096
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SINGULAR PERTURBATION PROBLEMS IN SHIP HYDRODYNAMICS

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CONTENTS

NOTATION

a_{ij}	added-mass coefficient
b_{ij}	damping coefficient
$b(x,z)$	hull offset
c_{ij}	restoring-force coefficient
$C(x)$	contour of body in the cross-section at x
$F_j^m(t)$	force in j-th mode due to body motion
$F_j^w(t)$	force in j-th mode due to incident waves
g	gravity constant
$g(x,z;\varepsilon)$	y offset of camber surface
$G(x,z)$	$g(x,z;\varepsilon)/\varepsilon$
$h(x,z;\varepsilon)$	half-thickness of body (equal to $b(x,z)$ for a symmetrical body)
$H(x,z)$	$h(x,z;\varepsilon)/\varepsilon$
$H(x)$	part of free-surface deflection in problems in two dimensions (see Equation (5-15))
$\mathbf{H}$	projection of a body onto the $y = 0$ plane (the centerplane of a ship)
i	$\sqrt{-1}$
$\mathbf{i}$ $\mathbf{j}$ $\mathbf{k}$	unit vectors parallel to three Cartesian axes
k ℓ m	Fourier-transform variables corresponding to x , y , and z , respectively
$K_n(z)$	modified Bessel function of second kind
L	length of a ship, or the segment of the x axis between bow and stern cross-sections
m_j	($j = 1 , \ldots , 6$) a set of functions defined over the surface of a slender body (see (2-75))
$m(x)$	added mass per unit length of a slender body
n_j	($j = 1 , \ldots , 6$) a set of functions defined over the surface of a slender body, equal to the components of the unit normal vector for $j = 1 , 2 , 3$ (see (2-72))
$n(x)$	damping coefficient per unit length of a slender ship
$\mathbf{n}$	unit vector normal to body surface (usually taken positive into the body)
$\mathbf{N}$	unit vector in a plane $x = \text{constant}$, normal to body contour in that cross-section

p	pressure
r	radius coordinate in cylindrical coordinate system
$r_0(x,\theta)$	radius coordinate of a slender body
$\mathbf{r}$	$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
s	half-span of a horseshoe vortex or a lifting line
$s(x)$	local half-span in wing cross-section at x, or cross-section area of a slender body (taken as the cross-section area of the *submerged* part, for a ship)
S	half-span of a wing of large aspect ratio
$T(x)$	keel depth of a ship at cross-section at x
T_{ij}	transfer functions between motion variables and forces on body
U	speed of a body, or speed of an incident stream (the latter invariably being taken in the positive x direction)
$\mathbf{v}(x,y,z)$	fluid velocity at (x,y,z) with uniform stream at infinity, $U = 1$, flowing around a body
$x \quad y \quad z$	Cartesian coordinates
$X \quad Y \quad Z$	stretched Cartesian coordinates, *e.g.*, $x = X$, $y = \varepsilon Y$, and $z = \varepsilon Z$ for a slender-body problem, xyz being far-field coordinates, XYZ near-field
$Z_n(x,y;\varepsilon)$	terms in a near-field expansion of $\zeta(x,y;\varepsilon)$ (*Cf.* $\zeta_n(x,y;\varepsilon)$)
$\alpha_n(x,z;\varepsilon)$	$\phi_n(x,0,z;\varepsilon)$ in thin-body problem
$\gamma_n(x,z;\varepsilon)$	normal velocity component in the plane of a sheet of dipoles
δ	motion-amplitude parameter in ship-motion problems
ε	small parameter in most problems considered
$\zeta(x,y,t)$	displacement of the free surface
$\zeta_n(x,y;\varepsilon)$	terms in a far-field expansion of $\zeta(x,y;\varepsilon)$ (*Cf.* $Z_n(x,y;\varepsilon)$)
$\zeta(z)$	function mapping the complex variable z onto an auxilliary (ζ) plane (in Section 5)
$\eta(x,y)$	steady part of free-surface deflection in ship-motion problem
$\eta(x)$	free-surface deflection in problems in two dimensions
$\eta_0(x)$	part of free-surface deflection in low-speed problem in two dimensions (see (5-7))

θ	angle variable in cylindrical coordinate system
$\theta(x,y,t)$	time-dependent part of free-surface deflection in ship-motion problem
κ	g/U^2 , a wave number in steady-motion problems
$\lambda_n(z)$	density of dipoles on a line (see (2-40))
$\mu_n(z)$	density of dipoles on a line (see (2-40))
$\mu_n(x,z;\varepsilon)$	density of dipoles on a surface
ν	ω^2/g , a wave number in oscillation problems
$\xi_j(t)$	displacement in the j-th mode of motion (see Section 2.32)
ρ	water density
$\sigma_n(x;\varepsilon)$	density of sources on a line
$\sigma_n(x,z;\varepsilon)$	density of sources on a surface
$\phi(x,y,z,t)$	velocity potential (The arguments may vary, but ϕ generally denotes the complete potential function in a problem.)
$\phi_0(x,y)$	in Section 5.4, potential for the problem in which the free surface is replaced by a rigid wall
$\phi_n(x,y,z;\varepsilon)$	terms in a far-field expansion of $\phi(x,y,z;\varepsilon)$
$\phi_j(x,y,z)$	normalized potential functions (see (2-73), (3-28))
$\Phi_n(x,y,z;\varepsilon)$	terms in a near-field expansion of $\phi(x,y,z;\varepsilon)$
$\Phi_j(x,y,z)$	normalized potential functions (see (3-44))
$\chi(x,y,z)$	velocity potential for the perturbation of a unit-strength incident stream by a slender ship
$\psi_j(x,y,z)$	normalized potential functions (see (2-76))
$\psi(x,y,z,t)$	time-dependent part of velocity potential in ship-motion problem (with forward speed)
$\Psi_j(x,y,z)$	normalized potential functions (see (3-45))
ω	radian frequency of sinusoidal oscillations
$\Omega_j(x,y,z)$	normalized potential functions (see (3-46))

MISCELLANEOUS CONVENTIONS

1) *Velocity Potential:* The velocity is always the positive gradient of the potential function.

2) *Coordinates and Orientation:* In problems involving a steady incident flow, that flow is always in the positive x direction. The vertical axis is the y axis in 2-D problems, the z axis in 3-D problems.

3) *Time Dependence:* In problems of sinusoidal oscillation, the time dependence is always in the form of the exponential function, $e^{i\omega t}$. In such problems, the real part only is intended to be used, but we do not indicate this explicitly in general.

4) *Fourier Transforms:* These are denoted by an asterisk. For example,

$$\sigma^*(k) = \int_{-\infty}^{\infty} dx\; e^{-ikx}\sigma(x) \;;\quad \sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ikx}\sigma^*(k) \;;$$

$$\phi^{**}(k,\ell;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\; dy\; e^{-i(kx+\ell y)}\phi(x,y,z) \;.$$

5) *Principal-Value Integrals:* These are denoted by a bar through the integral sign:

$$⨍_{-\infty}^{\infty} \frac{d\xi\; f(\xi)}{\xi - x}$$

6) *Order Notation:* There are three symbols used: O , o , $\sim$.

a) " $y = O(x)$ " means: $|y/x| < M$ as $x \to 0$, where M is a constant not depending on x .

b) " $y = o(x)$ " means: $|y/x| \to 0$ as $x \to 0$.

c) " $y \sim f(x)$ " means: $|y - f(x)| = o(f(x))$ as $x \to 0$.

1 INTRODUCTION

This paper is a survey of a group of ship hydrodynamics problems that have certain solution methods in common.

The problems are all formulated as *perturbation problems*, that is, the phenomena under study involve small disturbances from a basic state that can be described adequately without any special difficulties. The methods of solution make explicit use of the fact that the disturbances of the basic state are small. Mathematically, this is formalized by the introduction of one or more small parameters which serve as measures of the smallness of various quantities. The solutions obtained will generally be more nearly valid for small values of the parameter(s).

However, the problems will also be characterized by the fact that they are ill-posed in the limit as the small parameter(s) approaches zero. Thus, we call them *singular perturbation problems*. Special techniques are needed for treating such problems, and we have two which are especially valuable:

1) The Method of Matched Asymptotic Expansions, and

2) The Method of Multiple-Scale Expansions.

The first has a well-developed literature, and it has been made particularly accessible to engineers by Van Dyke (1964). The second, which has a longer history, is perhaps less well-known, but we now have a textbook treatment of it too, thanks to Cole (1968). Because of the availability of such books, my treatment of the methods in general will be extremely terse.

The necessity for treating ship hydrodynamics problems as perturbation problems arises most often in the incredible difficulty of handling the boundary condition which must be satisfied at the free surface. Even after neglecting viscosity, surface tension, compressibility, the motion of the air above, and a host of lesser matters, one can still make little progress toward solving free-surface problems unless one assumes that disturbances are small — in some sense. Historically, it has commonly been assumed that the boundary conditions may be linearized; in fact, this has so commonly been assumed that many writers hardly mention the fact, let alone try to justify it.

The two methods emphasized in this paper can also be applied to problems involving an infinite fluid. In fact, neither method was applied specifically to free-surface problems until quite recent times. Section 2 of this paper is devoted to several infinite-fluid problems. My justification, quite frankly, is almost entirely on didactic grounds. The methods can be made much clearer in these simpler problems, and so I include them here, although in some cases the infinite-fluid problems can be treated adequately by more elementary methods.

Most of the material in this paper has appeared in print elsewhere. My intention has been to present a coherent account of the treatment of singular perturbation problems in ship hydrodynamics, and so I have reworked solutions by other people and put them into a common notation and a common format. In some cases, I have made conscious decisions to follow certain routes and to ignore others. I am sure that I have made many such decisions unconsciously too. I have tried to give credit where it is due, but I am also sure that I have committed some sins of omission in the references. I apologize to those whom I may have slighted in this way.

1.1 Nature of the Problems and Their Solutions

We never really derive the perturbation *solution* of the exact* problem; we derive, at best, an exact solution of a perturbation *problem*. That is, we formulate an exact boundary-value problem, simplify the problem, solve the simplified version, and then hope that that solution is an approximation to the solution of the exact problem.

Thus, there will almost always be open questions about the validity of our solutions, and these questions can only be resolved through comparisons with exact solutions and experiments. We can have little hope of being rigorous. In fact, it is difficult to provide completely convincing arguments for doing some of the things that we do; in many cases, our approach is justified by the fact that it works! Much progress has been made in this field by people who try approaches "to see what will happen."

This does not imply that we shoot in the dark. It does suggest that we often depend more on intuition (or experience, which is the same thing) than on mathematical logic in deciding how to solve problems. The small disturbance assumptions by which free-surface problems have traditionally been linearized must have been tried first on this basis. The predictions which result from making such assumptions agree fairly well with observations of nature, and so we are encouraged to go on making the same assumptions in new problems. We may expect to be successful sometimes.

There are also open questions about the uniqueness of solutions. Engineers do not often worry about such matters, but they should certainly be aware of certain

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*"Exact" means only that nonlinear boundary conditions are treated exactly. I neglect viscosity, surface tension, compressibility, etc., and still call the problem "exact."

situations in which the dangers of non-uniqueness are especially great. The history of the study of free-surface problems provides numerous examples of invalid solutions being published by authors who were not sufficiently careful on this score. We have learned to be careful about imposing a radiation condition when necessary, although newcomers to the field are still occasionally trapped.* Questions about stability of our solutions are not so well appreciated, but of course solution stability is just one aspect of solution uniqueness. A particularly startling example has been pointed out in recent years by Benjamin & Feir (1967): Ordinary sinusoidal waves in deep water are unstable. This has now been demonstrated both theoretically and experimentally. It comes as no great surprise to those experimenters who had tried to generate high-purity sinusoidal waves for ship-motions experiments, but it was certainly quite a surprise to the theorists, who apparently did not suspect any such phenomenon before its discovery by Benjamin and Feir.

Since we shall be considering small-perturbation problems, we may expect the solutions to appear in the form of series expressions (not necessarily power series!). Often, we are content to obtain one term in such a series. Practically never do we face the question of whether the series converges. In fact, we usually just hope that the series has some validity, at least in an asymptotic sense.

The question will arise from time to time, "How small must the small parameter be in order that a one- (or two-

*Within the last few years, a leading German journal published an article on wave resistance in water of finite depth, in which it was concluded that a body had identically zero resistance if it were symmetrical fore and aft. The author was, I believe, primarily a numerical analyst, not familiar with the pitfalls of free-surface problems. He did not impose a numerical condition equivalent to a radiation condition. (This is one reference that I intentionally omit.)

or three- or n-)term expansion give valid predictions?" In ship-hydrodynamics problems, it is quite safe to assert that the only answer to such a question must be based on experimental evidence. In fact, even in simple problems, the knowledge of a few terms is not likely to help much with this question. For example, suppose that one tries to solve the simple differential equation: $y''(x) + y(x) = 0$, by means of a series of odd powers of x . How does one know that a two-term approximation is accurate to within one per cent even if x is as large as unity? One might compute the third term, of course, and compare it with the second term, hoping to guess what the effect of further terms would be. If it were too difficult to compute that third term, one could only hope that the solution had some validity, and perhaps one would try to find some experimental evidence on which to hazard a guess about validity. So it is in our ship-hydrodynamics problems. It will be necessary to discuss this point further at an appropriate place.

A related question concerns the precise definition of the small parameters that we use to formulate the approximate problems. In this paper I avoid defining the small parameter quantitatively. It is usually unnecessary and it is dangerous. I shall return to this point also.

1.2 Matched Asymptotic Expansions

For most of our problems, the approach advocated by Van Dyke (1964) is entirely adequate. I shall assume that the reader is familiar with (or has access to) Van Dyke's book. Only a few definitions and concepts will be mentioned here.

Perhaps the simplest problem that demonstrates the applicability of the method of matched asymptotic expansions is the following: Find the solution of the differential equation,

$$\varepsilon \ddot{y} + 2\dot{y} + y = 0 ,$$

subject to the initial conditions:

$$y(0) = 1 \; ; \quad \dot{y}(0) = 0 .$$

The parameter ε is to be considered small, and, in fact, we want to know how the solution of this problem behaves as $\varepsilon \to 0$. Now, if we set $\varepsilon = 0$, the order of the differential equation is reduced, and two initial conditions cannot be satisfied. Therefore, one cannot obtain a series expansion for the solution by a simple iteration scheme which starts with the solution for the limit case, $\varepsilon = 0$.

The exact solution for this problem is:

$$y(t) = \frac{-p_2 e^{p_1 t} + p_1 e^{p_2 t}}{p_1 - p_2} ,$$

where

$$p_1 = \frac{-1 - \sqrt{1 - \varepsilon}}{\varepsilon} \simeq -2/\varepsilon \; ;$$

$$p_2 = \frac{-1 + \sqrt{1 - \varepsilon}}{\varepsilon} \simeq -1/2 .$$

If we consider that $t = O(1)$ as $\varepsilon \to 0$, then the following approximation is valid for $y(t)$:

$$y(t) \sim e^{-t/2}\{1 + \frac{\varepsilon}{8}(2 - t) + \frac{\varepsilon^2}{32}(6 - 3t + 2t^2) + \ldots\}$$

This approximation could be obtained step-by-step, iteratively:

$$2\dot{y}_n + y_n = -\varepsilon \ddot{y}_{n-1} ,$$

where $y(t) \sim \Sigma\, y_n(t)$. However, it is not uniformly valid at $t = 0$, and the constants cannot be determined. On the other hand, we could consider that $t = O(\varepsilon)$ as $\varepsilon \to 0$ and rearrange the exact solution accordingly. This is most easily done if we set $t = \varepsilon\tau$ and rewrite everything in terms of τ . The approximation for $y(t)$ is then:

$$y(t) \sim 1 + \varepsilon\left(\frac{1}{4} - \frac{\tau}{2}\right) + \varepsilon^2\left(\frac{3}{16} - \frac{\tau}{4} + \frac{\tau^2}{8}\right) + \ldots$$

$$- e^{-2\tau}\left\{\frac{\varepsilon}{4} + \varepsilon^2\left(\frac{3}{16} + \frac{\tau}{8}\right) + \ldots \right\} .$$

This approximation could be obtained completely from the differential equation by an iteration scheme in which we let $y(t) \sim \Sigma\, Y_n(\tau;\varepsilon)$, the individual terms satisfying the equation:

$$Y_n''(\tau) + 2Y_n'(\tau) = -\varepsilon\, Y_{n-1}(\tau) \qquad [Y_n' \equiv dY/d\tau]$$

and the conditions:

$$Y_1(0) = 1 \;;\quad Y_n(0) = 0 \;,\ n > 1 \;;\ Y_n'(0) = 0 \;,\ n \geq 1 \,.$$

However, this solution is not uniformly valid for $\tau \to \infty$; in fact, one would hardly suspect that it represents a solution decaying exponentially with time.

The difficulty arises because the problem is characterized by two time scales, $1/p_1$ and $1/p_2$, and the two are grossly different. One of the two exponentials in the exact solution decays very rapidly and the other decays at a moderate rate. The contrast in these two time scales, along with the fact that each has its dominant effect in a distinct range of time, allows us to apply the method of matched asymptotic expansions to this problem. The Van Dyke prescription for doing this is as follows:

Define *the* n *term outer expansion* of $y(t)$ as $[y_1(t) + \ldots + y_n(t)]$; define *the* m *term inner expansion* of $y(t)$ as $[Y_1(\tau) + \ldots + Y_m(\tau)]$. In the n term outer expansion, substitute $t = \varepsilon\tau$ and rearrange the result into a series ordered according to ε ; truncate this expression after m terms, which gives *the* m *term inner expansion of the* n *term outer expansion.* Similarly, in the m term inner expansion, substitute $\tau = t/\varepsilon$ and rearrange the result into a series ordered according to ε ; truncate this expression after n terms, which gives *the* n *term outer expansion of the* m *term inner expansion.* The matching rule states that:

The m *term inner expansion of the* n *term outer expansion = the* n *term outer expansion of the* m *term inner expansion.*

In the example discussed in the previous paragraphs, the outer solution could not be obtained by a simple iteration scheme. The matching principle can now be used to determine the constants in the outer solution, and so an iteration scheme is now available, requiring, however, that inner and outer expansions be obtained simultaneously. In the example, the inner solution could be obtained completely and independently of the outer, but this is an accident which occurred because of the simple nature of the problem above. Ordinarily, in cases in which one might consider using the method of matched asymptotic expansions, one must proceed step-by-step to find first a term in one expansion, then a term in the other expansion, and so on.

It is worthwhile to be fairly precise about certain definitions. We use the equivalence sign, " $\sim$," frequently. For example, we write:

$$\phi(x,y,z;\varepsilon) \sim \sum_{n=0}^{N} \phi_n(x,y,z;\varepsilon)$$

This means that:

$$\left| \phi - \sum_{n=0}^{N} \phi_n \right| = o(\phi_N) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for fixed values of } (x,y,z) \ .$$

Also, it implies that $\phi_{n+1} = o(\phi_n)$ as $\varepsilon \to 0$. The qualification that (x,y,z) should be fixed is very important. In the example above, we would have the equivalent statement for the outer expansion:

$$|y(t;\varepsilon) - \sum_{n=1}^{N} y_n(t;\varepsilon)| = o(y_N) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for fixed} \quad t \ ,$$

and, for the inner expansion:

$$|y(t;\varepsilon) - \sum_{n=1}^{N} Y_n(\tau;\varepsilon)| = o(Y_N) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for fixed} \quad \tau \ .$$

In the latter, we evaluate the difference on the left-hand side for smaller and smaller values of t ($= \varepsilon\tau$) as $\varepsilon \to 0$; in other words we restrict the range of t more and more as $\varepsilon \to 0$. This is in contrast to the interpretation of the outer expansion, in which we simply fix t at any value while we let $\varepsilon \to 0$. In even more physical terms, we may say that the inner expansion describes the solution during the time when the $e^{p_1 t}$ term is varying rapidly, and the outer expansion describes the solution when the $e^{p_1 t}$ term has effectively reached zero and the $e^{p_2 t}$ term is varying significantly. This separation into two distinct regimes is characteristic of problems in which we apply the method of matched asymptotic expansions. Of course, the real key to the success of the method is in the procedure by which the two aspects of the solution are matched to each other. After all, they do represent just two aspects of the same solution.

Usually, we insist that our asymptotic expansions be *consistent*. A precise definition of this term is awkward, but perhaps it is clear if we state that each term in such a series depends on ε in a simple way that cannot be broken down into simpler terms of different orders of magnitude. For example, the following two series are equal:

$$\begin{aligned}[1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \ldots] = [1 &+ \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 + \tfrac{1}{8}\varepsilon^3 + \ldots] \\ &+ [\tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 + \tfrac{1}{8}\varepsilon^3 + \ldots] \\ &+ [\tfrac{1}{2}\varepsilon^2 + \tfrac{1}{4}\varepsilon^3 + \ldots] \\ &+ [\tfrac{1}{2}\varepsilon^3 + \ldots] + \ldots \quad .\end{aligned}$$

On the right-hand side, let:

$$f_0(\varepsilon) = 1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 + \tfrac{1}{8}\varepsilon^3 + \ldots \quad ;$$

$$f_n(\varepsilon) = \frac{\varepsilon^n}{2} f_0(\varepsilon) \ , \quad \text{for} \quad n > 0 \ .$$

Then we can write:

$$\sum_{n=0}^{N} \varepsilon^n \sim \sum_{n=0}^{N} f_n(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \ .$$

These happen to be convergent series (if $\varepsilon < 1$), but we can interpret them as asymptotic series just as well. The series on the left is "consistent;" the one on the right is not, because individual terms have their own ε substructure.

The striving for consistency can become a religion, but it is not a reliable faith. Consistency (or the lack of it) tells us nothing about the relative accuracy of otherwise equivalent asymptotic expansions. In fact, we could define a third asymptotic series with terms given by:

$$g_0(\varepsilon) = 1/(1-\varepsilon) \;; \quad g_n(\varepsilon) = 0 \quad \text{for} \quad n > 0 \; .$$

This series is grossly inconsistent, but one term gives the *exact* answer for the sum of the previous series! Occasionally one can make educated guesses about such things, replacing a few consistently arranged terms by a simple, inconsistent expression having much greater accuracy in practical computations. Mathematically, these different asymptotic series are equivalent, and, if ε is small enough, they will all give the same numerical results. But we want in practice to be able to use values of ε that are sometimes not "small enough."

We shall work with consistent series, for the most part, in spite of such possibilities of improvement through the use of inconsistent series. Most newcomers to this field of analysis find that there is a considerable element of art in the application of the method of matched asymptotic expansions, and I personally consider that the improvement of the expansions through the development of inconsistent expansions is the highest form of this art. Except in one respect, I do not intend to pursue the possibilities of inconsistent expansions in this paper.

The exception that I make is the following: Many singular perturbation problems lead to asymptotic-expansion solutions of the form:

$$\sum_{n=0}^{N} \sum_{m=0}^{n} a_{nm}\varepsilon^n(\log \varepsilon)^m ,$$

where a_{nm} does not depend on ε . We can,
this out in a long string of terms quite con
ranged. However, my practice will be to tre

$$h_n(\varepsilon) \equiv \varepsilon^n \sum_{m=0}^{n} a_{nm}(\log \varepsilon)^m$$

as a single term (albeit inconsistent) in the series $\Sigma h_n(\varepsilon)$. An alternative way of describing this practice is to say that I consider $\log \varepsilon = O(1)$ as $\varepsilon \to 0$! I have encountered some practical problems which could apparently not be solved by the Van Dyke matching principle unless treated in this way, and I have never seen or heard of a problem in which this practice led to difficulties. There are some good arguments for proceeding in this way, but I know of no proof that either way is *the* correct way. (Some of my colleagues will call this a cheap trick, rather than a higher expression of an art form.)

The classical example in physics of this kind of mathematical problem is the boundary layer first described by Prandtl in 1904. The thickness of the boundary layer becomes smaller and smaller as the small parameter, $1/\sqrt{R}$ approaches zero (R is the Reynolds number), but the presence of the boundary layer cannot be neglected, because then the governing differential equation becomes lower order, and the body boundary conditions cannot all be satisfied. Unfortunately, Prandtl did not realize the generality of the analysis which he introduced into the viscous-fluid problem, and, lacking the modern formalism for treating such problems, he could not obtain higher-order approximations.

Perhaps I should include a discussion of Prandtl's problem in this paper, since it might be considered as a "singular perturbation problem in ship hydrodynamics." However, I shall not do this, for several reasons. Van Dyke's coverage of the problem is excellent, I think. Also, the analysis concerns only laminar boundary layers, and they are really of quite limited interest in ship hydrodynamics. Finally, the formal procedure breaks down completely at the leading edge of a body, and the singularities that occur there cause major difficulties in all attempts to use the formalism to obtain higher-order approximations.

One final point should be emphasized, even at the risk of insulting the intelligence of readers who have read this far. Whenever we write, " $\varepsilon \to 0$," we are implying the existence of a sequence of physical problems in which the geometry or some fundamental parameter varies. For example, in Prandtl's boundary-layer problem, we may consider that viscosity changes as $\varepsilon = 1/\sqrt{R} \to 0$. In the simple ordinary-differential-equation example presented above, we may think of a spring-mass system in which the mass is changed systematically from one experiment to the next. Later, when we treat slender-body theory, we consider a sequence of problems in which the body changes each time. The theory always implies the possible existence of such a series of problems, and the quality of the predictions improves as the problem more nearly fits the limit case. Thus, we shall be able to apply the results of slender-body theory to bodies which are not especially slender. In such cases, we may expect that the predictions will be less accurate than the predictions that we would make for a much more slender body. But we never know *a priori* how slender the body must be for a certain accuracy to be realized, and it would be wrong to assert that the theory applies only to needle-like bodies. All that we can say is that it would be more accurate for such bodies than for not-so-slender bodies.

1.3 Multiple-Scale Expansions

In the problems of the previous section, we had two greatly contrasting scales for the independent variable. The fact that enabled us to obtain two separate expansions was that each of the scales dominated the behavior of the solution in a particular region of space or a particular period of time. The major practical concern was to ensure that the separate expansions matched, because they really represented just different aspects of the same solution.

The present section is devoted to problems in which there are again two greatly contrasting scales. However, in these problems, it will not be possible to isolate the effects of each scale into a more or less distinct region of space or time. The effects of the two scales mingle together completely. However, we may still expect to be able to identify these effects somehow, just because the two scales are so different.

There are classical problems of this kind, the most famous being related to nonlinear effects on certain periodic phenomena. Cole (1968) discusses a number of these problems. Perhaps the simplest example of all is a linear one: Find approximate solutions for small ε in the problem of a linear oscillator with very small damping, where the differential equation might be written:

$$\ddot{y} + 2\varepsilon\dot{y} + y = 0 .$$

To be specific, let the solution satisfy the initial conditions: $y(0) = 1$ and $\dot{y}(0) = 0$. Physically, we expect that the system will oscillate with gradually decreasing amplitude. It would be desirable if the approximate solution at least did not contradict this expectation.

We might try representing $y(t;\varepsilon)$ by an asymptotic expansion with respect to ε: $y(t;\varepsilon) \sim \Sigma\, y_n(t;\varepsilon)$. We would find immediately that the first term in this expansion is just: $y_0(t;\varepsilon) = \cos t$. This seems quite reasonable, since it represents a steady oscillation at the frequency appropriate to the undamped oscillator. The second term in the expansions would be obtained from:

$$\ddot{y}_1 + y_1 = -2\varepsilon\dot{y}_0 = 2\varepsilon \sin t \,, \quad \text{with } y_1(0) = \dot{y}_1(0) = 0 \,.$$

It is impossible to obtain a steady-state particular solution of this problem. In fact, the solution is:

$$y_1(t;\varepsilon) = \varepsilon[\sin t - t \cos t] \,.$$

Thus, we obtain an expansion in which the second term grows linearly with time. One might expect that succeeding terms will grow even faster. This expansion is correct, and, for small values of t , it could be used for numerical predictions. But we would certainly prefer to obtain an expansion which is uniformly* valid, even for very large t .

The exact solution is easily found, of course. It is:

$$y(t;\varepsilon) = e^{-\varepsilon t}\left(\cos \sqrt{(1-\varepsilon^2)}\,t + \frac{\varepsilon}{\sqrt{(1-\varepsilon^2)}} \sin \sqrt{(1-\varepsilon^2)}\,t\right) \,.$$

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*Strictly speaking, the series really is uniformly valid except at $t = \infty$.

The approximate solution becomes worse and worse with increasing t because the frequency is wrong and because the exponential factor is expanded in a power series in t. If we watch the oscillating mass on a time scale appropriate to the period of the oscillation, we do not see the exponential decay and the slight shift of frequency caused by the damping. On the other hand, if we watch for a very long time, the effects of damping accumulate gradually. Thus, the effects of the "slow-time" scale, $1/\varepsilon$, persist throughout the history of the motion as observed on a real-time scale, but these effects never occur suddenly. It is this fact which enables us to separate them out of the real-time problem.

There seems to be less reliable formalism available for handling such problems than in the case of the method of matched asymptotic expansions. More is left to the insight and ingenuity of the individual problem solver. In the example discussed above, the procedure is fairly clear: Expand $y(t;\varepsilon)$ in a series such as this:

$$y(t;\varepsilon) \sim y_0(\hat{t},\tau;\varepsilon) + y_1(\hat{t},\tau;\varepsilon) + \dots ,$$

where we define:

$$\hat{t} = \varepsilon t \; ; \qquad t = \tau + f_1(\tau;\varepsilon) + f_2(\tau;\varepsilon) + \dots \quad ,$$

and the functions f_n are to be determined in such a way that the approximation is uniformly valid for all t. In treating this particular problem, Cole immediately assumes that $t \propto \tau$ and further that $t/\tau = 1 + O(\varepsilon^2)$. These extra assumptions speed the solution considerably, but it is not clear how one would know to make them if the exact solution were not available. The exact solution takes the form:

$$y(t;\varepsilon) = e^{-\hat{t}} [\cos \tau + (\hat{t}/\tau) \sin \tau] \quad ,$$

in terms of the new variables. (The factor $(\hat{t}/\tau)$ does not depend on t .) Here it is clear how the two time scales enter into the solution as well as the problem. One may expect the relationship between t and τ to be equivalent to the expansion of the quantity $\surd(1-\varepsilon^2)$. The reader is referred to Cole's book for further discussion of the solution of such problems.

One problem that will be discussed later is a close relative of the clasical problems mentioned above. The solution by Salvesen (1969) of the higher-order problem of the wave resistance of a submerged body leads to a situation in which the first approximation is periodic downstream and that period is modified in the third-order approximation. (Otherwise the waves downstream in the higher approximation would grow larger and larger, without limit.) A similar problem involves the oscillation of a body on the free surface, in which the wavelength of the radiated waves must be modified in the third approximation. For example, see Lee (1968).

A quite different application of this method is the problem of very low speed motion of a body under or on a free surface. The simplest such case has been discussed by Ogilvie (1968). For a translating submerged body, there are two kinds of length scales: length scales associated with body dimensions and submergence, and the length scale U^2/g , which is associated with the presence of the free surface. Presumably, the latter has effects primarily near the free surface, in a "boundary layer" with thickness which varies with U^2/g as that variable approaches zero. But the effects of the body dimensions are also important near the free surface (or at least near a part of it). Thus the effects of the two length scales cannot be separated into distinct regions. A brief discussion of this problem appears in Section 5.42 of the present paper.

There may be many other problems of ship hydrodynamics in which this approach would be valuable. For example, many authors have obtained approximate solutions of problems involving submerged bodies by alternately satisfying a body boundary condition, then the free-surface condition, then again the body condition, etc. At each stage, when one condition is being satisfied, the other is being violated, but it is assumed that the errors become smaller and smaller with each iteration. Such a procedure is discussed, for example, by Wehausen & Laitone (1960), who point out the usefulness of Kochin functions in such procedures. However, there is often a question about the precise nature of such expansions. In the first approximation, for example, the effects of the free-surface are likely to drop off exponentially with distance from the surface. This makes it inappropriate to treat depth of submergence as a large parameter in the usual manner, because exponentially small orders of magnitude are either trivial or exceedingly difficult to handle. I do not believe that anyone has yet shown how to treat this problem systematically.

2 INFINITE-FLUID PROBLEMS

It is mainly the presence of the free surface in our problems that forces us to seek ever more sophisticated methods of approximation. However, the nature of the approximations can often be appreciated more easily by applying those methods to infinite-fluid problems. In this section, I discuss a number of problems that are geometrically similar to the ship problems that are my real concern. In some cases, it must be realized that the methods used here are not necessarily the best methods for the infinite-fluid problems. However, without the complications which accompany the presence of the free surface, one can better understand the significance of the coordinate distortions, the repeated re-ordering of series, and the matching of expansions.

The reader who feels comfortable with matched asymptotic expansions is invited to skip this chapter.

2.1 Thin Body

A "thin body" has one dimension which is characteristically much smaller than the other dimensions. In aerodynamics, the common example is the "thin wing," and, in ship hydrodynamics, one frequently treats a ship as if it were thin. In such problems, the incident flow is usually assumed to approach the body approximately edge-on, and so the thinness assumption allows one to linearize the flow problem.

In this section, thin-body problems are treated by the method of matched asymptotic expansions. This is not the way thin-body problems are normally attacked, and, in fact, I do not recall ever having heard of such a treatment. At the outset, I must point out that there are good reasons why this has been the case. If the body is symmetrical about a plane parallel to the direction of the incident flow, one does not need inner and outer expansions for solving the problem. And if the body lacks such symmetry, the lowest-order problem cannot be solved analytically, and so the method of matched asymptotic expansions does not offer the possibility that one may be able to obtain higher-order approximations.

