$

d. $p(o) = 2$, $p(-1) = 3$, $p(-2) = 6$
$q'(o) = 2$, $q'(-1) = -1$, $q=(-2) = 2$

e. $f(t) = \frac{2}{2} e^{ot} + \frac{3}{-1} e^{-t} + \frac{6}{2} e^{-2t} = 1 - 3e^{-t} + 3e^{-2t}$

Theorem II

If $y(t) = \mathcal{L}^{-1}\left[\frac{p(s)}{q(s)}\right]$ where p(s) and q(s) are polynomials and the order of q(s) is greater than the order of p(s), then the terms in $y(t)$ corresponding to the repeated linear factor $(s-a)^r$ of $q(s)$ are;

$$e^{at}\left[\frac{\phi^{(r-1)}(a)}{(r-1)!} + \frac{\phi^{(r-2)}(a)}{(r-2)!}\frac{t}{1!} + \cdots\cdots + \frac{\phi'(a)t^{r-2}}{1!\,(r-2)!} + \phi(a)\frac{t^{r-1}}{(r-1)!}\right]$$

where $\phi(s)$ is the quotient of $p(s)$ and all the factors of $q(s)$ except $(s-a)^r$.

Example:

$$\mathcal{L}(f(t)) = \frac{s+3}{(s+2)^2\,(s+1)} \qquad \text{what is } f(t)?$$

a. $\phi(s) = \frac{s+3}{s+1}$; $\phi'(s) = \frac{(s+1)-(s+3)}{(s+1)^2} = \frac{-2}{(s+1)^2}$

b. $\phi(-2) = -1$; $\phi'(-2) = -2$

so, terms in $f(t)$ corresponding to $(s+2)^2$ are

$$e^{-2t}\left[\frac{-2}{1!} + \frac{-1t}{0!1!}\right] = -e^{-2t}(2+t)$$

then, as in the example of Theorem I;

$p(s) = s+3$; $q(s) = s^3 + 5s^2 + 8s + 4$

$q'(s) = 3s^2 + 10s + 8$

$p(-1) = 2$ $\quad q'(-1) = 1$

so; $f(t) = -e^{-2t}(2+t) + 2e^{-t}$

Theorem III

If $y(t) = \mathcal{L}^{-1}\left[\frac{p(s)}{q(s)}\right]$, where $p(s)$ and $q(s)$ are polynomials and the order of $q(s)$ is greater than the order of $p(s)$, then the terms in $y(t)$ corresponding to an unrepeated quadratic factor $\left[(s+a)^2 + b^2\right]$ of $q(s)$ are $\frac{e^{-at}}{b}(\phi_i \cos bt + \phi_r \sin bt)$ where ϕ_r and ϕ_i are respectively, the real and imaginary parts of $\phi(-a+ib)$, and $\phi(s)$ is the quotient of $p(s)$ and all the factors of $q(s)$ except $\left[(s+a)^2 + b^2\right]$.

Example: $\mathcal{L}(f(t)) = \dfrac{s}{(s+2)^2\,(s^2+2s+2)}$

a. Considering the linear factor as in the example of Theorem II

$$\phi(s) = \frac{s}{(s^2 + 2s + 2)} \quad ; \quad \phi'(s) = \frac{-s^2 + 2}{(s^2 + 2s + 2)^2}$$

$$\phi(-2) = -1 \quad ; \quad \phi'(-2) = -\frac{1}{2}$$

so, the terms in f(t) corresponding to the linear factor are;

$$e^{-2t}\,(-1/2 - t) = -\frac{(1 + 2t)\,e^{-2t}}{2}$$

b. considering, the quadratic factor

$$s^2 + 2s + 2 = (s + 1)^2 + 1^2$$

$$\phi(s) = \frac{s}{(s + 2)^2}$$

$$\therefore\ \phi(-a + ib) = \phi(-1 + i) = \frac{-1 + i}{\left[(-1 + i) + 2\right]^2}$$

$$= \frac{-1 + i}{(1 + i)^2} = \frac{-1 + i}{2i} = \frac{1}{2} + \frac{i}{2}$$

so; $\phi_r = \phi_i = 1/2$

so; the terms in f(t) corresponding to $(s^2 + 2s + 2)$ are

$$\frac{e^{-t}(\cos t + \sin t)}{2}$$

Now, adding the two partial inverses, we get

$$f(t) = \frac{-(1 + 2t)\,e^{-2t}}{2} + \frac{e^{-t}(\cos t + \sin t)}{2}$$

B. The Inversion Integral

When the function cannot be reduced to a form ammenable to inversion by tables of transforms or Heaviside methods, there remains a most powerful method for the evaluation of inverse transformations. The inversion is given by an integral in the complex s-plane,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\,F(s)\,ds$$

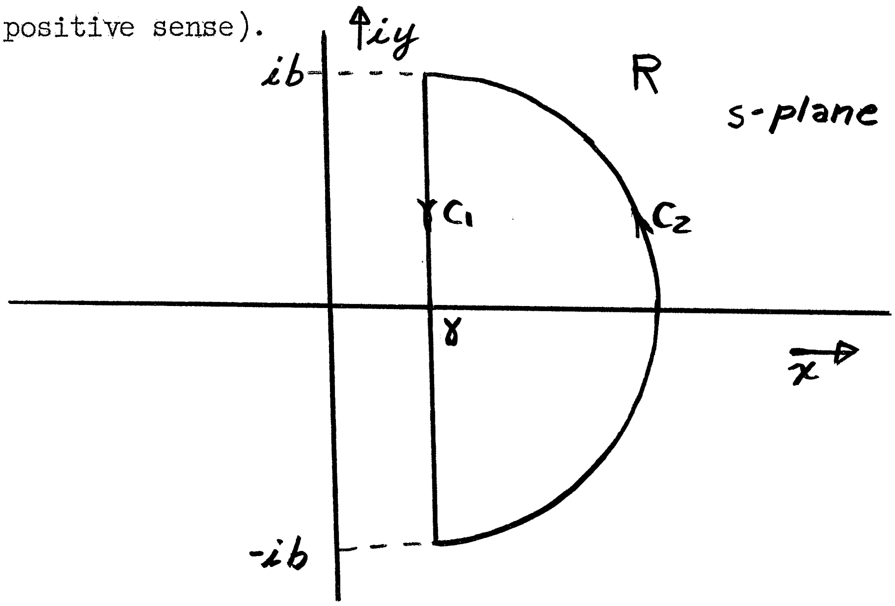
where γ is some real number so chosen that F(s) is analytic (see Appendix A) for $Re(s) \geq \gamma$, and the Cauchy principle value of the integral is to be taken, i.e.

$$f(t) = \frac{1}{2\pi i} \lim_{B \to \infty} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds.$$

Let us illustrate the formal origin of the inversion integral in the following way. In the complex plane let $\phi(z)$ be a function of z, analytic on the line $x = \gamma$, and in the entire half plane R to the right of this line. Moreover, let $|\phi(z)|$ approach zero uniformly as z becomes infinite through this half plane. Then if s_0 is any point in the half plane R, we can choose a semi-circular contour c, composed of c_1 and c_2, as shown below, and apply Cauchy's integral formula, (see Appendix B)

$$\phi(s_0) = \frac{1}{2\pi i} \oint_c \frac{\phi(z)}{z - s_0}\, dz$$

Here, $\phi(s)$ is analytic within and on the boundary c of a simply connected region R and s_0 is any point in the interior of R (Integration around C in positive sense).



Thus, Cauchy's integral formula yields

$$\phi(s) = \frac{1}{2\pi i} \oint_c \frac{\phi(z)dz}{z-s} = \frac{1}{2\pi i} \int_{\gamma+ib}^{\gamma-ib} \frac{\phi(z)dz}{z-s} + \frac{1}{2\pi i} \int_{c_2} \frac{\phi(z)dz}{z-s}$$

Now, for values of z on the path of integration, c_2, and for b sufficiently large,

$$|z - s| \geqq b - |s - \gamma| \geqq b - |s|$$

hence

$$\left| \int_{c_2} \frac{\phi(z)dz}{z-s} \right| \leqq \int_{c_2} \frac{|\phi(z)||dz|}{|z-s|}$$

$$\leqq \frac{M}{b - |s|} \int_{c_2} |dz|$$

$$= \pi \frac{bM}{b - |s|}$$

where M is the maximum value of $|\phi(z)|$ on c_2. As $b \to \infty$, the fraction $\frac{b}{b - |s|}$ approaches 1, and at the same time M approaches zero. Hence

$$\lim_{b \to \infty} \int_{c_2} \frac{\phi(z)dz}{z - s} = 0$$

and the contour integral reduces to

$$\phi(s) = \lim_{b \to \infty} \frac{1}{2\pi i} \int_{\gamma+ib}^{\gamma-ib} \frac{\phi(z)dz}{z - s} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(z)dz}{s - z}$$

now, taking the inverse transform of the above equation,

$$\mathcal{L}^{-1}\{\phi(s)\} = f(t) = \mathcal{L}^{-1}\left\{\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\phi(z)dz}{s - z}\right\}$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}^{-1}\left\{\frac{\phi(z)}{s - z}\right\} dz$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(z)\, \mathcal{L}^{-1}\left\{\frac{1}{s - z}\right\} dz$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(z)\; e^{tz}\, dz$$

since from our table of transforms

$$\mathcal{L}^{-1}\left\{\frac{1}{s - z}\right\} = e^{zt}$$

Our final equation after switching from z to s as dummy variable in the last integral

$$\mathcal{L}^{-1}\{\phi(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)\, e^{st}\, ds,$$

is just the inversion integral which we were establishing.

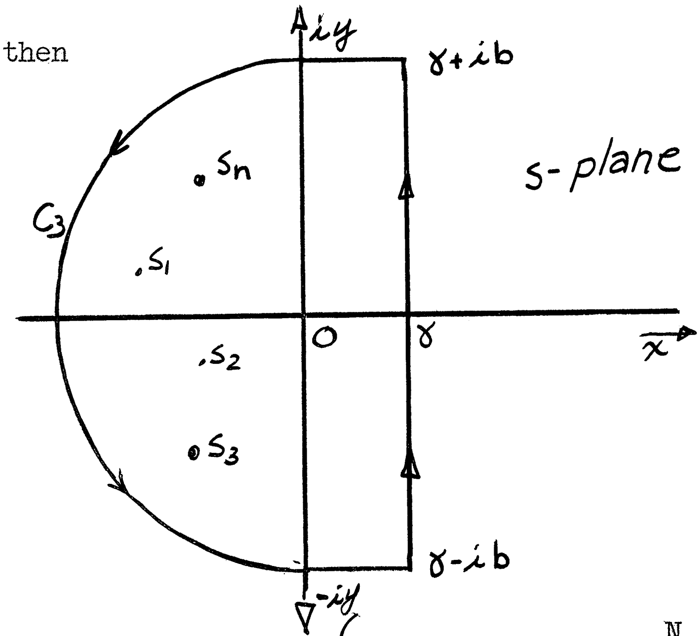
At this point, it would be advantageous to know how to evaluate the integral on the right. According to the residue theorem, the integral of $e^{st}\phi(s)$ around a path enclosing the isolated singular points s_1, s_2----s_n, of $e^{st}\phi(s)$ has the value

$$2\pi i\left[\rho_1(t) + \rho_2(t) + ----\rho_N(t),\right]$$

where $\rho_n(t)$ = the residue of $e^{st}\phi(s)$ at $s = s_n$.

For discussion of residues, see Appendix C; for singular points, Appendix D.

Let the path of integration be made up of the line segment $\gamma - ib$, $\gamma + ib$, and c_3, then



$$\frac{1}{2\pi i}\int_{\gamma-ib}^{\gamma+ib} e^{st}\phi(s)\, ds + \frac{1}{2\pi i}\int_{c_3} e^{st}\phi(s)\, ds = \sum_{n=1}^{N}\rho_n(t)$$

If the second integral around c_3 vanishes for $b \rightarrow \infty$, as often happens, we are led to the immediate result that

$$\mathcal{L}^{-1}\{\phi(s)\} = f(t) = \sum_{n=1}^{N}\rho_n(t).$$

Note that in the formal derivation of the inversion formula, we assumed that $\phi(s)$ (and therefore $e^{st}\phi(s)$) is analytic for $s \geq \gamma$, and that $\lim_{s\to\infty} |\phi(s)| = 0$ in that plane. In our discussion of the residue form of the inversion, we work in the left half-plane. This is because Laplace transforms have the property that they are analytic in a right half-plane, and that in that plane, $\lim_{s\to\infty} |\phi(s)| = 0$.

Questions of the validity of the above procedures, alterations of contour, and applications to problems are not dealt with here, as they are presented in detail in the references.