In fact, the problem of a thin body in an infinite fluid is not a genuine singular perturbation problem (although it may contain some sub-problems that are singular, such as the flow around the leading edge of an airfoil). However, I believe that the problem of a thin ship is singular; I shall discuss this in Section 4. There has been a considerable amount of misunderstanding as to what constitutes the near field and what constitutes the far field in the thin-ship wave-resistance problem, and the rectification of such misunderstanding requires a careful statement of the problem.

It is conceivable that this interpretation of the thin-ship problem may be useful in formulating a rational mathematical idealization of the maneuvering-ship problem.

For convenience, I separate the thin-body problem into two parts: a) the symmetrical-body problem, and b) the problem of a body of zero thickness. To treat an arbitrary thin body, with both thickness and camber, one should certainly consider both aspects at once. It is not really difficult to do this, and indeed the problem of an unsymmetrical body of zero thickness actually involves thickness effects (at higher orders of magnitude than in the symmetrical-body problem). I have kept the problems separate here only for clarity in discussing certain phenomena that occur.

2.11 *Symmetrical Body (Thickness Effects).* Let the body be defined by the equation:

$$y = \begin{cases} \pm h(x,z;\varepsilon) & \text{for } (x,0,z) \text{ in } \mathbf{H} , \\ 0 & \text{for } (x,0,z) \text{ not in } \mathbf{H} , \end{cases} \qquad (2\text{-}1)$$

where $\mathbf{H}$ is the part of the $y = 0$ plane which is inside the body. (It is the *centerplane* if the body is a ship.) The "thinness" of the body is expressed by writing:

$$h(x,z;\varepsilon) = \varepsilon H(x,z) , \qquad (2\text{-}2)$$

where ε is a small parameter and $H(x,z)$ is independent of ε. The body is immersed in an infinite fluid which is streaming past it with a speed U in the positive x direction. The flow, in the absence of the body, can be described by the velocity potential: Ux.

It will sometimes be convenient to say that the body is defined by the equation: $y = \pm h(x,z;\varepsilon)$, implying that the function $h(x,z;\varepsilon)$ is identically zero if $(x,0,z)$ is not in $\mathbf{H}$. Also, note that we shall frequently drop the explicit mention of the ε dependence.

As $\varepsilon \to 0$, the body shrinks down to a sheet of zero thickness aligned with the incident flow. Thus, the first term in an asymptotic expansion of the velocity potential in the far field is just the incident-stream potential. In general, let the far-field expansion be expressed as follows:

$$\phi(x,y,z;\varepsilon) \sim \sum_{n=0}^{N} \phi_n(x,y,z;\varepsilon) , \text{ where } \phi_{n+1} = o(\phi_n) \text{ as } \varepsilon \to 0 \text{ for fixed } (x,y,z) . \qquad (2\text{-}3)$$

Then we have:

$$\phi_0(x,y,z;\varepsilon) \quad = \quad Ux \quad . \tag{2-4}$$

The far field is the entire space except the $y = 0$ plane. Since the potential $\phi(x,y,z;\varepsilon)$ satisfies the Laplace equation throughout the fluid domain, the individual terms in the above expansion satisfy the Laplace equation in the far field:

$$\phi_{n_{xx}} + \phi_{n_{yy}} + \phi_{n_{zz}} \quad = \quad 0 \quad \text{for} \quad |y| > 0 \quad . \tag{2-5}$$

At infinity, we expect (on physical grounds) that:

$$\nabla(\phi - Ux) \rightarrow 0 \quad . \tag{2-6}$$

Therefore, for $n > 0$, every ϕ_n must be singular on the $y = 0$ plane or be a constant throughout space. The latter would be too trivial a result to consider, and so we assume that ϕ_n is indeed singular on the $y = 0$ plane.

But what kind of singularities will be needed? Because of the symmetry of the problem, it is not difficult to show that a sheet of sources will suffice. One can use Green's theorem to show this. Alternatively, one can use transform methods for solving the Laplace equation, which is practically equivalent to solving by separation of variables. Whatever method is used, the result is the same; $\phi_n(x,y,z;\varepsilon)$ has a representation:

$$\phi_n(x,y,z;\varepsilon) \quad = \quad -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_n(\xi,\zeta;\varepsilon)\ d\xi\ d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{1/2}} \quad , \tag{2-7}$$

where $\phi_n(x,z;\varepsilon)$ is an unknown source-density function. The outer expansion is just the sum of these:

$$\phi(x,y,z;\varepsilon) \quad \sim \quad Ux \quad - \quad \frac{1}{4\pi}\sum_{n=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_n(\xi,\zeta;\varepsilon)\ d\xi\ d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{1/2}} \, . \tag{2-8}$$

This is the most general possible outer expansion for this problem.

It will be necessary presently to know the inner expansion of the above outer expansion. To find it, define an inner variable:

$$Y = y/\varepsilon , \qquad (2\text{-}9)$$

substitute for y in the outer expansion, and re-order the resulting expression with respect to ε. A direct approach to this process is difficult, but the following method, in four steps, allows us to obtain the desired results to any number of terms in a fairly simple way:

1) Take the Fourier transform of ϕ_n with respect to x :

$$\phi_n^*(k;y,z;\varepsilon) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} d\zeta\ \sigma_n^*(k;\zeta;\varepsilon)\int_{-\infty}^{\infty}\frac{dx\ e^{-ikx}}{[x^2+y^2+(z-\zeta)^2]^{1/2}}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\ \sigma_n^*(k;\zeta;\varepsilon)\ K_0(|k|\sqrt{[y^2+(z-\zeta)^2]})$$

where K_0 is the modified Bessel function usually denoted this way, and $\sigma_n^*(k;z;\varepsilon)$ is the Fourier transform of the function $\sigma_n(x,z;\varepsilon)$. The convolution theorem was used in the first step above.

2) Take the Fourier Transform next with respect to z :

$$\phi_n^{**}(k;y;m;\varepsilon) = -\frac{1}{2\pi}\sigma_n^{**}(k,m;\varepsilon)\int_{-\infty}^{\infty} dz\ e^{-imz}\ K_0(|k|[y^2+z^2]^{1/2})$$

$$= -\frac{\sigma_n^{**}(k,m;\varepsilon)}{2(k^2+m^2)^{1/2}}\ e^{-(k^2+m^2)^{1/2}|y|} ,$$

where $\sigma_n^{**}(k,m;\varepsilon)$ is the double transform of $\sigma_n(x,z;\varepsilon)$.

3) Substitute: $y = \varepsilon Y$ and expand the exponential function into a power series:

$$\phi_n^{**}(k;y;m;\varepsilon) = -\frac{\sigma_n^{**}(k,m;\varepsilon)}{2(k^2+m^2)^{1/2}}\left[1 + \frac{1}{2!}\varepsilon^2Y^2(k^2+m^2) + \frac{1}{4!}\varepsilon^4Y^4(k^2+m^2)^2 + \ldots\right]$$

$$+ \frac{1}{2}\sigma_n^{**}(k,m;\varepsilon)\left[\varepsilon|Y| + \frac{1}{3!}\varepsilon^3|Y|^3(k^2+m^2) + \frac{1}{5!}\varepsilon^5|Y|^5(k^2+m^2)^2 + \ldots\right] .$$

4) Note that:

$$-\frac{\sigma_n^{**}(k,m;\varepsilon)}{2(k^2+m^2)^{1/2}} = \phi_n^{**}(k;0;m;\varepsilon) \qquad \alpha_n^{**}(k,m;\varepsilon) . \quad (2\text{-}10)$$

Also, we observe that, if $f^{**}(k,m)$ is the Fourier transform of $f(x,z)$, then $(k^2+m^2)f^{**}(k,m)$ is the Fourier transform of $-(f_{xx}+f_{zz})$. Defining the inverse transform of $\alpha_n^{**}(k,m;\varepsilon)$:

$$\alpha_n(x,z;\varepsilon) = \phi_n(x,0,z;\varepsilon) , \qquad (2\text{-}11)$$

and inverting the above series term-by-term, we obtain:

$$\phi_n(x,y,z;\varepsilon) \sim \alpha_n(x,z;\varepsilon) + \frac{1}{2}\varepsilon|Y|\sigma_n(x,z;\varepsilon) - \frac{1}{2!}\varepsilon^2|Y|^2(\alpha_{n_{xx}}+\alpha_{n_{zz}})$$

$$- \frac{1}{2\cdot 3!}\varepsilon^3|Y|^3(\sigma_{n_{xx}}+\sigma_{n_{zz}})$$

$$+ \frac{1}{4!}\varepsilon^4|Y|^4[\alpha_{n_{xxxx}}+2\alpha_{n_{xxzz}}+\alpha_{n_{zzzz}}] \qquad (2\text{-}12)$$

$$+ \ldots .$$

This is the inner expansion of a typical term in the outer expansion.

In order to combine the expansions of the separate terms into a single inner expansion of the outer expansion, let us assume that σ_n and α_n are both $O(\varepsilon^n)$. (It is not necessary to assume this; it is merely convenient.) Then we have for the desired expansion:

$$\begin{aligned}
\phi(x,y,z;\varepsilon) \sim\ & U\,x && O(1) \\
& + \alpha_1(x,z;\varepsilon) && O(\varepsilon) \\
& + \alpha_2(x,z;\varepsilon) + \tfrac{1}{2}|y|\sigma_1(x,z;\varepsilon) && O(\varepsilon^2) \\
& + \alpha_3(x,z;\varepsilon) + \tfrac{1}{2}|y|\sigma_2(x,z;\varepsilon) - \tfrac{1}{2}|y|^2(\alpha_{1_{xx}}+\alpha_{1_{zz}}) && O(\varepsilon^3) \\
& \qquad + O(\varepsilon^4) \ . && \text{(2-13)}
\end{aligned}$$

Note that we have reverted to far-field variables. We must here consider that $y = O(\varepsilon)$ in order to recognize the orders of magnitude as indicated above.

Next we must find the inner expansion of the exact solution. Substitute $y = \varepsilon Y$ in the formulation of the problem. The Laplace equation transforms as follows:

$$\phi_{YY} = -\varepsilon^2(\phi_{xx} + \phi_{zz}) \ . \qquad \text{(2-14)}$$

The kinematic condition on the body is:

$$\pm\phi_x h_x - \phi_y \pm \phi_z h_z = 0 \quad \text{on} \quad y = \pm h(x,z) \ ,$$

which transforms into:

$$\phi_Y = \pm\varepsilon^2(\phi_x H_x + \phi_z H_z) \quad \text{on} \quad Y = \pm H(x,z) \ . \qquad \text{(2-15)}$$

We assume that there exists a near-field asymptotic expansion of the solution:

$$\phi(x,y,z;\varepsilon) \sim \sum_{n=0}^{N} \Phi_n(x,Y,z;\varepsilon) \text{ , where } \Phi_{n+1} = o(\Phi_n) \text{ as } \varepsilon \to 0 \text{ , for fixed } (x,Y,z) \text{ .} \tag{2-16}$$

We could show carefully that:

$$\Phi_0(x,Y,z;\varepsilon) = Ux \text{ .}$$

(Perhaps it is obvious to most readers.) We then express the conditions on the near-field expansion as follows:

$$[L] \quad \Phi_{1_{YY}} + \Phi_{2_{YY}} + \Phi_{3_{YY}} + \dots \sim -\varepsilon^2 [\Phi_{1_{xx}} + \Phi_{1_{zz}} + \Phi_{2_{xx}} + \Phi_{2_{zz}} + \dots] \text{ ;} \tag{2-17}$$

$$[H] \quad \Phi_{1_Y} + \Phi_{2_Y} + \Phi_{3_Y} + \dots \sim \pm \varepsilon^2 [UH_x + \Phi_{1_x}H_x + \Phi_{1_z}H_z + \dots] \text{ on } Y = \pm H(x,z) \text{ .} \tag{2-18}$$

Solution of the Φ_1 *problem.* From the [L] condition above, it is clear that:

$$\Phi_{1_{YY}} = 0 \tag{2-19}$$

in the fluid domain. Therefore Φ_1 must be a linear function of Y . In view of the symmetry of the problem, we can set:

$$\Phi_1(x,Y,z;\varepsilon) = A_1(x,z;\varepsilon) + B_1(x,z;\varepsilon)|Y| \text{ , for } |Y| > H(x,z) \text{ .} \tag{2-20}$$

The body condition reduces to:

$$\Phi_{1_Y}(x,\pm H(x,z),z;\varepsilon) = \pm\varepsilon^2 UH_x(x,z) = \pm B_1(x,z;\varepsilon) = O(\varepsilon^2) \text{ .} \tag{2-21}$$

It appears that we have determined the value of $B_1(x,z;\varepsilon)$ — but this is wrong, as we shall see in a moment. The two-term inner expansion appears to be:

$$\phi(x,y,z;\varepsilon) \sim Ux + A_1(x,z;\varepsilon) + B_1(x,z;\varepsilon)|Y| .$$

Its outer expansion is obtained by setting $Y = y/\varepsilon$:

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\frac{1}{\varepsilon}B_1(x,z;\varepsilon)|y|} + \underset{O(\varepsilon^2)}{A_1(x,z;\varepsilon)} .$$

The order-of-magnitude estimates were obtained as follows: B_1 is $O(\varepsilon^2)$, from (2-21). If our expansion is consistent (as we insist), then A_1 is also $O(\varepsilon^2)$, by (2-20). Now, in the outer expansion of the inner expansion, the B_1 term is lower order than the A_1 term. The two-term outer expansion of the two-term inner expansion is:

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\frac{1}{\varepsilon} B_1(x,z;\varepsilon)|y|} .$$

On the other hand, the two-term inner expansion of the two-term outer expansion is, from (2-13),

$$\phi(x,y,z;\varepsilon) \sim Ux + \alpha_1(x,z;\varepsilon) .$$

There is no linear term here at all, and it seems that we cannot match the two expansions.

It is a very comforting feature of the method of matched asymptotic expansions that things go wrong this way when we have made unjustified assumptions. Our mistake was this: When we found that apparently $B_1 = \varepsilon^2 UH_x = O(\varepsilon^2)$, we eliminated the possibility that there might be a term which is

$O(\varepsilon)$ in the inner expansion*. Now we rectify this error. Once again, let Φ_1 be given by (2-20), but suppose that both "constants" are, in fact, $O(\varepsilon)$. The body boundary condition immediately yields the condition that:

$$B_1(x,z;\varepsilon) = 0 ,$$

and so we have:

$$\Phi_1(x,Y,z;\varepsilon) = A_1(x,z;\varepsilon) .$$

The inner expansion, to two terms, is now given by:

$$\phi(x,y,z;\varepsilon) \sim Ux + A_1(x,z;\varepsilon) .$$

When we match this to the inner expansion of the outer expansion, we find that:

$$A_1(x,z;\varepsilon) = \alpha_1(x,z;\varepsilon) = \phi_1(x,0,z;\varepsilon) .$$

(See (2-11).) Now we have matched the expansions satisfactorily, but the result is not yet of much use, since we do not know either function, A_1 or α_1 . It is worth noting, however, that the inner expansion can be rewritten:

$$\phi(x,y,z;\varepsilon) \sim Ux + \phi_1(x,0,z;\varepsilon) .$$

Thus, to two terms the inner expansion is determined entirely by the far-field solution, the latter being evaluated on the centerplane. In other words, in the near-field view, the fluid velocity (to this degree of approximation) is caused entirely by remote effects.

*This trouble would have been avoided if I had started by assuming that the expansion is a power series in ε , as many people do in such problems. However, that procedure can lead to even greater difficulties sometimes.

Solution of the Φ_2 *problem:* This is much more straightforward, and the results are more interesting. We may expect that $\Phi_2 = O(\varepsilon^2)$, since we still have the nonhomogeneous body condition to satisfy. In this case, then,

$$\Phi_2(x,Y,z;\varepsilon) \quad = \quad A_2(x,z;\varepsilon) + B_2(x,z;\varepsilon)|Y| \ ,$$

and the body condition requires that $B_2(x,z;\varepsilon) = \varepsilon^2 U H_x(x,z)$. The three-term inner expansion is:

$$\underset{}{\phi(x,y,z;\varepsilon)} \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\alpha_1(x,z;\varepsilon)} + \underset{O(\varepsilon^2)}{A_2(x,z;\varepsilon)} + \underset{O(\varepsilon^2)}{\varepsilon^2 U H_x(x,z)|Y|} \ .$$

The two-term outer expansion of this three-term inner expansion is:

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\alpha_1(x,z;\varepsilon)} + \underset{O(\varepsilon)}{Uh_x(x,z;\varepsilon)|y|} \ .$$

The three-term inner expansion of the two-term outer expansion is, from (2-13):

$$\phi(x,y,z;\varepsilon) \sim Ux + \alpha_1(x,z;\varepsilon) + \tfrac{1}{2}|y|\sigma_1(x,z;\varepsilon) \ .$$

(The α_2 in (2-13) is not carried over to the above expansion, since it originates in the third term of the outer expansion.) These two match if:

$$\sigma_1(x,z;\varepsilon) \quad = \quad 2Uh_x(x,z;\varepsilon) \quad = \quad O(\varepsilon) \ . \qquad \text{(2-22)}$$

Thus, finally, we have found $\sigma_1(x,z;\varepsilon)$, the source density in the first far-field approximation, as a function of the body geometry. It is the familiar result from thin-ship theory. In addition, we can now also write down

$\alpha_1(x,z;\varepsilon)$ by combining (2-7) and (2-11):

$$\alpha_1(x,z;\varepsilon) = -\frac{1}{2\pi}\iint_{\mathbf{H}} \frac{Uh_x(\xi,\zeta;\varepsilon)\ d\xi\ d\zeta}{[(x-\xi)^2 + (z-\zeta)^2]^{1/2}} = O(\varepsilon) .$$

We have the two-term outer expansion — with everything in it known — and the three-term inner expansion — with the "constant" $A_2(x,z;\varepsilon)$ not yet determined.

Solution of the higher-order problems: From the [L] condition, (2-17), it can be seen that $\Phi_2(x,Y,z;\varepsilon)$ is not linear in Y . However, the differential equaiton for Φ_2 is easily solved, the body boundary condition can be satisfied, and matching can be carried out with the outer expansion. The result is:

$$\Phi_3(x,Y,z;\varepsilon) = A_3(x,z;\varepsilon) + B_3(x,z;\varepsilon)|Y| - \tfrac{1}{2}\varepsilon^2 Y^2(\alpha_{1_{xx}}+\alpha_{1_{zz}}) ,$$

where:

$$B_3(x,z;\varepsilon) = \varepsilon^2[(\alpha_{1_x}H)_x + (\alpha_{1_z}H)_z] ,$$

$$A_3(x,z;\varepsilon) = \alpha_3(x,z;\varepsilon) .$$

We also obtain σ_2 , through the matching:

$$\sigma_2(x,z;\varepsilon) = 2[(\alpha_{1_x}h)_x + (\alpha_{1_z}h)_z] ,$$

and this information also gives us α_2 and A_2 .

Summary: Symmetrical Body. The results for both near- and far-field expansion are stated in terms of the far-field coordinates (the natural coordinates of the problem) in Table 2-1. In a sense, the results are rather trivial. There could be difficulties near the edges of $\mathbf{H}$, but, barring such possibilities, the inner expansion could be obtained from the outer expansion and then matched to the body boundary condition. This is actually the classical thin-ship approach. The outer expansion is uniformly valid near the thin body, except possibly near the edges.

In the classical approach to the thin-body problem, there is usually a legitimate question concerning the analytic continuation of the potential function into the region of space occupied by the body. Sometimes one avoids the problem by restricting attention to bodies which can legitimately be generated by a sheet of sources, but this is not very satisfying. The method of matched asymptotic expansions avoids the question altogether by eliminating the need to ask it. What we are really saying is this: From very far away, the disturbance appears as if it could have been generated by a sheet of sources, but close-up we allow for the possibility that this observation from afar may be somewhat inaccurate. In fact, there is no analytic continuation presumed in the present method.

One can show by the use of Green's theorem that the far-field picture is valid even if the analytic continuation is not possible. A particularly appealing (to me) version of such a proof has been provided by Maruo (1967) for the much more complicated problem of a heaving, pitching slender ship moving with finite forward speed on the surface of the ocean.

I suppose that the uniformity of the thin-body solution is the result of the fact that a well-posed potential problem can be stated by giving a Neumann boundary condition over a surface. The situation will be quite different when we consider slender-body theory; in the far field, it would be necessary to give boundary conditions on a line, and that does not lead to a well-posed potential problem in three dimensions. Similarly, we may expect trouble at the confluence of two boundary con-

TABLE 2-1

SYMMETRICAL THIN BODY

Near-Field (Inner) Expansion $\qquad y = O(\varepsilon)$

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\alpha_1(x,z;\varepsilon)} + \underbrace{\alpha_2(x,z;\varepsilon) + \tfrac{1}{2}\sigma_1(x,z;\varepsilon)|y|}_{O(\varepsilon^2)}$$

$$+ \underbrace{\alpha_3(x,z;\varepsilon) + \tfrac{1}{2}\sigma_2(x,z;\varepsilon)|y| - \tfrac{1}{2}|y|^2(\alpha_{1_{xx}}+\alpha_{1_{zz}})}_{O(\varepsilon^3)}$$

$$+ O(\varepsilon^4)$$

Far-Field (Outer) Expansion $\qquad y = O(1)$

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} - \frac{1}{4\pi}\sum_{n=1}^{N}\underset{O(\varepsilon^n)}{\iint_H \frac{\sigma_n(\xi,\zeta;\varepsilon)\,d\xi\,d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{1/2}}}$$

From Matching

$$\sigma_1(x,z;\varepsilon) = 2Uh_x(x,z;\varepsilon) \;;$$

$$\sigma_2(x,z;\varepsilon) = 2[(\alpha_{1_x}h)_x + (\alpha_{1_z}h)_z] \;;$$

etc.;

$$\alpha_n(x,z;\varepsilon) = -\frac{1}{4\pi}\iint_H \frac{\sigma_n(\xi,\zeta;\varepsilon)\,d\xi\,d\zeta}{[(x-\xi)^2 + (z-\zeta)^2]^{1/2}} \;.$$

ditions, and this indeed occurs when we try to treat a ship problem by the method discussed above. The free-surface conditions cannot be satisfied, and the difficulty can be traced back to the behavior of the far-field potential near the intersection of the centerplane and the undisturbed free surface.

2.12 *Unsymmetrical Body (Lifting Surface)* For the sake of simplicity, let the body have zero thickness. Then it can be represented as follows:

$$y = g(x,z;\varepsilon) = \varepsilon G(x,z) \quad \text{for} \quad (x,0,z) \quad \text{in} \quad \mathbf{H}, \qquad (2\text{-}23)$$

where $\mathbf{H}$ is now the projection of the body onto the $y = 0$ plane. Again, there is a uniform incident flow in the positive x direction.

The analysis is quite similar to the symmetrical-body case, at least in the near field, and so most of the details will be omitted here. In the near field, let there be an expansion:

$$\phi(x,y,z;\varepsilon) \sim \sum_{n=0}^{N} \Phi_n(x,Y,z;\varepsilon) ,$$

just as in (2-16). The first term is, again, $\Phi_0(x,Y,z;\varepsilon) = Ux$. The terms again satisfy the transformed Laplace equation, (2-17):

$$[L] \quad \Phi_{1YY} + \Phi_{2YY} + \Phi_{3YY} + \Phi_{4YY} + \cdots$$

$$\sim -\varepsilon^2(\Phi_{1xx} + \Phi_{1zz} + \Phi_{2xx} + \Phi_{2zz} + \cdots) ;$$

the body boundary condition is now:

$$[H] \quad \Phi_{1_Y} + \Phi_{2_Y} + \Phi_{3_Y} + \Phi_{4_Y} + \cdots \tag{2-24}$$

$$\sim \varepsilon^2\left(UG_x + (\Phi_{1_x}G_x + \Phi_{1_z}G_z) + (\Phi_{2_x}G_x + \Phi_{2_z}G_z) + \cdots\right)$$

$$\text{on} \quad Y = G(x,z) \ .$$

The solution for Φ_1 is generally an expression linear in Y, but, for the same reasons as in the symmetrical-body problem, only the "constant" term can ultimately be matched to the far-field solution, and so we take for Φ_1 :

$$\Phi_1(x,Y,z;\varepsilon) = A_1^{\pm}(x,z;\varepsilon) = O(\varepsilon) \ .$$

The superscript $\pm$ has been attached to the solution to indicate that this quantity may be different on the two sides of the body. This was not necessary in the previous problem, because of the symmetry, but in the present near-field problem the body completely isolates the fluid on its two sides and there is no reason to assume that A_1 is the same on both sides of the body. (It turns out, in fact, that $A_1^- = - A_1^+$.)

One next obtains:

$$\Phi_2(x,Y,z;\varepsilon) = A_2^{\pm}(x,z;\varepsilon) + B_2^{\pm}(x,z;\varepsilon)\ Y \ .$$

From the body boundary condition, the following is true:

$$\Phi_{2_Y}(x,G,z;\varepsilon) = B_2^{\pm}(x,z;\varepsilon) = \varepsilon^2 UG_x(x,z) \ . \tag{2-25}$$

Thus, we find that:

$$B_2^+(x,z;\varepsilon) = B_2^-(x,z;\varepsilon) \equiv B_2(x,z;\varepsilon) \ .$$

Similarly, one can proceed:

$$\Phi_3(x,Y,z;\varepsilon) = A_3^{\pm}(x,z;\varepsilon) + B_3^{\pm}(x,z;\varepsilon)\,Y - \tfrac{1}{2}\varepsilon^2 Y^2 (A_{1_{xx}}^{\pm} + A_{1_{zz}}^{\pm}) \;,$$

where:

$$B_3^{\pm}(x,z;\varepsilon) = \varepsilon^2 [(GA_{1_x}^{\pm})_x + (GA_{1_z}^{\pm})_z] \;.$$

It is interesting to note the following about the symmetry: It turns out that Φ_1 and Φ_2 are odd with respect to Y, but Φ_3 is neither even nor odd. The linear term in Φ_3, namely, $B_3^{\pm}(x,z;\varepsilon)Y$, is even, since it turns out that $B_3^{+} = - B_3^{-}$. Careful study of the Φ_2 problem shows that it actually implies that there is a generation of fluid in the body, but the rate of generation is higher order than the Φ_2 term. Physically, of course, there can be no fluid generated, and so a compensating source-like term appears in Φ_3.

The far field is again the entire space except for the plane $y = 0$. The relations (2-3) to (2-6) are again valid, as well as the discussion of them. But now it will not suffice to provide only source singularities on the centerplane; clearly we must also provide singularities which lead to antisymmetric potential functions. In fact, since the body has zero thickness, we shall expect the leading-order approximation to be strictly antisymmetric. These requirements are all met by a distribution of dipoles which are oriented with the y axis. The potential of such a sheet of dipoles can be expressed:

$$f(x,y,z) = \frac{y}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mu(\xi,\zeta)\,d\xi\,d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}} \;. \quad (2\text{-}26)$$

The inner expansion of such an integral can be obtained by the same Fourier-transform technique that was used before. One finds that:

$$f^{**}(k;y;m) = \frac{1}{2}(\operatorname{sgn} y)\ \mu^{**}(k,m)\ e^{|y|\,(k^2+m^2)^{1/2}} \quad .$$

The exponential function can be expanded into a series, which is then inverted term-by-term. Define a new function (*Cf*. (2-10)):

$$\gamma^{**}(k,m) = (k^2+m^2)^{1/2}\ \mu^{**}(k,m) = -\frac{(\mu_{xx}+\mu_{zz})^{**}}{(k^2+m^2)^{1/2}} \quad . \qquad (2\text{-}27)$$

The following relationships exist between the two functions $\mu(x,z)$ and $\gamma(x,z)$:

$$\mu(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\gamma(\xi,\zeta)\ d\xi\ d\zeta}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}} \quad ; \qquad (2\text{-}28)$$

$$\gamma(x,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{[\mu_{xx}+\mu_{zz}]\ d\xi\ d\zeta}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}} \quad . \qquad (2\text{-}29)$$

(Note the comparison between (2-28) and the relation between α_n and σ_n in Table 2-1. In fact, (2-29) gives the inversion of the formula in Table 2-1.) The inner expansion of $f(x,y,z)$ can now be written in terms of these two functions:

$$\begin{aligned} f(x,y,z) = \frac{1}{2}\{&\mu(x,z)(\operatorname{sgn} y) - \gamma(x,z)\,y - \frac{1}{2!}y^2(\operatorname{sgn} y)(\mu_{xx}+\mu_{zz}) \\ &+ \frac{1}{3!}y^3(\gamma_{xx}+\gamma_{zz}) + \frac{1}{4!}y^4(\operatorname{sgn} y)(\mu_{xxxx}+2\mu_{xxzz}+\mu_{zzzz}) + \ldots\} \quad . \end{aligned} \qquad (2\text{-}30)$$

This may be compared with (2-12).

Now let us assume that the two-term outer expansion is:

$$\phi(x,y,z;\varepsilon) \sim Ux + \frac{y}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mu_1(\xi,\zeta;\varepsilon)\ d\xi\ d\zeta}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \quad .$$

Furthermore, assume that μ_1 and γ_1 are both $O(\varepsilon)$. (If these assumptions are too restrictive, that fact will become clear in the subsequent steps of the method of matched asymptotic expansions.) Then the inner expansion of the two-term outer expansion is:

$$\phi(x,y,z;\varepsilon) \sim \underset{O(1)}{Ux} \pm \underset{O(\varepsilon)}{\mu_1(x,z;\varepsilon)} - \underset{O(\varepsilon^2)}{y\,\gamma_1(x,z;\varepsilon)} \mp \underset{O(\varepsilon^3)}{\tfrac{1}{2}y^2(\mu_{1_{xx}}+\mu_{1_{zz}})} \;.$$

I have kept four terms, as indicated by the order-of-magnitude notes under the terms. (Recall that $y = O(\varepsilon)$ in the inner expansion.

Matching with the appropriate forms of the outer expansion of the inner expansion, we find that:

$$A_1^{\pm}(x,z;\varepsilon) = \pm\,\mu_1(x,z;\ ) \quad ; \qquad (2\text{-}31)$$

$$B_2^{\pm}(x,z;\varepsilon) = -\,\varepsilon\,\gamma_1(x,z;\varepsilon) \quad . \qquad (2\text{-}32)$$

From (2-25), we find that:

$$\gamma_1(x,z;\varepsilon) = -\,\varepsilon U G_x(x,z) = -\,U g_x(x,z;\varepsilon) \;. \qquad (2\text{-}33)$$

It appears now that we could use this knowledge of γ_1 in (2-28) for determining μ_1 . But this is wrong. Note from (2-30) that $\gamma_1(x,z)$ is the normal velocity component on the $y = 0$ plane caused by the distribution of dipoles, $\mu_1(x,z)$, over that same plane. Now we would presumably restrict the dipole distribution to the region **H** , and so (2-29) is valid if the range of integration is reduced to just **H** , since the integrand is identically zero outside **H** . But the same is not true in (2-28). There is a generally non-zero normal component of velocity, $\gamma_1(x,z)$, over the entire plane, and the range of integration in (2-28) cannot be reduced to just **H** . Unfortunately, we know $\gamma_1(x,z)$ only on **H** , from (2-33), and so we have solved nothing.

This difficulty is hardly surprising, since we are really formulating here the classical lifting-surface problem, and its solution requires either the solution of a two-dimensional singular integral equation or the introduction of further simplifications — which will be discussed presently.

In the lifting-surface problem, we really should distribute dipoles over two regions, the centerplane **H** and the part of the plane $y = 0$ which is directly downstream of **H** . Let the latter be called **W** . Pressure must be continuous across **W** , since there is no body there to support a pressure jump. In the usual aerodynamics manner, one can then show that $\partial\mu_1/\partial x$ must be zero on **W** . In this way, the integration range in (2-29) can be reduced to an integral over just **H** .

Of course, lifting surface theory is usually worked out in terms of vorticity distributions. I happen to prefer using dipole distributions, mainly because then I do not have to worry about whether a vortex line might be ending in the fluid region. The connection is fairly simple between the two versions, of course. A single discrete *horseshoe vortex* extending spanwise between $z = s$ and $z = -s$ and downstream to $x = \infty$ corresponds to a sheet of dipoles of uniform density, spread over the plane region bounded by the vortex line. The potential function can be written, for unit vortex strength,

$$\phi(x,y,z) = \frac{y}{4\pi}\int_{-s}^{s} d\zeta \int_{0}^{\infty} \frac{d\xi}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}}$$

$$= \frac{y}{4\pi}\int_{-s}^{s} \frac{d\zeta}{y^2 + (z-\zeta)^2}\left[1 + \frac{x}{[x^2 + y^2 + (z-\zeta)^2]^{1/2}}\right]$$

$$= \frac{1}{4\pi}\left[\tan^{-1}\frac{y}{z-s} - \tan^{-1}\frac{y}{z+s} + \tan^{-1}\frac{y\sqrt{[x^2+y^2+(z-s)^2]}}{x(z-s)} - \tan^{-1}\frac{y\sqrt{[x^2+y^2+(z+s)^2]}}{x(z+s)}\right].$$

The normal velocity component in the plane of the vortex is:

$$\phi_y(x,0,z) = \frac{1}{4\pi}\left(\frac{1}{z-s}\left[1+\frac{[x^2+(z-s)^2]^{1/2}}{x}\right] - \frac{1}{z+s}\left[1+\frac{[x^2+(z+s)^2]^{1/2}}{x}\right]\right).$$

A *lifting line* can be described in a similar way if we allow the dipole density to vary with the spanwise coordinate, z . For simplicity, let us assume that $\mu(z) = \mu(-z)$, and that $\mu(s) = 0$. The potential for a lifting line is:

$$\phi(x,y,z) = \frac{y}{4\pi}\int_{-s}^{s} d\zeta\ \mu(\zeta)\int_{0}^{\infty}\frac{d\xi}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \tag{2-34}$$

$$= -\frac{y}{4\pi}\int_{0}^{s} ds'\ \mu'(s')\int_{-s'}^{s'} d\zeta\int_{0}^{\infty}\frac{d\xi}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}} \tag{2-35}$$

and the normal velocity component is:

$$\phi_y(x,0,z) = -\frac{1}{4\pi}\int_{-s}^{s}\frac{ds'\ \mu'(s')}{z-s'}\left[1+\frac{[x^2+(z-s')^2]^{1/2}}{x}\right]. \tag{2-36}$$

Note that this reduces to the result for the single horseshoe vortex if a) we set $\mu'(z) = \delta(z+s) - \delta(z-s)$* , and b) we integrate over a span from $-s-\beta$ to $s+\beta$, where β is a very small positive number. This may lend some credibility

* $\delta(z)$ is the usual Dirac delta function.

to the procedure frequently advocated by aerodynamicists in wing problems, *viz.*, when integrating by parts in the spanwise direction, extend the range of integration slightly beyond the wing tips so that quantities which become infinite at the tips do not yield infinite contributions that cannot be interpreted. (This is terrible mathematics, but apparently the physics is sound, since the results seem to be correct.)

Finally, we can use the above procedures to derive the corresponding expressions for a *lifting surface*. The important quantity is the normal velocity component, given by:

$$\phi_y(x,0,z) = -\frac{1}{4\pi}\int_L d\xi \int_{-s(\xi)}^{s(\xi)} d\zeta \, \frac{\mu_{xz}(\xi,\zeta)}{z-\zeta}\left\{1 + \frac{[(x-\xi)^2+(z-\zeta)^2]^{1/2}}{x-\xi}\right\}, \quad (2\text{-}37)$$

where L is the range of x covered by the lifting surface (the length of L being generally the chord length), and s(x) is the half-span at cross-section x . On **H** (the projection of the wing on the plane $y = 0$), the normal velocity component, ϕ_y , is known, either by direct application of the body boundary condition or by matching to a near-field solution, and we obtain the usual integral equation for a lifting surface.

We shall not be concerned here with the various methods of attempting directly to solve this integral equation, either by analytical or numerical methods. In fact, analytical methods do not exist, so far as I know, except for a few special geometries, such as elliptical planforms. The pair of equations (2-28) and (2-29) forms a remarkable analogy to a standard boundary-value problem in two dimensions which is analyzed thoroughly by Muskhelishvili (1953). One three-dimensional case has been solved analytically by a method that has some similarity to the standard methods for the 2-D problem; this was done by Kochin (1940). Even his circular-planform wing led to so much difficulty, it seems unlikely that it will be generalized to other planforms.

Analytical solutions have also been obtained for circular and then elliptic planforms by formulating the problem in terms of an acceleration potential in coordinate systems appropriate to such shapes of figures. This was all done long ago. See Kinner (1937) and Krienes (1940).

There are many numerical techniques for obtaining approximate solutions of this problem. However, I ignore these and proceed to analyze a special configuration which can be treated approximately as a limiting case of the general lifting-surface problem.

2.2 High-Aspect-Ratio Wing

It is an interesting historical fact that Prandtl's boundary-layer solution really contains the essence of the method of matched asymptotic expansions, but Prandtl failed to observe that the same technique would work in his lifting-line problem. In the boundary-layer problem, he really required the matching of two complementary, asymptotically valid, partial solutions. It was probably Friedrichs (1955) who first recognized that the high-aspect-ratio lifting-surface problem could be treated the same way. Van Dyke (1964) discusses the derivation of lifting-line theory in some detail from the point of view of matched asymptotic expansions. My presentation is not different from Van Dyke's in any startling ways. There are *some* differences, partly because I have in mind applications to planing problems eventually, partly because I am not an aeronautical (or aerospace) engineer at heart.

The conventional approach to solving the problem of a wing of high aspect ratio is to simplify (2-37) by arguments that relate the sizes of the terms involving $(x-\xi)^2$ and $(z-\zeta)^2$. (Quite comparable arguments are used in the conventional approach to the theory of slender wings.) If the radical in (2-37) can be simplified, then the ξ integration can be performed, and one is left with just the integral over ζ. In this way, the 2-D integral equation is reduced to a one-dimensional integral equation, which is of a standard form.

Using the method of matched asymptotic expansions, we return to the original formulation of the problem and derive a sequence of simpler problems, rather than try to work out approximate solutions of the integral equation. The large-aspect-ratio wing is "slender" in the spanwise direction. This means that cross-sections parallel to the $z = 0$ plane vary gradually in size and shape as z varies; in particular, the maximum dimension in the z direction, say $2S$ (the span), is much greater than the maximum dimension in the cross-sections. We shall make whatever further assumptions of this kind that we need in order to keep the solution well-behaved. The small paramter can be defined as the inverse of the aspect ratio, that is,

$$\varepsilon = 1/(AR) = (\text{area of } \mathbf{H})/4S^2 ,$$

where $\mathbf{H}$ is the projection of the wing onto the $y = 0$ plane. As before, it is not necessary to be so specific about the definition of ε, and in fact it may be misleading. A wing with aspect ratio equal to 100 might not be slender in the required sense if, for example, there were discontinuities in chord length in the spanwise direction. In any case, the wing shrinks down to a line, part of the z axis, as $\varepsilon \to 0$.

Let the body be defined by the following relation:

$$y = g(x,z) \pm h(x,z) \ , \qquad (2\text{-}38)$$

for $(x,0,z)$ in **H** . See Figure (2-1). It is not necessary that the body be a thin one, in the sense of the previous section. I do, however, specify that it should be symmetric with respect to z , for the sake of simplicity in what follows. Both of the functions $g(x,z)$ and $h(x,z)$ really depend on ε , of course*, but we shall generally omit explicit mention of the fact.

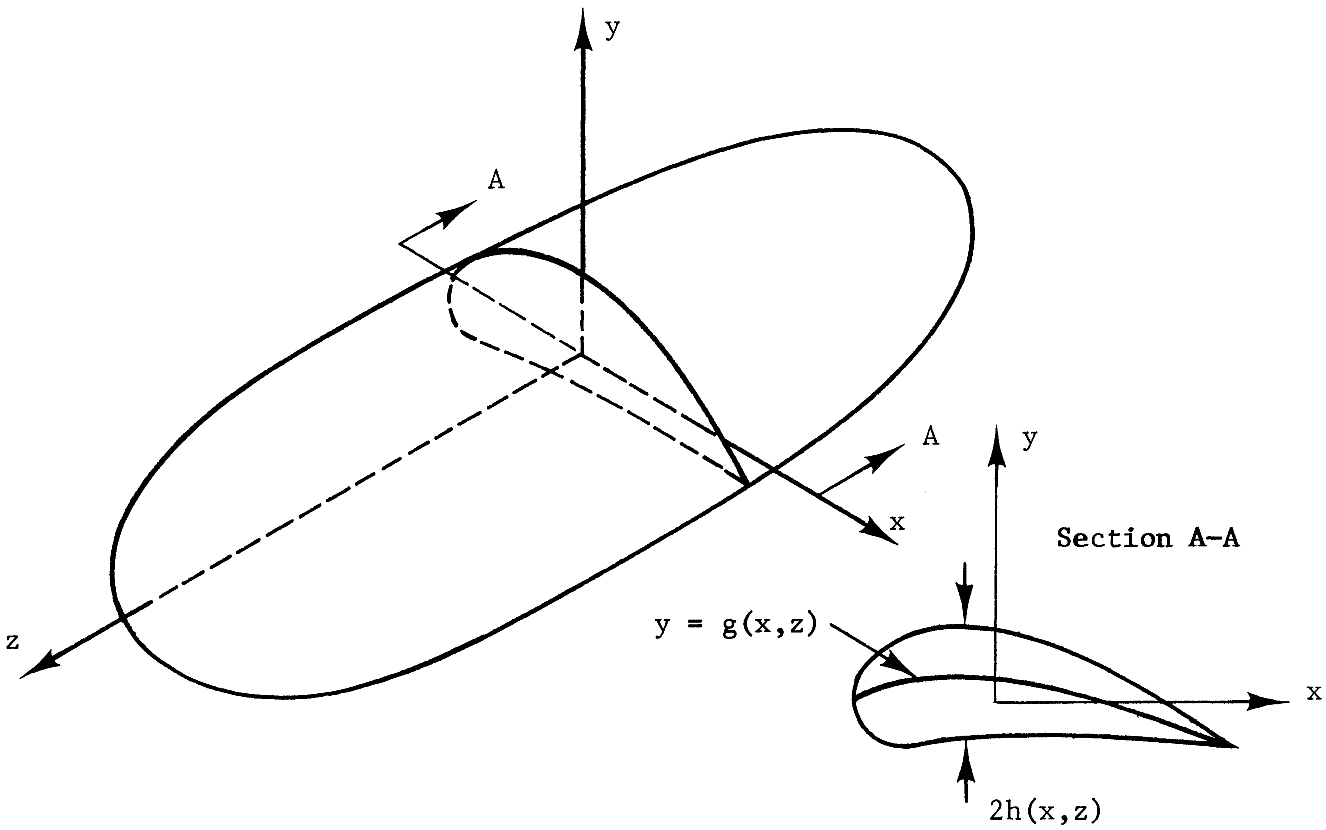

Figure (2-1). Coordinates for the High-Aspect-Ratio Wing.

There is an incident flow which, at infinity, is uniform in the x direction. Let the far-field solution be represented by the asymptotic expansion:

*In fact, g and h are both $O(\varepsilon)$.

$$\phi(x,y,z) \sim Ux + \sum_{n=1}^{N} \phi_n(x,y,z) , \quad \text{where} \quad \phi_{n+1} = o(\phi_n) \quad \text{as} \quad \varepsilon \to 0 , \quad \text{for fixed} \quad (x,y,z) . \tag{2-39}$$

(Again, the dependence on ε is suppressed in the notation.) Since the body shrinks to a line ($x = 0$, $y = 0$, $|z| < S$) in the limit as $\varepsilon \to 0$, the terms denoted by ϕ_n all represent flow perturbations which arise in the neighborhood of this singular line. They can be expressed in terms of singularities on that line, and the strengths of such singularities should be $o(1)$ as $\varepsilon \to 0$. In an ideal fluid, we could expect the occurrence of dipoles, quadripoles, etc., on the singular line. We also take the realistic point of view that viscosity cannot be completely neglected and that there may be some circulation as a result. In the usual aeronautical point of view, this implies that there may be a vortex line present, complete with a set of trailing vortices. In the point of view adopted in the previous section, I assume that there may be a sheet of dipoles behind the singular line. I also make the usual assumption that these wake dipoles (or vortices) lie in the plane $y = 0$. This part of the $y = 0$ plane ($0 < x < \infty$, $|z| < S$) will be denoted by **W** . (Note that **H** has all but disappeared in the far field view. It is only a line.)

We can now write the outer expansion in the following form:

$$\begin{aligned} \phi(x,y,z) \sim\ & Ux + \frac{y}{4\pi} \sum_{n=1}^{N} \int_{-S}^{S} \int_{0}^{\infty} \frac{\gamma_n(\zeta)\, d\xi\, d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}} \\ & + \frac{y}{4\pi} \sum_{n=1}^{N} \int_{-S}^{S} \frac{\mu_n(\zeta)\, d\zeta}{[x^2 + y^2 + (z-\zeta)^2]^{3/2}} \\ & + \frac{x}{4\pi} \sum_{n=0}^{N_N} \int_{-S}^{S} \frac{\lambda_n(\zeta)\, d\zeta}{[x^2 + y^2 + (z-\zeta)^2]^{3/2}} + \cdots . \end{aligned} \tag{2-40}$$

The first sum contains terms which are exactly of the form given in (2-34), that is, they represent a lifting line with a strength $\Sigma \gamma_n(z)$. The second and third sums represent lines of dipoles oriented vertically and longitudinally, respectively. It is implied above that the sums are asymptotic expansions, in our usual far-field sense.

We shall presently require the inner expansions of these terms. We obtain the inner expansions by assuming that $r = (x^2+y^2)^{1/2} = O(\varepsilon)$, which implies that *both* x and y are small.

Inner expansion of the lifting-line potential: Each of the double integrals containing a γ can be rewritten as a single integral:

$$\frac{y}{4\pi}\int_{-S}^{S}\int_{0}^{\infty}\frac{\gamma(\zeta)\,d\xi\,d\zeta}{[(x-\xi)^2+y^2+(z-\zeta)^2]^{3/2}}$$
$$= -\frac{1}{4\pi}\int_{-S}^{S}d\zeta\,\gamma'(\zeta)\left\{\tan^{-1}\frac{y}{z-\zeta}+\tan^{-1}\frac{y\sqrt{[x^2+y^2+(z-\zeta)^2]}}{x(z-\zeta)}\right\} \tag{2-41}$$

Now break this into two parts:

1) The first term in brackets on the right-hand side does not depend on x . As $y \to 0$ (*i.e.*, for $y = O(\varepsilon)$), its contribution can be represented:

$$\pm\frac{1}{4}\gamma(z) - \frac{y}{4\pi}⨍_{-S}^{S}\frac{d\zeta\,\gamma'(\zeta)}{z-\zeta}\left(1 + O(\varepsilon^2)\right),$$

where the double sign is chosen according to whether $y > 0$ or $y < 0$, respectively, and the special integral sign indicates that the Cauchy principal value is intended. This representation is valid only for $|z| < S$, but that is no restriction here. It may be noted that this term represents a distribution of vorticity extending to infinity both upstream and downstream. Thus, it leads to a discontinuity across the $y = 0$ plane, even upstream. The second term must compensate for this behavior, since there can be no discontinuities in the region $x < 0$.

2) The second term in brackets on the right-hand side of (2-41) must be considered carefully with respect to the branches of the square-root function. With a bit of effort, one can show that, as $r \to 0$, its contribution is:

$$\frac{\gamma(z)}{4}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right)\left(1 + O(\varepsilon^2)\right) \quad , \quad 0 < \tan^{-1}\frac{y}{x} < \pi \quad ;$$

$$\frac{\gamma(z)}{4}\left(3 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right)\left(1 + O(\varepsilon^2)\right) \quad , \quad \pi < \tan^{-1}\frac{y}{x} < 2\pi \quad .$$

Combining this result with the previous one, we find that the inner expansion of a lifting-line potential function can be written as follows:

$$\frac{y}{4\pi}\int_{-S}^{S}\int_{0}^{\infty}\frac{\gamma(\zeta)\,d\xi\,d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}}$$

$$\sim\left[\frac{1}{2}\gamma(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right] - \frac{y}{4\pi}\fint_{-S}^{S}\frac{d\zeta\,\gamma'(\zeta)}{z-\zeta}\right]\left[1 + O(\varepsilon^2)\right] \quad , \tag{2-42}$$

$$\text{for} \quad 0 < \tan^{-1}\frac{y}{x} < 2\pi \quad .$$

Inner expansion of the dipole-line potential: An integration by parts with respect to ζ transforms these integrals into an appropriate form so that one can let $r \to 0$ and thereby obtain the first terms in the desired expansions.

Typical terms in the second and third sums of (2-40) have the following inner expansions:

$$\frac{1}{4\pi}\int_{-S}^{S}\frac{[y\mu(\zeta) + x\lambda(\zeta)]\ d\zeta}{[x^2 + y^2 + (z-\zeta)^2]^{3/2}} \sim \left[\frac{y\mu(z) + x\lambda(z)}{2\pi(x^2 + y^2)}\right]\left[1 + O(\varepsilon^2 \log \varepsilon)\right] \quad . \qquad (2\text{-}43)$$

Note the occurrence of the logarithm of ε !

Inner expansion of the outer expansion: In order not to confuse the picture, I shall make more assumptions now, namely:

$$\gamma_1(z) = O(\varepsilon) \ ; \quad \mu_1(z) \ , \quad \lambda_1(z) \ , \quad \gamma_2(z) \ = \ O(\varepsilon^2) \ ;$$

also, all other terms in (2-40) are $o(\varepsilon^2)$. These statements can all be proven. The description of the problem is greatly simplified, however, by their being assumed now.

We can write the three-term outer expansion now:

$$\phi(x,y,z) \sim Ux + \phi_1(x,y,z) + \phi_2(x,y,z) \quad , \qquad (2\text{-}44)$$

where:

$$\phi_1(x,y,z) = \frac{y}{4\pi}\int_{-S}^{S}\int_{0}^{\infty}\frac{\gamma_1(\zeta)\ d\xi\ d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}} \ ; \qquad (2\text{-}44a)$$

$$\phi_2(x,y,z) = \frac{y}{4\pi}\int_{-S}^{S}\int_{0}^{\infty}\frac{\gamma_2(\zeta)\ d\xi\ d\zeta}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{3/2}} + \frac{1}{4\pi}\int_{-S}^{S}\frac{[y\mu_1(\zeta) + x\lambda_1(\zeta)]\ d\zeta}{[x^2 + y^2 + (z-\zeta)^2]^{3/2}} \quad . \qquad (2\text{-}44b)$$

The inner expansion of the one-term outer expansion is, of course:

$$\phi(x,y,z) \sim Ux \,, \qquad [O(\varepsilon)] \qquad (2\text{-}45a)$$

to any number of terms. (Recall that $x = O(\varepsilon)$ in the near field.) The inner expansion of the two-term outer expansion is:

$$\phi(x,y,z) \sim Ux + \frac{1}{2}\gamma_1(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right] \qquad [O(\varepsilon)]$$
$$- \frac{y}{4\pi}\int_{-S}^{S} \frac{d\zeta\, \gamma_1'(\zeta)}{z - \zeta} \,. \qquad [O(\varepsilon^2)] \qquad (2\text{-}45b)$$

Finally, the inner expansion of the three-term outer expansion is:

$$\phi(x,y,z) \sim Ux + \frac{1}{2}\gamma_1(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right] + \frac{y\mu_1(z) + x\lambda_1(z)}{2\pi(x^2 + y^2)} \qquad [O(\varepsilon)] \qquad (2\text{-}45c)$$
$$- \frac{y}{4\pi}\int_{-S}^{S} \frac{d\zeta\, \gamma_1'(\zeta)}{z - \zeta} + \frac{1}{2}\gamma_2(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right] \,. \qquad [O(\varepsilon^2)]$$

I have taken the trouble of writing out the inner expansion of the outer expansion in three ways just to point out how, in this problem, there is an additional term in the lowest-order expression each time we add another term *of higher order* in the outer expansion. Each of the three terms included in (2-44) contributes to the ε term in (2-45c). This phenomenon occurs frequently, and its occurrence is the reason that one must proceed step-by-step in the matching. In the present problem, one would be in some difficulty if he tried to write down an arbitrary number of terms in each expansion and immediately start matching.

Next we formulate the near-field problem. Instead of making the formal changes of variable, $x = \varepsilon X$ and $y = \varepsilon Y$, we shall simply understand now that, in the near field,

$x = O(\varepsilon)$ and $y = O(\varepsilon)$; also $\partial/\partial x = O(\varepsilon^{-1})$ and $\partial/\partial y = O(\varepsilon^{-1})$.

Of course, differentiation with respect to z does not affect orders of magnitude.

The Laplace equation can be written in the form:

$$\phi_{xx} + \phi_{yy} = - \phi_{zz} , \qquad (2\text{-}46)$$

where the right-hand side is ε^2 higher order than the left-hand side. The boundary condition on the body is:

$$0 = \phi_x(g_x \pm h_x) - \phi_y + \phi_z(g_z \pm h_z) \quad \text{on} \quad y = g \pm h . \qquad (2\text{-}47)$$

The last condition is equivalent to requiring that $\partial\phi/\partial n = 0$ on the body, where $\partial/\partial n$ denotes differentiation in the direction normal to the body surface. An alternative statement is the following:

$$\frac{\partial\phi}{\partial N} = \frac{(\pm g_x + h_x)\phi_x \mp \phi_y}{\sqrt{[1 + (g_x \pm h_x)^2]}} = - \frac{(h_z \pm g_z)\phi_z}{\sqrt{[1 + (g_x \pm h_x)^2]}} \quad \text{on} \quad y = g \pm h , \qquad (2\text{-}47')$$

where $\partial\phi/\partial N$ is the rate of change in a plane perpendicular to the z axis, measured in the direction normal to the body contour in that cross-section plane. Note that the left-hand side is $O(\phi/\varepsilon)$, since differentiation in the N direction has the same order-of-magnitude effect as differentiation with respect to x or y . The right-hand side, on the other hand, is $O(\phi\varepsilon)$, since g and h are both $O(\varepsilon)$.

Now let there be an inner expansion:

$$\phi(x,y,z) \sim \sum_{n=0}^{N} \Phi_n(x,y,z) , \quad \Phi_{n+1} = o(\Phi_n) \quad \text{as} \quad \varepsilon \to 0 , \qquad (2\text{-}48)$$

with $(x/\varepsilon, y/\varepsilon, z)$ fixed.

The first term in this expansion satisfies the conditions:

$$\Phi_{0_{xx}} + \Phi_{0_{yy}} = 0 \quad \text{in the fluid region;} \qquad (2\text{-}49)$$

$$\frac{\partial \Phi_0}{\partial N} = 0 \quad \text{on the body.} \qquad (2\text{-}50)$$

From (2-45a), it is clear that the one-term inner expansion, $\phi(x,y,z) \sim \Phi_0(x,y,z)$, must match the one-term outer expansion, $\phi(x,y,z) \sim Ux$. Thus, $\Phi_0(x,y,z)$ is the solution of a two-dimensional potential problem, and a rather conventional problem at that: In a section through the body drawn perpendicular to the spanwise axis, the potential satisfies the Laplace equation in two-dimensions, a homogeneous Neumann condition on the body, and a uniform-flow condition at infinity. The direction of the uniform flow is the same as the direction of the actual incident stream as viewed in the far field.

Since Φ_0 does satisfy the Laplace equation in two dimensions, the methods of complex-variable functions are available for determining its properties. In particular, if we assume that $\nabla\Phi_0$ is bounded everywhere in the fluid region and single-valued too, then Φ_0 can be expressed as the real part of an analytic function of a complex variable, the analytic function being such that its derivative can be expressed by a Laurent series. Thus, we can write for $\Phi_0(x,y;z)$:

$$\Phi_0(x,y;z) = Ux + \delta_0 \log r + \eta_0 \tan^{-1}\frac{y}{x} + A_{00} + \frac{A_{01}\cos\Theta}{r} + \frac{B_{01}\sin\Theta}{r} + \frac{A_{02}\cos 2\Theta}{r^2} + \frac{B_{02}\sin 2\Theta}{r^2} + \ldots, \qquad (2\text{-}51)$$

where $r = (x^2+y^2)^{1/2}$. The "constants" are all unknown functions of z , the spanwise coordinate. The first term represents a uniform stream at infinity, and I have already performed one matching to determine this term. The second and third terms represent a source and a vortex, respectively; the fourth term, a constant, is included for generality; the fifth and sixth terms represent a dipole; etc. Such an expansion as (2-51) is valid outside any circle about the origin which encompasses the body cross-section.

Every term in (2-51) must be of the same order of magnitude at a point in the near field, that is for $r = O(\varepsilon)$. If a term were of some other order of magnitude with respect to ε , the definition of "consistency" would eliminate it from this series. The orders of magnitude of most of the unknown constants can then be written down. Since the first term, Ux , is $O(\varepsilon)$, we can make the following statements:

$$\eta_0 \; , \; A_{00} = O(\varepsilon) \; ; \quad A_{01} \; , \; B_{01} = O(\varepsilon^2) \; ; \; A_{0n} \; , \; B_{0n} = O(\varepsilon^{n+1}) \; .$$

The term containing the logarithm does not fit the pattern quite so well — unless we follow my arbitrary practice of saying that $\log \varepsilon = O(1)$. (See the discussion of "consistency" in Section 1.2.) Then we can say that:

$$\delta_0 = O(\varepsilon) \quad .$$

The Laurent series expression for the near-field expansion is very convenient when it comes to finding the outer expansion of the inner expansion. All we need to do is to interpret r differently and re-arrange the terms according to their dependence on ε . Thus, if we consider that $r = O(1)$, the outer expansion of the one-term inner expansion is:

$$\phi(x,y,z) \sim \Phi_0(x,y;z) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\delta_0 \log r} + \underset{O(\varepsilon)}{\eta_0 \tan^{-1}\frac{y}{x}} + \underset{O(\varepsilon)}{A_{00}}$$

$$+ \underset{O(\varepsilon^2)}{\frac{A_{01}\cos\theta}{r}} + \underset{O(\varepsilon^2)}{\frac{B_{01}\sin\theta}{r}} + O(\varepsilon^3) \quad . \qquad (2\text{-}51')$$

This obviously matches the one-term outer expansion, with an asymptotically small error which is $O(\varepsilon)$.

We could keep two terms in (2-51'); that would be the two-term outer expansion of the one-term inner expansion, which would have to match the one-term inner expansion of the two-term outer expansion. From (2-51') and (2-45b), we thus construct the equality:

$$Ux + \delta_0 \log r + \eta_0 \tan^{-1}\frac{y}{x} + A_{00} = Ux + \frac{1}{2}\gamma_1(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right].$$

This can be true only if the following are separately true:

$$\delta_0 = 0 \quad ; \quad \eta_0 = -\frac{1}{2\pi}\gamma_1(z) \quad ; \quad A_{00} = \frac{1}{2}\gamma_1(z) \quad . \qquad (2\text{-}52)$$

The first of these three equations means only that there is no net source strength in the 2-D problem. The second relates the 2-D vortex strength, η_0 , to the dipole density, γ_1 , in the far field. The latter can obviously be interpreted also as a vortex strength. The third equality relates the "constant" term, A_{00} , in the near-field solution to the far-field solution's dependence on z , the spanwise coordinate. It is important to include such a term as this in the near-field solution, because it provides a three-dimensional effect in the otherwise two-dimensional problems.

Presumably, the near-field problem can be solved somehow. If the body is simple enough, an analytic solution may be obtainable; with the available powerful methods of the theory of functions of a complex variable, it is even reasonable to hope to find such solutions. However, even if numerical methods must be used, the solution can be found. Then all of the constants in (2-51) except A_{00} are known. The constant of most interest at this moment is η_0 ; it will be non-zero only if some mechanism has been included that can generate and determine a circulation around the body. I shall assume that a Kutta condition is available for this purpose, since the present section is concerned with wings. Then, with η_0 known, we can find the first approximation to the vorticity (and dipole density) in the far field, by means of (2-52). At the same time, A_{00} is determined.

Nothing more can be done now unless we find a higher-order term in either the near- or far-field expansion. It is interesting to pursue the near-field solution further first.

When we substitute the expansion, (2-48), into the Laplace equation, (2-46), and keep only leading-order terms, we obtain the partial differential equation for Φ_1 :

$$\Phi_{1_{xx}} + \Phi_{1_{yy}} = - \Phi_{0_{zz}} \text{, in the fluid domain.}$$

Now Φ_0 was found to be $O(\varepsilon)$, and we might reasonably expect that Φ_1 would be $O(\varepsilon^2)$. In fact, this turns out to be quite correct. In the equation just above, this means that the left-hand side is $O(1)$ and the right-hand side is $O(\varepsilon)$. Asymptotically, then, we have that:

$$\Phi_{1_{xx}} + \Phi_{1_{yy}} = 0 \quad \text{in the fluid domain.}$$

We again have a purely two-dimensional boundary-value problem to solve, if we can state the boundary conditions appropriately. From (2-47'), we find by the same arguments that:

$$\frac{\partial \Phi_1}{\partial N} = 0 \qquad \text{on the body.}$$

We do not know the conditions at infinity yet, but let us assume that the condition on Φ_1 is similar to that on Φ_0, i.e., the gradient of Φ_1 should be bounded.

This problem is identical to the Φ_0 problem, and so we can represent its solution outside of some circle by another series like the one in (2-51). We have not determined yet what the coefficients of the increasing terms are like, and so we allow two more arbitrary terms (the first two terms in the following):

$$\Phi_1(x,y;z) = \alpha_1 x + \beta_1 y + \delta_1 \log r + \eta_1 \tan^{-1}\frac{y}{x} + A_{10} + \frac{A_{11}\cos\theta}{r} + \frac{B_{11}\sin\theta}{r} + \frac{A_{12}\cos 2\theta}{r^2} + \frac{B_{12}\sin 2\theta}{r^2} + \dots \qquad (2\text{-}53)$$

All terms must be the same order of magnitude if $r = O(\varepsilon)$. Assuming that order to be ε^2, we have:

$$\alpha_1 , \beta_1 = O(\varepsilon) \; ; \; \delta_1 , \eta_1 , A_{10} = O(\varepsilon^2) \; ; \; A_{11} , B_{11} = O(\varepsilon^3) \; ; \text{etc.}$$

With this information in hand, we combine the first two terms in the inner expansion and then we obtain the outer expansion of the two-term inner expansion:

$$\begin{aligned} \phi(x,y,z) \sim\; & Ux && O(1) \\ & + \eta_0 \tan^{-1}\frac{y}{x} + A_{00} + \alpha_1 x + \beta_1 y && O(\varepsilon) \\ & + \frac{A_{01}\cos\theta}{r} + \frac{B_{01}\sin\theta}{r} + \delta_1 \log r + \eta_1 \tan^{-1}\frac{y}{x} + A_{10} && O(\varepsilon^2) \end{aligned} \qquad (2\text{-}54)$$

First we can keep just the first two orders of magnitude and match them with the two-term inner expansion of the two-term outer expansion, given in (2-45b). Using (2-52), we see that everything already matches except for the terms $\alpha_1 x + \beta_1 y$ and the integral in (2-45b). For these to match, we require that:

$$\alpha_1 = 0 \; ; \; \beta_1 = -\frac{1}{4\pi}\int_{-S}^{S} \frac{d\zeta \, \gamma_1'(\zeta)}{z - \zeta} \quad . \tag{2-55}$$

Physically, this means that the Φ_1 problem should have had as the condition at infinity,

$$| \Phi_1 - \beta_1 y | \to 0 \quad \text{as} \quad r \to \infty \quad ,$$

that is, there is a uniform stream at infinity, moving at a right angle to the actual incident uniform stream. This is the downwash velocity. With this condition at infinity known, the Φ_1 problem can be solved by the same method used for the Φ_0 problem, and all of the terms in (2-53) are then known, except A_{10} .

We have all of the information available to match the three-term outer expansion of the two-term inner expansion with the two-term inner expansion of the three-term outer expansion. Using (2-45c) and all of the terms in (2-54), we obtain the equation:

$$Ux + \eta_0 \tan^{-1}\frac{y}{x} + A_{00} + \beta_1 y + \frac{A_{01}\cos\theta}{r} + \frac{B_{01}\sin\theta}{r} + \delta_1 \log r + \eta_1 \tan^{-1}\frac{y}{x} + A_{10}$$

$$= Ux + \frac{1}{2}\gamma_1(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right] + \frac{y\mu_1(z) + x\lambda_1(z)}{2\pi(x^2 + y^2)}$$

$$- \frac{y}{4\pi}\int_{-S}^{S}\frac{d\zeta \, \gamma_1'(\zeta)}{z - \zeta} + \frac{1}{2}\gamma_2(z)\left[1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x}\right]$$

The unknown quantities are: A_{10} , μ_1 , λ_1 , γ_2 . This equation is satisfied only if:

$$\delta_1 = 0 \quad ; \quad \eta_1 = -\frac{1}{2\pi}\gamma_2(z) \quad ; \quad A_{10} = \frac{1}{2}\gamma_2(z) \; ;$$

$$\mu_1(z) = 2\pi B_{01} \quad ; \quad \lambda_1(z) = 2\pi A_{01} \quad . \tag{2-56}$$

From this matching step, we see that all quantities introduced so far are now completely known. There is no source strength in the second approximation*; there is a correction to the vorticity in the far-field description; there is a correction to the "constant" in the near-field problem; and the density of both vertical and longitudinal dipoles in the far field is known. It is interesting to note that the last were determined entirely from the lowest-order near-field solution, that is, from Φ_0 . When quadripoles first enter, it will be found that they too are determined in strength from Φ_0 solely.

The next term would be much more difficult to obtain, since, in the near field, it entails solving a Poisson equation in which the nonhomogeneous part depends on Φ_0 , which we have not obtained explicitly. (The right-hand side of (2-46) finally has an effect.) Also, spanwise effects occur in the body boundary condition, (2-47), for the first time. Therefore I shall put this problem to rest at this point. Table 2-2 shows the sequence of steps that we have followed in this problem.

One point in particular should be noted: The near-field problem was not linearized. If one can predict the flow around the two-dimensional forms which appear in the near-field problem, one is not limited to consideration of, say, thin wings. All that is necessary is that the spanwise

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*It would have been possible to eliminate the $\delta \log r$ terms in both problems above by noting that the body boundary conditions allow for no net source strength.

length be much greater than the dimensions in the two-dimensional problems and that there be gradual change in the body and flow geometry in the spanwise direction. Needless to say, the latter condition is usually violated at the wing tips, and so the analysis breaks down there. It may be hoped that the prediction of important physical quantities is not affected too seriously thereby, but higher and higher approximations certainly cannot be found until the extra singularities at the tips are removed somehow.

TABLE 2-2

HIGH-ASPECT-RATIO WING — SUMMARY

Terms	Far-Field Expansion	Near-Field Expansion	Quantity Determined by Matching
1	$\phi_0 = Ux$	Φ_0	1 Condition at infinity for Φ_0 problem
			2 Vorticity, $\gamma_1(z)$, in far field
2	$\phi_0 + \phi_1$	$\Phi_0 + \Phi_1$	3 Downwash velocity (condition at infinity for Φ_1 problem)
			4 Correction to vorticity in far field; densities of vertical, horizontal dipoles in far field
3	$\phi_0 + \phi_1 + \phi_2$		

2.3 Slender Body

In the previous section, we considered the flow around a slender body which was oriented with its long dimension perpendicular to the incident flow. Now we consider the flow around a slender body which is oriented with its long dimension approximately parallel to the incident flow. The same geometrical restrictions will be applied to the body in this problem, namely, that its transverse dimensions should be small compared with its long dimension and that cross-section shape, size, and orientation should vary gradually along the length.