IV. Table of Transforms.

F(s)	f(t)
1	Unit impulse at t = 0, $\delta(t)$
s	Unit doublet impulse at t = 0, $\delta_2(t)$
$\frac{1}{s}$	Unit step at t = 0, $u(t)$
$\frac{1}{(s+a)}$	e^{-at}
$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at} - e^{-bt}}{b-a}$
$\frac{s+c}{(s+a)(s+b)}$	$\frac{(c-a)e^{-at} - (c-b)e^{-bt}}{b-a}$
$\frac{s+c}{s(s+a)(s+b)}$	$\frac{c}{ab} + \frac{c-a}{a(a-b)} e^{-at} + \frac{c-b}{b(b-a)} e^{-bt}$
$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{e^{-at}}{(c-a)(b-a)} + \frac{e^{-bt}}{(a-b)(c-b)} + \frac{e^{-ct}}{(a-c)(b-c)}$
$\frac{s+d}{(s+a)(s+b)(s+c)}$	$\frac{(d-a)e^{-at}}{(b-a)(c-a)} + \frac{(d-b)e^{-bt}}{(a-b)(c-b)} + \frac{(d-c)e^{-ct}}{(a-c)(b-c)}$
$\frac{s^2+es+d}{(s+a)(s+b)(s+c)}$	$\frac{(a^2-ea+d)e^{-at}}{(b-a)(c-a)} + \frac{(c^2-ec+d)e^{-ct}}{(a-c)(b-c)} + \frac{(b^2-eb+d)e^{-bt}}{(a-b)(c-b)}$
$\frac{1}{s^2+b^2}$	$\frac{1}{b}\sin bt$

$\dfrac{s+d}{s^2+b^2}$	$\dfrac{1}{b}(d^2+b^2)^{1/2}\sin(bt+\psi)$ $\qquad \psi = \text{arc tan}\ \dfrac{b}{d}$
$\dfrac{s}{s^2+b^2}$	$\cos bt$
$\dfrac{1}{(s+a)^2+b^2}$	$\dfrac{e^{-at}\sin bt}{b}$
$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
$\dfrac{s+d}{(s+a)^2+b^2}$	$\dfrac{1}{b}\left[(d-a)^2+b^2\right]^{1/2} e^{-at}\sin(bt+\psi)$ $\psi = \text{arc tan}\left(\dfrac{b}{d-a}\right)$
$\dfrac{1}{s\left[(s+a)^2+b^2\right]}$	$\dfrac{1}{b_o^2} + \dfrac{1}{b_o b} e^{-at}\sin(bt-\psi)$ $\psi = \text{arc tan}\left(\dfrac{b}{-a}\right) \quad b_o^2 = a^2+b^2$
$\dfrac{s+d}{s\left[(s+a)^2+b^2\right]}$	$\dfrac{d}{b_o^2} + \dfrac{1}{bb_o}\left[(d-a)^2+b^2\right]^{1/2} e^{-at}\sin(bt+\psi)$ $\psi = \text{arc tan}\left(\dfrac{b}{d-a}\right) - \text{arc tan}\left(\dfrac{b}{-a}\right) \quad b_o^2 = a^2+b^2$
$\dfrac{1}{s^2-b^2}$	$\dfrac{1}{b}\sinh bt$
$\dfrac{s}{s^2-b^2}$	$\cosh bt$
$\dfrac{1}{s^n}$	$\dfrac{1}{(n-1)!}\, t^{n-1}$ (n is an integer >0)
$\dfrac{1}{s^\nu}$	$\dfrac{1}{\Gamma(\nu)}\, t^{\nu-1}$ $(\nu > 0)$ (ν may be non integer)
$\dfrac{1}{s^2(s+a)}$	$\dfrac{e^{-at}+at-1}{a^2}$
$\dfrac{s+d}{(s+a)\,s^2}$	$\dfrac{d-a}{a^2}\, e^{-at} + \dfrac{d}{a}\, t + \dfrac{a-d}{a^2}$

$\dfrac{s + d}{(s + a)^2}$	$\left[(d - a)t + 1\right] e^{-at}$
$\dfrac{1}{s(s + a)^2}$	$\dfrac{1 - (at + 1)\, e^{-at}}{a^2}$
$\dfrac{s + d}{s(s + a)^2}$	$\dfrac{d}{a^2} + \left[\dfrac{a - d}{a}\, t - \dfrac{d}{a^2}\right] e^{-at}$
$\dfrac{s^2 + es + d}{s(s + a)^2}$	$\dfrac{d}{a^2} + \left\{\left[\dfrac{ea - a^2 - d}{a}\right] t + \dfrac{a^2 - d}{a^2}\right\} e^{-at}$
$\dfrac{1}{s^2\,(s^2 + b^2)}$	$\dfrac{1}{b^2}\, t - \dfrac{1}{b^3} \sin bt$
$\dfrac{1}{s^3\,(s^2 + b^2)}$	$\dfrac{1}{b^4} (\cos bt - 1) + \dfrac{1}{2b^2}\, t^2$
$\dfrac{1}{s^2\,(s^2 - b^2)}$	$\dfrac{1}{b^3} \sinh bt - \dfrac{1}{b^2}\, t$
$\dfrac{1}{s^3\,(s^2 - b^2)}$	$\dfrac{1}{b^4} (\cosh bt - 1) - \dfrac{1}{2b^2}\, t^2$
$\dfrac{1}{(s^2 + b)^2}$	$\dfrac{1}{2b^3} (\sin bt - bt \cos bt)$
$\dfrac{s}{(s^2 + b^2)^2}$	$\dfrac{1}{2b}\, t \sin bt$
$\dfrac{s^2}{(s^2 + b^2)^2}$	$\dfrac{1}{2b} (\sin bt + bt \cos bt)$
$\dfrac{s^2 + b^2}{(s^2 + b^2)^2}$	$t \cos bt$
$\dfrac{1}{\left[(s+a)^2 + b^2\right]^2}$	$\dfrac{1}{2b^3}\, e^{-at} (\sin bt - bt \cos bt)$
$\dfrac{s + a}{\left[(s+a)^2 + b^2\right]^2}$	$\dfrac{1}{2b}\, te^{-at} \sin bt$

$\dfrac{(s+a)^2 - b^2}{\left[(s+a)^2 + b^2\right]^2}$	$te^{-at} \cos bt$
$\dfrac{1}{s^2} e^{-t_1 s}$	$(t - t_1)\, u\, (t - t_1)$
$\dfrac{1}{s^2} (t_1 s + 1)\, e^{-t_1 s}$	$tu(t - t_1)$
$\dfrac{1}{s^3} (t_1^2 s^2 + 2t_1 s + 2)\, e^{-t_1 s}$	$t^2 u(t - t_1)$

Appendix A

Analyticity

Let ω be a single valued complex function of z,

$\omega = f(z) = u(x,y) + iv(x,y)$

where u and v are real functions.

The definition of the limit of f(z) as z approaches z_o and the theorems on limits of sums, products and quotients correspond to those in the theory of functions of a real variable. The neighborhoods involved are now two-dimensional; however; and the condition

$$\lim_{z \to z_o} f(z) = u_o + iv_o$$

is satisfied if and only if the two-dimensional limits of the real functions u(x,y) and v(x,y) as $x \to x_o$, $y \to y_o$ have the values u_o and v_o respectively. Also, f(z) is continuous when $z = z_o$ if and only if u(x,y) and v(x,y) are both continuous at (x_o, y_o).

The derivative of ω at a point z is

$$\frac{d\omega}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided this limit exists. (it must be independent of direction).

Suppose one chooses a path on which $\Delta y = 0$ so that $\Delta z = \Delta x$. Then, since $\Delta\omega = \Delta u + i\Delta v$,

$$\frac{d\omega}{dz} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} + i\,\frac{\Delta v}{\Delta x}\right) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

or, if $\Delta x = 0$, so that $\Delta z = i\,\Delta y$, then

$$\frac{d\omega}{dz} = \lim_{\Delta y \to 0}\left(\frac{\Delta u}{i\Delta y} + \frac{i}{i}\,\frac{\Delta v}{\Delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts of the above equations, since we insist that the derivative must be independent of direction, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are known as the "Cauchy-Riemann conditions".

Now, the definition of analyticity is that "a function $f(z)$ is said to be analytic at a point z_o if its derivative $f'(z)$ exists at every point of some neighborhood of z_o". And, it is necessary and sufficient that $f(z) = u + iv$ satisfy the Cauchy-Riemann conditions in order for the function to have a derivative at point z.

Appendix B

Cauchy's Integral Formula

Theorem I: If $f(z)$ is analytic at all points within and on a closed curve, c, then

$$\oint_c f(z)\,dz = 0$$

Proof:

$$\oint_c f(z)dz = \oint_c (u + iv)\,(dx + idy) = \oint_c (udx - vdy) + i\oint_c vdx + udy$$

Applying Green's lemma to each integral,

$$\oint_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

but, because of analyticity the integrands on the right vanish identically, giving

$$\oint_c f(z)\,dz = 0$$

Theorem II

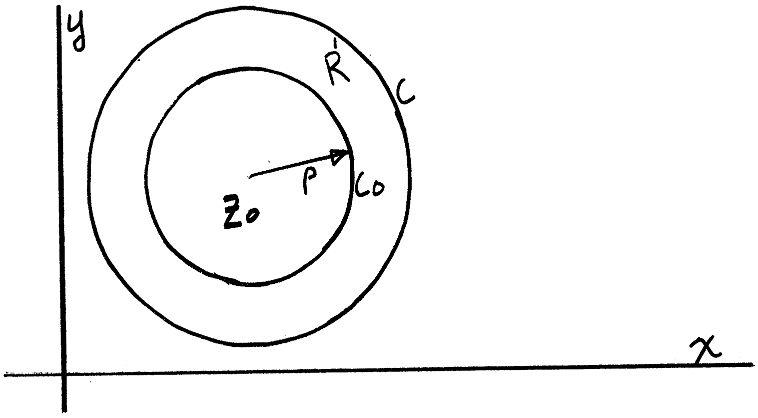
If $f(z)$ is analytic within and on the boundary c of a simply connected region R and if z_o is any point in the interior of R, then

$$f(z_o) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_o}\, dz$$

where the integration around c is in the positive sense.

Proof:

Let c_o be a circle with center at z_o whose radius ρ is sufficiently small that c_o lies entirely within R (see Figure below)



$f(z)$ is analytic everywhere within R, hence $\frac{f(z)}{z - z_o}$ is analytic everywhere within R except at $z = z_o$. By the "principle of deformation of contours", (see any complex variable book) we get

$$\oint_c \frac{f(z)}{z - z_o}\, dz = \oint_{c_o} \frac{f(z)}{z - z_o}\, dz = \oint_{c_o} \frac{f(z_o) + f(z) - f(z_o)}{z - z_o}\, dz$$

$$= f(z_o) \oint_{c_o} \frac{dz}{z - z_o} + \oint_{c_o} \frac{f(z) - f(z_o)}{z - z_o}\, dz$$

Consider the first integral,

$$\oint_{c_o} \frac{dz}{z - z_o}$$

and let $z - z_o = r\, e^{i\theta}$, $dz = r\, i\, e^{i\theta} d\theta$, getting

$$\int_0^{2\pi} \frac{r\, i\, e^{i\theta} d\theta}{r\, e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i$$

And, observe that

$$\left| \int_{c_0} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leqq \oint_{c_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz|$$

On c_0, $z - z_0 = \rho$

Also, $|f(z) - f(z_0)| < \epsilon$ provided $|z - z_0| \equiv \rho < \delta$

Choosing ρ to be less than δ, we write

$$\left| \oint_{c_0} \frac{f(z) - f(z_0)}{z - z_0} \right| dz < \oint_{c_0} \frac{\epsilon}{\rho} |dz| = \frac{\epsilon}{\rho} \oint_{c_0} |dz| = \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon$$

Since the integral on the left is independent of ϵ, yet cannot exceed $2\pi\epsilon$, which can be made arbitrarily small, it follows that the absolute value of the integral is zero.

$\therefore$ We have

$$\oint_{c_0} \frac{f(z)}{z - z_0} = f(z_0)\, 2\pi i + 0 \qquad \text{or,}$$

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} \, dz$$

which is "Cauchy's integral formula".

Appendix C

Calculation of Residues

I. Laurent series.

Theorem I:

If $f(z)$ is analytic throughout the closed region, R, bounded by two concentric circles, c_1 and c_2, then at any point in the annular ring bounded by the circles, $f(z)$ can be represented by the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

where a is the common center of the circles, and

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(\omega) \, d\omega}{(\omega - a)^{n+1}}$$

each integral being taken in the counter-clockwise sense around any curve c, lying within the annulus and encircling its inner boundary (for proof see any complex variable book). This series is called the Laurent series.

II. Residues

The coefficient, a_{-1}, of the term $(z - a)^{-1}$ in the Laurent expansion of a function, $f(z)$, is related to the integral of the function through the formula

$$a_{-1} = \frac{1}{2\pi i} \oint_c f(z)\, dz.$$

In particular, the coefficient of $(z - a)^{-1}$ in the expansion of $f(z)$ "around an isolated singular point" is called the "residue" of $f(z)$ at that point.

If we consider a simply closed curve c containing in its interior a number of isolated singularities of a function $f(z)$, then it can be shown that $\oint_c f(z)\, dz = 2\pi i \left[r_1 + r_2 + \text{-----} r_n \right]$ where r_n's are residues of $f(z)$ at the singular points within c.

III. Determination of Residues.

The determination of residues by the use of series expansions is often quite tedious. An alternative procedure for a simple or first order pole at $z = a$ can be obtained by writing

$$f(z) = \frac{a - 1}{z - a} + a_o + a_1(z - a) + \text{------}$$

and multiplying this by $(z - a)$, to get

$$(z - a)f(z) = a_1 + a_o(z - a) + a_1(z - a)^2 + \text{-----}$$

and letting $z \rightarrow a$, we get

$$a_{-1} = \lim_{z \to a} \left[(z - a)\, f(z) \right]$$

A general formula for the residue at a pole of order m is

$$(m - 1)!\, a_{-1} = \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z - a)^m\, f(z) \right]$$

For polynomials, the method of residues reduces to the Heaviside method for finding inverse Laplace transforms.

Appendix D

Regular and Singular Points

If $w = f(z)$ possesses a derivative at $z - z_0$ and at every point in some neighborhood of z_0, then $f(z)$ is said to be "analytic" at $z = z_0$, and z_0 is called a "regular point" of the function.

If a function is analytic at some point in every neighborhood of a point z_0, but not at z_0, then z_0 is called a "Singular point" of the function.

If a function is analytic at all points except z_0, in some neighborhood of z_0, then z_0 is an "isolated singular point".

About an isolated singular point z_0 a function always has a Laurent series representation =

$$f(z) = \frac{A-1}{z - z_0} + \frac{A-2}{(z - z_0)} + \text{-----} A_0 + A_1(z - z_0) + \text{-----} (0 < |z - z_0| < r_0)$$

where r_0 is the radius of the neighborhood in which $f(z)$ is analytic except at z_0. This series of negative powers of $(z - z_0)$ is called the "principle part" of $f(z)$ about the isolated singular point z_0. The point z_0 is an "essential singular point" of $f(z)$ if the principle part has an infinite number of non-vanishing terms. It is a "pole of order m" if $A_{-m} \neq 0$ and $A_{-n} = 0$ when $n > m$. It is called a "simple pole" when $m = 1$.

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FOURIER TRANSFORMS

I. Definitions

A. Basic Definitions

In addition to the Laplace transform there exists another commonly-used transform, or set of transforms, the Fourier transforms. At least five different Fourier transforms may be distinguished. Their definitions follow:

Finite Range Cosine transform

$$\mathcal{F}_c[n] = C_n[f] \equiv \int_0^{\pi} f(x) \cos nx \, dx \qquad (n = 0,1,2---)$$

Finite Range Sine transform

$$\mathcal{F}_s[n] = S_n[f] \equiv \int_0^{\pi} f(x) \sin nx \, dx \qquad (n = 1,2,3---)$$

Infinite Range Cosine transform

$$\mathcal{F}_c[r] = C_r[f] \equiv \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos rx \, dx \qquad (0 \leq r \leq \infty)$$

Infinite Range Sine transform

$$\mathcal{F}_s[r] = S_r[f] \equiv \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin rx \, dx \qquad (0 \leq r \leq \infty)$$

Infinite Range Exponential transform

$$\mathcal{F}_e[r] = \mathcal{E}_r[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{irx} \, dx \qquad (-\infty \leq r \leq \infty)$$

B. Range of Definition

In the infinite range transforms, the transform variable is continuous; in the case of the finite range transforms, the variable takes only positive integer values or zero. Considering the range of integration used in the definition of each transform, we see that the finite range transforms apply to functions defined on a finite interval, the infinite range sine and cosine transforms to functions defined on a semi-infinite interval, while the exponential transform applies to functions defined on the infinite interval.