Although both this section and the previous section concern slender bodies in an incident flow, convention says that only this section really presents "slender-body theory."

In ship hydrodynamics problems, slender-body theory has been applied mostly to nonlifting bodies, i.e., bodies not generating trailing vortex systems.* I shall limit myself here to such problems too. Specifically, I assume that there is no separation of the flow from the body; furthermore, there are no sharp edges at which a Kutta condition might be applied. The potential function should be continuous and single-valued throughout the fluid domain.

This restriction is not generally desirable. Certainly an important aspect of aerodynamics is the calculation of lift on a slender body which does generate a vortex wake; modern high-speed delta-wing aircraft and many slender missiles are genuine slender lifting bodies. There are several important ship-hydrodynamics problems which may ultimately

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*Obviously, a ship is a "lifting body," but I think it is commonly understood that the term implies a dynamic lift process, and that is the way I use it.

be best analyzed by a slender-wing* approach. Most important, perhaps, is the problem of a maneuvering ship. An attempt is made in this direction by Fedyayevskiy & Sobolev (1963), but it is not very successful because they use the conventional methods of slender-wing theory, and these break down in application to wings which are not more-or-less delta shaped.** A modern approach to slender-wing theory is given by Wang (1968).

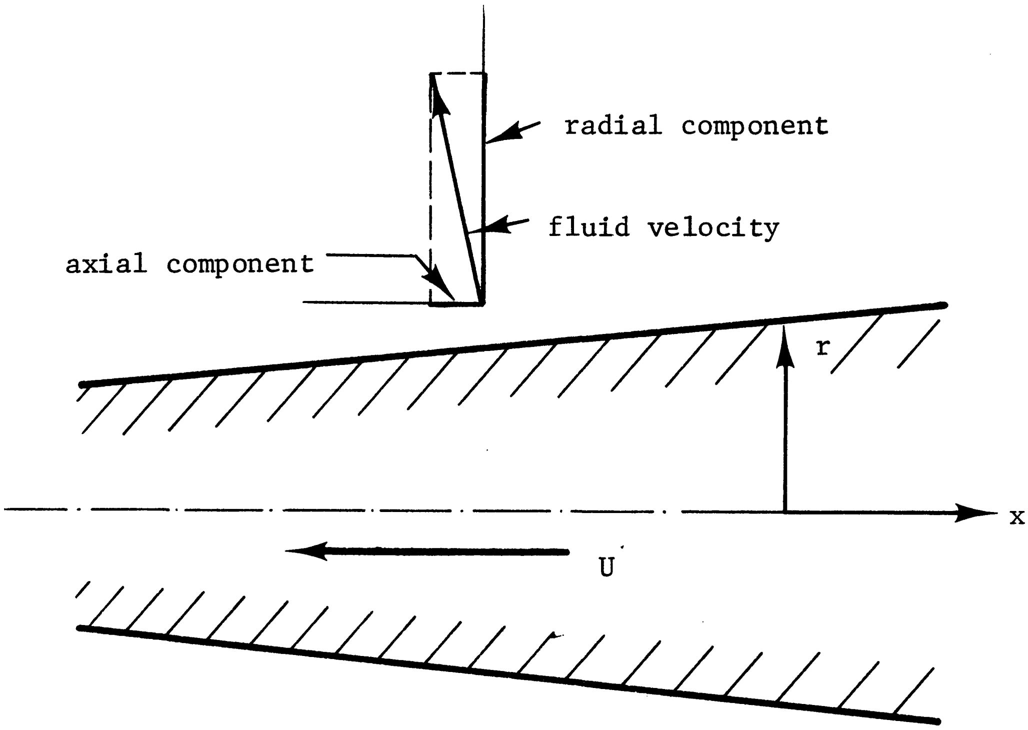

Figure (2-2). Fluid Velocity Near a Slender Body in Steady Motion.

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*"Slender wing", "wing of very low aspect ratio", and "slender lifting surface" are all equivalent terms in my usage.

**Conventional slender-wing theory can be used for wings in which the span increases monotonically downstream, ending in a squared-off trailing edge. If the incident stream is uniform and steady, the wing does not have to end at the location of the maximum span, but the part of the wing aft of this location must be uncambered. Not all of these condition are satisfied in the interesting ship maneuvering problems.

The physical ideas behind slender-body theory were developed fifty years ago, and the original way of looking at the problem is perhaps still the best way. Take a reference frame which is fixed with respect to the fluid at infinity. As a slender body moves past, one may imagine that its greatest effect on the fluid is to push it aside; the body also imparts to the fluid a velocity component in the axial direction, but this component should be quite small compared with the transverse component. *Both* components should be small compared with the forward speed of the body.

In modern slender-body theory, we attempt to formalize this estimate of the relative velocity-component magnitudes. We devise a procedure that automatically arranges velocities in the anticipated order:

1) Forward speed
2) Transverse perturbation
3) Longitudinal perturbation

When this pattern comes out of the boundary-value problem, we then investigate further to see what other patterns follow from the same assumptions. The whole body of assumptions, results, and intermediate mathematics constitute what we call "slender-body theory".

In aerodynamics, the original intuitive approach of Munk was not completely displaced until the late 1940's. The newer, more systematic approach which developed then is described well by Ward (1955). For the first time, it was possible to predict with some confidence how the flow around the various cross-sections interacted. There were some difficulties in principle, even with the new approach; what we now call the "outer expansion" of the problem was in effect forced to satisfy body boundary conditions. The difficulty is somewhat comparable to trying to force a Laurent-series solution to satisfy prescribed conditions which are stated on a contour inside the minimum circle of convergence. A readable, refreshing account of slender-body theory in the 1950's has been provided by Lighthill (1960).

During the early 1960's, slender-body theory was applied to ship hydrodynamics problems by several investigators. Probably the earliest to try this on a major scale was Vossers (1962); he attacked a variety of steady- and unsteady-motion problems by slender-body theory. He used a Green's function approach, which apparently avoids the fundamental difficulty in principle of the previous method. However, it is really too much to hope to obtain asymptotic estimates of five-fold integrals — without making mistakes. Apparently Vossers did hope for too much, but Joosen (1963) and (1964) corrected many of his mistakes. Newman (1964) also advocated the Green's-function approach and produced some interesting results.

The modern (*i.e.*, fashionable) alternative is to use the method of matched asymptotic expansions. In ship hydrodynamics, Tuck (1963a) first used this method in his doctoral thesis at Cambridge University. It avoids the difficulties in principle of Ward's approach, and it is easier to work with than the Green's-function method. Of course, the method of matched asymptotic expansions has its own set of difficulties of principle. However, it is the method that I shall pursue here.*

In any case, the analysis can be no better than the assumptions which are made at the beginning. Therefore I

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*A very recent account of slender-body theory, particularly with respect to its applications in ship hydrodynamics, has been published by Newman (1970). I think that his presentation and mine generally complement each other (and perhaps occasionally contradict too). Newman has provided a survey that seems comparable in intent to the one by Lighthill (1960), mentioned above, whereas I am trying to place slender-body theory into a hierarchy of singular perturbation problems. My emphasis is on the development and application of the method of solution.

shall be (perhaps painfully) explicit about the assumptions.

2.31 *Steady Forward Motion.* Let the body surface be specified by the equation:

$$r = r_0(x,\theta) , \quad x \text{ in } \Lambda ,$$

where $r = (y^2+z^2)^{1/2}$, and θ is an angle variable measured about the x axis. It will be assumed that $r = O(\varepsilon)$. In this section, I take the most conventional definition of θ , namely, that it be measured in a right-handed sense from the y axis. (In ship problems, it is more convenient to measure the angle from the negative vertical axis.) Λ is the part of the x axis which coincides with the longitudinal extent of the body; typically, one might take it to be the interval, $- L/2 < x < L/2$, but I shall not insist that the origin be located at the mid-length section. Figure 2-3 shows a typical cross-section.

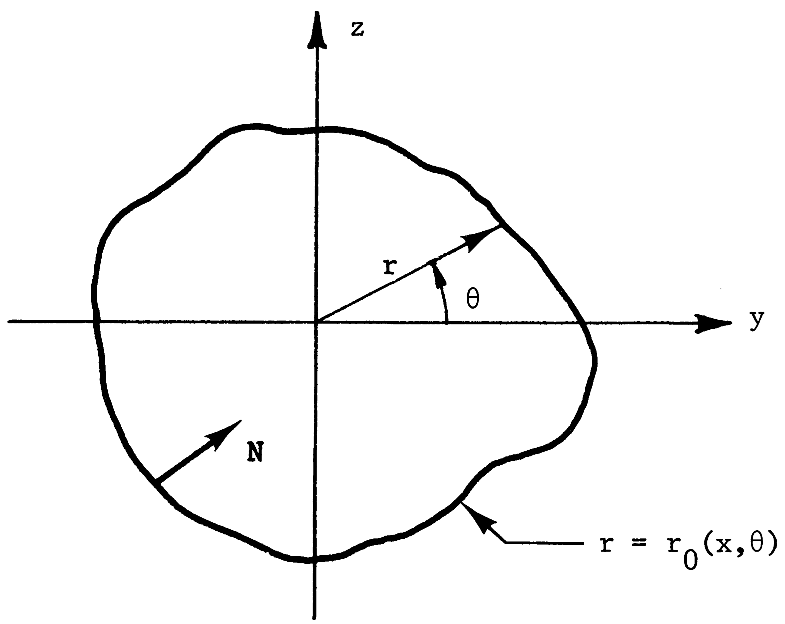

Figure (2-3). Cross-Section of the Slender Body.

As usual, assume that there exists a velocity potential, $\phi(x,y,z)$, which satisfies the Laplace equation. There is an incident stream which, in the absence of the body, is a uniform flow in the positive x direction, with the velocity potential Ux . It will be convenient to use cylindrical coordinates, (x,r,θ) , in which case the Laplace equation takes the form:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{xx} = 0 \text{ , for } r > r_0(x,\theta) \text{ .} \tag{2-57}$$

The kinematic boundary condition on the body can be written:

$$\phi_x r_{0x} - \phi_r + \frac{1}{r_0^2}\phi_\theta r_{0\theta} = 0 \quad \text{on} \quad r = r_0(x,\theta) \text{ .} \tag{2-58}$$

With respect to the physical arguments presented at the beginning of the section, note that our frame of reference is now moving with the body. Therefore, the velocity components should be ordered:

$$\partial\phi/\partial x \sim U = O(1) \text{ ; } \partial\phi/\partial y \text{ , } \partial\phi/\partial z \text{ , } \partial\phi/\partial r = o(1) \text{ ;}$$

$$\partial(\phi - Ux)/\partial x = o(\partial\phi/\partial r) \text{ .}$$

In each case, of course, the appropriate limit operation is that $\varepsilon \to 0$, where ε is the slenderness of the parameter. These order relations should be valid *near* the body.

Far away, there will be the uniform stream, which is $O(1)$, but there is no reason to assume that the perturbation velocity will have components with differing orders of magnitude.

These order-of-magnitude relations all come about automatically if, in the near field, we define new variables:

$$r = \varepsilon R \text{ , } \quad y = \varepsilon Y \text{ , } \quad z = \varepsilon Z \text{ ,}$$

and assume that differentiation with respect to x , Y , Z , R , and θ all have no effect on the order of magnitude of a quantity. Thus, suppose that the potential in the near field can be written: $\phi(x,y,z) = Ux + \Phi(x,Y,Z)$. Then the derivatives have the following orders of magnitude:

$$\frac{\partial\phi}{\partial x} = U + \frac{\partial\Phi}{\partial x} = O(1) + O(\Phi) \; ; \; \frac{\partial\phi}{\partial y} = \frac{\partial\Phi}{\partial y} = \frac{1}{\varepsilon}\frac{\partial\Phi}{\partial Y} = O(\Phi/\varepsilon) \; ;$$

$$\frac{\partial\phi}{\partial z} = O(\Phi/\varepsilon) \; ; \; \frac{\partial\phi}{\partial r} = O(\Phi/\varepsilon) \; .$$

It will turn out that $\Phi = O(\varepsilon^2)$. This means that the transverse velocity components, ϕ_y , ϕ_z , and ϕ_r are all $O(\varepsilon)$, that is, they are proportional to the slenderness parameter. Note also that a circumferential velocity component would be given by $(1/r)\partial\phi/\partial\theta = (1/\varepsilon R)\partial\Phi/\partial\theta = O(\Phi/\varepsilon)$, when we interpret $R = O(1)$ (that is, in the near field), and so circumferential and radial velocity components have the same order of magnitude. The perturbation of the longitudinal velocity component is $O(\Phi) = O(\varepsilon^2)$, which is, appropriately, a higher order of magnitude than that of the transverse velocity components.

In the far field, we assume that differentiation with respect to any of the natural space variables has no order-of-magnitude effect. Thus, we use the Cartesian coordinates (x,y,z) and the cylindrical coordinates (x,r,θ) in a very conventional manner. As $\varepsilon \to 0$, the slender body becomes more and more slender, shrinking down to a line which coincides with part of the x axis. (This is the line segment that I defined as Λ previously.) In the limit, there is no body at all and thus no disturbance of the incident uniform flow. In the far field, the disturbance is always $o(1)$. Therefore the far field consists of the entire space except the x axis, and the potential function must satisfy the

Laplace equation everywhere except possibly on the x axis.

At infinity, it is reasonable to require that the perturbation of the incident flow should vanish, which implies that the perturbation potential must be regular even at infinity. A velocity potential cannot be regular throughout space, including infinity, unless it is trivial. Therefore the velocity potential must be singular somewhere, and the only place in the far field where such behavior is permitted is on the x axis. Our far-field slender-body problems all reduce to finding appropriate singularity distributions on the x axis.

The Far-Field Singularity Distributions. In the far field, the first term in the asymptotic expansion for the potential function will be Ux . All of the following terms must represent flow fields for which the velocity approaches zero at infinity; they represent distributions of singularities on the x axis. The nature of the singularities can only be determined in the matching process, and so we must generally be prepared to handle all kinds of singularities.

One of the easier ways of doing this is to apply a Fourier transform to the Laplace equation, replacing the x dependence by a wave-number dependence. The resulting partial differential equation in two dimensions can be solved by separation of variables in cylindrical coordinates. When we require that the potential functions be single valued, we find that the solutions must all be products of:

$$K_n(|k|r) \quad \text{or} \quad I_n(|k|r) \qquad \text{and} \qquad \sin n\theta \quad \text{or} \quad \cos n\theta \quad ,$$

where K_n and I_n denote modified Bessel functions. Since I_n is poorly behaved when its argument is large, we reject it, so that the solution consists of terms:

$$K_n(|k|r)[\alpha \cos n\theta + \beta \sin n\theta] \quad .$$

The quantities α and β are constants with respect to r and θ, but they are both functions of k. They also depend on the index n, of course. The general solution is obtained by combining all such possible solutions. Any term in the far-field expansion of the potential function might be of the form:

$$\phi_m(x,y,z) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\ e^{ikx}\ K_n(|k|r)\ [a^*_{mn}(k)\cos n\theta + b^*_{mn}(k)\sin n\theta] \ , \tag{2-59}$$

where $a^*_{mn}(k)$ and $b^*_{mn}(k)$ are unknown functions. The most general far-field expansion comprises the incident-flow potential, Ux, and a sum of terms like the above, that is,

$$\phi(x,y,z) \sim Ux + \sum_{m=1}^{M} \phi_m(x,y,z) \quad \text{for fixed } (x,y,z) \text{ as } \varepsilon \to 0 \ . \tag{2-60}$$

It will be necessary to have the inner expansion of the outer expansion. This means that we must interpret r to be $O(\varepsilon)$ in the above expressions, instead of $O(1)$ as heretofore, and rearrange terms according to their dependence on ε. The easiest procedure is to replace any of the K_n functions in (2-59) by its series expansion for small argument. We obtain formulas such as the following:

<u>n = 0:</u>

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\ K_0(|k|r)\ a^*_{m0}(k) \tag{2-61a}$$

$$\sim -\frac{\log r}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx} a^*_{m0}(k) - \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx} a^*_{m0}(k)\ \log\frac{C|k|}{2} \quad ;$$

n > 0:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\ K_n(|k|r)\ a_{mn}^*(k)\binom{\cos}{\sin}n\theta \tag{2-61b}$$

$$\sim \frac{2^{n-1}(n-1)!}{2\pi r^n}\binom{\cos}{\sin}n\theta \int_{-\infty}^{\infty} \frac{dk}{|k|^n}\ e^{ikx} a_{mn}^*(k) \quad .$$

Physically, the n=0 integral represents the potential for a line of sources. This can be seen directly from (2-61a): As $r \to 0$, the function is proportional to $\log r$, which is the potential function for a source in two dimensions. However, the strength of the apparent 2-D source is a function of x . In fact, the integral defining that strength is identical to the integral which gives the inverse of a Fourier transform. Let $a_{m0}(x)$ be the function having $a_{m0}^*(k)$ as its Fourier transform, and further define:

$$\sigma_m(x) \equiv -\ 2\pi\ a_{m0}(x) \quad .$$

Then the result in (2-61a) can be rewritten:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\ K_0(|k|r)\ a_{m0}^*(k) \sim \frac{1}{2\pi}\sigma_m(x)\log r - \frac{1}{4\pi} f_m(x) \ , \tag{2-61a'}$$

where

$$f_m(x) = \int_{-\infty}^{\infty} d\xi\ \sigma_m'(\xi)\ \log 2|x-\xi|\ \mathrm{sgn}\ (x-\xi) \quad . \tag{2-61a''}$$

By manipulating the full integral containing K_0 , one can also show that:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\ K_0(|k|r)\ a_{m0}^*(k) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\sigma_m(\xi)\ d\xi}{[(x-\xi)^2 + r^2]^{1/2}} \ , \tag{2-62}$$

which is easily recognized as the potential function for a line distribution of sources.

Similarly, the other integrals can be interpreted in terms of dipoles, quadripoles, etc. In particular, we see that for $n = 1$ the inner expansion of the integral reduces to the potential in two dimensions for a dipole. We may consider the variable x as a parameter, and then we have a different 2-D dipole strength at each x .

The Sequence of Near-Field Problems. In the near field, we can formalize our procedure by making the changes of variables already mentioned, $r = \varepsilon R$, $y = \varepsilon Y$, $z = \varepsilon Z$, then assuming that differentiation with respect to R , Y , or Z does not affect orders of magnitude. Instead of doing this, I shall simply retain the ordinary variables, r , y , and z , and I ask the reader to recall that differentiation with respect to any of these three variables causes a change in order of magnitude. Thus, for example, $\partial\phi/\partial r = O(\phi/\varepsilon)$ in the near field.

In cylindrical coordinates, the Laplace equation and the body boundary condition can be written as follows:

$$[L] \qquad \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -\phi_{xx} ; \qquad (2\text{-}57')$$

$$[H] \qquad \frac{\partial\phi}{\partial N} = \mathbf{N}\cdot\nabla\phi = \frac{-\phi_r + (1/r_0^2)r_{0\theta}\phi_\theta}{\sqrt{[1 + (r_{0\theta}/r_0)^2]}} = \frac{\phi_x\, r_{0x}}{\sqrt{[1 + (r_{0\theta}/r_0)^2]}}$$

$$\text{on } r = r_0(x,\theta) . \qquad (2\text{-}58')$$

The definition of $\mathbf{N}$ is analogous to that in (2-47'). It is a unit vector lying in the cross-section plane at some x , perpendicular to the contour of the body in that cross-section. It has the three components:

$$N = \frac{(0, -1, r_{0\theta}/r_0)}{\sqrt{[1 + (r_{0\theta}/r_0)^2]}} ,$$

measured in the x, r, and θ directions, respectively. Equation (2-58'), like (2-58), expresses the fact that $\partial\phi/\partial n = 0$, where **n** is the unit vector normal to the body surface.

Let the inner expansion be expressed as follows:

$$\phi(x,y,z) \sim \sum_{n=0}^{N} \Phi_n(x,y,z) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for fixed} \quad (x, y/\varepsilon, z/\varepsilon) \quad .$$

Substitute this expansion into the [L] and [H] conditions above:

$$[L] \quad \nabla^2_{y,z}(\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \dots) = -(\Phi_{0_{xx}} + \Phi_{1_{xx}} + \dots) ;$$

$$[H] \quad \frac{\partial\Phi_0}{\partial N} + \frac{\partial\Phi_1}{\partial N} + \frac{\partial\Phi_2}{\partial N} + \dots = -\frac{r_{0_x}(\Phi_{0_x} + \Phi_{1_x} + \dots)}{\sqrt{[1 + (r_{0\theta}/r_0)^2]}} ;$$

The operator $\nabla^2_{y,z}$ is the 2-D Laplacian in the y-z plane, that is,

$$\nabla^2_{y,z} \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} .$$

It can be proven that the first term in the expansion, Φ_0, represents just the uniform stream:

$$\Phi_0(x,y,z) = Ux .$$

This appears so obvious that I pass on immediately to the Φ_1 problem. From the [L] and [H] conditions, we find:

$$[L_1] \quad \nabla^2_{y,z}\Phi_1 = 0 \quad \text{in the fluid domain;}$$

$$[H_1] \quad \frac{\partial \Phi_1}{\partial N} = -\frac{U\, r_{0_x}}{\sqrt{[1 + (r_{0\theta}/r_0)^2]}} \quad \text{on} \quad r = r_0 \quad . \qquad (2\text{-}63)$$

Finding Φ_1 is strictly a problem in two dimensions. In fact, it is just the problem that the early aerodynamicists put forth intuitively at the beginning of their slender-body analysis. (It was also the end of their analysis!) For an arbitrary body shape, we might have to solve this boundary-value problem numerically; that is not much of a problem today. However, we are not yet ready to work with numbers, because the formulation of the problem is not quite complete: we have not specified the behavior of Φ_1 at infinity. To do so requires that we match the unknown solution of this problem to the far-field expansion.

First, note what (2-63) tells us about the order of magnitude of Φ_1 . The right-hand member is $O(\varepsilon)$ and the left-hand member is $O(\Phi_1/\varepsilon)$ (because of the differentiation in the transverse direction), which together imply:

$$\Phi_1 = O(\varepsilon^2) \quad .$$

Actually, (2-63) says only that Φ_1 cannot be *higher* order than ε^2 ; it could be *lower* order if the matching introduced some effect that required ε^2 to be $o(\Phi_1)$, but this does not happen.

This Φ_1 problem is remarkably similar to the Φ_0 problem in Section 2.2. If we can assume that $\nabla\Phi_1$ is bounded at infinity, then we can express Φ_1 in a series just like the one in (2-51). Whether $\nabla\Phi_1$ really is bounded at infinity can only be determined from the matching, of course, but we go ahead with the assumption, trusting that our method will show us if we have made unwarranted assumptions.

It should be noted too that there are important differences between this problem and the problem of Section 2.2. The Neumann-type of condition on the body was homogeneous there, but it is not homogeneous here. Thus, one may expect that there may be a non-zero net source strength inside the body in the present problem. What happens at infinity is also different. In the earlier problem, the potential had to represent a uniform flow at infinity, and we supposed that there might be the proper circumstances that a circulation flow could occur. In the present problem, the uniform flow at infinity has been included in Φ_0 , and so we might expect that Φ_1 will represent a flow with velocity vanishing at infinity, and there appears to be no reason to expect a circulation in the 2-D problem.

It would be tedious to go through the same arguments that were used previously, and so I shall only summarize the results that would be obtained after a careful matching process. In the near field, Φ_1 does indeed yield a velocity field which is bounded in magnitude at infinity, and there is no circulation. Thus, it can be represented by the series:

$$\Phi_1(x,y,z) = C_1 + \frac{A_{10}}{2\pi} \log r + \frac{A_{11}\cos\theta + B_{11}\sin\theta}{2\pi\, r} + \ldots . \tag{2-64}$$

The "constants" are all functions of x . In the near field, all terms must be the same order of magnitude, by definition, and so A_{11} and B_{11} are $O(\varepsilon A_{10})$. (I am, as usual, ignoring quantities which are $O(\log \varepsilon)$.) In the matching, the $1/r$ terms are lost in the first round, and the $\log r$ and constant terms are forced to match the inner expansion of the outer expansion.

In the outer expansion, (2-60), only a line of sources in the ϕ_1 term of (2-60) can match the near-field expansion properly. That is, in (2-59) and (2-60), we have the following:

$$a^*_{1n}(k) = b^*_{1n}(k) = 0 \quad \text{except for } n = 0 \ .$$

The two-term outer expansion and its two-term inner expansion are:

$$\phi(x,y,z) \sim Ux + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} K_0(|k|r)\, a^*_{10}(k) \qquad (2\text{-}65a)$$

$$\sim Ux + \frac{1}{2\pi}\sigma_1(x)\log r - \frac{1}{4\pi} f_1(x) \ , \qquad (2\text{-}65b)$$

where (2-61a') has been used to express the latter.

Matching between the near-field and far-field then shows that:

$$A_{10} = \sigma_1(x) \qquad (2\text{-}66a)$$

$$C_1 = -\frac{1}{4\pi} f_1(x) \ . \qquad (2\text{-}66b)$$

In obtaining an actual solution, one proceeds through the following steps: 1) Matching shows that Φ_1 represents a flow with bounded velocity at infinity. 2) Then the Φ_1 problem is completely formulated and can be solved. 3) From the solution of the Φ_1 problem, the function $A_{10}(x)$ can be determined, which, through the matching, gives $\sigma_1(x)$, and the far-field two-term expansion is known. 4) From the matching relation for $C_1(x)$, along with formula (2-61a"), the near-field potential is known completely to two terms, and the $C_1(x)$ term includes the most important effects of interaction among sections. This sequence of steps shows what an intimate relationship exists between near- and far-field expansions.

The source strength, $\sigma_1(x) = A_{10}(x)$, can be computed without the necessity of solving the flow problem. In the near-field picture, draw a circle which encloses the body section. The net flux rate across this circle is just A_{10}. From the body boundary condition, (2-63), one can show that there is a net flux rate across the body surface, and it is given by $Us'(x)$, where:

$$s(x) = \int_0^{2\pi} d\theta \; r_0^2(x,\theta) = \text{cross-section area at } x . \tag{2-67a}$$

The two fluxes must be equal, and so we find that:

$$\sigma_1(x) = A_{10}(x) = Us'(x) \tag{2-67b}$$

Thus, the source strength is proportional to the rate of change with x of the body cross-sectional area.

I shall not pursue the solution to higher order of magnitude, although there is no insuperable difficulty in doing so. Rather, I prefer to point out several interesting facts about the solution and then close this section.

In the far field, the solution to two terms is axially symmetric, although the body is not a body of revolution. The near-field two-term expansion is not symmetric in this way unless the body is circular and is aligned with the incident flow. However, the near-field solution can be represented by the series,

$$\phi(x,y,z) \sim Ux + \frac{\sigma_1(x)}{2\pi} \log r - \frac{1}{4\pi} f_1(x) + \frac{A_{11}\cos\theta + B_{11}\sin\theta}{2\pi r} + \ldots ,$$

and, at large r, the axially symmetric terms dominate this series.

If the far-field expansion is carried to three terms, it will be found that the third term can be interpreted in terms of a line of dipoles, both vertically and horizontally oriented. Such terms will be of the form given in (2-61b), with $n = 1$; they contain unknown functions, $a^*_{21}(k)$ and $b^*_{21}(k)$, which must be determined through matching. These unknown functions will depend entirely on the solution of the Φ_1 problem discussed above. In fact, one finds explicitly that:

$$A_{11}(x) = \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ikx} a^*_{21}(k) \; ; \quad B_{11}(x) = \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ikx} b^*_{21}(k) .$$

Thus, the two-term inner expansion contains enough information to determine the strength of the dipoles which appear in the third term of the far-field expansion. The same inner expansion would determine the strengths of quadripoles in the fourth term of the far-field expansion, etc., etc.

On the other hand, the far-field expansion (even at the second term) contains much information about three-dimensional effects, information which is largely lacking in the near-field expansion. I have already pointed out that only the "constant" term contains important information about 3-D effects in the two-term near-field expansion. The rest of the Φ_1 solution depends on just the shape of the local section and the local rate of change of section shape and size. If higher-order near-field terms are found, it will be seen that they are influenced even by the two-term outer expansion. In fact, the "constant" term in Φ_1 can be interpreted as a modification to the incident stream, caused by the presence of all the other cross-sections of the body. The effects of this extra incident flow on the transverse velocity field are not perceived until one finds a higher order expansion of the solution in the near field.

The briefest account of slender-body theory would be seriously lacking without mention of the possibly catastrophic effects of body ends. If a body has a blunt end, then $s(x)$ increases linearly in some neighborhood of the end. Accordingly, $s'(x)$ is discontinuous, jumping from a value of zero just beyond the end to a finite value at the end. This is an obvious violation of our assumptions about "slenderness." But trouble develops even without a blunt-ended body. For example, if the tip is pointed (but not cusped), there will still be a stagnation point right at the point. Thus this case violates the assumption that longitudinal perturbation of the incident flow velocity is a second-order quantity.

Sometimes these end effects can be overlooked with impunity. There are major examples later in this paper. However, even when we have such luck, we must be prepared to have higher-order expansions go awry.

2.32 *Small-Amplitude Oscillations at Forward Speed.* In this section, we consider the same kind of body as in Section 2.31, namely, a slender body which is aligned approximately with an incident stream. However, now we formulate a time-dependent problem in which the body performs small-amplitude oscillations while it moves through the fluid.

It would be entirely feasible to consider the general problem in which the body oscillates with the six degrees of freedom of a rigid body. (We could even include more degrees of freedom by allowing deformations of the body.) However, the major concepts should be clear if we allow only two degrees of freedom, a) a lateral translation, comparable to the heave or sway of a ship, and b) a rotation, like the pitch or yaw of a ship.

In this section, I shall depart from my usual approach and first treat the problem for a perfectly general body, then introduce the slenderness property at the very end. This introduces a bit of variety, but more important is the fact that some general properties of the physical system can be pointed out, without any confusion over the effects of assuming slenderness of the body.

We use two coordinate systems: Oxyz is fixed in the body with its origin at the center of gravity, and O'x'y'z' is an inertial system which moves with the mean motion of the body center of gravity. With respect to the stationary fluid at infinity, the mean motion is a translation at speed U in the negative x' direction; thus, in the O'x'y'z' system, there appears to be a flow past the body in the positive x' direction.

The two reference systems differ because the body oscillates in the z direction, the instantaneous displacement being denoted by $\xi_3(t)$, and rotates about the y axis, the angular displacement being denoted by $\xi_5(t)$.

In a more general problem, we could let $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$ denote surge, sway, and heave (displacements along the x , y , and z axes, respectively) and $\xi_4(t)$, $\xi_5(t)$, and $\xi_6(t)$ denote roll, pitch, and yaw (rotations about the x , y , and z axes, respectively). It will be assumed explicitly that $\xi_j(t)$ is a small quantity, so that squares and products can be neglected in comparison with the quantity itself. Furthermore, it will be assumed that $\xi_j(t)$ varies sinusoidally in time and it will be represented by the real part of a complex function varying as $e^{i\omega t}$. We shall not usually bother to indicate that only the real part of a complex quantity is to be implied. Thus we can write:

$$\dot{\xi}_j(t) = i\omega\xi_j(t) \quad . \tag{2-68}$$

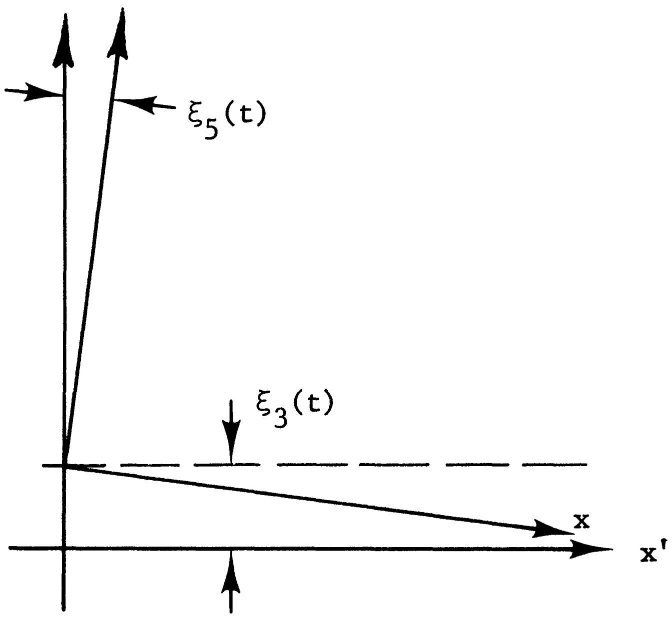

Figure (2-4). Two Coordinate Systems for Oscillation Problem.

The relationship between the two coordinate systems is as follows (See Figure (2-4)):

$$\begin{aligned} x &= x' \cos \xi_5 - (z'-\xi_3) \sin \xi_5 \simeq x' - z'\xi_5 \quad ; \\ y &= y' \quad ; \\ z &= x' \sin \xi_5 + (z'-\xi_3) \cos \xi_5 \simeq x'\xi_5 + z' - \xi_3 \quad . \end{aligned} \tag{2-69}$$

The absolute velocity of the center of gravity is:

$$
\begin{aligned}
- U \mathbf{i}' + i\omega\xi_3\mathbf{k}' &= (- U \cos \xi_5 - i\omega\xi_3 \sin \xi_5)\, \mathbf{i} \\
&\quad + (- U \sin \xi_5 + i\omega\xi_3 \cos \xi_5)\mathbf{k}
\end{aligned}
\qquad (2\text{-}70)
$$

$$
\simeq \; - U \mathbf{i} + (i\omega\xi_3 - U\xi_5)\, \mathbf{k} \qquad (2\text{-}70')
$$

where $(\mathbf{i},\mathbf{j},\mathbf{k})$ are unit vectors in the Oxyz system, and $(\mathbf{i}',\mathbf{j}',\mathbf{k}')$ are unit vectors in the O'x'y'z' system.

Let the body surface be defined by the equation:

$$
S(x,y,z) = 0 . \qquad (2\text{-}71)
$$

Denote the unit vector normal to the surface, inwardly directed, by $\mathbf{n}$:

$$
\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} . \qquad (2\text{-}72a)
$$

It is convenient to make a number of other definitions, as follows:

1) n_j : Extend the above definition of n_j to $j = 4,5,6$ as follows:

$$
\mathbf{r} \times \mathbf{n} = n_4\mathbf{i} + n_5\mathbf{j} + n_6\mathbf{k} \qquad (2\text{-}72b)
$$

where

$$
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} .
$$

In particular, note that:

$$
n_3 = \mathbf{n} \cdot \mathbf{k} \quad \text{and} \quad n_5 = zn_1 - xn_3 . \qquad (2\text{-}72')
$$

2) ϕ_1 : This is a normalized velocity potential. It satisfies:

$$\left.\begin{aligned} \phi_{i_{xx}} + \phi_{i_{yy}} + \phi_{i_{zz}} &= 0 \quad \text{in fluid region;} \\ \frac{\partial \phi_i}{\partial n} &= n_i \quad \text{on} \quad S(x,y,z) = 0 \; ; \\ |\nabla \phi_i| &\to 0 \quad \text{at infinity.} \end{aligned}\right\} \qquad (2\text{-}73)$$

3) $\mathbf{v}(x,y,z)$: This is a normalized fluid velocity, equal to the fluid velocity at (x,y,z) when an incident stream flows past the body, the stream having unit velocity, $\mathbf{i}$, at infinity. It can be represented as follows:

$$\mathbf{v}(x,y,z) = \nabla[x - \phi_1(x,y,z)] \; . \qquad (2\text{-}74)$$

4) m_i : This quantity is related to the rate of change of $\mathbf{v}(x,y,z)$ in the neighborhood of the body, as follows:

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} \equiv \mathbf{m} \equiv -(\mathbf{n}\cdot\nabla)\mathbf{v} \; ; \qquad (2\text{-}75a)$$

$$m_4\mathbf{i} + m_5\mathbf{j} + m_6\mathbf{k} \equiv -(\mathbf{n}\cdot\nabla)(\mathbf{r}\times\mathbf{v}) \; . \qquad (2\text{-}75b)$$

In particular, note that:

$$m_3 = \frac{\partial \phi_{1_z}}{\partial n} \equiv \phi_{1_{zn}} \; ; \qquad (2\text{-}75a')$$

$$\begin{aligned} m_5 &= -\frac{\partial}{\partial n}(\mathbf{j}\cdot\mathbf{r}\times\mathbf{v}) = -\frac{\partial}{\partial n}(zv_1 - xv_3) \\ &= -\frac{\partial}{\partial n}[z(1-\phi_{1_x}) + x\phi_{1_z}] = -n_3 + \frac{\partial}{\partial n}(z\phi_{1_x} - x\phi_{1_z}) \; . \end{aligned} \qquad (2\text{-}75b')$$

5) ψ_i : This is another useful normalized velocity potential. It is related to m_i the way ϕ_i is related to n_i . It satisfies:

$$\left.\begin{aligned} \psi_{i_{xx}} + \psi_{i_{yy}} + \psi_{i_{zz}} &= 0 \quad \text{in fluid region;} \\ \frac{\partial \psi_i}{\partial n} &= 0 \quad \text{on} \quad S(x,y,z) = 0 \; ; \\ |\nabla \psi_i| &\to 0 \quad \text{at infinity.} \end{aligned}\right\} \qquad (2\text{-}76)$$

In particular, it can be seen that these conditions are satisfied for $i = 3,5$ if:

$$\psi_3(x,y,z) = \phi_{1_z}(x,y,z) \; ; \qquad (2\text{-}76')$$

$$\psi_5(x,y,z) = -\phi_3(x,y,z) - (z\phi_{1_x} - x\phi_{1_z}) \; . \qquad (2\text{-}76'')$$

(The last term does satisfy the Laplace equation.)

Now we can write down the velocity potential for the combined translation and oscillation in terms of the above-defined quantities. It is a well-known fact of classical hydrodynamics that the fluid motion can be expressed as a superposition of six separate motions, each of which would be caused by the motion of the body in one of the rigid-body degrees of freedom. However, it is essential for the use of this fact that the description be made in terms of a coordinate system fixed with respect to the body. Note that there is no linearization implicit in this superposition, in the sense that there is no requirement that motions be small in any way. In the body-fixed reference frame, the velocity potential is:

$$[-U \cos \xi_5 - i\omega\xi_3 \sin \xi_5]\phi_3(x,y,z)$$
$$+ \; [-U \sin \xi_5 + i\omega\xi_3 \cos \xi_5]\phi_5(x,y,z) \; + \; i\omega\xi_5\phi_5(x,y,z) \quad .$$

The nature of the superposition is obvious when we compare the first two coefficients here with (2-70). However, it must also be recalled that the velocity potential obtained in this way gives the *absolute* velocity of the fluid, that is, the gradient of this potential is the velocity in a reference frame fixed to the fluid at infinity.* Thus, we must add to this potential an extra term to provide for the apparent incident stream in the observation reference frame. The latter has the velocity potential Ux' , and so the complete potential is:

$$\begin{aligned}\phi(x',y',z',t) &= Ux' - (U\cos\xi_5 + i\omega\xi_3\sin\xi_5)\,\phi_1(x,y,z) \\ &\quad + (-U\sin\xi_5 + i\omega\xi_3\cos\xi_5)\,\phi_3(x,y,z) + i\omega\xi_5\,\phi_5(x,y,z)\end{aligned} \tag{2-77}$$

$$\begin{aligned}&\simeq Ux' - U\,\phi_1(x,y,z) \\ &\quad + [i\omega\,\phi_3(x,y,z)]\,\xi_3(t) \\ &\quad + [i\omega\,\phi_5(x,y,z) - U\,\phi_3(x,y,z)]\,\xi_5(t) \quad .\end{aligned} \tag{2-77'}$$

The potential ϕ has been defined basically in terms of the inertial reference frame, although most of the right-hand side here is expressed in terms of the body-fixed system. Note that not only the incident stream is defined in terms of primed coordinates, but also the body motion is really defined in those coordinates as well; in particular, heave motion is a translation of the body along an axis fixed with respect to the fluid at infinity.

The Bernoulli equation must be used for computing the pressure:

$$-\frac{p}{\rho} = \phi_t + \frac{1}{2}\left(\phi_{x'}^2 + \phi_{y'}^2 + \phi_{z'}^2\right) \quad .$$

*This can be concluded also by recalling the definition of ϕ_j : its gradient vanishes at infinity.

The linear approximations of the derivatives here are as follows:

$$\phi_t = [(i\omega)^2\phi_3 + (i\omega U)\phi_{1_z}]\xi_3(t) + [(i\omega)^2\phi_5 - (i\omega U)\phi_3 + (i\omega U)(z\phi_{1_x} - x\phi_{1_z})]\xi_5(t) \quad ;$$

$$\phi_{x'} = U[1 - \phi_{1_x}] + [i\omega\phi_{3_x}]\xi_3(t) + [i\omega\phi_{5_x} - U\phi_{3_x} - U\phi_{1_z}]\xi_5(t) \quad ;$$

$$\phi_{y'} = - U[\phi_{1_y}] + [i\omega\phi_{3_y}]\xi_3(t) + [i\omega\phi_{5_y} - U\phi_{3_y}]\xi_5(t) \quad ;$$

$$\phi_{z'} = - U[\phi_{1_z}] + [i\omega\phi_{3_z}]\xi_3(t) + [i\omega\phi_{5_z} - U\phi_{3_z} + U\phi_{1_x}]\xi_5(t) \quad .$$

Some simplification has been done through the dropping of quadratic terms in ξ_j . Substituting these expressions into the Bernoulli equation and simplifying somewhat, one finds that:

$$-\frac{p}{\rho} = \frac{U^2}{2}\mathbf{v}^2 + [(i\omega)^2\phi_3 + (i\omega U)(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)]\xi_3(t) + [(i\omega)^2\phi_5 + (i\omega U)(\psi_5 + \mathbf{v}\cdot\nabla\phi_5) - U^2(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)]\xi_5(t) \quad .$$

In the terms containing ξ_j , one can use primed and unprimed coordinates interchangeably, since the difference leads to terms of higher order.

The force (moment) corresponding to the j-th mode of oscillation is given by:

$$F_j(t) = \int_S dS\, n_j\, p(x,y,z,t) \;\doteq\; \sum_i T_{ji}\, \xi_i(t) + F_{j0} \quad ,$$

where S is the surface of the body at any instant and F_{j0} is the steady force component. (For $j = 1,2,3$, the latter is zero.) The "transfer functions" T_{ij} are:

$$T_{33} = -\rho\int_S dS\, n_3\, [(i\omega)^2\phi_3 + (i\omega U)(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)] \quad ;$$

$$T_{35} = -\rho \int_S dS\ n_3\ [(i\omega)^2 \phi_5 + (i\omega U)(\psi_5 + \mathbf{v}\cdot\nabla\phi_5) - U^2(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)] \quad ;$$

$$T_{53} = -\rho \int_S dS\ n_5\ [(i\omega)^2 \phi_3 + (i\omega U)(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)] \quad ;$$

$$T_{55} = -\rho \int_S dS\ n_5\ [(i\omega)^2 \phi_5 + (i\omega U)(\psi_5 + \mathbf{v}\cdot\nabla\phi_5) - U^2(\psi_3 + \mathbf{v}\cdot\nabla\phi_3)] \quad .$$

These formulas can be simplified considerably, even before we introduce the slenderness approximation. We use two theorems: One is an extension of Stokes' theorem, proven by Tuck (See Ogilvie & Tuck (1969)):

$$\int_S dS\ n_j\ (\mathbf{v}\cdot\nabla\phi_i) = -\int_S dS\ m_j\ \phi_i \quad .$$

The other theorem is Green's theorem; in applying it, we note that all of the functions decrease sufficiently rapidly far away that there is no need to account for effects at infinity.* Thus, in T_{33} and T_{35} we have:

$$\int_S dS\ n_3(\psi_3 + \mathbf{v}\cdot\nabla\phi_3) = \int_S dS\ (\phi_{3_n}\psi_3 - \psi_{3_n}\phi_3) = 0$$

Similarly, in T_{55} ,

$$\int_S dS\ n_5(\psi_5 + \mathbf{v}\cdot\nabla\phi_5) = 0 \quad .$$

In T_{35} , we manipulate one integral as follows:

---

*ϕ_1 and ϕ_3 appear to represent dipoles at infinity; thus, both are proportional to $1/r^2$ as $r \to \infty$. ϕ_5 appears to represent a quadripole, thus is proportional to $1/r^3$ at infinity.

$$\int_S dS\ n_3(\psi_5 + \mathbf{v}\cdot\nabla\phi_5) = \int_S dS\ (n_3\psi_5 - m_3\phi_5)$$

$$= \int_S dS\ (n_3\psi_5 - \psi_{3_n}\phi_5 + \psi_3\phi_{5_n} - \psi_3 n_5)$$

$$= - \int_S dS\ n_3\ \phi_3 + \int_S dS\ z\ (n_3\phi_{1_x} - n_1\phi_{1_z}) \quad .$$

Similarly, in T_{53} and T_{55}, we find:

$$\int_S dS\ n_5(\psi_3 + \mathbf{v}\cdot\nabla\phi_3) = \int_S dS\ n_3\ \phi_3 - \int_S dS\ z\ (n_3\phi_{1_x} - n_1\phi_{1_z}) \quad .$$

The last integral in the last two expressions can be rewritten:

$$\int_S dS\ z\ (n_3\phi_{1_x} - n_1\phi_{1_z}) = \mathbf{j}\cdot\int_S dS\ z\ \mathbf{n}\times\nabla\phi_1$$

$$= \mathbf{j}\cdot\int_S dS\ [\mathbf{n}\times\nabla(z\phi_1) - \mathbf{n}\times(\phi_1\mathbf{k})]$$

$$= \int_S dS\ n_1\ \phi_1 \quad ,$$

the last equality following from application of Stokes' theorem to the first term. Combining all of these results, we find for the T_{ji} :

$$T_{33} = - \rho(i\omega)^2 \int_S dS\ n_3\ \phi_3 \quad ;$$

$$T_{35} = - \rho(i\omega)^2 \int_S dS\ n_3\ \phi_5 + \rho(i\omega U) \int_S dS\ (n_3\phi_3 - n_1\phi_1) \quad ;$$

$$T_{53} = - \rho(i\omega)^2 \int_S dS\ n_3\ \phi_5 - \rho(i\omega U) \int_S dS\ (n_3\phi_3 - n_1\phi_1) \quad ;$$

$$T_{55} = - \rho(i\omega)^2 \int_S dS\ n_5\ \phi_5 + U^2 \int_S dS\ (n_3\phi_3 - n_1\phi_1) \quad .$$

These results have been obtained with no assumptions made about the shape of the body. The only assumption was that the sinusoidal oscillations had very small amplitude.

Now, finally, let us assume that the body is slender. The only effect is that we lose the terms containing $n_1\phi_1$. For a slender body, n_3 and n_5 are $O(1)$ as the slenderness parameter, ε , approaches zero, whereas n_1 is $O(\varepsilon)$. From (2-73), we see that ϕ_1 is therefore higher order than ϕ_3 and ϕ_5 by a factor of ε . Thus:

$$\left[\int_S dS\; n_1\phi_1 \Big/ \int_S dS\; n_3\phi_3\right] = O(\varepsilon^2) \ .$$

Seldom in practical problems do we ever retain terms with such a great difference in orders of magnitude, and so we neglect the terms containing $n_1\phi_1$ if the body is slender.

In the ship-motion problem, the quantity corresponding to T_{33} will be

$$(i\omega)^2[a_{33} + \frac{1}{i\omega} b_{33}] \ ,$$

where a_{33} and b_{33} are the heave added-mass and damping coefficients, respectively.* The other T_{ij}'s have a similar interpretation in terms of pitch added-moment-of-inertia and damping coefficients, cross-coupling coefficients, etc. We note that there are three kinds of terms here:

a) Terms independent of U . These are all of the same form:

*In the ship-motion problem, ϕ_j is complex. Here, of course, ϕ_j is purely real, and so there is no analog to b_{ij} .

$$T_{ij}^{(0)} = - (i\omega)^2 \int_S dS \; n_i \; \phi_j \quad . \tag{2-78}$$

b) Terms proportional to U. These occur only in the cross terms, T_{ij}, with $i \neq j$. For a slender body, we have:

$$T_{35} = T_{35}^{(0)} - (U/i\omega) \, T_{33}^{(0)} \quad ; \tag{2-79}$$

$$T_{53} = T_{53}^{(0)} + (U/i\omega) \, T_{33}^{(0)} \quad . \tag{2-80}$$

c) A term proportional to U^2. This occurs only in T_{55}:

$$T_{55} = T_{55}^{(0)} + (U/i\omega)^2 \, T_{33}^{(0)} \quad . \tag{2-81}$$

Even at zero forward speed, there is coupling between the heave and pitch modes, unless the body is symmetrical fore-and-aft. If the body *is* symmetrical, one can show that $T_{35}^{(0)}$ and $T_{53}^{(0)}$ are zero. But even in this case, the existence of forward speed causes a loss of symmetry, and so a pure-heave motion causes a pitch moment, and a pure-pitch motion causes a heave force. The symmetry between T_{35} and T_{53} should be noted: The speed-independent parts are equal, whereas the speed-dependent parts are exactly opposite.

One remarkable fact is that there is no interaction between the oscillatory motion and the perturbation of the uniform stream by the steady forward motion. If the above formulas are derived from the kinetic-energy formula by use of the Lagrange equations, this fact is perhaps obvious. When we derive expressions for force and moment on an oscillating ship, it is anything but obvious.

For the sake of completeness, I write out here the final formulas for the T_{ij}'s for a slender body in an infinite fluid. We note first that, by the same procedures used in the steady-forward-motion problem, the following is true to a first approximation:

$$\phi_{jyy} + \phi_{jzz} = 0 \quad \text{, in the near field.}$$

From (2-72'), it is rather obvious that, for a slender body,

$$n_5 = -xn_3[1 + O(\varepsilon^2)] \quad ,$$

and thus, from (2-73):

$$\phi_5 = -x\phi_3[1 + O(\varepsilon^2)] \quad .$$

Now let:

$$m(x) \equiv \rho \int_{C(x)} d\ell \, n_3\phi_3 \equiv \text{added mass per unit length} \quad , \qquad (2\text{-}82)$$

where $C(x)$ is the contour around the body in the cross-section at x . Then clearly:

$$T_{33}^{(0)} = T_{33} = -\rho(i\omega)^2 \int_L dx \int_{C(x)} d\ell \, n_3\phi_3 = -(i\omega)^2 \int_L dx \, m(x) \quad ,$$

where L is the domain of the length of the body. Similarly, we obtain:

$$T_{35}^{(0)} = T_{53}^{(0)} = \rho(i\omega)^2 \int_L dx \, x \int_{C(x)} d\ell \, n_3\phi_3 = (i\omega)^2 \int_L dx \, x \, m(x) \quad ;$$

$$T_{55}^{(0)} = -(i\omega)^2 \int_L dx \, x^2 m(x) \quad .$$

Collecting these results, we have:

$$\left.\begin{aligned}
T_{33} &= -(i\omega)^2 \int_L dx\; m(x) \quad ; \\
T_{35} &= (i\omega)^2 \int_L dx\; x\; m(x) - \frac{U}{i\omega} T_{33} \; ; \\
T_{53} &= (i\omega)^2 \int_L dx\; x\; m(x) + \frac{U}{i\omega} T_{33} \; ; \\
T_{55} &= -(i\omega)^2 \int_L dx\; x^2 m(x) + \left(\frac{U}{i\omega}\right)^2 T_{33} \; .
\end{aligned}\right\} \qquad (2\text{-}83)$$

3 SLENDER SHIP

Of all the problems discussed in this paper, the slender-ship problem has led to the most important practical consequences. Therefore it is not unreasonable to devote the longest chapter to the problem. Even so, some aspects will not be covered; perhaps the most important missing example is the case of sinkage and trim of a ship.

In the four sections, two steady-motion and two unsteady-motion problems are discussed. The first steady-motion problem is the wave-resistance problem, that is, the problem of a ship in steady forward motion on the surface of an infinite ocean. In the second section, the problem treated is essentially the same, but the Froude number is assumed to be related to the slenderness parameter in such a way that Froude number approaches infinity as slenderness approaches zero; this rather unnatural relationship is discussed at some length. In the third section, I discuss in some detail the problem of heave and pitch motions of a ship at zero forward speed; the results are not at all surprising, but the method is quite clear in this case, which helps one in approaching the final section. It is concerned with the problem which is the combination of the first and third problems: heave and pitch motions of a ship with forward speed.

3.1 The Moderate-Speed, Steady-Motion Problem

The theory presented here is due to Tuck (1963a)*. The analysis — as far as I carry it here — is not very much

*This reference is not readily available, but the material which is of interest here can also be found in Tuck (1963b), Tuck Tuck (1964a), and Tuck (1964b), all of which are gathered into Tuck (1965a).

more difficult than the analysis of the infinite-fluid problem, and so it will only be sketched here.

The theory is attractive for its simplicity and its elegance, but unfortunately it has not been successful in predicting wave resistance. The reasons are not entirely clear, although they have been discussed for many years. See, for example, Kotik & Thomsen (1963). The difficulty could very well be that real ships are just not slender enough for a one-term expansion (or perhaps any number of terms) to give an accurate prediction of wave resistance. This is the old question, "How small must the 'small' parameter be?" Another possibility is that the error arises because the lowest-order slender-body theory places the source of the disturbance precisely on the level of the undisturbed free surface, and so there are no attenuation effects due to finite submergence of parts of the hull. (These two possible causes of error are not entirely separate.) Still another possible cause is considered in Section 3.2.

The hull surface will be specified by the equation:

$$r = r_0(x,\theta) \quad . \tag{3-1}$$

Now it will be convenient to measure θ from the negative z axis, since most ships are symmetrical about the midplane. We assume that $r_0 = O(\varepsilon)$ and that $\partial^n r_0/\partial x^n = O(\varepsilon)$, as needed.

There is a velocity potential satisfying the Laplace equation and the same kinematic body boundary condition, (2-5b), as in the infinite-fluid problem. The incident stream is again taken in the positive x direction, that is, with velocity potential Ux. The two free-surface conditions are:

$$g\zeta + \frac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] = \frac{1}{2}U^2 , \quad \text{on} \quad z = \zeta(x,y) ; \tag{3-2}$$

$$\zeta_x \phi_x + \zeta_y \phi_y - \phi_z = 0 , \quad \text{on} \quad z = \zeta(x,y) . \tag{3-3}$$

Finally, there is a radiation condition to be satisfied.

As usual, we assume that there is a far-field expansion:

$$\phi(x,y,z) \sim \sum_{n=0}^{N} \phi_n(x,y,z) , \text{ where } \phi_{n+1} = o(\phi_n) \text{ as } \varepsilon \to 0 , \text{ for fixed } (x,y,z), \tag{3-4}$$

and a near-field expansion:

$$\phi(x,y,z) \sim \sum_{n=0}^{N} \Phi_n(x,y,z) , \text{ where } \Phi_{n+1} = o(\Phi_n) \text{ as } \varepsilon \to 0 , \text{ for fixed } (x,y/\varepsilon,z/\varepsilon) . \tag{3-5}$$

These expansions are substituted into all of the exact conditions, from which we obtain two sequences of problems which must be solved simultaneously.

In the far field, the first term in the expansion for ϕ must be just the incident uniform-stream potential, Ux, since the body vanishes as $\varepsilon \to 0$ and the asymptotic representation $\phi \sim \phi_0 = Ux$ satisfies the free-surface conditions (trivially). The second term represents a line of singularities on the x axis. One really ought to allow the most general possible kind of singularities on this line, but it is no surprise to find that just sources are sufficient at first, and so we consider the special case of a line of sources on the free surface. One can show that higher-order singularities could not be matched to the near-field solution. Alternatively, one can construct a far-field solution using Green's theorem and show that it really represents just a line distribution of sources. See, for example, Maruo (1967).

One can use the classical Havelock source potential to express the desired potential for a line of sources, but Tuck's procedure is more convenient in the slender-body problem: Apply a double-Fourier transform operation to the Laplace equation, reducing it to an ordinary differential equation with z as independent variable:

$$-(k^2 + \ell^2)\phi^{**}(k,\ell;z) \; + \; \phi^{**}_{zz}(k,\ell;z) \; = \; 0 \; ,$$

where k and ℓ are the transform variables, and the asterisks denote the transforms. Assume for the moment that the line of sources is located at $z = z_0 < 0$. The above differential equation can be solved generally, with a different solution above and below $z = z_0$. The solution in the upper region is forced to satisfy the linearized free-surface condition, the solution in the lower region must vanish at great depths, and the two must have the discontinuity at $z = z_0$ appropriate to the source singularities. Finally, one may allow $z_0 \to 0$. In physical variables, the result is:

$$\begin{aligned}\phi_1(x,y,z) \; = \; & -\frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk \; e^{ikx} \; \sigma^*(k) \; K_0(|k|r) \\ & - \lim_{\mu\to 0} \frac{U^2}{4\pi^2}\int_{-\infty}^{\infty} dk \; e^{ikx} \; k^2\sigma^*(k) \int_{-\infty}^{\infty} \frac{d\ell \; e^{i\ell y + z\sqrt{(k^2+\ell^2)}}}{\sqrt{(k^2+\ell^2)} \; [g\sqrt{(k^2+\ell^2)} - (Uk - i\mu/2)^2]} \; ,\end{aligned} \qquad (3\text{-}6)$$

where μ denotes a fictitious Rayleigh viscosity, guaranteeing that the proper radiation condition is satisfied, and $\sigma^*(k)$ is the Fourier transform of $\sigma(x)$, the source density*.

*Define $\sigma(x) \equiv 0$ for values of x ahead of and behind the ship.

The two-term outer expansion is:

$$\phi(x,y,z) \sim Ux + \phi_1(x,y,z) \quad ,$$

which has the two-term inner expansion:

$$\phi(x,y,z) \sim Ux + \frac{1}{\pi}\sigma(x)\log r - \frac{1}{2\pi} f(x) - g(x) \quad , \qquad (3\text{-}7)$$

where

$$f(x) = \int_{-\infty}^{\infty} d\xi\, \sigma'(\xi) \log 2|x-\xi| \operatorname{sgn}(x-\xi) \qquad (3\text{-}8a)$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \sigma^*(k) \log\frac{C|k|}{2} \quad ; \qquad (3\text{-}8b)$$

$$g(x) = \lim_{\mu\to 0} \frac{U^2}{4\pi^2}\int_{-\infty}^{\infty} dk\, e^{ikx}\, k^2\sigma^*(k) \int_{-\infty}^{\infty} \frac{d\ell}{\sqrt{(k^2+\ell^2)}\,[g\sqrt{(k^2+\ell^2)} - (Uk-i\mu/2)^2]} \qquad (3\text{-}9)$$

The expansion should be compared with the corresponding expansion for a line of sources in an infinite fluid, as given in (2-65). We now have an extra term, $g(x)$, and the terms containing $\sigma(x)$ and $f(x)$ differ by a factor of two from the earlier result. The latter variation is not important; it results from the fact that the line of sources was taken at $z = z_0 < 0$, and those sources merged with their images when we let $z_0 \to 0$.

The most interesting feature of this inner expansion of the two-term outer expansion is that the wave effects are all contained in $g(x)$ — a function of just x . In the infinite-fluid problem, all 3-D effects in the near field were included (in the first approximation) in the single function of x , $f_1(x)$. We now have a generalization of this for the free-surface problem.

In the near field, it is easy to show that the first term in the asymptotic expansion of the potential is again just the uniform-stream potential, Ux . The next term, Φ_1 , must satisfy the Laplace equation in two dimensions (in the cross-plane) and the same body boundary condition as before, (2-63):

$$\frac{\partial \Phi_1}{\partial N} = - \frac{U r_{0_x}(x,\theta)}{[1+(r_{0_\theta}/r_0)^2]^{1/2}} \quad \text{on} \quad r = r_0(x,\theta) \quad . \qquad (3\text{-}10)$$

As in the infinite-fluid problem, this condition suggests that

$$\Phi_1 = O(\varepsilon^2) \quad ,$$

since $r_0 = O(\varepsilon)$ and $\partial/\partial N = O(\varepsilon^{-1})$.

Now consider the free-surface conditions. In the Bernoulli equation, note the orders of magnitude:

$$g\zeta + U\Phi_{1_x} + \tfrac{1}{2}(\Phi_{1_x}^2 + \Phi_{1_y}^2 + \Phi_{1_z}^2) + \ldots = 0 \quad \text{on} \quad z = \zeta(x,y) \quad .$$

$$O(\zeta) \quad O(\varepsilon^2) \quad O(\varepsilon^4) \; O(\varepsilon^2) \; O(\varepsilon^2)$$

The term containing $\Phi_{1_x}^2$ can be dropped, but the others containing Φ_1 are all the same order of magnitude, and we have no reason to suppose that the ζ term is higher order. In the kinematic condition, note the orders of magnitude:

$$U\zeta_x + \Phi_{1_x}\zeta_x + \Phi_{1_y}\zeta_y - \Phi_{1_z} + \ldots = 0 \quad \text{on} \quad z = \zeta(x,y) \quad .$$

$$O(\zeta) \quad O(\zeta\varepsilon^2) \quad O(\zeta) \quad O(\varepsilon)$$

Clearly, we can drop the term containing Φ_{1_x} , but no others.

Now we must relate the order of magnitude of ζ with the order of magnitude of Φ_1 . From the kinematic condition, one might suppose that $\zeta = O(\varepsilon)$. However, the dynamic condition then implies that $\zeta \sim 0$, which means only that ζ is higher order than we assumed. In fact, the only assumption which is consistent with both conditions is that:

$$\zeta = O(\varepsilon^2) \quad .$$

The kinematic condition then reduces to:

$$\Phi_{1_z} = 0 \quad \text{on} \quad z = 0 \quad ; \tag{3-11}$$

thus, Φ_1 represents the flow which would occur in the presence of a rigid wall at $z = 0$. From the dynamic free-surface condition, we can compute the first approximation to the wave shape:

$$g\zeta(x,y) \quad \sim \quad -\left(U\,\Phi_{1_x} + \frac{1}{2}\,\Phi_{1_y}^2\right)\Big|_{z=0} \quad . \tag{3-12}$$

It may appear to be a paradox that we have a flow without waves, from which we compute a wave shape! But, like all paradoxes, it is a matter of interpretation and understanding. We shall return to this point presently.

Since Φ_1 satisfies the Laplace equation in two dimensions a rigid-wall condition on $z = 0$, it can be continued analytically into the upper half space as an even function of z . All of the arguments used in the infinite-fluid problem can then be carried over directly. In particular, at large distance from the origin, we can write, as in (2-64),

$$\Phi_1 \quad \sim \quad C_1 + \frac{A_{10}}{2\pi}\log r + O\left(\frac{1}{r}\right) \quad , \quad \text{as} \quad r \to \infty \quad .$$

The two-term inner expansion can be matched to the two-term outer expansion. We obtain:

$$A_{10} = 2\,\sigma(x) \quad ;$$

$$C_1 = -\frac{1}{2\pi} f(x) - g(x) \quad .$$

Note that there is again a factor of 2 difference from the infinite-fluid results, (2-66). Of course, the term $g(x)$ is new here.

We can again determine A_{10} and thus σ in terms of body shape, without the necessity of solving the near-field hydrodynamic problem. By the simple flux argument, we find that:

$$A_{10} = 2\,U\,s'(x) \quad , \tag{3-13a}$$

where $s(x)$ is the cross-sectional area *of the submerged part* of the hull. With this convention, we find that

$$\sigma(x) = U\,s'(x) \quad , \tag{3-13b}$$

just as in the infinite-fluid problem. Again, we have been able to determine the complete two-term outer expansion without explicitly solving the near-field problem. This occurs because the source-like behavior which dominates far away from the body (still in the near-field sense) can be found simply in terms of the rate of change of cross-section, and it provides all the information needed for determining the two-term far-field expansion.

Enough information is now available to determine a first approximation of the wave resistance. It can be computed in either of two ways: 1) integrate the near-field pressure over the hull surface, or 2) use the far-field expansion and the momentum theorem. In either case, one obtains:

$D_W \equiv$ wave resistance

$$\sim -\frac{1}{2}\rho U^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds(x)\; ds(\xi)\; Y_0(\kappa|x-\xi|) \quad ,$$

where

$$s(x) = \text{immersed cross-section area,}$$

$$\kappa = g/U^2 \ ,$$

$$ds(x) = \frac{ds(x)}{dx}\, dx \ ,$$

$Y_0(z)$ = Bessel function of the second kind, of order zero, argument z .

This is the slender-body wave-resistance formula which is so notoriously inaccurate. At speeds for which one would hope to use it, it gives values that are too high by a factor of 3 or more. Generally, one could not (and should not) expect to correct such errors by including higher-order terms, and so it is rather futile to pursue this analysis further.

Streamlines, Waves, Pressure Distributions. I mentioned previously the apparent paradox of prescribing a rigid-wall free-surface condition, then using the solution of that problem to compute wave shapes, as in formula (3-12). Such a procedure really can be quite rational.