C. Existence conditions.

As an existence condition for all these transforms it is customarily required that the function be absolutely integrable over the range, i.e.

$$\int_{\text{range}} |f(x)| \, dx \quad \text{exists}$$

Note that although for the derivations to follow, the more stringent conditions of continuity or sectional continuity are imposed upon the function, absolute integrability is all that is required in the general case.

II. Some Fundamental Properties

A. Transforms of Derivatives of Functions

Consider the finite range cosine transform of the derivative f' of the function f,

$$C_n\left[f'\right] = \int_0^{\pi} f'(x) \cos nx \, dx$$

Integrating by parts,

$$C_n\left[f\right] = f(x) \cos nx \Big|_0^{\pi} + n \int_0^{\pi} f(x) \sin nx \, dx$$

$$= f(\pi) \cos n\pi - f(0^+) + n\, S_n\left[f\right]$$

and since n is an integer,

$$C_n\left[f'\right] = \left[(-1)^n f(\pi) - f(0^+)\right] + nS_n\left[f\right]$$

Consider also,

$$S_n\left[f'\right] = \int_0^{\pi} f'(x) \sin nx \, dx$$

$$= f(x) \sin nx \Big|_0^{\pi} - n \int_0^{\pi} f(x) \cos nx \, dx$$

$$= -nC_n\left[f\right]$$

Now take for f, f = g'; we get by iteration

$$C_n\left[f'\right] = C_n\left[g''\right] = \left[(-1)^n g'(\pi) - g'(0^+)\right] + n\, S_n\left[g'\right]$$

$$= \left[(-1)^n g'(\pi) - g'(0^+)\right] - n^2\, C_n\left[g\right]$$

Similarly,

$$S_n\left[g''\right] = -n\, C_n\left[g'\right]$$
$$= -n\left[(-1)^n g(\pi) - g(0)\right] - n^2 S_n\left[g\right]$$
$$= n\left[g(0) - (-1)^n g(\pi)\right] - n^2 S_n\left[g\right]$$

Now consider the infinite range cosine transform

$$C_r\left[f'\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos rx\, dx$$

and again integrating by parts, and assuming $\lim\limits_{x\to\infty} f(x) = 0$, which is a consequence of our condition of absolute integrability, we get

$$C_r\left[f'\right] = \sqrt{\frac{2}{\pi}}\left[-f(0)\right] + r\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin rx\, dx$$
$$= -\sqrt{\frac{2}{\pi}}\, f(0) + r\, S_r\left[f\right]$$

and also

$$S_r\left[f'\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin rx\, dx$$
$$= -r\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos rx\, dx = -r\, C_r\left[f\right]$$

Iterating once, we find

$$C_r\left[f''\right] = -\sqrt{\frac{2}{\pi}}\, f'(0) + rS_r\left[f'\right]$$
$$= -\sqrt{\frac{2}{\pi}}\, f'(0) - r^2 C_r\left[f\right]$$

Similarly,

$$S_r\left[f''\right] = -r\, C_r\left[f'\right]$$
$$= -r\sqrt{\frac{2}{\pi}}\, f(0) - r^2 S_r\left[f\right]$$

Finally, consider

$$\mathcal{E}_r\left[f'\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{irx}\, dx$$

and assuming $\lim\limits_{x\to\infty} f(x) \to 0$

$$\mathcal{E}_r\left[f'\right] = -\frac{ir}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{irx}\, dx$$

Iterating,

$$\mathcal{E}_r\left[f''\right] = -r^2\mathcal{E}_r\left[f\right]$$

In each case we have assumed continuity for f' and f" in order to perform the indicated parts integrations. One may proceed with the iterations, obtaining relations involving transforms of higher derivatives. Further properties are derivable with similar ease, the procedure usually involving an integration by parts.

B. Relations among Infinite Range Transforms.

It is interesting to note some relations among the infinite range transforms. Recalling the identity

$$e^{irx} = \cos rx + i \sin rx$$

we find that

$$\begin{aligned}
\mathcal{E}_r[f] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \cos rx \, dx + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \sin rx \, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(x) \cos rx \, dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x) \cos rx \, dx \\
&\quad + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{0} f(x) \sin rx \, dx + \frac{i}{\sqrt{2\pi}}\int_{0}^{\infty} f(x) \sin rx \, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(-x) \cos rx \, dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x) \cos rx \, dx \\
&\quad + \frac{i}{\sqrt{2\pi}}\int_{0}^{\infty} f(x) \sin rx \, dx - \frac{i}{\sqrt{2\pi}}\int_{0}^{\infty} f(-x) \sin rx \, dx
\end{aligned}$$

or

$$\mathcal{E}_r[f] = \frac{1}{2}\left\{C_r\left[f(-x)\right] + C_r\left[f(x)\right] + i\, S_r\left[f(x)\right] - i\, S_r\left[f(-x)\right]\right\}$$

which is not very interesting except when f(x) is either even or odd on the infinite interval; if even, i.e. if $f(x) = f(-x)$, then the exponential transform reduces to the cosine transform; if odd, i.e., if $f(x) = -f(-x)$, then the exponential transform reduces to the sine transform, with a factor $\sqrt{-1}$.

C. Transforms of Functions of Two Variables.

The transforms may also be used with functions of two or more variables; for example, if f is a function of x and y, defined for $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, then,

$$S_m\left[f\right] = \int_0^{\pi} f(x,y)\sin mx\,dx$$

$$S_n\left[f\right] = \int_0^{\pi} f(x,y)\sin ny\,dy$$

$$S_{m,n}\left[f\right] = \int_0^{\pi} S_n\left[f\right]\sin my\,dy = \int_0^{\pi} S_m\left[f\right]\sin n\pi\,dx$$

$$= \int_0^{\pi}\int_0^{\pi} f(x,y)\sin mx\sin ny\,dx\,dy$$

Furthermore,

$$S_{m,n}\left[\frac{\partial^2 f}{\partial x^2}\right] = m\left\{S_n\left[f(0,y) - (-1)^m\; S_n\left[f(\pi,y)\right]\right]\right\} - m^2 S_{m,n}\left[f\right]$$

so that if

$$f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) \text{ then,}$$

$$S_{mn}\left[\frac{\partial^2 f}{\partial x^2}\right] = -m^2\; S_{mn}\left[f\right]$$

$$S_{mn}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = -(m^2 + n^2)\, S_{mn}\left[f\right]$$

Similar formulae may be derived for $C_{m,n}$ and extensions can be worked out in analogy to the single-variable properties. These transforms of more than one variable amount to transforms of transforms, obtained by taking the transform of the function with respect to a single variable, and subsequently taking the transform of this transformed function with respect to another variable. In fact, if the boundary conditions in the various dimensions are not all of the same type, more than one type of transformation may be used. (one fairly common combination is the Fourier plus the Laplace transformation).

D. Fourier Exponential Transforms of Functions of Three Variables.

Consider a function of three variables, $f(x_1, x_2, x_3)$, piecewise continuous and absolutely integrable over the infinite range with respect to each variable. We may apply the exponential transform with respect to each variable, defining the three-times transformed function.

$$\mathcal{E}_{\underline{k}}\left[f\right] = g(k_1,k_2,k_3) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x_1+k_2x_2+k_3x_3)} f(x_1,x_2,x_3)dx_1dx_2dx_3$$

Using vector notation, this may be written

$$\mathcal{E}_{\underline{k}}\left[f\right] = g(\underline{k}) = \frac{1}{(2\pi)^{3/2}}\int_{\underline{x}} e^{i\underline{k}\cdot\underline{x}}\ f(\underline{x})\ d^3x$$

where k has components k_1, k_2, k_3,

x has components x_1,x_2,x_3, and $d^3x = dx_1dx_2dx_3$, and the integration is to be taken over the full range, $-\infty$ to ∞ of each variable. The inverse transformation gives back f(x),

$$f(\underline{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\underline{k}} e^{-i\underline{k}\cdot\underline{x}}\ g(\underline{k})\ d^3k$$

Properties: say $\mathcal{E}_{\underline{k}}\left[f\right] = g(\underline{k})$; $\mathcal{E}_{\underline{k}}\left[\underline{F}\right] = \underline{G}\ (\underline{k})$; then

1. $\mathcal{E}_{\underline{k}}\left[\underline{\nabla} f\right] = -\ i\ \underline{k}\ g\ (\underline{k})$

2. $\mathcal{E}_{\underline{k}}\left[\underline{\nabla}\cdot F\right] = -i\ \underline{k}\ \circ\ \underline{G}\ (\underline{k})$

3. $\mathcal{E}_{\underline{k}}\left[\nabla\times\underline{F}\right] = -i\ \underline{k}\ \times\ G(\underline{k})$

4. $\mathcal{E}_{\underline{k}}\left[\nabla^2 f\right] = -k^2 f$

(From a glance at formulae 1 to 4, we see that under this transformation, the vector operator ∇ operating on a function transforms into the vector i$\underline{k}$ times the transformed function).

III. Summary of Fourier Transform Formulas

A. Finite Transforms (Functions defined on any finite range can be transformed into functions defined on the interval $0 \leq x \leq \pi$)

1. Definitions

a. $S_n[y] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} y(x) \sin nx \, dx \qquad n = 1,2,\ldots$

b. $C_n[y] = \frac{2}{\pi} \int_0^{\pi} y(x) \cos nx \, dx \qquad n = 0,1,\ldots$

2. Inversions ($0 \leq x \leq \pi$)

a. $y(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} S_n[y] \sin nx$

b. $y(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} C_n[y] \cos nx + \frac{1}{\pi} C_o[y]$

3. Transforms of Derivatives

a. $S_n[y'] = -n \, C_n[y] \qquad n = 1,2,\ldots$

b. $C_n[y'] = n \, S_n[y] - y(0) + (-1)^n y(\pi) \qquad$ $n = 0,1,2,\ldots$ (note that functions must be known on boundaries)

c. $S_n[y''] = -n^2 S_n[y] + n\left[y(0) + (-1)^n y(\pi)\right] \qquad$ $n = 1,2,\ldots$ (note that functions must be known on boundaries)

d. $C_n[y''] = -n^2 C_n[y] - y'(0) + (-1)^n y(\pi) \qquad$ $n = 0,1,2,\ldots$ (note that derivative must be known on the boundaries)

4. Transforms of Integrals

a. $S_n\left[\int_0^x y(\xi) d\xi\right] = \frac{1}{n} C_n[y] - \frac{(-1)^n}{n} C_o[y] \qquad n = 1,2,\ldots$

b. $C_n\left[\int_0^x y(\xi) \, d\xi\right] = \frac{-1}{n} S_n[y]$

$C_o\left[\int_0^x y(\xi) \, d\xi\right] = \pi \, C_o[y] - C_o[xy]$

5. Convolution Properties

a. Define convolution of $f(x)$, $g(x)$ $(-\pi \leq x \leq \pi)$

$$p^* q \equiv \int_{-\pi}^{\pi} p(x - \xi)\, q(\xi)\, d\xi = q^* p$$

b. Transforms of Convolutions

Define extension of $f(x)$, where $f(x)$ defined an range $0 \leq x \leq \pi$

Odd extension: $f_1(-x) = -f_1(x)$

$f_1(x + 2\pi) = f_1(x)$

Even extension: $f_2(-x) = f_2(x)$; $f_2(x + 2\pi) = f_2(x)$

1. $2\, S_n[f]\, S_n[g] = -C_n[f_1{}^* g_1]$
2. $2\, S_n[f]\, C_n[g] = S_n[f_1{}^* g_2]$
3. $2\, C_n[f]\, C_n[g] = C_n[f_1{}^* g_2]$

6. Derivatives of Transforms

a. $\dfrac{d}{dn} S_n[y] = C_n[xy]$

b. $\dfrac{d}{dn} C_n[y] = -S_n[xy]$

(Here the differentiated transforms must be in a form valid for n a continuous variable instead of only for integral n).

B. Transforms on Infinite Intervals.

It must be true that $\int_0^{\infty} y(x)\, dx$ or $\int_{-\infty}^{\infty} y(x)\, dx$ exists.

1. Definitions:

a. $S_r[y] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y(x) \sin rx\, dx \qquad r \geq 0$

b. $C_r[y] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y(x) \cos rx\, dx \qquad r \geq 0$

c. $\mathcal{E}_r[y] = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x)\, e^{irx}\, dx \qquad -\infty \leq r \leq \infty$

2. Inversions

a. $y(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty S_r[y] \sin rx \, dx \qquad x > 0$

$= \sqrt{\frac{2}{\pi}} \; S_x\left[S_r[y]\right]$

b. $y(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty C_r[y] \cos rx \, dx \qquad x > 0$

$= \sqrt{\frac{2}{\pi}} \; C_x\left[C_r[y]\right]$

c. $y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{E}_r[y] \, e^{-irx} \, dx = \mathcal{E}_{-x}\left[\mathcal{E}_r[y]\right] = \mathcal{E}_x\left[\mathcal{E}_{-r}[y]\right]$

(The Cauchy principal value of the integral is to be taken).

3. Transforms of derivatives

a. $S_r[y'] = -r \, C_r[y]$

b. $C_r[y'] = r \, S_r[y] - y(0)$

c. $\mathcal{E}_r[y'] = -ir \, \mathcal{E}_r[y]$

d. $S_r[y''] = -r^2 S_r[y] + ry(0)$

e. $C_r[y''] = -r^2 C_r[y] - y'(0)$

f. $\mathcal{E}_r[y''] = -r^2 \mathcal{E}_r[y]$

g. $\mathcal{E}_r \frac{d^n y}{dx^n} = (-ir)^n \mathcal{E}_r[y]$

4. Transforms of Integrals

a. $S_r\left[\int_0^x y(\xi) \, d\xi\right] = \frac{1}{r} C_r[y]$

b. $C_r\left[\int_0^x y(\xi) \, d\xi\right] = -\frac{1}{r} S_r[y]$

(In a and b, require $\int_0^x y(\xi) d\xi$ be sect. cont. and $\to 0$ as $x \to \infty$).

c. $\mathcal{E}_r\left[\int_c^x y(\xi) \, d\xi\right] = \frac{1}{r} \mathcal{E}_r[y]$, c is any lower limit

(in c. require $\int_{-\infty}^x y(\xi) \, d\xi$ to be sect. cont. and $\to 0$ as $x \to \infty$).

d. $S_r\left[\int_0^x \int_0^\lambda y(\xi)\, d\xi\, d\lambda\right] = \frac{-1}{r^2} S_r[y]$

e. $C_r\left[\int_0^x \int_0^\lambda y(\xi)\, d\xi\, d\lambda\right] = \frac{-1}{r^2} C_r[y]$

f. $\mathcal{E}_r\left[\int_0^x \int_0^\lambda y(\xi)\, d\xi\, d\lambda\right] = \frac{-1}{r^2} \mathcal{E}_r[y]$

5. Some Relations Between Transforms for real $y(x)$,

a. $C_r[y] + i\, S_r[y] = \mathcal{E}_r[y]$

or

$$\begin{cases} C_r[y] = \mathcal{R}e\,(\mathcal{E}_r[y]), \\ S_r[y] = \mathcal{I}m\,(\mathcal{E}_r[y]) \end{cases}$$

b. For $y(x) = y(-x)$, $y(x) = \mathcal{O}(e^{-\epsilon t})$ $\epsilon > 0$

$\mathcal{E}_r[y] = 2\, L[y]$, where Laplace transform variable is taken as ir.