Once a velocity potential is known everywhere, it is a fairly simple task for a computer to figure out the velocity field and to produce streamlines. Figure (3-1) shows the

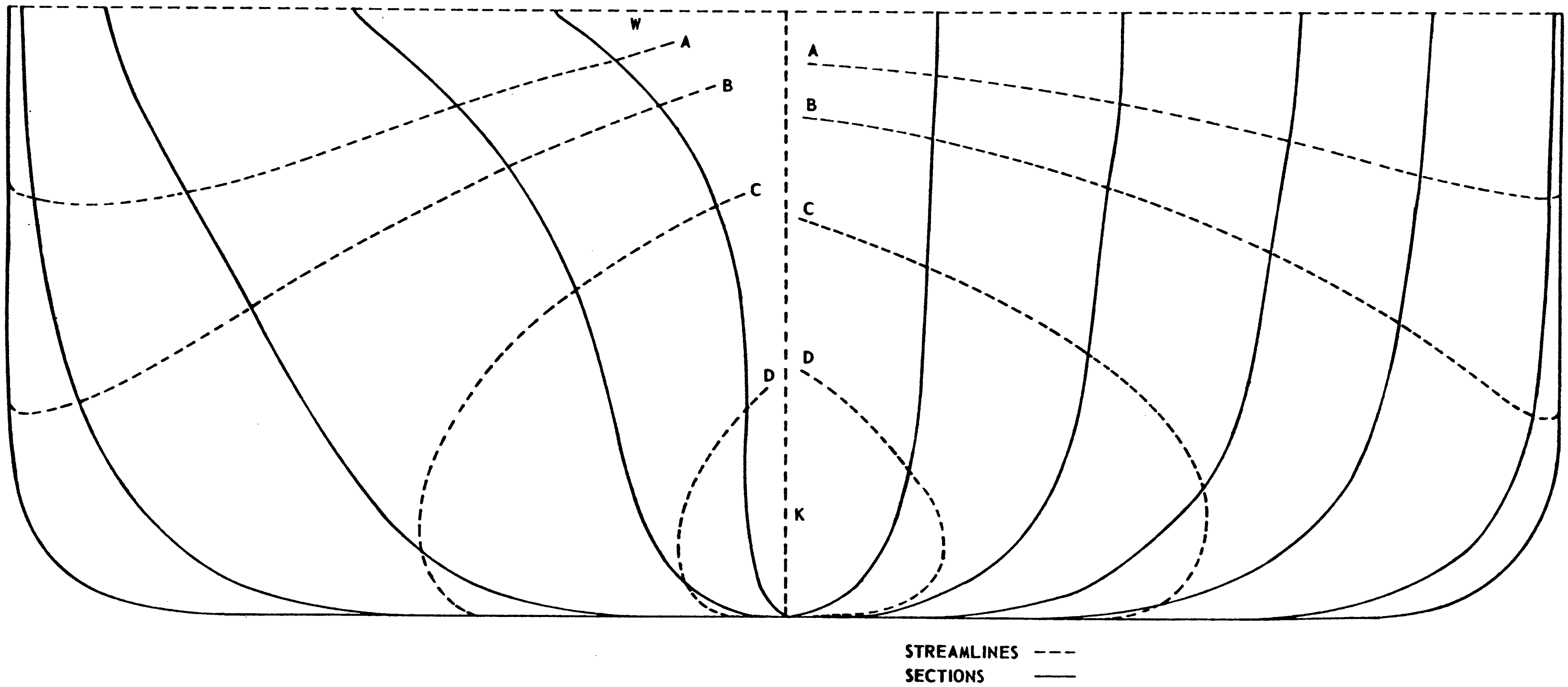

Figure (3-1). Steady-Motion Streamlines on Ship Hull According to First-Order Slender-Body Theory (Body-Plan View). From Tuck & Von Kerczek (1968).

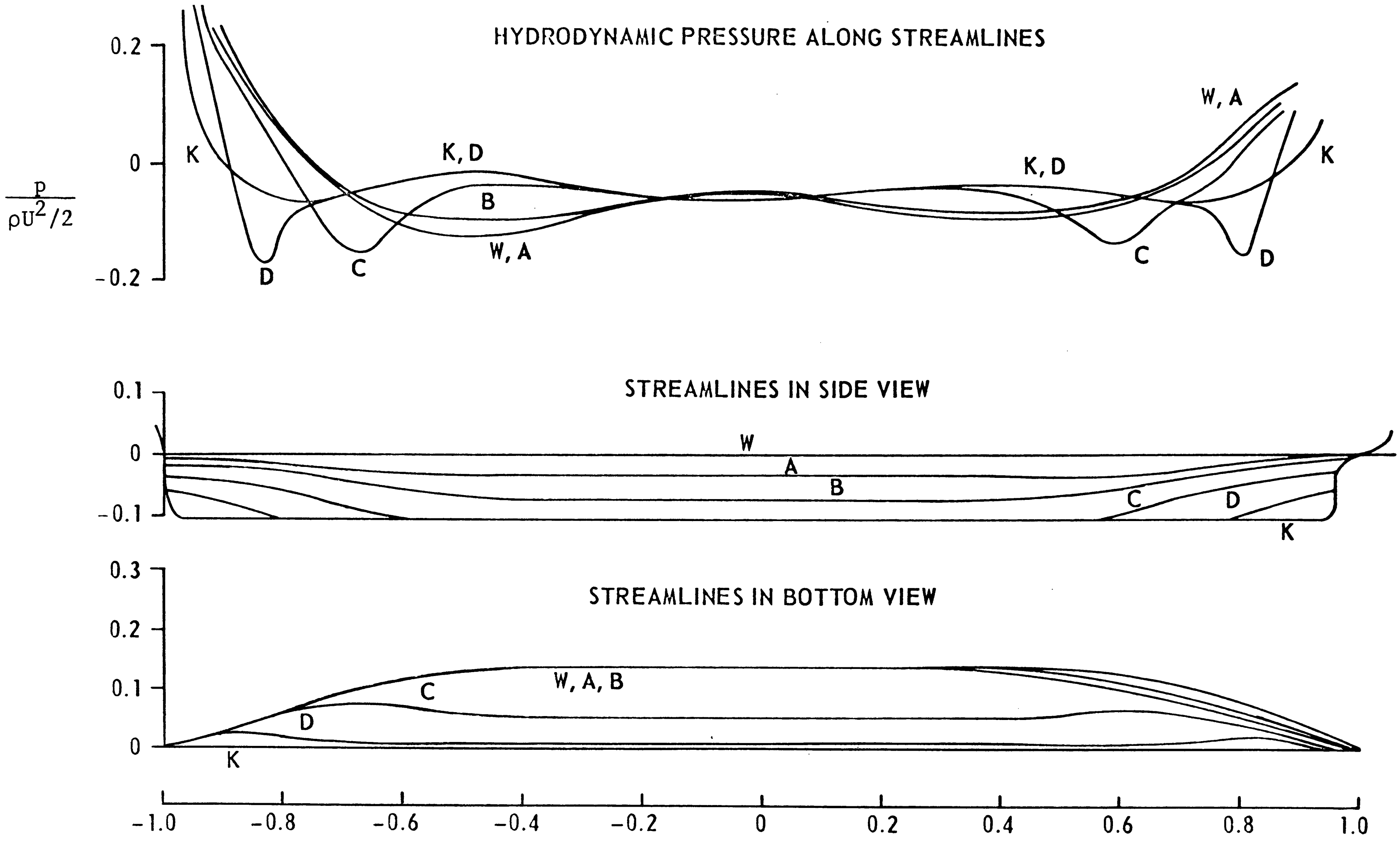

Figure (3-2). Steady-Motion Streamlines and Hydrodynamic Pressure on Ship Hull According to First-Order Slender-Body Theory (Side and Plan Views).

From Tuck & Von Kerczek (1968).

streamlines around a Series 60 hull, calculated from the near-field slender-body solution by Tuck and Von Kerczek (1968). The upper boundary of the figure is the rigid-wall streamline. Figure (3-2) shows the same streamlines in two other views. These drawings are accurate (in principle) to order ε . This means, loosely speaking, that they show the streamlines on a scale which is appropriate for measuring beam and draft of the ship. Thus, we see that some of the streamlines start near mid-draft, pass under the bottom, then return to approximately their original depth. These are variations which show on a scale intended for measuring quantities which are $O(\varepsilon)$.

The wave height, on the contrary, is $O(\varepsilon^2)$, as we found earlier. Therefore it should not show in these figures. Our assumptions have led to the conclusion that wave height is small compared with beam and draft. Thin-ship theory, on the other hand, predicts that wave height and beam are comparable — without being very explicit about the ratio of wave height to draft.

In the section of Figure (3-2) showing hydrodynamic pressure along streamlines, only the waterplane curve (denoted by **W**) is really consistent. On any streamline, the pressure will vary mostly because of the changing hydrostatic head along the streamline. Such pressure variations are $O(\varepsilon)$. If we were to work out a second-order theory and plot the streamlines, the shift in streamline position from first-order theory to second-order theory would lead to a hydrostatic pressure change which is $O(\varepsilon^2)$. This is the same as the order of magnitude of the hydrodynamic pressure, but it is ignored in the figure.

On the other hand, if we were inside the ship measuring pressure at a point on the hull, we would not care which streamline went past that point. We could use the Bernoulli equation to estimate the pressure at any point, and the estimate consistent to order ε^2 would be found from the equation:

$$0 = \frac{p}{\rho} + gz + U\Phi_{1_x} + \frac{1}{2}(\Phi_{1_y}^2 + \Phi_{1_z}^2) \quad .$$

3.2 The High-Speed, Steady-Motion Problem

In the preceding analysis, we have said nothing explicit about the speed other than assuming that it was finite. The first term in the velocity-potential expansions was Ux, and all other terms were assumed to be small in comparison.

In principle, there is no reason to provide or allow a connection between Froude number and our slenderness expansion parameter. However, the practical manner in which a perturbation analysis is used may justify our making such an unnatural connection. In practice, we work out an *asymptotic* expansion, which provides a description that becomes approximately valid (in a certain sense) as the small parameter approaches zero. But we use the expansion under conditions in which the small parameter is quite finite, and we just hope that the resulting error is not too big. The size of that error may depend on other parameters of the problem, and we may possibly reduce the error by allowing such other parameters to vary simultaneously with the basic slenderness parameter.

In the steady-motion problem that we have been considering, the small parameter ε could be thought of as the beam/length ratio. There is a completely different length scale in the problem, namely, $U^2/g = F^2L$, where F is the Froude number and L is ship length. This length is proportional to the wavelength of a wave with propagation speed equal to ship speed. When we assume that $F = O(1)$ as $\varepsilon \to 0$, we imply that the speed is such as to produce waves which can be measured on a scale appropriate for measuring ship length, and we imply that this speed is unrelated to slenderness.

If we are interested in problems of very-low-speed ships or very-high-speed ships, in which the generated waves are, respectively, much shorter or much longer than ship length, it is entirely conceivable that our severely truncated asymptotic expansions may be made even more inaccurate by the extreme values of Froude number. We may increase the practical accuracy by assuming, say, that wavelength approaches zero or infinity, respectively as $\varepsilon \to 0$. This is not to imply that there really is a connection between speed and slenderness. It is done only in the hope that wavelength and ship length may be more accurately represented *when we use the theory with a finite value of* ε .

Formally, the low-speed problem may be treated simply as a special case of Tuck's analysis, as described in Section 3.1. One finds that the appropriate far-field problem contains a rigid-wall free-surface boundary condition (in the first approximation). Thus, both near- and far-field approximations are without real gravity-wave effects. However, this formal approach is quite improper. The difficulty is so serious that we devote a special section later to the low-speed problem. It is perhaps the most singular of all of our singular perturbation problems. The difficulty, in essence, is that we have treated all perturbation velocity components as being small compared with U , and this leads to nonsense if we allow U to approach zero.

At high speed, a slender-body theory can be developed along lines paralleling Tuck's analysis. This has been done by Ogilvie (1967). The resulting near-field and far-field boundary-value problems are quite different from Tuck's, however. No numerical results have been obtained yet from this analysis.

Near-field and far-field regions are defined just as in the previous slender-body problem. In the far-field, the velocity-potential expansion starts with the uniform-stream term, Ux , followed by a term representing a line of singularities. The near-field expansion also starts with the uniform-stream term, followed by a term which satisfies the Laplace equation in two dimensions.

The differences appear first in the boundary conditions satisfied by these expansions. The proper way of setting up these conditions is to nondimensionalize everything and then assume that Froude number, F , is related to the slenderness parameter, ε , in such a way that $F \to \infty$ as $\varepsilon \to 0$. It is easier just to let the gravity constant, g , approach zero in this limit. The only interesting new case, it turns out, is: $g = O(\varepsilon)$. We now assume this to be the case.

Since g appears only in the dynamic free-surface boundary condition, the body boundary condition will be the same as in the moderate-speed problem, Equation (3-10), and in the infinite fluid problem, Equation (2-63).

In the far field, the disturbance vanishes as $\varepsilon \to 0$. Therefore the free-surface disturbance is $o(1)$. If we let the expansion of the velocity potential, $\phi(x,y,z)$, be expressed:

$$\phi(x,y,z) \sim Ux + \sum_{n=1}^{N} \phi_n(x,y,z) \quad , \quad \text{for fixed} \quad (x,y,z) \quad ,$$

the dynamic and kinematic free-surface conditions are, approximately:

$$\left.\begin{aligned} 0 &\simeq U\zeta_x - \phi_{1_z} \quad , \\ 0 &\simeq g\zeta + U\phi_{1_x} \quad , \end{aligned}\right\} \quad \text{on} \quad z = 0 \quad . \tag{3-14}$$

We do not know the relative orders of magnitude of ζ and ϕ , *a priori*, but a study of the possibilities shows that only one combination is possible, namely, that ζ and ϕ_1 are the same order of magnitude. Then, in the dynamic condition, the term containing g is higher order than the other term, and it can be neglected in the first approximation, that is,

$$\phi_{1_x} = 0 \quad , \quad \text{on} \quad z = 0 \quad , \tag{3-14}$$

which implies also that

$$\phi_1 = 0 \quad \text{on} \quad z = 0 \quad . \tag{3-15}$$

Thus, the free surface acts like a pressure-relief surface, with no restraining effect of gravity (to this order of magnitude).

This condition points to a fundamentally different kind of solution from that of the previous problems. If we continue the function ϕ_1 analytically into the upper half-space, it must be odd with respect to the surface $z = 0$. Thus ϕ_1 cannot represent a line of sources. The least singular solution represents a line of dipoles, oriented vertically. Assuming that ϕ_1 will consist *only* of such dipoles, we can write it:

$$\phi_1(x,y,z) = \frac{\sin\theta}{r}\int_0^\infty d\xi\ \mu(\xi)\left[1 + \frac{x-\xi}{[(x-\xi)^2 + r^2]^{1/2}}\right] , \qquad (3\text{-}16)$$

where $y = r\cos\theta$ and $z = r\sin\theta$. The two-term outer expansion and the two-term inner expansion of the two-term outer expansion are, respectively:

$$\begin{aligned}\phi(x,y,z) &\sim Ux + \phi_1(x,y,z)\\ &\sim Ux + \frac{2\sin\theta}{r}\int_0^x d\xi\ \mu(\xi) .\end{aligned} \qquad (3\text{-}17)$$

I am now assuming that the bow of the ship is located at $x = 0$; then, in matching to the near-field solution, we can show that the dipole density must be zero upstream of the ship bow. This expansion is unaffected by the downstream dipoles.

In the near field, we assume the usual expansion:

$$\phi(x,y,z) \sim Ux + \sum_{n=1}^{N}\Phi_n(x,y,z) \quad \text{for fixed } (x,y/\varepsilon,z/\varepsilon) .$$

The term Φ_1 satisfies the 2-D Laplace equation:

$$\Phi_{1_{yy}} + \Phi_{1_{zz}} = 0 .$$

The body boundary condition suggests that $\Phi_1 = O(\varepsilon^2)$, just as it did before in Equation (3-10).

From the dynamic free-surface condition,

$$0 \simeq g\zeta + U\Phi_{1_x} + \frac{1}{2}(\Phi_{1_y}^2 + \Phi_{1_z}^2) \qquad \text{on} \qquad z = \zeta(x,y) \quad ,$$

we see that $\zeta = O(\varepsilon)$ (since $g = O(\varepsilon)$). This causes a new problem. We would like, as usual, to change this condition at $z = \zeta(x,y)$ to a modified condition at $z = 0$. But this is not possible. For example, the term Φ_{1_x} would be transformed:

$$\underset{O(\varepsilon^2)}{\Phi_{1_x}(x,y,\zeta(x,y))} = \underset{O(\varepsilon^2)}{\Phi_{1_x}(x,y,0)} + \underset{O(\varepsilon)}{\zeta(x,y)}\,\underset{O(\varepsilon)}{\Phi_{1_{xz}}(x,y,0)} + \ldots \quad .$$

Every term, in fact, will be the same order of magnitude, and so this ordinary kind of expansion fails. We must continue to apply the condition on the actual (unknown) location of the free surface.*

The kinematic free-surface condition is also nonlinear and must be satisfied on the unknown location of the free surface:

$$0 \simeq U\zeta_x + \Phi_{1_y}\zeta_y - \Phi_{1_z} \qquad \text{on} \qquad z = \zeta(x,y) \quad .$$

Each term here is $O(\varepsilon)$, and so none can be ignored.

We are left in the rather uncomfortable position of having to solve a nonlinear problem just to obtain a first approximation to the near-field potential function. However, that nonlinear problem is a two-dimensional problem, which is not an insignificant advantage, and, as we shall see, it is possible in principle to predict the location of the free surface, thus avoiding the necessity of searching for it.

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*If we expand: $\zeta \sim \Sigma\zeta_n$, we could apply the condition on $z = \zeta_1$, then apply the usual kind of transformation, as above, so that conditions on higher-order terms would be applied on *a priori* known surfaces.

We do not have a condition to apply at infinity in the Φ_1 (near-field) problem. It is not so straightforward in this case to predict the form of the solution as $r \to \infty$, but Ogilvie (1967) showed that:

$$\Phi_1(x;y,z) = \frac{A_{11} \sin \theta}{r} [1 + O(1/r)] \quad \text{as} \quad r \to \infty ,$$

where A_{11} is a constant to be determined. There is no source-like behavior. This might have been expected, of course, since the inner expansion of the outer expansion, (3-17), showed the characteristics of a two-dimensional dipole. An intermediate expansion can be used to show that these statements are correct.

A numerical procedure for solving this problem may be the following: Suppose that at some x we know the value of Φ_1 on the free surface, $z = \zeta(x,y)$, and that we also know $\zeta(x,y)$ at that x. Using Green's theorem, we can write:

$$\Phi_1(x;y,z) = \frac{1}{2\pi} \int \left[\frac{\partial \Phi_1}{\partial N} \log r' - \Phi_1 \frac{\partial}{\partial N} (\log r') \right] d\ell' ,$$

where $r'^2 = [(y-y')^2 + (z-z')^2]$, and the integration is carried out in the cross-section, with (y',z') ranging over the body contour, the free-surface contour, and a closing contour at infinity. The last of these contours contributes nothing and can be ignored. We assumed that Φ_1 is known on the free surface, and, from the body boundary condition, we know $\partial\Phi_1/\partial N$ on the hull. If we let the field point, $(x;y,z)$, approach the hull surface, we obtain an integral equation, with Φ_1 unknown on the hull and $\partial\Phi_1/\partial N$ unknown on the free surface. This is not quite the usual form for an integral equation, but it should be possible to solve it approximately by essentially standard numerical methods. Then the Green's-theorem integral can be used to express Φ_1 at all points in that cross-section. Thus, solution of an integral equation in one dimension allows the potential to be found.

This procedure has not used the information contained in the free-surface conditions. Usually, we look on the free surface conditions as complications that cause tremendous difficulty in the finding of solutions. Now we take an opposite point of view: Supposing that we have solved the above problem at some x, we use the kinematic conditions to predict the value of ζ just downstream:

$$\zeta(x+\Delta x,y) = \zeta(x,y) + \Delta x\, \zeta_x(x,y) + \ldots$$

$$= \zeta(x,y) + \frac{\Delta x}{U}\,[\Phi_{1_z}(x,y,\zeta(x,y)) - \Phi_{1_y}\zeta_y] + \ldots \quad .$$

Similarly, we predict the value of Φ_1 on the free surface just downstream:

$$\Phi_1\Big|_{\zeta(x+\Delta x,y)} = \Phi_1 + (\Delta x)\,[\Phi_{1_x} + \zeta_x \Phi_{1_z}] + \ldots \quad ,$$

where the right-hand side is evaluated at $(x,y,\zeta(x,y))$, and the dynamic boundary condition is used to evaluate Φ_{1_x}.

Now we are ready to start over. Presumably having solved the problem at some x, we have used the free-surface conditions to formulate the equivalent problem at $x + \Delta x$. The most serious difficulty may very well be in starting the whole process, and there seems to be no elegant prescription for carrying out that essential first step; in some problems, it is possible that a linearized solution may suffice for a start, but this is not certain. Another serious difficulty may be the stability of the method.

This analysis has led to the possibility of predicting waves with amplitude which is $O(\varepsilon)$, that is, waves comparable in amplitude to ship beam and draft. Such a possibility makes the analysis worth further investigation, but it is also the cause of the major difficulty, viz., the necessity of solving a nonlinear problem in the near field.

When the above analysis was offered for publication in 1967, one of the referees called attention to the fact that the conclusions seemed to be quite at variance with those of Rispin (1966) and Wu (1967). Simple observation shows that, at very great distance, the dominant fluid motion should be gravity-related free-surface waves, whereas my high-Froude-number analysis predicts no true wave motion in the far field. Actually, all aspects of the problem are in complete harmony if we consider a "far-far field" in which distance from the ship is $O(\varepsilon^{-1})$, that is, much greater than ship length. The two free-surface conditions then fall into the usual linearized format, and we would expect to find progressive waves in such a region.

This is quite reasonable. At very high Froude number, one expects typical waves to be very long — in this case, considerably longer than the ship. The appropriate distortion of coordinates is an isotropic *compression* in scale far, far away, in contrast to our usual anisotropic *stretching* of coordinates in the cross-plane near the body. In their two-dimensional planing problems, Rispin (1966) and Wu (1967) performed just such a distortion. Their problem is discussed at some length later, when we come to two-dimensional problems.

The present problem is an interesting case in which an inconsistent expansion might be useful in the far field. Suppose that we arbitrarily replace the free-surface condition, (3-15), by the usual moderate-speed condition,

$$U^2\phi_{xx} + g\phi_y = 0 , \quad \text{on} \quad z = 0 .$$

If Froude number is indeed very high, then this condition is quite equivalent to (3-15). But the potential function which satisfies this condition does not represent just the simple line of dipoles implied by (3-16). There will be all of the well-known extra terms involving the free surface. If such an inconsistent far-field solution can be matched to the near-field solution, then the waveless far-field solution obtained previously can be avoided. Perhaps this is worth further study.

3.3 Oscillatory Motion at Zero Speed

A systematic study of the zero-speed ship-motions problem by means of the method of matched asymptotic expansions does not yield any results that were not obtained previously by simpler means. However, it is instructive to consider this problem by this method because the results *are* rather obvious and it is then clear how the formalism is used in place of some common physical arguments. Then, in the more complicated forward-speed problem, in which physical insight is less reliable, the same formalism can be applied with reasonable faith in its predictions.

Only the slender-body idealization of a ship has led to useful prediction methods in the ship-motion problem*.

*Note that "strip theory" is a special case of "slender-body theory."

The thin-ship model, which was intensively studied from the late 1940's until the early 1960's, was useful for certain restricted aspects of the problem. For example, the damping of heave and pitch motions, as predicted by thin-ship theory, is fairly accurate. But the complete theory is deficient. A straightforward one-parameter analysis leads to the prediction of resonances in heave and pitch with no added-mass or damping effects, as shown by Peters & Stoker (1954). (See also Peters & Stoker (1957) and Stoker (1957).) A multi-parameter thin-ship analysis is apparently satisfactory in principle, as demonstrated by Newman (1961), but no one has used it for prediction purposes. It is too complicated.

Slender-body theory at one time appeared to have comparable difficulties, but these have been largely removed in recent years, and a theory which is essentially rational now exists and is fairly successful in predicting ship motions.

In early versions of the slender-body theory of ship motions, all inertial effects (both ship and fluid) were lost in the lowest-order approximation, along with hydrodynamic damping effects. The theory was even more primitive than the classical Froude-Krylov approach. Excitation was computed from the pressure field of the waves, undisturbed by the presence or motions of the ship, and the restoring forces were simply the quasi-static changes in buoyancy and moment of buoyancy. Even the mass of the ship was supposed to be negligible in the lowest-order theory.

These deficiencies are removed by assuming that the frequency of motion is high, in an asymptotic sense. That is, if one assumes that the frequency of sinusoidal oscillation is $O(\varepsilon^{-1/2})$ *, then the ship inertia force is the same order of magnitude as the excitation and the buoyancy restoring forces. The hydrodynamic force and moment also enter into the calculation of ship motions at the lowest order of magnitude. This was all recognized, for example, by Newman & Tuck (1964). However, correcting the slender-body theory in this way was rejected by many workers on the ground that the resulting theory would be valid only for very short incident waves, whereas the most important ship motions are known to occur when the waves have wavelengths comparable to ship length.

The choice was this: 1) Follow the reasonable usual assumptions of slender-body theory and obtain a rather useless theory**. 2) Accept the formal assumption that frequency is high and obtain a much more interesting theory — which turns out to be very similar to the intuitive but quite successful "strip theory" of ship motions. In what follows, I make the second choice.

The reasons for the success of this choice have become clear in the last few years. In one of the most important practical problems, namely, the prediction of heave and pitch motions in head seas, we can truly say that we are dealing with a high-frequency phenomenon. Because of the Doppler shift in apparent wave frequency, fairly long waves

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* ε is the usual slenderness parameter.

**Newman & Tuck showed, for example, that the lowest-order perturbation potential resulting from ship oscillations satisfies a rigid-wall free-surface condition, even with forward speed included. Maruo (1967) has the same result for the forced-oscillation problem. Newman & Tuck performed calculations with a second-order theory for the zero-speed case and found practically no change in their predictions due to second-order effects. They did not make such calculations in the forward-speed problem.

are encountered at rather high frequencies; the waves are long enough to cause large excitation forces, and the frequencies are high enough to cause resonance effects. At zero speed, on the other hand, incident waves with frequency near the resonance frequencies of a ship are likely to be much shorter in length than the ship, and so their net excitation effect is much reduced through interference. For typical ships on the ocean, most of the heave and pitch motion at zero speed is caused by waves with length comparable to ship length, and so the frequencies of such motion are well below the resonance frequencies. Thus, at zero speed, prediction of ship motions can be treated largely on a quasi-static basis; the system response is "spring-controlled" rather than "mass-controlled."

The problem is very much like the simple spring-mass problem discussed in Section 1.2. If the mass of a spring-mass system is very small, we can ignore inertia effects at low frequency. Thus, if the system is described by the differential equation:

$$m\ddot{y} + ky = F\,e^{i\omega t} ,$$

the exact and approximate solutions, given by:

$$y_{ex} = Fe^{i\omega t}/(k-m\omega^2) \quad , \quad y_{ap} = Fe^{i\omega t}/k \quad ,$$

respectively, are approximately equal if ω is small enough. If we solve this equation on the understanding that ω is *very large*, we must keep all quantities in the exact solution. But that solution will reduce *numerically* to the approximate solution if we evaluate it with a small value of ω .

We could say that the solution obtained on the assumption of high frequency becomes inconsistent if we apply it to problems at low frequency, but, if the appropriate small parameter is small enough, an inconsistent approximation is no worse numerically than a consistent approximation.

Once more I would warn against trying to make absolute judgments of what is "small" and what is "not small." I avoid careful definitions of my small parameters largely for this reason; if the definition is not precise, one can never be tempted to put numbers into the definition! In the problem ahead, we cannot possibly judge analytically how "slender" the ship must be or how "high" the frequency must be for the results to have some validity.

In all of the discussions of ship motions, I use the same notation as in the study of oscillatory motion in an infinite fluid. See Section 2.32.

The ship in its mean position will be defined by the equation:

$$S_0(x,y,z) = -z + d(x,y) = 0 \ , \tag{3-18}$$

where $d(x,y) = O(\varepsilon)$; the instantaneous hull position is defined by the following equation:

$$S(x,y,z,t) = -z + d(x,y) + \xi_3(t) - x\,\xi_5(t) = 0 \ . \tag{3-19}$$

The ship is heading toward negative x (although it does not matter in the zero-speed case). Upward heave and bow-up pitch are considered positive.

We assume that all motions have very small amplitude. Symbolically, we write that:

$\xi_i(t) = O(\varepsilon\delta)$ as either ε or δ approaches zero,

where ε is the usual slenderness parameter, and δ is a "motion-amplitude" parameter. This convenient assumption allows us to vary the motion amplitude for a *given* ship (*i.e.*, for fixed ε), and it also guarantees that the motions are small compared with the ship beam and draft, even as the latter approach zero as $\varepsilon \to 0$. Velocity potential, wave height, motion variables, and all other dependent variables may be expected to have double asymptotic expansions, valid as $\varepsilon \to 0$ and $\delta \to 0$. We shall consistently carry terms which are linear in δ . The steady-motion problems already treated correspond to the $\delta = 0$ case; at zero speed, the $\delta = 0$ case is trivial. The problem ahead is to solve the linear motions problem — "linear" in terms of motion amplitude. With respect to the slenderness parameter, we shall consistently carry up to ε^2 terms.

It should be noted that the slenderness assumption is not needed in formulating a linear motions problem at zero forward speed; it is convenient, however, in practical application of the theory.

All motions are assumed to be sinusoidal at radian frequency ω . I use a complex exponential notation, so that: $\dot{\xi}_i(t) = i\omega\xi_i(t)$. Also, it is assumed that $\omega = O(\varepsilon^{-1/2})$, and so symbolically we can write: $\partial/\partial t = O(\varepsilon^{-1/2})$.

The potential function, $\phi(x,y,z,t)$, satisfies the Laplace equation and the following boundary conditions:

$$[A] \quad 0 = g\zeta + \phi_t + \frac{1}{2}[\phi_x^2+\phi_y^2+\phi_z^2] , \quad \text{on} \quad z = \zeta(x,y,t) ; \qquad (3\text{-}20a)$$

$$[B]\quad 0 = \phi_x\zeta_x + \phi_y\zeta_y - \phi_z + \zeta_t \ , \quad \text{on} \quad z = \zeta(x,y,t) \quad ; \qquad (3\text{-}20b)$$

$$[S]\quad 0 = \phi_x S_x + \phi_y S_y - \phi_z + S_t \ , \quad \text{on} \quad S(x,y,z,t) = 0 \quad . (3\text{-}21)$$

We consider first the problem of a ship which is forced by some external means to heave and pitch in calm water. In the far field, the slenderness assumption leads us to expect that the potential function can be represented by a line of singularities on the x axis. From previous experience, we might hope that a line of sources would suffice in the first approximation; this turns out to be correct. Since these sources represent an oscillating ship, the strengths of the sources will also vary sinusoidally. Suppose that there is a source distribution on the x axis:

$$\mathit{Re}\ \{\sigma(x)\ e^{i\omega t}\} \quad , \quad -\infty < x < \infty \quad .$$

Define $\sigma(x)$ to be identically zero beyond the ends of the ship. Obviously, $\sigma(x) = o(1)$ as $\delta \to 0$, since there is no fluid motion at all for $\delta = 0$. Therefore, in the first approximation, we may linearize the free-surface conditions. More precisely, we could assume the existence of asymptotic expansions, $\phi \sim \Sigma\phi_n$ and $\zeta \sim \Sigma\zeta_n$, and let the first term in each be $o(1)$ as $\delta \to 0$. The linearized free-surface conditions take their usual form:

$$[A]\quad 0 = g\zeta + \phi_t \ , \qquad \text{on} \quad z = 0 \ ;$$

$$[B]\quad 0 = -\phi_z + \zeta_t \ , \qquad \text{on} \quad z = 0 \ .$$

These can be combined into the following:

$$\phi_z - \nu\phi = 0 \ , \qquad \text{on} \quad z = 0 \ , \qquad (3\text{-}22)$$

where $\nu = \omega^2/g = O(\varepsilon^{-1})$. In the far field, it is very difficult to guess how differentiation alters orders of magnitude. If the oscillation frequency is very high, then the resulting waves are very short; it would be reasonable, perhaps, to try stretching the coordinates, and there would be no obvious basis for doing this anisotropically. The approach which I take here is somewhat different: Solve the above-stated linear problem exactly, then observe the behavior of the solution for high frequency of oscillation. In other words, the problem is not stated in a consistant manner, but when we have the solution we rearrange it and make it consistent.

The desired potential function can be written in the following form:

$$\phi(x,y,z,t) = Re\ \{\phi(x,y,z)\ e^{i\omega t}\} \quad , \qquad (3\text{-}23)$$

where:

$$\phi(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\ \sigma(\xi) \int_{0}^{\infty} \frac{k\ dk}{k-\nu}\ e^{kz}\ J_0(k\sqrt{[(x-\xi)^2+y^2]}) \qquad (3\text{-}23a)$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk\ e^{ikx}\ \sigma^*(k) \int_{-\infty}^{\infty} \frac{d\ell\ e^{i\ell y+z\sqrt{(k^2+\ell^2)}}}{\sqrt{(k^2+\ell^2)}-\nu} \quad . \qquad (3\text{-}23b)$$

The form in (3-23a) can be obtained readily by superposing a distribution of free-surface sources; J_0 is the ordinary Bessel function of order zero, and the wiggly arrow shows that the integral is to be interpreted as a contour integral, indented at the pole in the obvious sense indicated. Form (3-23b) is obtained by a transform method; $\sigma^*(k)$ is the Fourier transform of $\sigma(x)$; details may be found in Ogilvie and Tuck (1969). Again, the inner integral is to be interpreted as a contour integral; there are two poles in this case. In both formulas, the path of the contour has been chosen so that the solution has a satisfactory behavior at infinity, *viz.*, it represents outgoing waves.