6. Convolution Properties

a. $2\, S_r[f]\, S_r[g] = C_r\left[\int_0^\infty g(\xi)\, (f(x+\xi) - f_1(x-\xi))\, d\xi\right]$

b. $2\, S_r[f]\, C_r[g] = S_r\left[\int_0^\infty g(\xi)\, (f(x+\xi) + f_1(x-\xi))\, d\xi\right]$

$= C_r\left[\int_0^\infty f(\xi)\, (g_2(x-\xi) - g(x+\xi))\, d\xi\right]$

c. $2\, C_r[f]\, C_r[g] = C_r\left[\int_0^\infty g(\xi)\, (f_2(x-\xi) + f(x+\xi))\, d\xi\right]$

where extensions defined

$y_1(-x) = -y(x)$; $y_2(-x) = y(x)$, all x.

d. $\mathcal{E}_r[f]\, \mathcal{E}_r[g] = \mathcal{E}_r\left[\int_{-\infty}^\infty f(\xi)\, g(x-\xi)\, d\xi\right]$

7. Derivatives of Transforms

a. $\frac{d}{dr} S_r[y] = C_r[xy]$

b. $\frac{d}{dr} C_r[y] = -S_r[xy]$

c. $\frac{d}{dr} \mathcal{E}_r[y] = i\, \mathcal{E}_r[xy]$

IV. Types of Problems to which Fourier Transforms are Applicable.

A. General Discussion.

It is to our great advantage to have some inkling as to just which transform to use where. We have noted that finite-range transforms are useful on functions defined over a finite range, $\mathcal{F}_c[r]$ and $\mathcal{F}_s[r]$ are useful on functions defined over semi-infinite intervals, and $\mathcal{F}_e[r]$ on functions defined over the infinite range. Still more can be said.

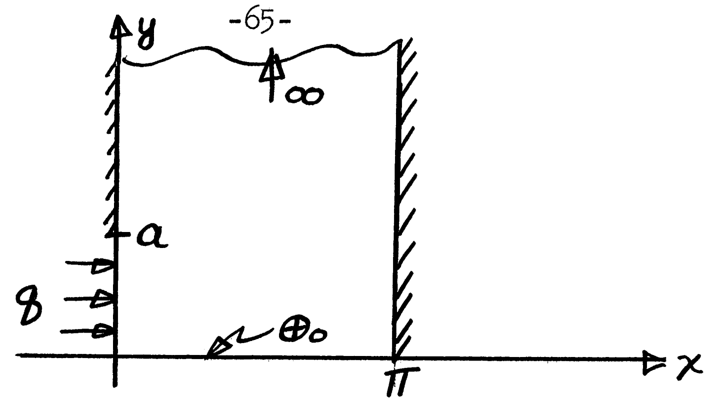
First, it goes almost without saying that if it can be avoided, it is undesirable to introduce an unknown quantity into an equation. Now, if an equation in f, which is defined $0 \leq x \leq \pi$ contains a differential operator which one wishes to reduce, say $\frac{d^2f}{dx^2}$ and $f(\pi)$ and $f(0)$ are known, while $f'(\pi)$ and $f'(0)$ are not, then clearly S_n is used, for in so doing we introduce $f(\pi)$ and $f(0)$ and need not know the value of f' at any point. We would not use C_n for $f'(0)$ and $f'(\pi)$ are unknown and would enter the transformed equations as unknowns, which would not be solved for until later in the work. On the other hand, if $f'(0)$ and $f'(\pi)$ are known, one uses C_n for the same reason. The situation is similar with respect to the infinite range transforms; use $\mathcal{F}_s[r]$ reduce $\frac{d^2f}{dx^2}$ when $f(0)$ is known, $\mathcal{F}_c[r]$ when $f'(0)$ is known. No such question arises with respect to $\mathcal{F}_e[r]$.

We have noted at the start that the functions to which the Fourier transforms are applicable are usually required to be absolutely integrable. This kind of knowledge of a function is usually evident from the physical meaning of the function, before the function itself is known. The Laplace transform, on the other hand merely requires that the function be of exponential order, i.e.,

$$|f(x)| < Me^{\alpha x} \qquad M \text{ \& } \alpha \text{ any real numbers}$$

B. An Example of the Use of Fourier Transforms

Consider the following steady-state heat conduction problem in a medium with no internal heat generation.



Face x=0 has a heat flux q, $0 < y < a$, and is insulated, $y > a$, and face $x=\pi$ is insulated for all y. Face y=0 is held at temperature Φ_o, all x, $0 < x < \pi$. The slab extends $0 \leq y \leq \infty$. The equation to be solved is Laplace's equation with boundary conditions.

$$\nabla^2 \Phi = 0$$

$$-k\frac{\partial \Phi}{\partial x}\bigg|_{x=0} \begin{cases} q & 0 < y < a \\ 0 & y > a \end{cases}$$

$$\frac{\partial \Phi}{\partial x}\bigg|_{x=\pi} = 0$$

$$\Phi(x,0) = \Phi_o$$

We propose to do the problem by the method of Fourier transforms, but intuitively we know $\lim\limits_{y\to\infty} \Phi = \bar{\Phi} \neq 0$ and, therefore, the transform of Φ does not exist. However, the function $\Phi - \bar{\Phi} \equiv \Theta$ is such that $\lim\limits_{y\to\infty} \Theta = \lim\limits_{y\to\infty} \Phi - \bar{\Phi} = \bar{\Phi} - \bar{\Phi} = 0$ and the transform may (in fact, does) exist. Let us, therefore, substitute in the above problem

$$\Phi = \Theta + \bar{\Phi}$$

to obtain

$$\nabla^2 \Theta = 0$$

$$-k\frac{\partial \Theta}{\partial x}\bigg|_{x=0} = \begin{cases} q & 0 < y < a \\ 0 & y > a \end{cases}$$

$$\frac{\partial \Theta}{\partial x}\bigg|_{x=\pi} = 0$$

$$\Theta(x,0) + \bar{\Phi} = \Phi_o \; ; \; \Theta(x,0) = \Phi_o - \bar{\Phi} \equiv \Theta_o$$

The structure of the problem is not essentially changed, except that now Θ_0 is not known since $\bar{\Phi}$ is not known.

We must reduce the operators $\frac{\partial^2\Theta}{\partial x^2}$ and $\frac{\partial^2\Theta}{\partial y^2}$. In x, we known $\left.\frac{\partial\Theta}{\partial x}\right|_{x=0}$ and $\left.\frac{\partial\Theta}{\partial x}\right|_{x=\pi}$. Thus, a finite range Fourier cosine transform is indicated (see section IIA) In y, we known $\Theta(x,0) = \Theta_0$ and that $\lim_{y\to\infty}\Theta = 0$. Therefore, an infinite range Fourier sine transform is indicated. Denote x-transformed functions by superscript f_n, y-transformed functions by superscript F. Recall

$$\left(\frac{\partial^2\Theta}{\partial x^2}\right)^f = -n^2\Theta^f + (-1)^n \left.\frac{\partial\Theta}{\partial x}\right|_{x=\pi} - \left.\frac{\partial\Theta}{\partial x}\right|_{x=0}$$

and

$$\left(\frac{\partial^2\Theta}{\partial y^2}\right)^F = -r^2\Theta^F + r\,\Theta(x,0)$$

It is irrelevant in which order the transformations are applied or inverted, although one order may prove nicer than another. Let us transform first with respect to x.

$$-n^2\Theta^{fn} + (-1)^n \left.\frac{\partial\Theta}{\partial x}\right|_{x=\pi} - \left.\frac{\partial\Theta}{\partial x}\right|_{x=0} + \frac{d^2\Theta^{fn}}{dy^2} = 0$$

$$\left.\frac{\partial\Theta}{\partial x}\right|_{x=0} = \begin{cases} -q/k & 0 < y < a \\ 0 & y > a \end{cases}$$

$$\left.\frac{\partial\Theta}{\partial x}\right|_{x=\pi} = 0$$

$$\Theta^{fn}(0) = \begin{cases} \pi & n=0 \\ 0 & n=1,2,\ldots \end{cases}$$

then with respect to y: (Churchill pag. 300, formula 3)

$$-n^2\Theta^{fn\,F} + (-1) \left.\frac{\partial\Theta}{\partial x}\right|^F_{x=\pi} - \left.\frac{\partial\Theta}{\partial x}\right|^F_{x=0} - r^2\Theta^{fnF} + r\,\Theta^{fn}(0) = 0$$

$$\left.\frac{\partial\Theta}{\partial x}\right|^F_{x=0} = -q/k\ \frac{(1-\cos ar)}{r}$$

(See Erdélye, p. 63, formula 1)

$$\left.\frac{\partial \Theta}{\partial x}\right|_{x=\pi}^{F} = 0$$

$$\Theta^{fn}(0) = 0 \qquad n = 1,2,3,\ldots$$

Making substitutions, this yields the single algebraic equation.

$$(-n^2 - r^2)\Theta^{fnF} + \frac{q}{k}\frac{(1 - \cos ar)}{r} = 0; \quad (n=1,2,\ldots)$$

$$-r^2\Theta^{foF} + \frac{q}{k}\frac{(1 - \cos ar)}{r} + \pi\Theta_0 r = 0; \quad (n = 0)$$

Solve for Θ^{fnF}:

$$\Theta^{foF} = \frac{q}{k}\frac{(1 - \cos ar)}{r^3} + \frac{\pi\Theta_0}{r}; \quad (n=0)$$

$$\Theta^{fnF} = \frac{q}{k}\frac{(1 - \cos ar)}{r(r^2 + n^2)}; \quad (n=1,2,\ldots)$$

We propose to invert first with respect to r, but we would run into difficulties for n=0. Let us, therefore, integrate the x-transformed equations directly for n=0 to get Θ^{fo}.

We have

$$\frac{d^2\Theta^{fo}}{dy^2} + \frac{q}{k} = 0 \qquad 0 < y < a$$

$$\frac{d^2\Theta^{fo}}{dy^2} = 0 \qquad y > a$$

$$\Theta^{fo}(y=0) = \pi\Theta_0;$$

$$\lim_{y\to\infty}\Theta^{fo} = 0$$

integrating,

$$\Theta^{fo} = -\frac{q}{2k}y^2 + C_1 y + C_2 \qquad 0 < y < a$$

$$\Theta^{fo} = C_3 y + C_4 \qquad y > a$$

Now $\lim_{y\to\infty}\Theta^{f} = 0$, so $C_3 = C_4 = 0$.

Also

$$\Theta^{fo}(0) = \pi\Theta_0 = C_2$$

It is necessary to cook up another condition to get C_1. In a problem of this type, we must require $\frac{\partial\Theta}{\partial y}$ and Θ to be continuous, therefore $\frac{\partial\Theta^{fo}}{\partial y}$

and Θ^{fo} are continuous. Apply these conditions at y=a.

$$\left.\frac{d\Theta^{fo}}{dy}\right|_{a-} = \left.\frac{d\Theta^{fo}}{dy}\right|_{a+}$$

$$-\frac{qa}{k} + C_1 = 0; \quad C_1 = \frac{qa}{k}$$

$$\Theta^{fo}(a-) = \Theta^{fo}(a+)$$

$$\frac{-qa^2}{2k} + \frac{qa^2}{k} + \pi\Theta_o = 0.$$

Somewhat surprisingly, applying this last condition yields

$$\Theta_o = \frac{-qa^2}{2\pi k}$$

Thus

$$\Theta^{fo} = \begin{cases} -\dfrac{qa}{2k}(y-a)^2 & y \leq a \\ 0 & y \geq a \end{cases}$$

We have Θ^{fo}. Let us invert Θ^{fnF} to get Θ^{fn}

$$\Theta^{fnF} = \frac{q}{k}\,\frac{(1 - \cos ar)}{r(n^2 + r^2)} \qquad (n=1,2,\ldots).$$

The inverse of $\dfrac{1}{r(n^2 + r^2)}$ is $\dfrac{1}{n^2}(1 - e^{-ny})$ (See Erdelye, p.65, formula 20).

Also, a property of the Fourier sine and cosine transforms is

$$F^{-1}\left[g(r)\cos ar\right] = \frac{1}{2}\left[G(y+a) + G(y-a)\right]$$

and it is also true that for the sine transform, if $F^{-1}\left[g^f\right] = G(y)$, then

$G(-y) = -G(y)$. Therefore,

$$F^{-1}\left[\frac{1}{r(r^2 + n^2)}(1 - \cos ar)\right] =$$

$$= \begin{cases} \dfrac{1}{n^2}(1 - e^{-ny}) - \dfrac{1}{2n^2}\left[1 - e^{-n(y+a)} + 1 - e^{-n(y-a)}\right] \\ \dfrac{1}{n^2}(1 - e^{-ny}) - \dfrac{1}{2n^2}\left[1 - e^{-n(y+a)} - 1 + e^{-n(a-y)}\right] \end{cases}$$

$$= \begin{cases} \dfrac{1}{n^2}\left[1 - e^{-ny} - 1 + e^{-ny}\,\dfrac{(e^{na} + e^{-na})}{2}\right] & y > a \\ \dfrac{1}{n^2}\left[1 - e^{-ny} + \dfrac{e^{-n(y+a)} - e^{-n(a-y)}}{2}\right] & y < a \end{cases}$$

$$= \begin{cases} \dfrac{e^{-ny}}{n^2}\,(\cosh na - 1) & y > a \\ \dfrac{1}{n^2}\,(1 - e^{-ny} - e^{-na}\sinh ny) & y < a \end{cases}$$