We need the inner expansion of this potential function, that is, we must find its behavior as $r = (y^2+z^2)^{1/2} \to 0$. The basic idea here in finding the inner expansion is to use the second form of solution, convert the contour integral into an integral along a closed contour, and use the calculus of residues. The integrand of the inner integral has four singularities, located at $\ell = \pm\ell_0$ and at $\ell = \pm i|k|$, where $\ell_0 = (\nu^2-k^2)^{1/2}$. The first two are simple poles, but the second two are branch points. We "connect" the latter via the point at infinity; see Figure (3-3). It is drawn for the case that $|k| < \nu$; if $|k| > \nu$, all four singularities are purely imaginary. The contour is closed as shown if $y > 0$. (Otherwise, the contour is closed below.) The integrals along the large circular arcs approach zero as the radius of the arcs approaches infinity. Then the inner integral in $\phi(x,y,z)$ is equal to $2\pi i$ times the residue at $\ell = -\ell_0$, less the value of the contour integral down and back up the imaginary axis. The latter can be shown to be $O(\varepsilon)$, and so the inner integral in $\phi(x,y,z)$ is:

$$\int_{-\infty}^{\infty} \frac{d\ell\, e^{i\ell y+z\sqrt{k^2+\ell^2}}}{(k^2+\ell^2)^{1/2}-\nu} = -\frac{2\pi i\nu}{(\nu^2-k^2)^{1/2}}\, e^{\nu z-iy\sqrt{\nu^2-k^2}} + O(\varepsilon) .$$

Next, we assume that the source distribution is smooth enough that $\sigma(x)$ does not vary rapidly on a length scale comparable with ship beam. This assumption implies that

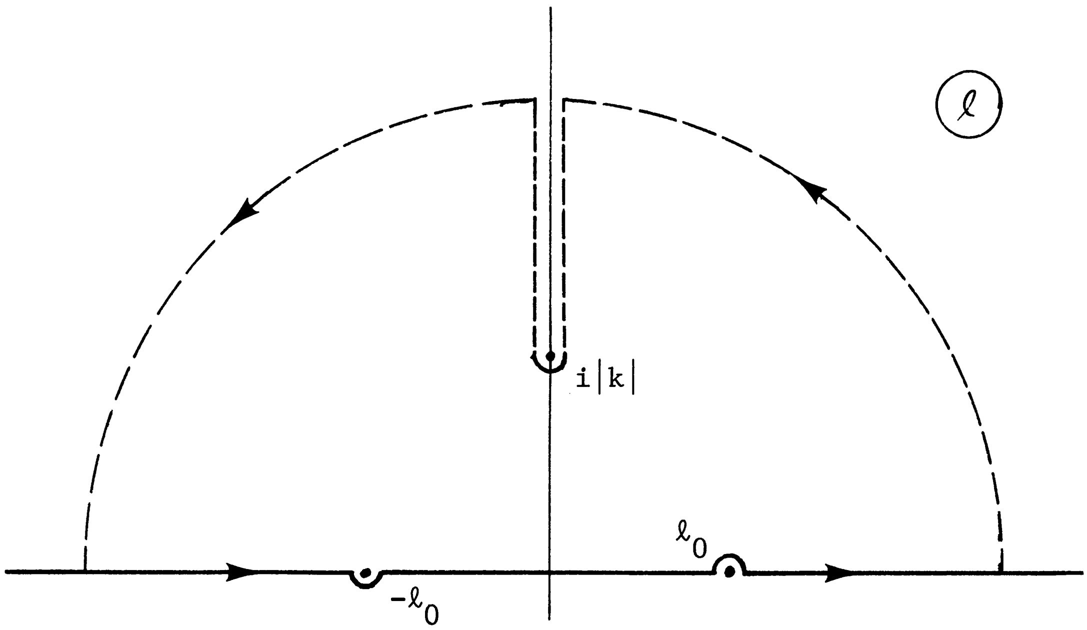

Figure (3-3). Contour of Integration Defining the Velocity Potential of a Line of Pulsating Sources: Zero-Speed Case.

$\sigma^*(k)$ decreases rapidly with increasing values of k, and so the value of the above inner integral — a function of k — does not really matter except when k is small in magnitude. Accordingly, we expand the above expression in a manner appropriate for small $|k|$. We obtain.

$$\phi(x,y,z) \simeq \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk\, e^{ikx}\sigma^*(k)\left[\frac{2\pi i\nu}{(\nu^2 - k^2)^{1/2}}\, e^{\nu z - iy\sqrt{\nu^2-k^2}}\right]$$

$$\simeq \frac{i}{2\pi}\, e^{\nu z - i\nu y}\int_{-\infty}^{\infty} dk\, e^{ikx}\sigma^*(k)\{1 + \ldots\}$$

$$= i\, e^{\nu z - i\nu y}\sigma(x) + \ldots \quad . \qquad (3\text{-}24)$$

With the time dependence reintroduced, we have:

$$\phi(x,y,z,t) \simeq \mathcal{Re}\{i\, \sigma(x)\, e^{\nu z}e^{i(\omega t - \nu y)}\} + \ldots \quad . \qquad (3\text{-}25)$$

This approximation represents a travelling wave; for $y > 0$, in particular, the wave is moving away from the line of sources. For $y < 0$, we must start over, closing the contour for the ℓ integration on the lower side of the ℓ plane. It turns out that the result is the same if only we replace y by $|y|$. Thus, we have an outgoing wave for $y < 0$ also. In both cases, the outgoing wave has the form appropriate for a gravity wave in two dimensions.

In the approximations above, it is necessary to require that r be not extraordinarily large; if one assumes that $r = O(1)$ and $\omega = O(\varepsilon^{-1/2})$, then the above results follow logically. Thus the very simple approximation above is valid even in part of the far field. It is an example of the well-known physical principle that nearly unidirectional waves can be generated if the wave generator is much larger than a wavelength.

If we let $r = O(\varepsilon)$, no change occurs in this approximation. Since $\nu = O(\varepsilon^{-1})$, it is not permissible to expand the exponential functions even when y and z are $O(\varepsilon)$. The only effect of passing from far field to near field now is to change the scale of the observed wave motion.

This far-field analysis has provided information that was probably quite obvious intuitively: In the near-field, the condition at infinity is that there should be outgoing, two-dimensional, gravity waves*. With this information in hand, we can move on to the formulation and solution of the near-field problem.

*I cannot imagine that anyone would ever have doubted this fact, even without the above analysis to show it. But in the forward-speed problem, the condition at infinity in the near field is not at all obvious, and such an analysis seems necessary.

In the near-field, we make the usual slender-body assumptions:

$$\frac{\partial}{\partial y} \, , \, \frac{\partial}{\partial z} \, , \, \frac{\partial}{\partial r} \quad = \quad O(\varepsilon^{-1}) \quad .$$

To a first approximation, the potential function satisfies the Laplace equation in two dimensions:

$$\phi_{yy} + \phi_{zz} \sim 0 \quad ,$$

and the linear free-surface condition

$$\phi_z - \nu\phi \sim 0 \quad \text{on} \quad z = 0 \quad . \qquad (3\text{-}22)$$

With the assumptions made above, the two terms here are of the same order of magnitude. (If we did not assume high frequency, we would obtain just the rigid-wall boundary condition, $\phi_z = 0$.) This condition implies that we shall be solving a gravity-wave problem in two dimensions. At infinity, we know from the far-field solution that the appropriate condition is an outgoing-wave requirement. All that remains is to put the body boundary condition, (3-21), into the appropriate form.

Let $\partial/\partial N$ denote differentiation in the direction normal to the body contour in a cross section. Then, from (3-19) and (3-21),

$$\frac{\partial\phi}{\partial N} = \frac{\phi_z - \phi_y S_y}{(1+d_y{}^2)^{1/2}} = \frac{\phi_x S_x + S_t}{(1+d_y{}^2)^{1/2}}$$

$$= \frac{\dot{\xi}_3 - x\dot{\xi}_5 - \xi_5\phi_x + d_x\phi_x}{(1+d_y{}^2)^{1/2}} \sim \frac{\dot{\xi}_3 - x\dot{\xi}_5}{(1+d_y{}^2)^{1/2}} \quad . \qquad (3\text{-}26)$$

The last simplification involves an error which is $O(\varepsilon^2)$ higher order than the retained terms. To the same approximation, we can write (see (2-72')):

$$n_3 = \mathbf{n} \cdot \mathbf{k} \sim \frac{1}{(1+d_y^{\,2})^{1/2}} \; ; \quad n_5 \sim -\, x n_3 \; .$$

Thus, the boundary condition is:

$$\frac{\partial \phi}{\partial N} \sim n_3 \dot{\xi}_3 + n_5 \dot{\xi}_5 \; , \qquad \text{on} \quad z = d(x,y) \; . \tag{3-27}$$

As in the infinite-fluid problem (*Cf.* (2-73)), we can define normalized potential functions, $\phi_i(x,y,z)$:

$$\phi_{i_{yy}} + \phi_{i_{zz}} = 0 \; , \qquad \text{in the fluid region;} \tag{3-28a}$$

$$\frac{\partial \phi_i}{\partial N} = n_i \; , \qquad \text{on} \quad z = d(x,y) \; ; \tag{3-28b}$$

$$\phi_{i_z} - \nu \phi_i = 0 \; , \qquad \text{on} \quad z = 0 \; , \tag{3-28c}$$

where $\nu = \omega^2/g$. In the present case, the functions satisfy the 2-D Laplace equation and a 2-D body boundary condition, and they must satisfy the linearized free-surface condition. Instead of the previous simple condition at infinity, we must impose the 2-D outgoing-wave radiation condition and a condition of vanishing disturbance at great depths. Thus, the boundary-value problem is much more complicated than in the infinite-fluid case, but, thanks to the slenderness assumption, we have only 2-D problems to solve, and, thanks to the small-amplitude assumption, the problems are linear.

The actual velocity potential function can now be expressed:

$$\phi(x,y,z,t) \sim Re\left[\sum_{j=3,5} i\omega\xi_j(t)\phi_j(x,y,z)\right] \quad . \qquad (3\text{-}29)$$

It must be observed that each ϕ_j is complex, because of the radiation condition. It is necessary to devise an appropriate numerical scheme for solving these problems. Both mapping techniques and integral-equation methods have been successfully applied. Note, incidentally, that the heave/pitch problem requires solution of just the ϕ_3 problem, since the slenderness assumption allows the approximation to be made that $\phi_5 \simeq -x\phi_3$.

The result of this analysis is a pure strip theory, that is, the flow appears to take place in cross-sections as if each cross-section were independent of the others. It is consistent to follow the solution of this problem with a computation of the pressure field at each cross-section, from which force-per-unit-length, then force and moment on the ship can be found after appropriate integrations. We obtain the following formulas for the force and moment on the ship resulting from the motion of the ship:

$$F_j^m(t) = -\rho\int_{S_0} ds\; n_j\left[g(\xi_3 - x\xi_5) + (i\omega)^2(\xi_3\phi_3 + \xi_5\phi_5)\right] , \quad (3\text{-}30)$$

where $j = 3$ for heave force and $j = 5$ for pitch moment, and the symbol S_0 denotes that the integration is to be taken over the hull surface in its mean or undisturbed position, which is specified by Equation (3-18). The first term, involving g , is just a buoyancy effect. The following terms are purely hydrodynamic; they will be expressed

in terms of added-mass and damping coefficients, as follows:
Let:

$$m(x) + \frac{1}{i\omega} n(x) \equiv \rho \int_{C(x)} d\ell \; n_3 \, \phi_3 \quad , \tag{3-31}$$

where $C(x)$ is the contour of the immersed part of the cross-section of x . *Cf.* (2-82). We call $m(x)$ the "added mass per unit length" and $n(x)$ the "damping coefficient per unit length." Using the slender-body approximations that $\phi_5 \simeq -x\phi_3$ and $n_5 \simeq -xn_3$, we find for $F_j^m(t)$:

$$F_3^m(t) = -\rho g \int_{S_0} dS \; n_3(\xi_3 - x\xi_5) - (i\omega)^2 \int_L dx \; (\xi_3 - x\xi_5)[m(x) + n(x)/i\omega] \; ;$$

$$F_5^m(t) = \rho g \int_{S_0} dS \; xn_3(\xi_3 - x\xi_5) + (i\omega)^2 \int dx \; x(\xi_3 - x\xi_5)[m(x) + n(x)/i\omega] \; . \tag{3-32}$$

Finally, we abbreviate these formulas:

$$F_j^m(t) = - \sum_{i=3,5} [(i\omega)^2 a_{ji} + (i\omega) b_{ji} + c_{ji}] \xi_i(t) \quad , \tag{3-33}$$

where:

$$a_{33} = \int_L dx \; m(x) \; ; \qquad b_{33} = \int_L dx \; n(x) \; ;$$

$$a_{35} = a_{53} = -\int_L dx \; x \, m(x) \; ; \qquad b_{35} = b_{53} = -\int_L dx \; x \, n(x) \; ;$$

$$a_{55} = \int_L dx \; x^2 m(x) \; ; \qquad b_{55} = \int_L dx \; x^2 n(x) \; ;$$

$$c_{33} = \rho g \int_{S_0} dS \; n_3 = 2\rho g \int_L dx \; b(x,0) \; ;$$

$$c_{35} = c_{53} = - \rho g \int_{S_0} dS \; x \, n_3 = - 2\rho g \int_L dx \; x \; b(x,0) \; ;$$

$$c_{55} = \rho g \int_{S_0} dS \; x^2 n_3 = 2\rho g \int_L dx \; x^2 b(x,0) \; ;$$

$b(x,z)$ is the hull offset at a point (x,z) on the centerplane.

The wave-excitation problem can be formulated as a singular perturbation problem, but such a problem has never been satisfactorily solved, even for the zero-speed case. Fortunately, another approach is available for obtaining the wave excitation; this is the very elegant theorem proven by Khaskind (1957). It allows one to compute the wave excitation force, including the effects of the diffraction wave, without solving the diffraction problem. Since we thus avoid the singular perturbation problem altogether, only the final results are presented here. (Reference may be made to Newman (1963) for details of the zero-speed case.) Let the incident wave have the velocity potential:

$$\phi_0(x,z,t) = \frac{igh}{\omega} e^{\nu z + i(\omega t - \nu x)} \quad ;$$

the corresponding wave shape is given by:

$$\zeta_0(x,t) = h \, e^{i(\omega t - \nu x)} \quad .$$

This is the head-seas case. For an arbitrary body, the heave force due to the incident waves is:

$$F_3^W(t) = \rho g h \, e^{i\omega t} \int_{S_0} dS \, e^{\nu z - i\nu x} \{(1 - \nu\phi_3)n_3 + i\nu\phi_3 n_1\} \quad .$$

If the body is a slender ship, with axis parallel to the wave-propagation direction, this formula simplifies to the following:

$$F_3^W(t) \simeq \rho g h \, e^{i\omega t} \int_L dx \, e^{-i\nu x} \int_{C(x)} d\ell \, n_3 \, e^{\nu z} \, (1 - \nu\phi_3) \quad . \qquad (3\text{-}34a)$$

The corresponding expression for pitch moment on a slender body is:

$$F_5^W(t) \simeq \rho g h \, e^{i\omega t} \int_L dx \, e^{-i\nu x}(-x) \int_{C(x)} d\ell \, n_3 \, e^{\nu z} \, (1 - \nu\phi_3) \quad . \qquad (3\text{-}34b)$$

In the expression $(1-\nu\phi_3)$ in the integrand, the first term leads to the force (moment) which would exist if the presence of the ship did not alter the pressure distribution in the wave; in other words, it gives the so-called "Froude-Krylov" excitation. This fact can be proven by applying Gauss' theorem to the integral. Dynamic effects in the wave ("Smith effect") are properly accounted for. The second term gives all effects of the diffraction wave.

A final rewriting of the wave-force formula is worthwhile. The above approximate expression for $F_3^W(t)$ can be manipulated into the following:

$$F_3^W(t) \simeq 2\rho g \int_L dx \, b(x,0)\zeta_0(x,t) \left[1 - \frac{\nu}{b(x,0)} \int_{-T(x)}^{0} dz \, e^{\nu z} \, b(x,z)\right]$$

$$+ \; i\rho\omega \int_L dx \, \zeta_{0_t}(x,t) \int_{C(x)} d\ell \, n_3 \, \phi_3 \, e^{\nu z} \quad .$$

The first term shows the Froude-Krylov force quite explicitly; the product of $\zeta_0(x,t)$ and the quantity in brackets is often called an "effective waveheight," the second factor being a quantitative representation of the Smith effect. The second integral term has been expressed in terms of the vertical speed of the wave surface, $\zeta_{0_t}(x,t)$. This term should be compared with the force expression for the calm-water problem, (3-30). For a slender body, the hydrodynamic part of the latter can be written, for $j = 3$,

$$- i\rho\omega \int_{S_0} dS\ n_3[\dot{\xi}_3(t)\phi_3+\dot{\xi}_5(t)\phi_5] \simeq - i\rho\omega \int_L dx\ [\dot{\xi}_3(t)-x\dot{\xi}_5(t)] \int_{C(x)} d\ell\ n_3\phi_3 \quad .$$

The last quantity in brackets is the vertical speed of the cross-section at any particular x . Comparison with the second term of $F_3^W(t)$ shows that the latter is almost exactly the same as the hydrodynamic force that we would predict if each section of the ship had a vertical speed $- \zeta_{0_t}(x,t)$. This analogy would be exact, in fact, if the exponential factor, $e^{\nu z}$, were not present in the $F_3^W(t)$ formula.

Except for that factor, what we have found is that Korvin-Kroukovsky's well-known "relative-velocity hypothesis" is approximately correct according to the analysis above. The hypothesis is particularly accurate for very long waves, in which case $e^{\nu z} \simeq 1$ over the depth of the ship, but it is less accurate for short waves. Again, it should be noted that we have no absolute basis for saying whether a particular wave is short or long in this respect. In computing the Froude-Krylov part of the force, it is well-known that the exponential-decay factor *must* be included in practically all cases of practical interest; this has been amply demonstrated experimentally. It suggests that one should be wary of dropping the exponential factor in the diffraction-wave force expression.

Summary. In the far field, we assumed that the effects of the heaving/pitching ship could be represented by a line of pulsating singularities located at the intersection of the ship centerplane and the undisturbed free surface. For a first approximation, we tried using just sources, and these were sufficient to allow matching with the near-field solution. In particular, the inner expansion of the outer expansion showed that the near-field expansion would satisfy a two-dimensional outgoing-wave radiation condition, at least in the first approximation. With this fact established, we formulated the near-field problem; it reduced ultimately to the determination of a velocity potential in two dimensions, the potential satisfying a linear free-surface condition and an ordinary kinematic body boundary condition, as well as the outgoing-wave condition. This is a standard problem which must generally be solved numerically with the aid of a large computer; such programs exist. The force and moment were expressed as integrals of added-mass-per-unit-length and damping-per-unit-length, both of which could be found from the velocity potential for the 2-D problem. Finally, the determination of the wave excitation force and moment was carried out by application of the Khaskind formula, which permits us to avoid the singular perturbation problem involved in solving for the diffraction wave.

3.4 Oscillatory Motion with Forward Speed

The problem of predicting the hydrodynamic force on an oscillating ship with forward speed is not fundamentally much different from the same problem in the zero-speed case. It is considerably more complex, to be sure, but no new assumptions are needed.

The approach here is that of Ogilvie and Tuck (1969). Alternative approaches have been devised by numerous other authors; some of these were mentioned in the last section. The distinguishing characteristics of the Ogilvie-Tuck approach are: 1) application of the method of matched asymptotic expansions, and 2) assumption that frequency is high in the asymptotic sense that $\omega = O(\epsilon^{-1/2})$, while Froude number is $O(1)$. Also, the problem is broken down into a series of linear problems by the use of a "motion-amplitude" parameter, δ, which is a measure of the amplitude of motion relative to the size of ship beam and draft.

The reference frame is assumed to move with the mean motion of the center of gravity of the ship. Thus it appears that there is a uniform stream at infinity, and we take this stream in the positive x direction. The z axis points upward from an origin located in the plane of the undisturbed free surface, and the y axis completes the right-handed system. (Positive y is measured to starboard.)

Let the velocity potential be written:

$$\phi(x,y,z,t) = Ux + U\chi(x,y,z) + \psi(x,y,z,t) , \qquad (3\text{-}35a)$$

where $U[x + \chi(x,y,z)]$ is the solution of the steady-motion problem discussed in Section 3.1. For the moment, we simply assume that $\psi(x,y,z,t)$ includes everything that must be added to the steady-motion potential so that $\phi(x,y,z,t)$ is the solution of the complete problem. We shall also divide the free-surface deformation function into two parts:

$$\zeta(x,y,t) \;=\; \eta(x,y) + \theta(x,y,t) \quad , \tag{3-35b}$$

where $\eta(x,y)$ is the free-surface shape in the steady-motion problem (the $\zeta(x,y)$ of Section 3.1), and $\theta(x,y,t)$ is whatever must be added so that $\zeta(x,y,t)$ is the complete free-surface deformation.

The body surface is defined mathematically just as in Section 3.3 for the zero-speed problem; see (3-18) and (3-19). The same assumptions are made about orders of magnitude:

$$\xi_i(t) \;=\; O(\varepsilon\delta) \quad ; \qquad \omega \;=\; O(\varepsilon^{-1/2}) \quad .$$

From these assumptions and the subsequent analysis, it turns out that

$$\psi(x,y,t) \;=\; O(\varepsilon^{3/2}\delta) \quad , \qquad \theta(x,y,t) \;=\; O(\varepsilon\delta) \quad ,$$

as either ε or $\delta \to 0$. We can look on the complete solution as a double expansion in ε and δ . From this point of view, the expansion for the potential can be written:

$$\begin{aligned}\phi(x,y,z,t) \;=\; & \{\underset{O(\delta^0\varepsilon^0)}{Ux} + \underset{O(\delta^0\varepsilon^2)}{U\chi_1(x,y,z)} + \ldots\} \\ & + \{\underset{O(\delta^1\varepsilon^{3/2})}{\psi_1(x,y,z,t)} + \underset{O(\delta^1\varepsilon^2)}{\psi_2(x,y,z,t)} + \ldots\} + o(\delta) \quad .\end{aligned} \tag{3-36}$$

The order of magnitude of the term $U\chi_1(x,y,z)$ was found in Section 3.1. The order of magnitude of ψ_1 may be somewhat surprising. Physically, it implies that the effects of ship oscillations dominate the effects of steady forward motion — in the first approximation. These orders of magnitude were derived by Ogilvie & Tuck. Here, I shall not prove them, but I hope to make them appear plausible. It should be noted that *the high-frequency assumption was made just so that the orders of magnitude would come out this way.* (*Cf.* the discussion in Section 2.3, in which it was pointed out that the formalism for the steady-motion slender-body problem is established to force certain expected results to come out of the analysis. We are doing the same here, forcing strip theory to come out as the first approximation.)

The linearity of the ψ_1 problem permits us to assume that the time dependence of ψ_1 and of the corresponding first term in a θ expansion can be represented by a factor $e^{i\omega t}$.

In order to find any effects of interaction between steady forward motion and oscillatory motion, it is necessary to solve for the term $\psi_2(x,y,z,t)$. Thus, we must retain two terms in the time-dependent part of the potential function. (The problem is still linear, however, in terms of δ .) It is not convenient to be repeatedly attaching subscripts to the symbols, and so I shall simply write out equations and conditions which are asymptotically valid to the order of magnitude appropriate to keeping ε^2 terms in the expansion of $\psi(x,y,z,t)$.

In the far field, the effect of the oscillating ship can be represented in terms of line distributions of singularities. Again, we try to get along with just a distribution of sources, and we are successful if we allow for the existence of both steady and pulsating sources. The steady-source distribution is exactly the same as in the steady-motion problem. Let the density of the unsteady sources be given by $\sigma(x)e^{i\omega t}$; define $\sigma(x) \equiv 0$ for the values of x beyond the bow or stern. The corresponding potential function must satisfy the Laplace equation in three dimensions, a radiation condition, and the usual linearized free-surface condition:

$$(i\omega)^2\psi + 2i\omega U\psi_x + U^2\psi_{xx} + g\psi_z = 0 \qquad \text{on } z = 0 \quad . \tag{3-37}$$

Then it can be shown that:

$$\psi(x,y,z,t) \sim -\frac{e^{i\omega t}}{4\pi^2}\int_{-\infty}^{\infty} dk\, e^{ikx}\sigma^*(k)\int_C \frac{d\ell\, \exp\,[i\ell|y|+z\sqrt{k^2+\ell^2}]}{\sqrt{k^2+\ell^2} - \nu(1+Uk/\omega)^2} \quad , \tag{3-38}$$

where $\sigma^*(k)$ is the Fourier transform of $\sigma(x)$, and the contour **C** is taken as in Figure (3-4), where k_1 and k_2 are the real roots* $(k_1 < k_2)$ of the equation:

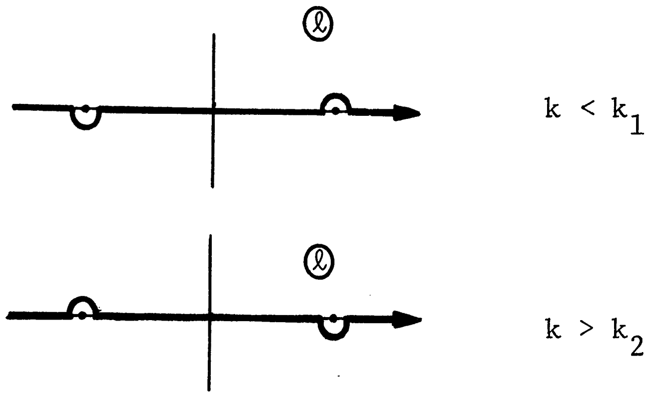

Figure (3-4). Contour of Integration Defining the Velocity Potential of a Line of Pulsating Sources: Forward-Speed Case.

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*There are two real roots if $\tau \equiv \omega U/g > 1/4$; the other two roots are a complex pair. Since we assume that $\omega = O(1/\sqrt{\varepsilon})$, then also $\tau = O(1/\sqrt{\varepsilon})$, and we are assured that $\tau \gg 1/4$. However, if $\tau \downarrow 1/4$, the complex pair come together, and our estimates are all very bad. Of course, it is well known that the ship-motion problem is singular at $\tau = 1/4$. For still smaller values of τ , there are four real roots of the above equation, and the solution can again be interpreted physically and mathematically. From experimental evidence, it appears that our final formulas can be applied for any forward speed, at least in head seas, but the presence of a singularity at $\tau = 1/4$ shows that this is accidental. Our theory is a high-frequency, finite-speed theory, and it really should not be possible to let U vary continuously down to zero.

$$\left[1 + \frac{U\,k}{\omega}\right]^4 - \left[\frac{k}{\nu}\right]^2 = 0 \quad ,$$

and the contour is indented as shown at the poles on the real axis in the ℓ plane. The contour **C** extends from $-\infty$ to $+\infty$. The poles in the ℓ plane all fall on the imaginary axis if $k_1 < k < k_2$, and then **C** is the entire real axis, with no special interpretations being necessary.

The above expression for $\psi(x,y,z,t)$ is a one-term outer expansion, but it is not a *consistent* one-term expansion. It is shown by Ogilvie & Tuck that a much simpler expression is possible if $r = (y^2+z^2)^{1/2}$ is $O(1)$ as $\varepsilon \to 0$; emphasis should be placed here on the restriction that r is not extraordinarily large. If $\sigma^*(k)$ is restricted in a rather reasonable way, it follows that:

$$\psi(x,y,z,t) \sim \frac{i}{2\pi}\, e^{i\omega t}\int_{-\infty}^{\infty} dk\; e^{ikx}\sigma^*(k)\; e^{\nu(z-i|y|)(1+Uk/\omega)^2} \; . \qquad (3\text{-}39)$$

We can take this as our one-term outer expansion of $\psi(x,y,z,t)$.

The inner expansion of this expression is obtained by letting $r = O(\varepsilon)$. Then we find that:

$$\psi(x,y,z,t) \sim i\, e^{i\omega t}\, e^{\nu(z-i|y|)}\left[\sigma(x) - 2i(\omega U/g)(z-i|y|)\sigma'(x)\right] . \qquad (3\text{-}40)$$

Since $\nu r = O(1)$, it is not appropriate to expand the exponential function further. This is a two-term inner expansion of the outer expansion of ψ ; the first term represents an outgoing, two-dimensional, gravity wave, just as in the zero-speed problem (see (3-25)), but the second term represents a wave motion in which the amplitude increases linearly with distance from the x axis. The latter is a rather strange kind of potential function; it represents a wave which becomes larger and larger, without limit, at large distance. However, one must remember that this is the inner expansion of the outer expansion of $\psi(x,y,z,t)$; it means that there are waves near the x axis which seem to increase in size *when viewed in the near field.* At very great distances, one must revert to the previous integral expressions for $\psi(x,y,z,t)$.

We must next find an inner expansion which satisfies conditions appropriate to the near field and which matches the above far-field expansion. One finds readily that:

$$\psi_{yy} + \psi_{zz} = 0 \qquad \text{in the fluid region,}$$

to the order of magnitude that we consistently retain. Thus, the partial differential equation is again reduced to one in two dimensions, and so we seek to restate all boundary conditions in a form appropriate to a 2-D problem.

The body boundary condition must be carefully expressed in terms of a relationship to be satisfied on the instantaneous position of the body. This condition can then be restated as a different condition to be applied on the mean position of the hull. It can be shown that:

$$\frac{\partial\psi}{\partial N} \sim \underset{[\varepsilon^{1/2}\delta]}{\frac{\dot{\xi}_3 - x\dot{\xi}_5}{(1+d_y^2)^{1/2}}} + \underset{[\varepsilon\delta]}{\frac{-U\xi_5 + U(\xi_3 - x\xi_5)(h_{0_y}\chi_{yz} - \chi_{zz})}{(1+d_y^2)^{1/2}}} \text{ on } z = d(x,y) \; . \tag{3-41}$$

The derivative on the left has the same meaning as in the previous slender-body analyses: It is the rate of change in a cross-section plane, in the direction normal to the hull contour in that plane. The first term on the right-hand side is the same as in the zero-speed problem; see (3-26). The quantity $-U\xi_5$ has a simple physical interpretation: it is a cross-flow velocity caused by the instantaneous angle of attack. The remaining terms all arise as a correction on the steady-motion potential function, $U\chi$; the latter satisfies a boundary condition on the mean position of the ship, which is not generally the actual position of the ship, and so it must be modified.

Intuitive derivations of strip theory usually omit the terms involving χ . However, in a consistent slender-body derivation, they are the same order of magnitude as the angle-of-attack term. (This says nothing about which is the more nearly valid appraoch!)

The free-surface condition reduces ultimately to:

$$\underset{[\varepsilon^{1/2}\delta]}{\psi_{tt} + g\psi_z} \sim \underset{[\varepsilon\delta]}{- 2U\psi_{tx}} - 2U\chi_y\psi_{ty} - U\chi_{yy}\psi_t \quad \text{on} \quad z = 0 \quad . \tag{3-42}$$

The orders of magnitude are noted, again on the basis of information not derived here. This condition can be compared with the linear condition used in the far field, (3-37). The two terms on the left here are obviously the same as the terms $(i\omega)^2\psi+g\psi_z$ in (3-37), and the first term on the right here, $-2U\psi_{tx}$, is the same as the term $2i\omega U\psi_x$ in (3-37). The other two terms on the right-hand side here are basically nonlinear in origin; they involve interactions between the oscillation and the steady perturbation of the incident stream. The term $U^2\psi_{xx}$ which appears in (3-37) is missing here because it is $O(\varepsilon^{3/2}\delta)$ in the near field by our reckoning.

Again it is worthwhile to compare this boundary condition with its nearest equivalent in other versions of slender-body theory or strip theory of ship motions. If we did not assume that frequency is very large, slender-body theory would require in the first approximation that $\psi_z = 0$, since the other terms are all higher order. This is just the free-surface boundary condition obtained in this problem by Newman and Tuck (1964) and by Maruo (1967). Higher order approximations would involve nonhomogeneous Neumann conditions on $z = 0$. On the other hand, in most derivations of strip theory, it is assumed that the free-surface condition is: $\psi_{tt}+g\psi_z = 0$ on $z = 0$. This agrees with the lowest-order condition obtained by Ogilvie and Tuck, as given above. However, the assumption of this boundary condition in the usual strip-theory derivation is quite arbitrary, and no means is available to extend it to higher-order approximations. The assumptions made by Ogilvie and Tuck were chosen explicitly so that the simplest approximation would be just strip theory, and we see here that that goal was achieved. This basis for choosing assumptions was selected only because strip theory had proven to be the most accurate procedure available for predicting ship motions.*

The method of solution used by Ogilvie and Tuck is to find several functions each of which satisfies some part of the nonhomogeneous conditions. In particular, let the solution be expressed in the following form:

$$\psi(x,y,z,t) = \sum_j [i\omega\Phi_j + U\Psi_j + (i\omega)^2 U\Omega_j]\xi_j(t) , \qquad (3\text{-}43)$$

where $j = 3$ and 5 , and Φ_j , Ψ_j , and Ω_j satisfy the following conditions, respectively:

*In other words, we stopped fretting about how irrational strip theory was and set out to derive it formally!

$$\Phi_{j_{yy}} + \Phi_{j_{zz}} = 0 \; ; \; \Phi_{j_n} = n_j \quad \text{on} \quad z = d(x,y) \; ; \quad \Phi_{j_z} - \nu\Phi_j = 0 \quad \text{on} \quad z = 0 \; ; \tag{3-44}$$

$$\Psi_{j_{yy}} + \Psi_{j_{zz}} = 0 \; ; \; \Psi_{j_n} = m_j \quad \text{on} \quad z = d(x,y) \; ; \quad \Psi_{j_z} - \nu\Psi_j = 0 \quad \text{on} \quad z = 0 \; ; \tag{3-45}$$

$$\Omega_{j_{yy}} + \Omega_{j_{zz}} = 0 \; ; \; \Omega_{j_n} = 0 \quad \text{on} \quad z = d(x,y) \; ;$$
$$\Omega_{j_z} - \nu\Omega_j = -(1/g)[2\Phi_{j_x} + 2\chi_y\Phi_{j_y} + \chi_{yy}\Phi_j] \quad \text{on} \quad z = 0 \quad . \tag{3-46}$$

The quantities n_j were defined previously, in (2-72), as the six components of a generalized normal vector. Also, the quantities m_j were defined earlier, by (2-75). In the present notation, let $\mathbf{v}(x,y,z)$ (See (2-74)) be defined by:

$$\mathbf{v}(x,y,z) = \nabla[x + \chi(x,y,z)] \quad .$$

Then m_j is again given by the previous formulas. Now it requires just a bit of manipulating to show that the assumed solution above indeed satisfies the body and free-surface boundary conditions; I omit the proof.