Now, lacking a known inversion to invert with respect to n, we use the series form

$$\Theta = \frac{1}{\pi}\,\Theta^{fo} + \frac{2}{\pi}\sum_{n=1}^{\infty}\Theta^{fn}\cos nx$$

$$= \begin{cases} \dfrac{-qa}{2\pi k}(y-a)^2 + \dfrac{2q}{\pi k}\displaystyle\sum_{n=1}^{\infty}\dfrac{(1-e^{-ny} - e^{-na}\sinh ny)}{n^2}\cos nx & y < a \\ \dfrac{2q}{\pi k}\displaystyle\sum_{n=1}^{\infty}\dfrac{e^{-ny}}{n^2}(\cosh na - 1)\cos nx & y > a \end{cases}$$

and recall that

$$\Phi = \Theta + \bar{\Phi}$$

$$\Phi_o = \Theta_o + \bar{\Phi} = -\frac{qa^2}{2\pi k} + \bar{\Phi}$$

$$\bar{\Phi} = \Phi_o + \frac{qa^2}{2\pi k}$$

$$\Phi = \Theta + \Theta_o + \frac{qa^2}{2\pi k}$$

V. Inversion of Fourier Transforms.

A. Inversion of Finite Range Transforms

Inversions of the finite range transforms are easily seen to be a consequence of the completeness and orthogonality of the cosines in the case of the cosine transform and of the sine in the case of the sine transform on the interval of integration. Indeed, one sees that the integrals defining C_n

and S_n are just the Fourier coefficients for expansions of f in a cosine series or a sine series. Thus their inversions are given by

$$f(x) = C_n^{-1}\left[C_n\right] = \frac{1}{\pi} C_o\left[f\right] + \frac{2}{\pi}\sum_{n=1}^{\infty} C_n\left[f\right]\cos nx \qquad (0 \leq x \leq \pi)$$

$$f(x) = S_n^{-1}\left[S_n\right] = \frac{2}{\pi}\sum_{n=1}^{\infty} S_n\left[f\right]\sin nx \qquad (0 \leq x \leq \pi)$$

Two facts, though obvious, should be noted with regard to these transforms. If the function is defined over some range other than $0 \leq x \leq \pi$, say $0 \leq x \leq L$, there arises no difficulty since one can define a new variable, say $\xi = \frac{\pi x}{L}$, such that when $x = L$, $\xi = \pi$, and $f(x) = g(\xi) = f(\frac{L}{\pi}\xi)$ and proceed. If the function is extended out of the range $0 \leq x \leq \pi$, the inversion of the cosine transform is the even extension, i.e.,

$$C_n^{-1}\left[C_n\ (x)\right] = C_n^{-1}\left[C_n\ (-x)\right]$$

while the inversion of the sine transform is the odd extension, i.e.

$$S_n^{-1}\left[S_n\ (x)\right] = -\ S_n^{-1}\left[S_n\ (-x)\right]$$

B. Inversion of Infinite Range Transforms.

Inversions of the infinite range transforms follow from the Fourier integral theorem in various forms. The inversion of the cosine transform, for example, arises from the formula

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos rx\int_0^{\infty} f(y)\cos ry\ dy\ dr$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos rx\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(y)\cos ry\ dy\ dr$$

The interior integral is just what we above defined as $C_r\left[f\right]$, thus

$$f(x) = C_r^{-1}\left[C_r\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} C_r\left[f\right]\cos rx\ dr$$

The other inversions follow immediately in the same way from other forms of the Fourier integral theorem. It is to our great advantage to note that, with the normalization factor $\sqrt{\frac{2}{\pi}}$ or $\frac{1}{\sqrt{2\pi}}$ inserted as above, the inversion integral is just the transform of the transform, i.e.,

$$f(x) = C_r^{-1}\left[C_r\right] = C_x\left[C_r\left[f\right]\right]$$

Similar formulae for the sine and exponential transforms are

$$f(x) = S_r^{-1}\left[S_r\right] = S_x\left[S_r\left[f\right]\right]$$

$$f(x) = E_r^{-1}\left[E_r\right] = E_{-x}\left[E_r\left[f\right]\right]$$

Knowing this fact doubles the utility of a table of transforms since it can be used backwards as well as forwards. That is, given a transform one wishes to invert, one may first look for it among tabulated transformed functions; not finding an inversion there, one may equally well look for his transform among the tabulated functions, if it is found there, the inversion of the given transform is the transform of the tabulated function.

There are tables of both the finite range transforms and the infinite range transforms, useful for the purpose of inverting these transforms. However, this is just one way of obtaining an inversion (the easiest, of course). In the case of the finite range transforms, where the inversion is a Fourier series, and one does not know how to sum it, that is, get the inversion in closed form, then the truncated series in a useful approximation to the inverse.

In the case of the infinite range transforms, the inversion integral is subject to evaluation by the methods of complex integration and residue theory.

C. Inversion of Fourier Exponential Transforms.

We have seen that if the transform of a function F(x) is

$$f(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\, e^{irx}\, dx$$

then (under proper conditions on F) the function can be recovered from its transform through the inversion formula,

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(r)\, e^{-irx}\, dr.$$

Note that in the inversion formula, r is a real variable. Let us change the variable r to a new (complex) variable, ir = s, and say $\phi(s) = f(r)$.

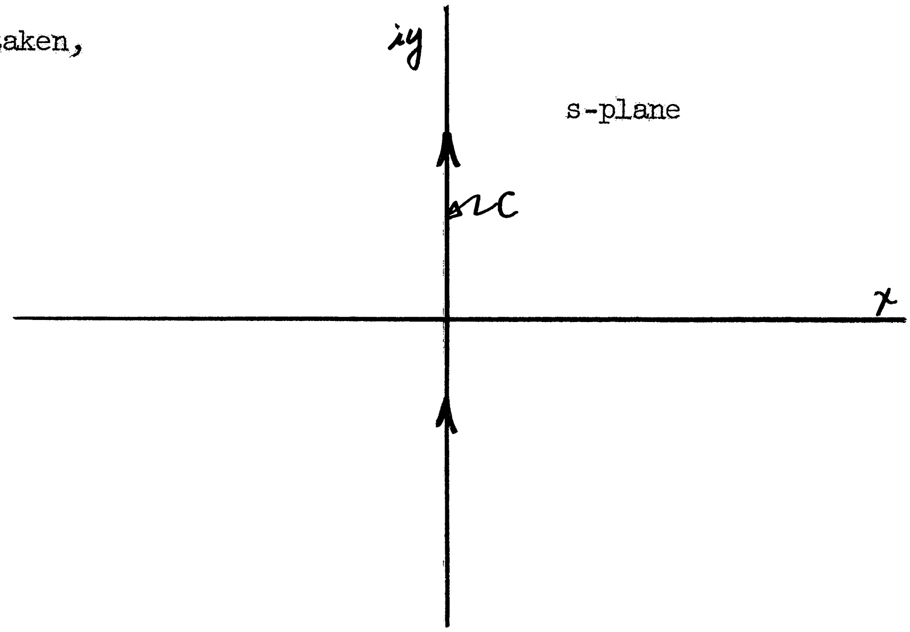
Then

$$\phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\, e^{sx}\, dx$$

and the inversion is (sine   idr = ds)

$$F(x) = -\frac{i}{\sqrt{2\pi}} \int_c \phi(s)\, e^{-sx}\, ds$$

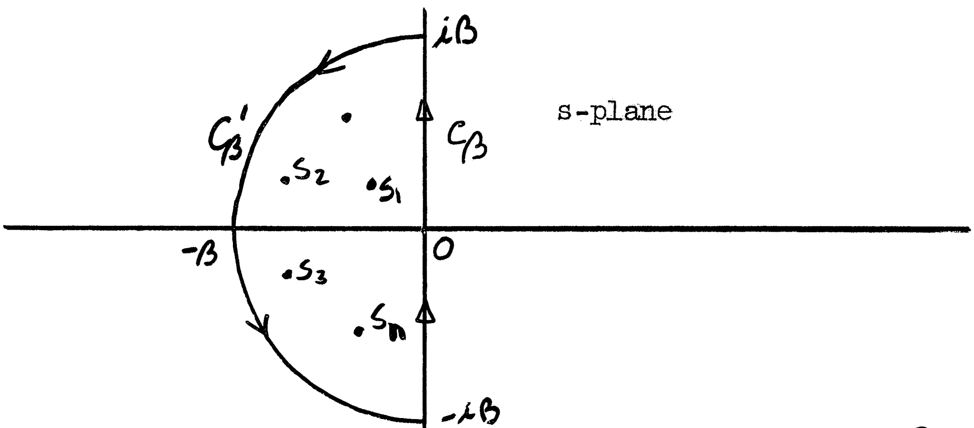
where c is the curve in the s-plane below, and the Cauchy Principal value is to be taken,



that is

$$F(x) = -\frac{i}{\sqrt{2\pi}} \lim_{\beta\to\infty} \int_{-i\beta}^{i\beta} \phi(s)\, e^{-sx}\, ds$$

Suppose $\phi(s)$ is analytic in the left half plane $\mathcal{R}e(s) < 0$, except at a finite number of isolated singular points s_n. Let us close the path c_β, $-\beta \leq \mathcal{I}m\, s \leq \beta$, with a semicircle in the left half-plane, choosing β



so large as to include all finite singular points in the plane $\mathcal{R}e(s) < 0$. By Cauchy's residue theorem, then we have

$$\int_{c\beta} e^{-sx}\phi(s)\,ds + \int_{c'\beta} e^{-sx}\phi(s)\,ds = 2\pi i \sum_{j=1}^{k} \rho_j$$

where ρ_j denotes the residue of $e^{-sx}\phi(s)$ at the singular point s_j, and we have assumed that there are k such singular points.

Since we have hypothesized that β be so large that c'_β include all finite singular points in the left half plane, in the limit as $\beta \to \infty$, the right side remains constant, and we have

$$\lim_{\beta\to\infty} \int_{c\beta} e^{-sx}\phi(s)\,ds + \lim_{\beta\to\infty} \int_{c\beta} e^{-sx}\phi(s)\,ds = 2\pi i \sum_{j=1}^{k} \rho_j$$

or

$$\lim_{\beta\to\infty} \int_{-i\beta}^{i\beta} e^{-sx}\phi(s)\,ds = 2\pi i \sum_{j=1}^{k} \rho_j - \lim_{\beta\to\infty} \int_{c'\beta} e^{-sx}\phi(s)\,ds$$

Thus

$$F(x) = -\frac{i}{\sqrt{2\pi}} \lim_{\beta\to\infty} \int_{-i\beta}^{i\beta} e^{sx}\phi(s)\,ds =$$

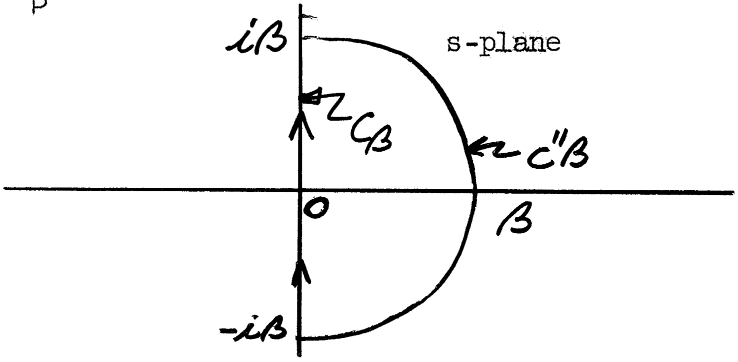
$$\sqrt{2\pi} \sum_{j=1}^{k} \rho_j + \frac{i}{\sqrt{2\pi}} \lim_{\beta\to\infty} \int_{c'\beta} e^{-sx}\phi(s)\,ds$$

Many times, the limit on the right hand side is zero, or is easy to evaluate, so that the above formula is a very useful device for inverting the transform.

Reasoning in similar fashion, but completing the path with a semi-circle in the right half plane $\mathcal{Re}(s) > 0$, we obtain a similar formula.

$$F(x) = -\sqrt{2\pi} \sum_{j=1}^{k'} \rho'_j - \frac{i}{\sqrt{2\pi}} \lim_{\beta\to\infty} \int_{c''\beta} e^{-sx}\phi(s)\,ds$$

where the curve c''_β is shown below,



One may find that for $x < 0$, the limit

$$\lim_{\beta\to\infty} \int_{c'\beta} e^{-sx} \phi(s)\ ds = 0$$

so that the first formula becomes

$$F(x) = \sqrt{2\pi} \sum_{j=1}^{k} \rho_j$$

(recall that ρ_j are residues at singular points in the left half-plane $\mathrm{Re}(s) < 0$). Again, one may find that for $x > 0$, the limit

$$\lim_{\beta\to\infty} \int_{c''\beta} e^{-sx} \phi(s)\ ds = 0$$

so that the second formula becomes

$$F(x) = -\sqrt{2\pi} \sum_{j=1}^{k'} \rho_{j'}.$$

At $x = 0$, $F(0) = \frac{1}{2}\left[F(0^+) + F(0^-)\right]$

where $F(0^+) = \lim_{\epsilon\to 0} F(\epsilon)$,

and $F(0^-) = \lim_{\epsilon\to 0} F(-\epsilon)$.

Appendix

VI. Transform Tables

$$S_n(y) = \int_0^a y(x) \sin \frac{n\pi x}{a}\ dx \qquad n \text{ a positive integer}$$

$$C_n(y) = \int_0^a y(x) \cos \frac{n\pi x}{a}\ dx \qquad n \text{ a non-negative integer}$$

$y(x)$	$S_n(y)$
1	$\frac{a}{n\pi}\left[1 - (-1)^n\right]$
x	$\frac{a^2}{n\pi}(-1)^{n+1}$
x^2	$\frac{a^3}{n\pi}(-1)^{n+1} - 2\left(\frac{a}{n\pi}\right)^3\left[1 - (-1)^n\right]$

$y(x)$	$S_n(y)$	
e^{cx}	$\dfrac{n\pi a}{n^2\pi^2 + c^2a^2}\left[1 - (-1)^n e^{ca}\right]$	
$\sin \omega x$	$\dfrac{a}{2}$	$\left(n = \dfrac{\omega a}{\pi}\right)$
	$\dfrac{n\pi a}{n^2\pi^2 - \omega^2 a^2}\,(-1)^{n+1} \sin \omega a$	$\left(n \neq \dfrac{\omega a}{\pi}\right)$
$\cos \quad x$	0	$\left(n = \dfrac{\omega a}{\pi}\right)$
	$\dfrac{n\pi a}{n^2\pi^2 - \omega^2 a^2}\left[1 - (-1)^n \cos \omega a\right]$	$\left(n \neq \dfrac{\omega a}{\pi}\right)$
$\sinh cx$	$\dfrac{n\pi a}{n^2\pi^2 + c^2a^2}\,(-1)^{n+1} \sinh c\ a$	
$\cosh cx$	$\dfrac{n\pi a}{n^2\pi^2 + c^2a^2}\left[1 - (-1)^n \cosh ca\right]$	
$a - x$	$\dfrac{a^2}{n\pi}$	
$x(a - x)$	$2\left(\dfrac{a}{n\pi}\right)^3\left[1 - (-1)^n\right]$	
$\dfrac{\sin \omega(a - x)}{\sin \omega a}$	$\dfrac{n\pi a}{n^2\pi^2 - \omega^2 a^2}$	$\left(n \neq \dfrac{\omega a}{\pi}\right)$
$\dfrac{\sinh c(a - x)}{\sinh ca}$	$\dfrac{n\pi a}{n^2\pi^2 + c^2a^2}$	

$y(x)$	$C_n(x)$	
1	a	$(n=0)$
	0	$(n=1,2,\ldots)$
x	$\dfrac{1}{2}a^2$	$(n=0)$
	$\left(\dfrac{a}{n\pi}\right)^2\left[(-1)^n - 1\right]$	$(n=1,2,\ldots)$

$y(x)$	$C_n(x)$	
x^2	$\frac{1}{3}a^3$	$(n=0)$
	$\frac{2a^3}{n^2\pi^2}(-1)^n$	$(n=1,2,\ldots)$
e^{cx}	$\frac{a^2c}{n^2\pi^2+c^2a^2}\left[(-1)^n e^{ca}-1\right]$	
$\sin\omega x$	0	$\left(n=\frac{\omega a}{\pi}\right)$
	$\frac{a^2\omega}{n^2\pi^2-\omega^2a^2}\left[(-1)^n\cos\omega a-1\right]$	$\left(n\neq\frac{\omega a}{\pi}\right)$
$\cos\omega x$	$\frac{1}{2}a$	$\left(n=\frac{\omega a}{\pi}\right)$
	$\frac{a^2w}{n^2\pi^2-\omega^2a^2}(-1)^{n+1}\sin\omega a$	$\left(n\neq\frac{\omega a}{\pi}\right)$
$\sinh cx$	$\frac{a^2c}{n^2\pi^2+c^2a^2}\left[(-1)^n\cosh ca-1\right]$	
$\cosh cx$	$\frac{a^2c}{n^2\pi^2+c^2a^2}(-1)^n\sinh ca$	
$(x-a)^2$	$\frac{4}{3}a^3$	$(n=0)$
	$2\frac{a^3}{n^2\pi^2}$	$(n=1,2,\ldots)$
$\frac{\cos\omega(a-x)}{\sin\omega a}$	$\frac{a^2\omega}{n^2\pi^2+\omega^2a^2}$	$\left(n\neq\frac{\omega a}{\pi}\right)$
$\frac{\cosh c(a-x)}{\sinh ca}$	$\frac{a^2c}{n^2\pi^2+c^2a^2}$	

For a few additional transforms of this type, see Churchill or Sneddon (Reference 2 and 3)

For transforms of the form $\int_{-\infty}^{\infty} y(x)\,e^{inx}\,dx$, $\int_{0}^{\infty} y(x)\sin nx\,dx$, and $\int_{0}^{\infty} y(x)\cos nx\,dx$, see Erdelyi (reference 1).

References

1. Sneddon, Ian N. "Fourier Transforms", McGraw Hill, 1951.

2. Churchill, Ruel V., "Operational Mathematics", McGraw Hill, 1958.

3. Erdelyi, Volume I, "Bateman Mathematical Tables".

MISCELLANEOUS IDENTITIES, DEFINITIONS, FUNCTIONS AND NOTATIONS

I. Leibnitz's Rule:

If $f(x) = \int_{a(x)}^{b(x)} g(x,y)\, dy$

Then

$$\frac{d}{dx} f(x) = g\left[x,b(x)\right] b'(x) - g\left[x,a(x)\right] a'(x) + \int_{a(x)}^{b(x)} \frac{2\ g\ (x,y)}{2x}\, dy$$

II. General solution of first order linear differential equation.

$$\frac{dy}{dx} + a(x)\, y = f(x)$$

with boundary condition

$$y(x_o) = y_o$$

The procedure is to find an integrating factor. Define h such that

$$\frac{dh}{dx} = a(x). \quad \text{Thus}$$

$$h = \int_c^x a(x')\, dx'$$

The integrating factor will be e^h, since

$$\frac{d(e^h y)}{dx} = e^h \frac{dy}{dx} + e^h y \frac{dh}{dx} = e^h \frac{dy}{dx} + e^h a(x)y = e^h f(x)$$

Then

$$\int_{y_o e^{h_o}}^{ye^h} d(e^h y) = \int_{x_o}^{x} e^{h'} f(x')\, dx'$$

where $h_o = \int_c^{x_o} a(x')\, dx$

$$h' = \int_c^{x_o} a(x'')\, dx''$$

Thus,

$$ye^{h} - y_o e^{ho} = \int_{x_o}^{x} e^{h'} f(x') \, dx'$$

Hence

$$y = y_o \, e^{ho-h} + \int_{x_o}^{x} e^{h'-h} f(x') \, dx'$$

Recalling that

$$h = \int_{c}^{x} a(x') \, dx'$$

$$h' - h = \int_{c}^{x'} a(x'') \, dx'' - \int_{c}^{x} a(x'') \, dx' = -\int_{x'}^{x} a(x'') \, dx''$$

Finally,

$$y = y_o \, e^{ho-h} + \int_{x_o}^{x} f(x') \, e^{-\int_{x'}^{x} a(x'')dx''} \, dx'$$

(Note that the constant c appearing as lower limit in the integral of the integrating factor is <u>not</u> a boundary condition: it disappears in the final solution).

III. Identities in Vector Analysis

Below, underscored quantities are vectors, and ∇ is the vector differential operator $\nabla = \hat{\underline{i}} \frac{\partial}{\partial x} + \hat{\underline{j}} \frac{\partial}{\partial y} + \hat{\underline{k}} \frac{\partial}{\partial z}$, and $\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$ are unit vectors in x, y, z directions respectively.

1) $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

2) $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot b)$

3) $$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = \underline{a} \cdot \underline{b} \times (\underline{c} \times \underline{d})$$
$$= \underline{a} \cdot (\underline{b} \cdot \underline{dc} - \underline{b} \cdot \underline{cd})$$
$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot d) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$$

4) $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d}) = (\underline{a} \times \underline{b} \cdot \underline{d})\,\underline{c} - (\underline{a} \times \underline{b} \cdot \underline{c})\,\underline{d}$

5) $\underline{\nabla}(\phi + \psi) = \underline{\nabla}\phi + \underline{\nabla}\psi$

6) $\underline{\nabla}(\phi\psi) = \phi\underline{\nabla}\psi + \psi\underline{\nabla}\phi$

7) $\underline{\nabla}(\underline{a} \cdot \underline{b}) = (\underline{a} \cdot \underline{\nabla})\,\underline{b} + (\underline{b} \cdot \underline{\nabla})\,\underline{a} + \underline{a} \times (\underline{\nabla} \times b) + \underline{b} \times (\underline{\nabla} \times \underline{a})$

8) $\underline{\nabla} \cdot (\underline{a} + \underline{b}) = \underline{\nabla} \cdot \underline{a} + \underline{\nabla} \cdot \underline{b}$

9) $\nabla \times (\underline{a} + \underline{b}) = \underline{\nabla} \times \underline{a} + \underline{\nabla} \times \underline{b}$

10) $\underline{\nabla} \cdot (\phi\,\underline{a}) = \underline{a} \cdot \underline{\nabla}\phi + \phi\,\underline{\nabla} \cdot \underline{a}$

11) $\underline{\nabla} \times (\phi\,\underline{a}) = (\underline{\nabla}\phi) \times \underline{a} + \phi\,\underline{\nabla} \times \underline{a}$

12) $\underline{\nabla} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot \underline{\nabla} \times \underline{a} - \underline{a} \cdot \underline{\nabla} \times \underline{b}$

13) $\underline{\nabla} \times (\underline{a} \times \underline{b}) = \underline{a}\,\underline{\nabla} \cdot \underline{b} - \underline{b}\,\underline{\nabla} \cdot \underline{a} + (\underline{b} \cdot \underline{\nabla})\,\underline{a} - (\underline{a} \cdot \underline{\nabla})\,\underline{b}$

14) $\underline{\nabla} \times \underline{\nabla} \times \underline{a} = \underline{\nabla}(\underline{\nabla} \cdot \underline{a}) - \nabla^2 \underline{a}$

15) $\underline{\nabla} \times \underline{\nabla}\phi = 0$

16) $\underline{\nabla} \cdot \underline{\nabla} \times \underline{a} = 0$

If $\underline{r} = \hat{\underline{i}}\,x + \hat{\underline{j}}\,y + \hat{\underline{k}}\,z$

17) $\underline{\nabla} \cdot \underline{r} = 3, \underline{\nabla} \times \underline{r} = 0$

If V represents a volume bounded by a closed surface S with unit vector $\hat{\underline{n}}$ normal to S and directed positively outwards, then,

18) $\int_V \underline{\nabla}\phi \, dv = \int_S \phi\,\hat{\underline{n}}\, d a$

19) $\int_V \underline{\nabla} \cdot \underline{a}\, d v = \int_S \underline{a} \cdot \hat{\underline{n}}\, d a$ (Gauss' Theorem)

20) $\int_V f\,\underline{\nabla} \cdot \underline{g}\, d v = \int_S f\,\underline{g} \cdot \hat{\underline{n}}\, d s - \int_V \underline{g} \cdot \underline{\nabla} f\, d v$

21) $\int_V \underline{g}\cdot\nabla f\,d\,v = \int_S f\,\underline{g}\cdot\hat{\underline{n}}\,d\,s - \int_V f\,\nabla\cdot\underline{g}\,d\,v$

22) $\int_V (\nabla \times \underline{a})\,d\,v = \int_S \hat{\underline{n}}\times\underline{a}\,d\,a$

23) $\int_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\,dv = \int_S (\Phi\nabla\Psi - \Psi\nabla\Phi)\cdot\underline{n}\,d\,s$ (Green's Theorem)

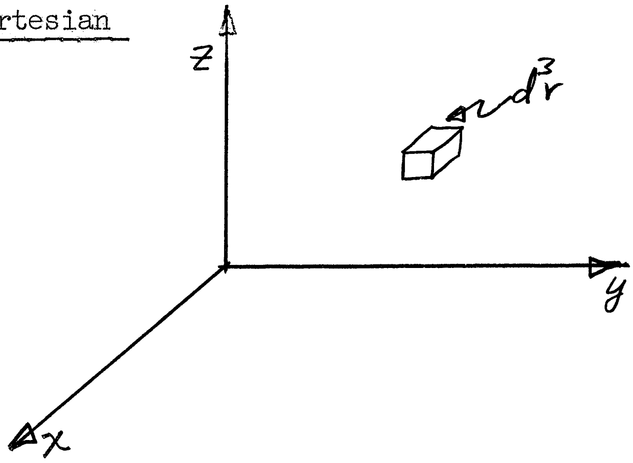
If S is an unclosed surface bounded by contour C, and $d\underline{s}$ is an increment of length along C.

24) $\int_S \hat{\underline{n}}\times\nabla\phi\,d\,a = \int_C \phi\,d\,\underline{s}$

25) $\int_S \nabla\times\underline{a}\cdot\underline{n}\,d\,a = \int_C \underline{a}\cdot d\,\underline{s}$ (Stokes' Theorem)

IV. Cartesian, Cylindrical, and Spherical Coordinate Systems

Cartesian

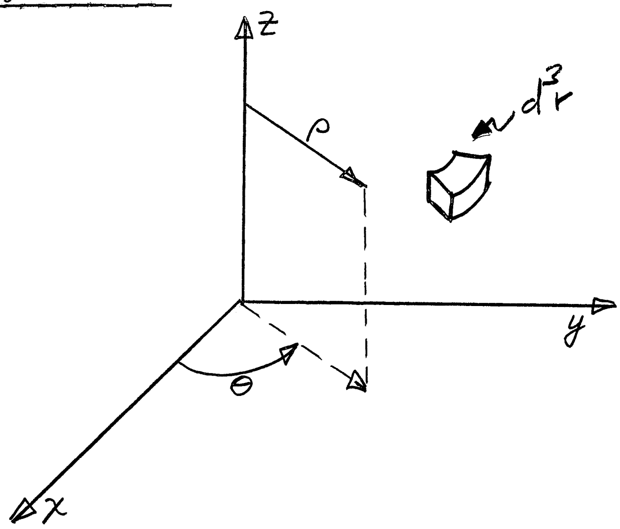


$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$d^3r = dx\,dy\,dz$$

Cylindrical

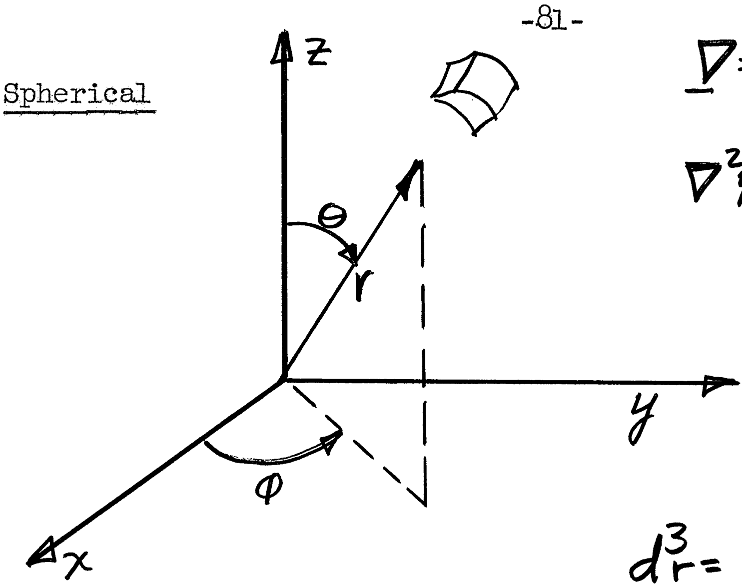


$$\nabla = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$d^3r = \rho\,d\rho\,d\theta\,dz$$

Spherical



$$\underline{\nabla} = \frac{\partial}{\partial r}\underline{\hat{r}_0} + \frac{1}{r}\frac{\partial}{\partial \theta}\underline{\hat{\theta}_0} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\underline{\hat{\Phi}_0}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$d^3r = r^2\sin\theta\, dr\, d\theta\, d\phi$$

V. Index Notation

A short note will be given on this notation which greatly simplifies certain mathematical problems (to mention one advantage). What is involved is essentially just the adoption of a convention. The convention used here suffices for work in rectangular Cartesian coordinates. For more general coordinate systems a more elaborate convention is needed; it is explained in reference works, see for example 1, 2, 3.

Consider a simple example which illustrates the utility and application of the index notation. Suppose one has a set of three equations

$$u = a_x x + a_y y + a_z z$$
$$v = b_x x + b_y y + b_x z$$
$$w = c_x x + c_y y + c_z z$$

By defining

$$u = u_1 , \quad v = u_2 , \quad w = u_3;$$
$$x = x_1 , \quad y = x_2 , \quad z = x_3;$$
$$a_x = a_{11} , \quad a_y = a_{12} , \quad a_z = z_{13};$$
$$b_x = a_{21} , \quad b_y = a_{22} , \quad b_z = a_{23};$$
$$c_x = a_{31} , \quad c_y = a_{32} , \quad c_z = a_{33};$$

These may be written

$$u_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$u_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$u_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

or

$$u_1 = \sum_{\alpha=1}^{3} a_{1\alpha}x_\alpha$$
$$u_2 = \sum_{\alpha=1}^{3} a_{2\alpha}x_\alpha$$
$$u_3 = \sum_{\alpha=1}^{3} a_{3\alpha}x$$

or

$$u_i = \sum_{\alpha=1}^{3} a_{i\alpha}x_\alpha \qquad (i = 1,2,3)$$

To this point we have effected considerable simplification of the original equations. By the introduction of the "summation convention", we can go still further. Notice that there are two kinds of indices on the right, i, which occurs <u>once</u> in the product, and α, which occurs <u>twice</u>. Index i is called a "single-occuring" index; α is called a "doubly-occuring" index. <u>The convention to be introduced is</u>:

a. <u>Doubly-occuring</u> indices are to be given all possible values and the results summed within the equation

b. <u>Singly-occuring</u> indices are to be assigned one value in an equation, but as many equations are to be generated as there are available values for the index.

Thus by Part a we may drop the sum symbol, and by Part b we may drop the parenthesis denoting values for i. With the summation convention in force, we have

$$u_i = a_{i\alpha}x_\alpha$$

which unambiguously represents the original equations, <u>if α and i have the same range</u>, which we shall assume.

A nice but unnecessary finishing touch can be put on the convention which seems to make things clearer in the work: for all singly occuring indices use lower-case Roman letters; for all doubly-occuring indices, use lower-case Greek letters.

We remark that it is possible to have any number of indices on a quantity.

VI. Examples of the Use of Index Notation

A. Some Handy Symbols

1. $\delta_{ij} = 1 \qquad i = j$

$\qquad = 0 \qquad i \neq j$

This is the Kronecker delta.

2. $\mathcal{E}_{ijk} = 1, \qquad i \neq j,\ i \neq k,\ j \neq k,\ i,j,k$ in cyclic order

$\qquad = -1, \qquad i \neq j,\ i \neq k,\ j \neq k,\ i,j,k$ in anticyclic order

$\qquad = 0, \qquad i = j$ or $i = k$, or $j = k$

This is called the Levy-Civita Tensor density.

B. Some Relationships Expressed in Index Notation

1. Dot product (a scalar)

$$\underline{a}\cdot\underline{b} = a_\alpha b_\alpha$$

2. Cross-product (a vector; consider i^{th} component)

$$(\underline{a} \times \underline{b})_i = \epsilon_{i\alpha b}\, a_\alpha b_\beta = -\epsilon_{i\beta\alpha} b_\beta a_\alpha$$

3. Triple product (a scaler)

$$\underline{a}\cdot\underline{b}\times\underline{c} = a_\alpha \epsilon_{\alpha\beta\gamma} b_\beta c_\gamma$$

4. Gradient (φ is a scalar)

$$(\underline{\nabla}\varphi)_i = \frac{\partial \varphi}{\partial x_i}$$

5. Divergence ($\underline{V}$ is a vector)

$$\nabla \cdot \underline{V} = \frac{\partial V_\alpha}{\partial x_\alpha}$$

6. Curl ($\underline{V}$ is a vector)

$$(\underline{\nabla} \times \underline{V})_i = \epsilon_{i\alpha\beta} \frac{\partial V_\alpha}{\partial x_\beta}$$

7. Laplacian

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\alpha}$$

8. $$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

C. Some Identities in δ_{ij} and ϵ_{ijk} in 3-D Space

1. $\delta_{\alpha\alpha} = 3$

2. $\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma} = 6$

3. $\epsilon_{i\alpha\beta}\epsilon_{j\alpha\beta} = 2\delta_{ij}$

4. $\epsilon_{ij\beta}\epsilon_{kl\beta} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$

5. $$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$$

VII. The Dirac Delta "Function"

The Dirac delta "function" symbolizes an integration operation and in this sense is not strictly, in the interpretation of Professor R. V. Churchill, a function. Thus the quotation marks around the word "function". It cannot be the end result of a calculation, but is meaningful only if an integration is to be carried out over its argument. We define the Dirac δ-"function" as follows:

$$\delta(x) \equiv 0, \qquad x \neq 0$$

$$\int_{-\epsilon}^{\epsilon} \delta(x)\, dx = 1, \qquad \epsilon > 0$$

$$\int_{-\epsilon}^{\epsilon} f(x)\, \delta(x)\, dx = f(0)$$

Very often it is convenient to think of the δ-"function" as a function zero everywhere except where its argument is zero, but which is so large at that point where its argument vanishes that its integral over any region of which that point is an interior point, is equal to unity. Mathematicians shudder at the idea of the δ-"function", but physicists have used them for years (carefully), finding them of great utility.

Schiff's "Quantum Mechanics" lists some properties of the Dirac δ:

$$\delta(x) = \delta(-x)$$

$$\delta'(x) = -\delta'(-x) \qquad \left(\delta'(x) = \frac{d\delta(x)}{dx}\right)$$

$$x\delta(x) = 0$$

$$X\delta'(x) = -\delta(x)$$

$$\delta(ax) = \frac{1}{a}\delta(x) \qquad a > 0$$

$$\delta(x^2 - a^2) = \frac{2}{a}\Big[\delta(x - a) + \delta(x + a)\Big] \qquad a > 0$$

$$\int \delta(a - x)\,\delta(x - b)\,dx = \delta(a - b)$$

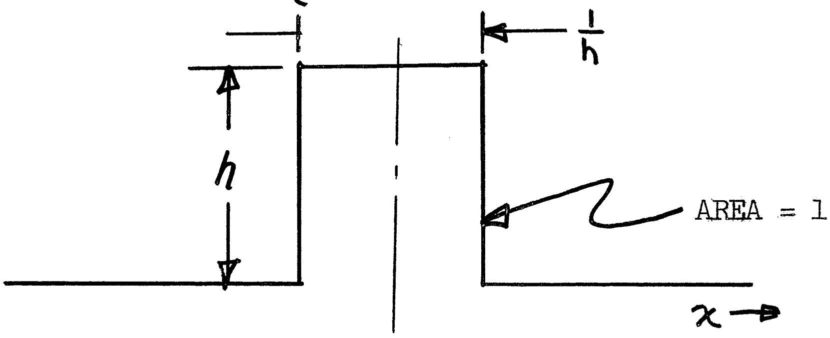
$$f(x)\,\delta(x - a) = f(a)\,\delta(x - a)$$

Professor Churchill uses as a δ-"function" the operation

$$\lim_{h \to 0} \int_{-\epsilon}^{\epsilon} f(x)\,U(h,x)\,dx \,, \qquad \epsilon > 0$$

where U(h,x) is the function

$$U(h,x) \equiv \begin{cases} \frac{1}{h}\,; & -\frac{h}{2} \le x \le \frac{h}{2} \\ 0\,; & x < -\frac{h}{2},\ x > \frac{h}{2} \end{cases}$$



Here, note that the limit is taken <u>after</u> integration. Other such representations are common, like that given by Schiff;

$$\delta(x) = \lim_{g\to\infty} \frac{\sin gx}{\pi x}$$

which means not that the limit is to be taken exactly as shown, but rather that it is taken <u>after</u> integration, i.e., with this representation,

$$\int_{-\epsilon}^{\epsilon} \delta(x)\, dx = \lim_{g\to\infty} \int_{-\epsilon}^{\epsilon} \frac{\sin gx}{\pi x}\, dx$$

$$\int_{-\epsilon}^{\epsilon} f(x)\, \delta(x)\, dx = \lim_{g\to\infty} \int_{-\epsilon}^{\epsilon} f(x) \frac{\sin gx}{\pi x}\, dx.$$

Schiff gives still another representation, in terms of an integral;

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi}\, d\xi = \lim_{g\to\infty} \frac{1}{2\pi} \int_{-g}^{g} e^{ix\xi}\, d\xi$$

$$= \lim_{g\to\infty} \frac{1}{2\pi} \frac{2 \sin gx}{x}$$

$$= \lim_{g\to\infty} \frac{\sin gx}{\pi x}$$

thus

$$\int_{-\epsilon}^{\epsilon} f(x)\, \delta(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i\xi x} f(x)\, d\xi dx.$$

VIII. <u>Gamma Functions</u>.

A. Definitions

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}\, t^{x-1}\, dt$$

B. Properties

a. $\Gamma(x + 1) = x\Gamma(x)$

b. $\Gamma(n) = (n - 1)!$ $\quad (\Gamma(1) = 1)$ $\quad$ (n positive integer)

c. $\Gamma(x)\Gamma(1 - x) = \dfrac{\pi}{\sin \pi x}$

d. $\Gamma(x)\Gamma(x + 1/2) = 2^{1-2x}\, \pi^{1/2}\, \Gamma(2x)$

e. $\Gamma \dfrac{(1 - b)}{(1 - a)} = \dfrac{1 - a}{1 - b} \circ \dfrac{2 - a}{a - b} \cdot \dfrac{3 - a}{3 - b} \ldots$

f. $\Gamma(1/2) = \sqrt{\pi}$

Since by (a) one may reduce $\Gamma(x)$ to a product involving Γ of some number between 1 and 2, a handy table for calculations is one like that found at the end of Chemical Rubber Integral Tables, for $\Gamma(x)$ $1 \leq x \leq 2$.

References

1. H. Margenau and G. M. Murphy; "The Mathematics of Physics and Chemistry" D. Van Nostrand Company, Inc., New York, 1956, pp. 93-98.

2. Whittaker and Watson, "Modern Analysis", 4th Edition, Cambridge University Press (1927), Chapter VII.

IX. Error Function

A.
$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2}\, d\lambda$$

B.
$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\lambda^2}\, d\lambda$$

NOTES AND CONVERSION FACTORS

I Electrical Units

A. The Electrostatic CGS System

1. The electrostatic cgs unit of charge, sometimes called the escoulomb or statcoulomb, is that "point" charge which repels an equal "point" charge at a distance of 1 cm. in a vacuum, with a force of 1 dyne.

2. The electrostatic cgs unit of field strength is that field in which 1 escoulomb experiences a force of 1 dyne. It, therefore, is 1 dyne/escoulomb.

3. The electrostatic cgs unit of potential difference (or esvolt) is the difference of potential between two points such that 1 erg of work is done in carrying 1 escoulomb from one point to the other. It is 1 erg/escoulomb.

B. The Electromagnetic CGS System

1. The unit magnetic pole is a "point" pole which repels an equal pole at a distance of 1 cm, in a vacuum, with a force of 1 dyne.

2. The unit magnetic field strength, the oersted, is that field in which a unit pole experiences a force of 1 dyne. It therefore is 1 dyne/unit pole.

3. The absolute unit of current (or abampere) is that current which in a circular wire of 1-cm radius, produces a magnetic field of strength 2π dynes/(unit pole) at the center of the circle. One abampere approximately equals 3×10^{10} esamperes or 10 amp.

4. The electromagnetic cgs unit of charge (or abcoulomb) is the quantity of electricity passing in 1 sec. through any cross section of a conductor carrying a steady current of 1 abampere. One abcoulomb equals 10 coulombs.

5. The electromagnetic cgs unit of potential difference (or abvolt) is a potential difference between two points, such that 1 erg of work is done in transferring abcoulomb from one point to the other. One abvolt = 10^{-8} volt = approximately $1/(3 \times 10^{10})$ esvolt.

C. Practical Electrical Units and Their Equivalents in the Absolute System

	Practical	Electrostatic cgs	Electromagnet cgs
Quantity	1 coulomb	3×10^9 escoulombs	1/10 abcoulomb
Current	1 ampere	3×10^9 esamperes	1/10 abcoulomb
Potential Difference	1 volt	1/300 esvolt	10^8 abvolts
Electrical field strength	1 volt/cm	1/300 dyne/escoulomb	10^8 abvolts/cm

D. Some Energy Relationships

1 esvolt x 1 escoulomb = 1 erg

1 abvolt x 1 abcoulomb = 1 erg

1 volt x 1 coulomb = 10^7 ergs = 1 joule

The electron volt equals the work done when an electron is moved from one point to another differing in potential by 1 volt.

$$1 \text{ electron volt} = 4.80 \times 10^{-10} \text{ escoulomb} \times 1/300 \text{ esvolt}$$

$$= 1.60 \times 10^{-12} \text{ erg.}$$

II. Physical Constants and Conversion Factors; Dimensional Analysis

Numerical Constants

e (base of natural logarithm)	2.718
$\log_e 10$	2.303
π^2	9.870

Lengths, Areas

Micron	μ	10^{-4} cm
Angstrom unit	A	10^{-8} cm
X-unit	XU	10^{-11}cm
Wavelength of 1-volt photon	—	12,396 A
Calcite grating space at 20°C	d	3.036 A
Separation of electron and proton in ground state of H	a_o	0.5291×10^{-8} cm
Compton wavelength	h/m_oc	2.426×10^{-10} cm
"Conventional electron radius"	e^2/m_oc^2	2.8175×10^{-13} cm
De Broglie wave of 1-volt electron	h/m_ov	12.26 A
Barn	b	10^{-24} cm^2

Masses and Mass Equivalents

Electrons	m_o	9.107×10^{-28} gm
		5.488×10^{-14} AMU
1/16 mass of O^{16} = Atomic mass unit	AMU	1.6595×10^{-24} gm
Proton	M_H	1.6722×10^{-24} gm
		1.00758 AMU
Neutron	M_N	1.6744×10^{-24} gm
		1.00894 AMU
Deuteron	M_D	3.343×10^{-24} gm
		2.01417 AMU
Alpha particle	M_α	6.642×10^{-24} gm
		4.00279 AMU
H^1 atom		1.00812 AMU
H^2 atom		2.01472 AMU
He^4 atom		4.00389 AMU
Proton mass over electron mass	M_H/m_o	1836.1

Energies and Speeds

Electron volt 4.80 escoulomb x 1/300 esu	ev	1.602×10^{-12} erg
Million electron volts	Mev	1.074×10^{-3} AMU
Energy equivalent of electron mass	m_oc^2	0.5110 Mev
Ionization energy of H atom	—	13.60 ev
Speed of 1-volt electron	—	5.931×10^7 cm/sec
Speed of light	c	2.9979×10^{10} cm/sec
	c^2	8.9874×10^{20} (cm/sec)^2

Other Electronic and Atomic Constants

Electronic charge	e	4.802×10^{-10} esu 1.602×10^{-20} emu
Charge/mass for electron	e/m_o	5.273×10^{17} esu/gm 1.759×10^{7} emu/gm
Planck's constant	h	6.624×10^{-27} erg sec, or 4.135×10^{-15} ev sec
Unit of angular momentum	$h/2\pi$	1.054×10^{-27} erg sec
Duane's constant	h/e	1.379×10^{-17} erg sec/esu
Rydberg constant		
For H^1 atom		109,678 cm^{-1}
For infinite		109,737 cm^{-1}
Bohr magneton	μ_B	9.271×10^{-20} erg/oersted
Fine-structure constant	α	$7.297 \times 10^{-3} = 1/137.04$

Constants Needed in Kinetics and Radiation Theory

Gas constant	R	8.314×10^{7} erg/(mole ^{o}C)
Boltzmann's constant (gas constant for 1 molecule)	$k = R/N$	1.380×10^{-16} erg/(^{o}K molecule)
Molar volume of perfect gas at $0^{o}C$ and 760 mm of Hg	V_m	22415 cm^3/mole
Faraday*	F	9652.2 emu/equivalent
Avogadro's number	$N = F/e$	6.025×10^{23} molecules/mole
Number of molecules in 1 cm^3 of perfect gas at $0^{o}C$ and 760 mm of Hg	F/eV_m	2.687×10^{19} molecules/cm^3
Average kinetic energy of a molecule at $0^{o}C$ and 760 mm of Hg ($T_o = 273.16^{o}K$)	$3/2\ kT_o$	5.655×10^{-14} erg
Stefan-Boltzmann constant	$\sigma = \dfrac{2\pi^5 k^4}{15\ h^3 c^2}$	5.669×10^{-5} erg/($cm^2 deg^4 sec$)
First radiation constant	c_1	4.99×10^{-15} erg cm
Second radiation constant	$c_2 = hc/k$	1.439 cm ^{o}C

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* Based on the "physical" scale of atomics weights, in which $O^{16} = 16$ exactly. On the chemical scale the value 16 refers to the natural mixture of oxygen isotopes. The two differ by 0.018 per cent

Table 1
Names of Units

Quantity	Symbol	System: Rationalized MKS	Unrationalized MKS	Unrationalized Electrostatic (CGS)	Unrationalized Electromagnetic (CGS)
Force	F	Newton	Newton	Dyne	Dyne
Length	ℓ	Meter	Meter	Centimeter	Centimeter
Time	t	Second	Second	Second	Second
Energy	W	Joule	Joule	Erg	Erg
Charge	Q	Coulomb	Coulomb	Statcoulomb	Abcoulomb
Electric potential	Φ	Volt	Volt	Statvolt	Abvolt
Electric field		$\frac{\text{Volt}}{\text{Meter}}$	$\frac{\text{Volt}}{\text{Meter}}$	$\frac{\text{Statvolt}}{\text{Centimeter}}$	$\frac{\text{Abvolt}}{\text{Centimeter}}$
Electric flux		Coulomb	(No name)	Esu of* ___	Emu of* ___
Electric flux density	D	$\frac{\text{Coulomb}}{\text{Meter}^2}$	(No name)	Esu of* ___	Emu of* ___
Capacitance	C	Farad	Farad	Statfarad	Abfarad
Current	I	Ampere	Ampere	Statampere	Abampere
Magnetic potential	Ψ	Ampere-turn	Pragilbert	Esu of* ___	Gilbert
Magnetic field intensity	H	$\frac{\text{Ampere-turn}}{\text{Meter}}$	Praoersted	Esu of* ___	Oersted
Magnetic flux	ϕ	Weber	Weber	Esu of* ___	Maxwell
Magnetic flux density	B	$\frac{\text{Weber}}{\text{Meter}^2}$	$\frac{\text{Weber}}{\text{Meter}^2}$	Esu of* ___	Gauss
Inductance	L	Henry	Henry	Stathenry	Abhenry

* Insert name of quantity. Esu is an abbreviation for electrostatic unit, and emu, electromagnetic unit. Thus Esu of ___ would be read as "electrostatic unit of charge (or other quantity)". The prefixes stat and ab are conveniently used on single-word description only.

Table 2
Conversion Factors

Quantity	Symbol	Multiply Number of	By	To obtain Number of
Force	F	Dynes	10^{-5}	Newtons
		Pounds	4.448	
		Ounces	0.2780	
Length	ℓ	Centimeters	10^{-2}	Meters
		Feet	0.3048	
		Inches	2.540×10^{-2}	
		Mils	2.540×10^{-5}	
Area	A	Centimeter^2	10^{-4}	Meters^2
		Feet^2	0.09290	
		Inches^2	6.452×10^{-4}	
		Mils^2	6.452×10^{-10}	
		Circular mils	5.067×10^{-10}	
Torque	T	Dyne-centimeters	10^{-7}	Newton-meters
		Pound-feet	1.356	
		Ounce-inches	7.062×10^{-3}	
Energy	W	Ergs	10^{-7}	Joules
		Foot-pounds	1.356	
		British thermal units	1.055×10^{3}	
		Kilowatt hours	3.600×10^{6}	
Charge	Q	Statcoulombs	3.335×10^{-10}	Coulombs
		Abcoulombs	10	
Electric potential	Φ	Statvolts	299.8	Volts
		Abvolts	10^{-8}	
Electric field intensity	$\mathcal{E}$	Statvolts/centimeter	2.998×10^{4}	Volts/meter
		Abvolts/centimeter	10^{-6}	
		Volts/centimeter	100	
		Volts/mil	3.937×10^{4}	

Table 2
(con'd)

Quantity	Symbol	Multiply Number of	By	To obtain Number of
Electric flux	ψ	Unrationalized MKS units of electric flux	7.958×10^{-2}	Coulombs
		Esu of electric flux	26.54×10^{-12}	
		Emu of electric flux	0.7985	
Electric flux density	D	Unrationalized MKS units of electric flux density	7.958×10^{-2}	Coulombs/meter^2
		Esu of electric flux density	26.54×10^{-8}	
		Emu of electric flux density	7.958×10^{3}	
Capacitance	C	Statfarads	1.112×10^{-12}	Farads
		Abfarads	10^{9}	
Current	I	Statamperes	3.335×10^{-10}	Amperes
		Abamperes	10	
Magnetic potential	ψ	Pragilberts	7.958×10^{-2}	Ampere-turn
		Esu of magnetic potential	26.54×10^{-12}	
		Gilberts	0.7958	
Magnetic field intensity	H	Praoersteds	7.958×10^{-2}	Ampere-turns/meter
		Esu of magnetic field intensity	26.54×10^{-10}	
		Oersteds	79.58	
Magnetic flux density	B	Esu of magnetic flux density	2.998×10^{6}	Webers/meter
		Gauss	10^{-4}	
Inductance	L	Stathenrys	8.988×10^{11}	Henrys
		Abhenrys	10^{-9}	

Table 3
Dimensional Analysis, Using, F, L, T, and Q

Mechanical Quantity	Symbol	Unit	Dimension
Force	f	newton	F
Length	x	meter	L
Time	t	second	T
Velocity	v	meter/second	LT^{-1}
Acceleration	a	meter/second^2	LT^{-2}
Mass	M	kilogram	$FL^{-1}T^2$
Spring constant (translation)	$K_T = \frac{f}{x}$	meter/newton	FL^{-1}
Damping constant (translation)	$R_T = \frac{f}{v}$	newton-second/meter	$FL^{-1}T$
Torque	T	newton-meter	FL
Angle	θ	radian	
Angular velocity	ω	radian/second	T^{-1}
Angular acceleration	α	radian/second^2	T^{-2}
Moment of inertia	I	kilogram-meter^2	FLT^2
Spring constant (rotation)	$K_R = \frac{T}{\theta}$	newton-meter	FL
Damping constant (rotation)	$R_R = \frac{T}{\omega}$	newton-meter-second	FLT
Energy	W	joule	FL
Power	P	watt	FLT^{-1}

Electric or Magnetic Quantity	Symbol	Unit	Dimension
Charge	Q	coulomb	Q
Permittivity	ϵ	farad/meter	$F^{-1}L^{-2}Q^2$
Electric field intensity	$\mathcal{E}$	volt/meter	FQ^{-1}
Electric potential	Φ	volt	FLQ^{-1}
Electric flux density	D	coulomb/meter^2	$L^{-2}Q$
Electric flux	ψ	coulomb	Q
Capacitance	C	farad	$F^{-1}L^{-1}Q^2$
Current	i	ampere	$T^{-1}Q$
Magnetic flux density	B	weber/meter^2	$FL^{-1}TQ^{-1}$
Permeability	μ	henry/meter	FT^2Q^{-2}
Magnetic field intensity	H	ampere/meter	$L^{-1}T^{-1}Q$
Magnetic potential	ψ	ampere	$T^{-1}Q$
Magnetic flux (or flux linkage)	ϕ (or λ)	weber	$FLTQ^{-1}$
Inductance	L	henry	FLT^2Q^{-2}
Magnetic pole (a mathematical concept)	p	weber	$FLTQ^{-1}$
Resistance	R	ohm	$FLTQ^{-2}$

Table 4
Dimensional Analysis, Using M, L, T, and

Mechanical Quantity	Symbol	Unit	Dimension
Mass	M	kilogram	M
Length	x	meter	L
Time	t	second	T
Velocity	v	meter/second	LT^{-1}
Acceleration	a	meter/second^2	LT^{-2}
Force	f	newton	MLT^{-2}
Spring constant (translation)	$K_T = \frac{f}{x}$	newton/meter	MT^{-2}
Damping constant (translation)	$R_T = \frac{f}{v}$	newton-second/meter	MT^{-1}
Torque	T	newton-meter	ML^2T
Angle	θ	radian	
Angular velocity	ω	radian/second	T^{-1}
Angular acceleration	α	radian/second^2	T^{-2}
Moment of intertia	I	kilogram-meter^2	ML^2
Spring constant (rotation)	$K_R = \frac{T}{\theta}$	newton-meter	ML^2T^{-2}
Damping constant (rotation)	$R_R = \frac{T}{\omega}$	newton-meter-second	ML^2T^{-1}
Energy	W	joule	ML^2T^{-2}
Power	P	watt	ML^2T^{-3}

Electric or Magnetic Quantity	Symbol	Unit	Dimension
Permeability	μ	henry/meter	μ
Current	i	ampere	$M^{1/2}L^{1/2}T^{-1}\mu^{-1/2}$
Magnetic flux density	B	weber/meter^2	$M^{1/2}L^{-1/2}T^{-1}\mu^{1/2}$
Magnetic field density	H	ampere/meter	$M^{1/2}L^{-1/2}T^{-1}\mu^{-1/2}$
Magnetic potential		ampere	$M^{1/2}L^{1/2}T^{-1}\ {}^{-1/2}$
Magnetic flux (or flux linkage)	ϕ (or λ)	weber	$M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$
Inductance	L	henry	$L\mu$
Charge	Q	coulomb	$M^{1/2}L^{1/2}\mu^{-1/2}$
Permittivity	ϵ	farad/meter	$L^{-2}T^{2}\mu^{-1}$
Electric field intensity	$\mathcal{E}$	volt/meter	$M^{1/2}L^{1/2}T^{-2}\mu^{1/2}$
Electric potential	Φ	volt	$M^{1/2}L^{3/2}T^{-2}\mu^{1/2}$
Electric flux density	D	coulomb/meter^2	$M^{1/2}L^{-3/2}\mu^{-1/2}$
Electric flux	ψ	coulomb	$M^{1/2}L^{1/2}\mu^{-1/2}$
Capacitance	C	farad	$L^{-1}T^{2}\mu^{-1}$
Magnetic pole (a mathematical concept)	p	weber	$M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$
Resistance	R	ohm	$LT^{-1}\mu$

Contents

APPENDICES

Errata - II

NOTES ON MATHEMATICS (AMERICAN NUCLEAR SOCIETY PROJECT)

Pg. 8 - Next to last line:

Orthagonal should be orthogonal

Pg. 11-c: $F(x) = \frac{ao}{2} + \dots$ not $\frac{ao}{Z}$

Pg. 12-g: $F(x) = \frac{ao}{2} + \Sigma \dots$ not $\frac{ao}{Z}\Sigma$

Pg. 13: II: Legendre Polynomials

$$H(x,y) = (\quad) = \sum_{\ell=0} (\quad)$$

Previously omitted

Also the line below this, should read $P_\ell = \frac{1}{\ell!}$ (the factorial was unclear).

Pg. 16: should read (absolute signs were missing)

1. $P_\ell^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^{|m|}}{dx^{|m|}} P_\ell(x) \quad |m| < \quad)$

[not $(0 < m < \ell)$]

2. $P_\ell^m(x) = [\quad] \qquad (0 < m < \ell)$

Pg. 24: c: should read:

$$Jn(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x)^{2r+n}}{2^{2r+n}\, r!\, \Gamma(1 + n + r)}$$

[was missing]

Pg. 25: No. 4-c:

$\lim_{x \to 0} J_n(x)$ not $\lim_{x \to \infty} J_n(x)$

Pg. 28-c:

$$I_\nu = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \sum_{r=0}^{\infty} \frac{(1/4\, x^2)^r}{r!\, \Gamma(\nu + 1 + r)}$$

Pg. 51: III - line 20

$$f(z) = \frac{a_{-1}}{z - a} + \ldots \qquad \text{not} \quad \frac{a-1}{z-a} + .$$

Line 22:

$$(z - a) f(z) = a_{-1} + \cdot\cdot \qquad \text{not} \quad \underline{a}_1 + \cdot\cdot$$

Pg. 52: line 14

$$f(z) = \frac{A_{-1}}{z - z_0} + \frac{A_{-2}}{(z - z_0)^2}$$ was missing

Page 56: 4th line from bottom:

should be $|X| \to \infty$, not $X \to \infty$.

Page 59: line 6,

$\underline{k}$, not k.

Line 7:

$\underline{x}$, not x

Page 60: should read

1. Definitions

a) $S_n[y] = \int_0^\pi y(x) \sin nx\, dx$

b) $C_n[y] = \int_0^\pi y(x) \cos nx\, dx$

2. Inversions

a) $y(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} (\qquad)$

b) $y(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} (\qquad)$

Pg. 64: line 17.

after $J_s[r]$ insert the word to .

Pg. 66: line 3.

We know $\frac{\partial\theta}{2x}\Big|x = \theta$ not we known $\frac{\partial\theta}{2x}\Big|x = \theta$

Line 7, should read fn, not f_n

Line 8, all f should be fn

Line 17,

$$\theta^{fn}(p) = \left\{\begin{matrix}\pi & \theta_o \\ 0 & \end{matrix}\right. \quad \text{not} \int_0^\pi$$

Pg. 68. Last two equations should have after them

$y > a$

$y < a$

Pg. 69: line 5, should read

Now, lacking a known inversion $\underbrace{\text{in closed form}}_{\text{(this part missing)}}$

Pg. 77: I : Leibnitz Rule

$$\frac{d}{dx} f(x) = (\quad) + \int_{a(x)}^{b(x)} \frac{dg(x,y)}{dx} dy \quad , \quad \text{not} \int_{a(x)}^{b(x)} \frac{2y(x,y)}{dx} dy$$

Pg. 79: No. 19

$$\int \nabla \cdot \underline{a} \; dv = \int_s a \cdot \hat{\underline{n}} \; d \; s \qquad \text{(not da)}$$

Pg. 80: No. 22

$$\int_v (\nabla \times \underline{a})dv = \int_s \underline{n} \times \underline{a} \; ds \qquad \text{(not da)}$$

Pg. 91: Gas constant $R = 8.314 \times 10^7$ erg/mole °K not /mole °C