The above near-field solution must match the far-field solution, which has an inner expansion given in (3-40). In connection with the latter, a comment was made earlier that the near-field solution would have to represent a wave motion in which one component grows linearly in amplitude as $|y| \to \infty$. Now we can see that just such an interpretation must be given to the Ω_j functions, for otherwise we cannot possibly find solutions to the problems set above for Ω_j. The nonhomogeneous free-surface condition on Ω_j can be compared to the free-surface condition that would result if a pressure distribution were applied to the free surface. In fact, if a pressure field were applied externally on $z = 0$, the pressure being given by:

$$p(x,y,t) = i\rho\omega U\xi_j(t)[2\Phi_{jx} + 2\chi_y\Phi_{jy} + \chi_{yy}\Phi_j] \quad ,$$

then the potential function would have to be $(i\omega)^2 U\Omega_j\xi_j(t)$, with $\Omega_j(x,y,z)$ satisfying the conditions stated previously. This "pressure distribution" is periodic in time, and it is also periodic in y as $|y| \to \infty$; the latter comes from the term containing Φ_{jx}. Furthermore, the time and space periodicities are related to each other in just the way that one would expect for a plane gravity wave. This can be proven by studying the boundary-value problem for Φ_j. Thus, there is an effective pressure distribution over an infinite area, and it excites waves just the right combination of frequency and wavelength so that we have a resonance response. In an ordinary two-dimensional problem, there would be no solution satisfying all of these conditions. However, our solution need not be regular at infinity; it must only match the far-field expansion. And the far-field expansion predicts an appropriate singular behavior at infinity. It is shown by Ogilvie and Tuck that the solution of this inner problem does exactly match the above far-field solution. The way the pieces of the puzzle all fit together is rather typical of the method of matched asymptotic expansions, and it indicates at least that the manipulations of asymptotic relations were probably done correctly! (It still says nothing about the correctness of the assumptions.)

There is no benefit to be derived by repeating here the solution of the above detailed problems. Rather, we jump to the results for the heave force and the pitch moment, and we do little more than compare these results with the comparable formulas in two previous problems:

CASE 1 : The oscillating slender body, with forward speed, in an infinite fluid (Section 2.32)

CASE 2 : The oscillating slender body (ship), at zero forward speed, on a free surface (Section 3.3)

In all cases, let the force (moment) be expressed in the form:

$$F_j^m(t) = -\sum_{i=3,5}\left((i\omega)^2 a_{ji} + (i\omega) b_{ji} + c_{ji}\right)\xi_i(t) \quad .$$

We define c_{ji} to be independent of frequency and of forward speed. (We must make some such arbitrary convention, or the separation into a_{ji} and c_{ji} components is not unique.) With this convention, c_{ji} represents just the buoyancy restoring force (moment). Thus, $c_{ji} = 0$ for all j,i in case 1; in cases 2 and 3, c_{ji} is given by:

$$[c_{ji}] = 2\rho g \int_L dx \, \{1,-x\}\begin{pmatrix} 1 \\ -x \end{pmatrix} b(x,0) \quad .$$

Table 3-1 shows a_{ji} and b_{ji} for the three problems. In cases 1 and 2, the results have been obtained from Sections 2.32 and 3.3, respectively. For case 3, the present problem, the lengthy derivation will be found in Ogilvie and Tuck (1969). Some points should be noted:

1. All of the terms in Case 3 include the corresponding Case 2 terms, *i.e.*, the added mass and damping at forward speed can be computed in terms of the added mass and damping at zero speed, plus a speed-dependent component. Formally, we could also say that Case 1 includes all of the Case 2 terms, with $n(x)$ set equal to zero. From this point of view, the only differences among the three cases are the forward-speed effects.

[Text continued on page 142.]

TABLE 3-1

ADDED-MASS AND DAMPING COEFFICIENTS IN THREE PROBLEMS

	CASE 1 - Body with Forward Speed in Infinite Fluid	CASE 2 - Ship with Zero Forward Speed on Free Surface	CASE 3 - Ship with Forward Speed on Free Surface
a_{33}	$\int_L dx\ m(x)$	$\int_L dx\ m(x)$	$\int_L dx\ m(x)$
b_{33}	0	$\int_L dx\ n(x)$	$\int_L dx\ n(x)$
a_{35}	$-\int_L dx\ x\ m(x)$	$-\int_L dx\ x\ m(x)$	$-\int_L dx\ x\ m(x) + (U/\omega^2)b_{33} - (2\rho\nu U/\omega)\ Im\ \{ \mathbf{I} \}$
b_{35}	$- U\ a_{33}$	$-\int_L dx\ x\ n(x)$	$-\int_L dx\ x\ n(x) - U\ a_{33} - (2\rho\nu U)\ Re\ \{ \mathbf{I} \}$
a_{53}	$-\int_L dx\ x\ m(x)$	$-\int_L dx\ x\ m(x)$	$-\int_L dx\ x\ m(x) - (U/\omega^2)b_{33} + (2\rho\nu U/\omega)\ Im\ \{ \mathbf{I} \}$
b_{53}	$+ U\ a_{33}$	$-\int_L dx\ x\ n(x)$	$-\int_L dx\ x\ n(x) + U\ a_{33} + (2\rho\nu U)\ Re\ \{ \mathbf{I} \}$
a_{55}	$\int_L dx\ x^2 m(x) - (U/\omega)^2 a_{33}$	$\int_L dx\ x^2 m(x)$	$\int_L dx\ x^2 m(x)$
b_{55}	0	$\int_L dx\ x^2 n(x)$	$\int_L dx\ x^2 n(x)$

Continued . . . .

TABLE 3-1
(Continued)

Note 1) In all cases, m(x) and n(x) are defined:

$$m(x) + \frac{1}{i\omega} n(x) \equiv \rho \int_{C(x)} d\ell \; n_3 \phi_3 \quad ,$$

where C(x) is the wetted part of the cross-section contour at x , and n_3 and ϕ_3 have the same meaning as in Sections 2.32 and 3.3. In CASE 1, ϕ_3 is a real quantity, and so $n(x) = 0$.

Note 2) The quantity **I** in CASE 3 is defined as follows: Let $\phi = \phi_3$ and let ϕ_∞ be a 2-D potential function which is sinusoidal in y , such that $|\phi - \phi_\infty| \to 0$ as $y \to \infty$.
Then:

$$\mathbf{I} \equiv \int_L dx \left[\int_{b(x,0)}^{\infty} dy \left[\phi^2(x,y,0) - \phi_\infty^2(x,y,0)\right] - \frac{i}{2\nu}\phi_\infty^2(x,b(x,0),0) \right] \quad ,$$

where b(x,z) gives the hull offset corresponding to the point (x,0,z) on the centerplane.

2. The coupling coefficients b_{35} and b_{53} include a forward-speed term $\mp Ua_{33}$ in both Case 1 and Case 3. This means, first of all, that there can be some damping even in the infinite-fluid problem. Secondly, it means that this contribution to the damping coefficients is not altered by the presence of the free surface. Note that in neither case is it necessary to ignore the steady perturbation of the incident stream (the χ terms in (3-41), for example) in order to obtain this result.

3. The other coupling coefficients, a_{35} and a_{53}, contain similar speed-dependent terms in Case 3; they arise at the same point in the analysis as the terms discussed in 2. above. We could arbitrarily include such terms, $\pm(U/\omega^2)b_{33}$, in Case 1 too, without causing any errors, since b_{33} is zero anyway in Case 1.

4. In Case 1, there is a speed-dependent term in a_{55} which is lacking in Case 3. The reason for the lack is that such a term is higher order in terms of ε in the ship problem, because of the assumption that $\omega = O(\varepsilon^{-1/2})$. There was no need for a high-frequency assumption in Case 1, and so the extra term could legitimately be retained.

5. If, in Case 3, one arbitrarily includes the forward-speed term, $-(U/\omega)^2 a_{33}$, in the a_{55} coefficient, making it identical to the Case 1 coefficient, then it is consistent to modify b_{55} in a similar way, namely, by changing it to:

$$b_{55} = \int_L dx\, x^2 n(x) - (U/\omega)^2 b_{33} \quad .$$

The relationship between these forward-speed effects is quite the same as that discussed above in paragraphs 2. and 3. In the b_{55} coefficient of Case 1, we could also introduce an extra term, $-(U/\omega)^2 b_{33}$, without causing any error, since b_{33} is zero anyway in this case. Thus we can maintain the symmetry between Case 1 and Case 3.

6. The only forward-speed terms not yet discussed are those in Case 3 which involve the integral **I** . They arise from the inclusion of the functions Ω_j in the potential function, as in (3-43), and the necessity for including those functions is a consequence of the fact that the right-hand side of (3-42), the free-surface condition, is not zero. Now, the right-hand side of (3-42) represents an interaction between the forward motion and the oscillation. One might try to simplify matters by assuming that one can neglect the effects of χ , the perturbation of the incident stream by the body. But this reduces (3-42) to the following:

$$\underset{O(\varepsilon^{1/2}\delta)}{\psi_{tt} + g\psi_z} \simeq \underset{O(\varepsilon\delta)}{-\,2U\psi_{tx}} , \quad \text{on} \quad z = 0 . \tag{3-47}$$

The right-hand side is still not zero, and we would still have the Ω_j functions to contend with. In fact, it may be recalled that this remaining term on the right-hand side was the one that caused the major trouble in interpreting the Ω_j problems. Neglect of the χ terms leads to the condition on Ω_j (*Cf*. (3-46)):

$$\Omega_{jz} - \nu\Omega_j = -(2/g)\Phi_{jx} , \quad \text{on} \quad z = 0 , \tag{3-48}$$

and it is the one remaining right-hand term which causes the solution for Ω_j to diverge at infinity. The usual procedure at this point is to set $\Omega_j = 0$, turn the other way, and just ignore these problems. The results are in remarkably good agreement with experimental observations, and one still wonders how this can be rationalized mathematically.

Finally, we should at least mention the problem of predicting wave excitations in the forward-speed problem. The singular perturbation problem involved in solving for the diffraction waves has not been satisfactorily worked out yet, at least, not in a manner compatible with the approach presented above.

One might hope to avoid the diffraction problem by using the Khaskind relations, as in the zero-speed problem. (See Section 3.3.) In fact, Newman (1965) has derived what I call the Khaskind-Newman relations. These provide a generalization of Khaskind's formula, relating the wave excitation on a moving ship to the problem of forced oscillations of the ship *when the ship is moving in the reverse direction.* Unfortunately for our purposes, Newman's derivation is based on an *a priori* linearization of the free-surface, in the sense that our terms involving χ can be neglected. Therefore, the appropriate diffraction problem cannot really be avoided in this way. Also, it is necessary to have available the potential function for the forced-motion problem, and this includes at least a part of the Ω_j functions even if the χ dependence is ignored.

In a not-yet published paper, Newman has applied the Khaskind-Newman relations in the forward-speed problem by arbitrarily ignoring the Ω_j functions in the forced-motion potential function. He finds for the heave excitation force:

$$F_3^W(t) \simeq \rho g h(1 + U\omega_o/g)e^{i\omega t}\int_L dx\; e^{-i\nu_0 x}\int_{C(x)} d\ell\; n_3\; e^{\nu_0 z}$$

$$\cdot \left[1 - \nu_0\Phi_3 + \frac{U\nu_0}{i\omega}\,(\chi_z+\Psi_3) - \frac{U\nu_0}{\omega}\right] ,$$

where, as before, ω is the frequency of oscillation (that is, the frequency of encounter) and $\nu = \omega^2/g$; the frequency measured in an earth-fixed reference frame is denoted by ω_0 , and we define $\nu_0 = \omega_0^2/g$. The actual wavelength of the incident waves is $\lambda = 2\pi/\nu_0$. The two frequencies are related as follows: $\omega = \omega_0 + U\omega_0^2/g$. These formulas are all valid for the head-seas case only.

This formula should be compared with (3-34a), which was the corresponding result in the zero-speed problem. The first term in brackets yields the Froude-Krylov force, and the second term yields a pure-strip-theory prediction of the diffraction wave force, which can be interpreted approximately in terms of the relative-motion hypothesis. The remaining terms represent an interaction between forward speed and the incident waves.

Again, it should be pointed out that more than just nonlinear effects have been neglected in setting Ω_j equal to zero. In fact, the usual linear free-surface condition for ship-motions problems can be written:

$$\psi_{tt} + g\psi_z \;=\; - 2U\psi_{tx} = U^2\psi_{xx} \quad , \qquad \text{on} \quad z = 0 \quad .$$

(*Cf.* (3-37) and (3-47).) Even the inclusion of the Ω_j terms still omits some effects usually considered as linear, namely, the effects of the term $-U^2\psi_{xx}$ in this boundary condition. These effects are higher order in the theory presented here solely because of the high-frequency assumption.

4 THIN-SHIP THEORY AS AN OUTER EXPANSION

It has already been shown how one can view a symmetrical thin-body problem in terms of inner and outer expansions; the usual description of the flow around such a body is really just the first term of an outer or far-field expansion. It was not at all obvious that one *had* to use such a powerful method on such a problem, but it was clear that one *could* do this. Probably the only advantage of doing so in the infinite-fluid case was that one could avoid possible questions about the validity of analytically continuing the potential function inside the body surface. On the other hand, one had then to face all kinds of difficulties in principle in justifying use of matched asymptotic expansions. It was a rather academic exercise.

The situation may be quite different in the thin-ship problem. The purpose of this chapter is to show one can obtain the first results of thin-ship theory in the same way as for the infinite-fluid problem but that a second-order solution leads to fundamental difficulty. The latter appears to suggest that a combination thin-body/slender-body approach may be appropriate. A limited amount of other evidence may be cited to support this idea.

I wish to emphasize that there are no new results in this chapter. It is all a matter of *interpretation*. Perhaps someone will be able to show that the problem discussed here has a trivial explanation. On the other hand, perhaps someone will be stimulated to do further research on the subject. In either case, I shall be happy with the outcome.

The problem may be partially stated just as the infinite-fluid, thin-body problem was stated. Let there be a velocity potential, $\phi(x,y,z)$, which satisfies the Laplace equation,

[L] $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$,

everywhere in the fluid domain and the body boundary condition,

[H] $0 = \phi_x h_x \mp \phi_y + \phi_z h_z$, on $y = \pm h(x,z) = \varepsilon H(x,z)$.

Now we add on the two free-surface conditions:

[A] $\frac{1}{2} U^2 = g\zeta + \frac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2]$, on $z = \zeta(x,y)$;

[B] $0 = \phi_x \zeta_x + \phi_y \zeta_y - \phi_z$, on $z = \zeta(x,y)$.

Also, we must specify a radiation condition.

In the far field, where $y = O(1)$, we assume the existence of the expansions:

$$\left.\begin{aligned} \phi(x,y,z) &\sim \sum_{n=0}^{N} \phi_n(x,y,z) , \\ \zeta(x,y) &\sim \sum_{n=1}^{N} \zeta_n(x,y) , \end{aligned}\right\} \quad \text{for fixed } (x,y,z) ;$$

in the near field, where $y = O(\varepsilon)$, there are expansions:

$$\left.\begin{aligned} \phi(x,y,z) &\sim \sum_{n=0}^{N} \Phi_n(x,y,z) , \\ \zeta(x,y) &\sim \sum_{n=1}^{N} Z_n(x,y) , \end{aligned}\right\} \quad \text{for fixed } (x,y/\varepsilon,z) .$$

We assume right away that:

$$\phi_0(x,y,z) = \Phi_0(x,y,z) = Ux .$$

In the far field, the ship vanishes as $\varepsilon \to 0$, and so we take the entire space outside of the plane $y = 0$ (below the free surface) as the far field. It is easily seen that the second term in the outer expansion must be of the form:

$$\phi_1(x,y,z) = -\frac{1}{4\pi}\iint_{\mathbf{H}} \sigma_1(\xi,\zeta)G(x,y,z;\xi,0,\zeta)\ d\xi\ d\zeta \ , \qquad (4\text{-}1)$$

where $\mathbf{H}$ is the portion of the centerplane of the ship below $z = 0$, $\sigma_1(x,z)$ is an unknown source density, and $G(x,y,z;\xi,\eta,\zeta)$ is the usual Green's function for a linearized problem of steady motion with a free surface. It has the important property:

$$G_{xx} + \kappa G_z = 0 \ , \quad \text{on} \quad z = 0 \ , \qquad (4\text{-}2)$$

where $\kappa = g/U^2$. Of course, the potential function, ϕ_1 , also has this property:

$$\phi_{1_{xx}} + \kappa\phi_{1_z} = 0 \ , \quad \text{on} \quad z = 0 \ .$$

For later convenience, we define:

$$\alpha_1(x,z) \equiv \phi_1(x,0,z) \ , \qquad (4\text{-}3)$$

and so $\alpha_1(x,z)$ has the property too:

$$\alpha_{1_{xx}} + \kappa\alpha_{1_z} = 0 \ , \quad \text{on} \quad z = 0 \ . \qquad (4\text{-}4)$$

With $\phi_1(x,y,z)$ given by (4-1), the two-term outer expansion is:

$$\phi(x,y,z) \sim Ux + \phi_1(x,y,z) \ ,$$

and the inner expansion of the two-term outer expansion is:

$$\phi(x,y,z) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\alpha_1(x,z)} + \underset{O(\varepsilon^2)}{\tfrac{1}{2}|y|\sigma_1(x,z)} + \ldots \ .$$

I have taken my usual liberty of indicating unproven orders of magnitude. I am not really *assuming* these orders of magnitude; I am saying that one can prove that these are correct, and I display them here now simply as an aid to the reader.

Now consider the near field. Just as in the infinite-fluid problem, one may stretch coordinates, $y = \varepsilon Y$, and follow through the consequences. This is effectively what I do, without writing the change of variable explicitly. Thus, the Laplace equation yields the condition:

$$\Phi_{1_{yy}} = 0 ,$$

and so Φ_1 must be a linear function of y . The same analysis as used in the infinite-fluid problem, Section 2.11, leads to the conclusion that Φ_1 is even more restricted than this. It must be a constant with respect to y . Thus, let:

$$\Phi_1(x,y,z) \equiv A_1(x,z) .$$

The two-term inner expansion is then:

$$\phi(x,y,z) \sim Ux + A_1(x,z) .$$

Matching gives the unsurprising result that:

$$A_1(x,z) = \alpha_1(x,z) . \tag{4-5}$$

In other words, once again the inner expansion starts out simply as the inner expansion of the outer expansion; it is not necessary to formulate a near-field problem to obtain this result.

The same arguments lead to the prediction that:

$$\Phi_2(x,y,z) = A_2(x,z) + U\, h_x(x,z)\,|y| . \tag{4-6}$$

Thus the three-term inner expansion is formed, and it can be matched with the three-term inner expansion of the two-term outer expansion, yielding the familiar result once again that:

$$\sigma_1(x,z) \quad = \quad 2Uh_x(x,z) \quad .$$

(See (2-22).) This obviously had to come out this way, since we have not yet introduced any effects of the presence of the free surface. It should be noted that only the function $A_2(x,z)$ is not already determined. (Knowledge of $\sigma_1(x,z)$ allows us to express $\alpha_1(x,z)$ explicitly, from (4-1) and (4-3).)

A systematic treatment of the free-surface conditions leads to the following:

$$[A] \quad 0 = gZ_1 + U\Phi_{1_x} \qquad O(\varepsilon)$$
$$+ gZ_2 + U\Phi_{2_x} + UZ_1\Phi_{1_{xz}} + \tfrac{1}{2}(\Phi_{1_x}^2+\Phi_{1_z}^2) + \tfrac{1}{2}(\Phi_{2_y}^2) \qquad O(\varepsilon^2)$$
$$+ \ldots , \qquad \text{on} \quad z = 0 \ ;$$

$$[B] \quad 0 = UZ_{1_x} - \Phi_{1_z} \qquad O(\varepsilon)$$
$$+ UZ_{2_x} - \Phi_{2_z} - Z_1\Phi_{1_{zz}} + \Phi_{1_x}Z_{1_x} + \Phi_{2_y}Z_{2_y} \qquad O(\varepsilon^2)$$
$$+ \ldots , \qquad \text{on} \quad z = 0 \ .$$

The lowest-order conditions in [A] and [B] together require that:

$$\Phi_{1_{xx}} + \kappa\Phi_{1_z} = 0 , \quad \text{on} \quad z = 0 \quad .$$

We see that this is automatically satisfied by our $\Phi_1(x,y,z) = A_1(x,z) = \alpha_1(x,z)$. (See (4-4).) The first term in the expansion for wave shape in the near field is also determined:

$$Z_1(x,y) \quad = \quad -\frac{U}{g}\,\Phi_{1_x}(x,0) \quad .$$

This really says only that the free surface appears in the near field to be raised (or lowered) by just the limiting value (as $y \to 0$) of $\zeta(x,y)$ in the far field. Again, a rather trivial result.

When we consider the ε^2 terms in the free-surface conditions, it is a different matter. The two conditions can be combined into the following:

$$0 = \Phi_{2_{xx}} + \kappa\Phi_{2_z} - \frac{U}{g}(\alpha_{1_x}\alpha_{1_{xz}})_x + \frac{1}{2U}(\alpha_{1_x}^2+\alpha_{1_z}^2)_x + \frac{U}{2}(h_x^2)_x - \frac{1}{U}\alpha_{1_x}\alpha_{1_{zz}} + \frac{1}{U}\alpha_{1_{xx}}\alpha_{1_x} \mp \frac{g}{U}h_x Z_{2_y} . \tag{4-7}$$

In condition [A], we note that differentiation of the ε^2 terms with respect to y yields:

$$0 = gZ_{2_y} \pm U^2 h_{xx} .$$

Therefore, in the complicated free-surface condition above, (4-7), only the first two terms involve y ; all of the other terms are functions of just x . From (4-7) and (4-6), we can thus write the following:

$$0 = [h_{xxx}(x,0) + \kappa h_{xz}(x,0)]\, U|y| + (\text{a function of } x).$$

This must be true for any y , and so we obtain the condition:

$$0 = h_{xxx} + \kappa h_{xz} , \quad \text{on } z = 0 .$$

If the ship is wall-sided at $z = 0$, the second term is separately zero, and so we would have to require that $h_{xxx} = 0$ at $z = 0$.

Now this is clearly unacceptable. Why should our theory work only for such a special case? (The waterline is made of circular arcs in this case.)

As a result of our having stretched the coordinates, we came to the prediction that the fluid velocity near the thin body consists of a tangential component which is essentially independent of the local conditions plus a normal component which depends only on local conditions. Near the free surface, such results are simply untenable.

I present here a formalism which apparently avoids this difficulty. Again, I point out that no new results are obtained. However, it does seem possible that the procedure might be fruitful if studied further.

The idea is to define a third region, complete with its own asymptotic expansions of ϕ and ζ. This region will be essentially the same as the near field in a slender-body analysis, that is, it is a region in which $y = O(\varepsilon)$ *and* $z = O(\varepsilon)$ as $\varepsilon \to 0$. It follows from this assumption that $\partial/\partial y$ and $\partial/\partial z$ both have the effect of changing orders of magnitude by a factor $1/\varepsilon$. What is most important is that this region is interposed between the thin-body near field and the free surface. Thus, it is no longer necessary or even proper to try to make the previous inner expansion satisfy the free-surface conditions.

We expect, as usual, that the first term in the expansion of ϕ in this new near field will be just Ux. Furthermore, we can expect the next term to be rather trivial, since the second term in the previous near-field expansion did actually satisfy the free-surface condition. Using the usual arguments of slender-body theory, we find in fact that the three-term expansion of ϕ in this new near field is:

$$\phi(x,y,z) \sim \underset{O(1)}{Ux} + \underset{O(\varepsilon)}{\alpha_1(x,0)} + \underset{O(\varepsilon^2)}{Uh_x(x,0)|y|} - \underset{O(\varepsilon^2)}{\frac{1}{\kappa} z\alpha_{1_{xx}}(x,0)} \; .$$

The corresponding wave shape is found to be:

$$\zeta(x,y) \sim -\frac{U}{g}\,\alpha_{1_x}(x,0) \qquad O(\varepsilon)$$

$$\left.\begin{aligned}&-\frac{U}{g}\left[Uh_{xx}(x,0)\,|y| + \frac{U^3}{g^2}\,\alpha_{1_x}(x,0)\;\alpha_{1_{xxx}}(x,0)\right]\\ &-\frac{1}{2g}\left[\alpha_{1_x}^2(x,0) + U^2h_x^2(x,0) + \left(\frac{U^2}{g}\right)^2\alpha_{1_{xx}}^2(x,0)\right]\end{aligned}\right\} O(\varepsilon^2)$$

$$+ \;\ldots\;.$$

It can be shown in straightforward fashion that these results match the far-field expansion as $\sqrt{(y^2+z^2)} \to \infty$ and they match the previous (thin-body) near-field expansion as $z \to -\infty$. Furthermore, they satisfy the free-surface conditions without the necessity for imposing unacceptable restrictions on the body shape. There is just one aspect that requires special care: The free-surface conditions cannot be satisfied on the surface $z = 0$ in this near field. The reason is that the first term of the ζ expansion is $O(\varepsilon)$, and differentiation with respect to z is assumed to change orders of magnitude by $1/\varepsilon$. Thus, suppose that we want to evaluate some function $f(z)$ on $z = \zeta$ in terms of its value (and values of its derivatives) on $z = 0$. The usual procedure is to write:

$$\begin{array}{ccccccccc} f(\zeta) & = & f(0) & + & \zeta f'(0) & + & \frac{1}{2}\,\zeta^2 f''(0) & + & \ldots\;. \\ & & O(f) & & O(\varepsilon)\cdot O(f/\varepsilon) & & O(\varepsilon^2)\cdot O(f/\varepsilon^2) & & \end{array}$$

With our set of assumptions, this expansion is useless; we cannot terminate it. The one simplification which is admissible here is to evaluate $f(z)$ and its derivatives on $z = Z_1$, where $\zeta = Z_1 + o(\varepsilon)$.*

*The same difficulty arises also in Sections 3.2 and 5.42.

I have not worked out any more terms in any of these expansions, but I suspect that the next term in this near-field expansion will be much more interesting. In the far field, it is well-known that the third term in the expansion of the potential function will include the effects of what appears to be a pressure distribution over the free surface. It was shown by Wehausen (1963) that at the intersection of the undisturbed free surface and the hull surface the solution is singular, and he represented the singular part by a line integral taken along this line of intersection. From the point of view of the method of matched asymptotic expansions, it should be possible to represent the far-field effects of that line integral in terms of an equivalent line of singularities on the x axis. The strength of the singularities would be determined, as usual, by matching the solution to the near-field expansion. At this stage, thin-ship theory will have become a singular perturbation problem.

5 STEADY MOTION IN TWO DIMENSIONS (2-D)

Sometimes we study two-dimensional problems with the intent of incorporating the solutions into approximate three-dimensional solutions, as in the treatment of high-aspect-ratio wings and in slender-body theory. And sometimes we investigate two-dimensional problems simply because the corresponding three-dimensional problems are too difficult.

The problems discussed in this section are in the second group. It is not likely that any of these problems and their solutions will have practical application before several more years have passed, even in the context of strip theories. Here are some of the most fundamental difficulties related to the presence of the free surface.

The first two subsections concern a 2-D body which pierces the free surface. Such a problem is intrinsically nonlinear. We might try to formulate the problem as a perturbation problem, in this case involving a perturbation of a uniform stream. However, there must be a stagnation point somewhere on the body, and at that point the perturbation velocity is equal in magnitude to the incident stream velocity. It is not small! If the stagnation point is near the free surface, the free-surface conditions cannot be linearized. We must find methods which are adaptable to highly nonlinear problems.

Such a method is the classical hodograph method, used since the nineteenth century for solving free-streamline problems. But it introduces a new difficulty: It cannot be used to treat free streamlines which are affected by gravity, which means that only infinite-Froude-number problems can be treated directly. This leads to a further great difficulty, which is discussed in some detail in Section 5.1.

In Section 5.3, a brief discussion is presented of the problem studied by Salvesen (1969). It contains two aspects of interest: It is a case in which the free-surface conditions can be linearized because of the depth of the moving body, and I have already commented in the *Introduction* that there are very interesting fundamental questions involved in such procedures. Also, it presents a clear example of the classical phenomenon discussed in the section on multiple scale expansions: The wavelength obtained in the first approximation must be modified in subsequent approximations, or the solution becomes unbounded at infinity — where we know perfectly well that the waves are bounded in amplitude.

Finally, Section 5.4 describes two recent attempts to approach the problem of extremely low-speed motion. The difficulty is basically this: In the usual linearization, we assume that all velocity components (at least in the vicinity of the free surface) are much smaller than the forward speed — which becomes nonsense if we subsequently decide to let U, the forward speed, approach zero. What is needed is a perturbation scheme in which somehow the small parameter is proportional to U. Then it is certainly permissible to allow U to approach zero. Section 5.41 shows a very straightforward procedure for doing this; however, it leads to a sequence of Neumann problems, and so the wave nature of the fluid motion is lost. In Section 5.42, an alternative method is discussed. It is an application of the multi-scale expansion procedure to which Section 1.3 was devoted.

5.1 Gravity Effects in Planing

Before we try to treat this problem properly, let us consider briefly a well-known approach to the 2-D planing problem and determine why it is not completely satisfactory. In the middle 1930's, A. E. Green wrote several papers on the subject, and the essence of his approach is well-presented by Milne-Thomson (1968). A flat plate is located with its trailing edge at the origin of coordinates, as shown in Figure (5-1). There is an incident stream with speed U coming from the left, and, at infinity upstream, there is a free surface at $y = h$. The effects of gravity are neglected. The fluid is assumed to leave the trailing edge smoothly (a Kutta condition), and a jet of fluid is deflected forward and upward by the plate. In the absence of gravity, the jet never comes down to trouble us again. In the figure, A marks the leading edge of the plate and C marks the stagnation point.

The physical plane shown in Figure (5-1) is also the complex $z = x + iy$ plane. Let $F(z) = \phi(x,y) + i\psi(x,y)$ be the complex velocity potential for this problem. Then $F(z)$ effects a mapping of the z plane onto an F plane, as shown in Figure (5-2), in which points are marked to correspond to Figure (5-1). It is assumed that $\phi = 0$ and $\psi = 0$ at the stagnation point. Furthermore, we have set $\psi = Ua$ on the upstream free-surface streamline, IJ, which implies that a is the thickness of the jet and that Ua is the rate at which fluid leaves in the jet. Of course, $F(z)$ is not known yet.

We can also consider that the z plane is mapped by the function $w(z) = dF/dz$. $w(z)$ is the "complex velocity," that is, $w = u-iv$, where u and v are the velocity components in the x and y directions, respectively. The entire fluid region is mapped by $w(z)$ into the region bounded by a half-circle and its diameter, as shown in Figure (5-3). Again, points are marked to correspond to Figure (5-1). The diameter is the image of the planing surface, on which the *direction* of the velocity vector is known, and the circle is the image of the entire free surface, on which the *magnitude* of the velocity vector is known (from the

Bernoulli equation). Again, we note that the mapping function itself is not yet known.

The functions $w(z)$ and $F(z)$ are, of course, very simply related, although neither is known explicitly yet. In order to obtain another relationship, one introduces the $\zeta = \xi + i\eta$ plane, in which the fluid domain is mapped into the lower half-space, as shown in Figure (5-4). We can write out the explicit expressions for mapping the F and w planes into the ζ plane. The first is accomplished by means of the Schwarz-Christoffel transformation:

$$\frac{dF}{d\zeta} = \frac{Ua}{\pi(b + c)} \frac{\zeta - c}{\zeta + b} \equiv H(\zeta) \quad ,$$

which can actually be integrated, yielding:

$$F(z(\zeta)) = \frac{U\,a}{\pi} \left(\frac{\zeta - c}{b + c} - \log \frac{\zeta + b}{b + c} \right) \quad .$$

The second mapping can be shown to take either of the equivalent forms:

$$\begin{aligned} w(z(\zeta)) &= U\, e^{i\alpha} \frac{\zeta - c}{(1 - \zeta c) + i\sqrt{(1-c^2)}\sqrt{(\zeta^2-1)}} \\ &= U\, e^{i\alpha} \frac{(1 - \zeta c) - i\sqrt{(1-c^2)}\sqrt{(\zeta^2-1)}}{\zeta - c} \quad . \end{aligned} \qquad (5\text{-}1)$$

The solution is then completed by using the relationship between F and w, along with these expressions, to obtain the relationship between z and ζ. Since:

$$\frac{dz}{d\zeta} = \frac{dz}{dF}\frac{dF}{d\zeta} = \frac{H(\zeta)}{w(z(\zeta))} \quad ,$$

of which the right-hand side is known, we can integrate to obtain:

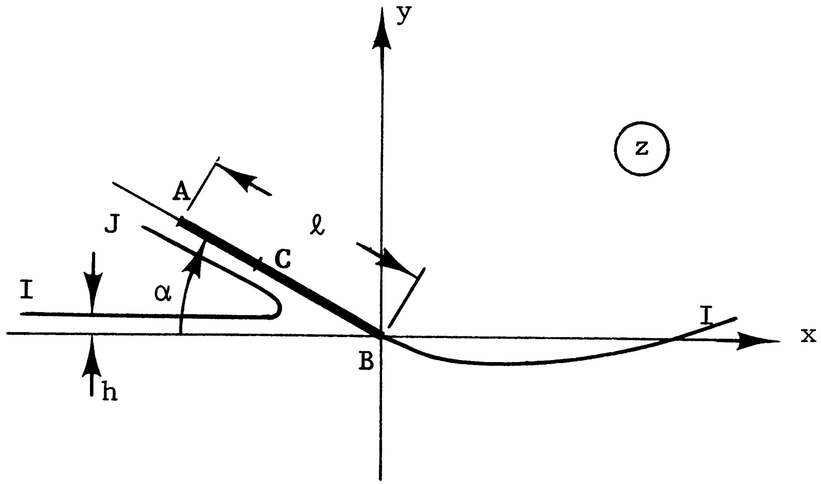

Figure (5-1). Planing Problem in the Physical Plane.

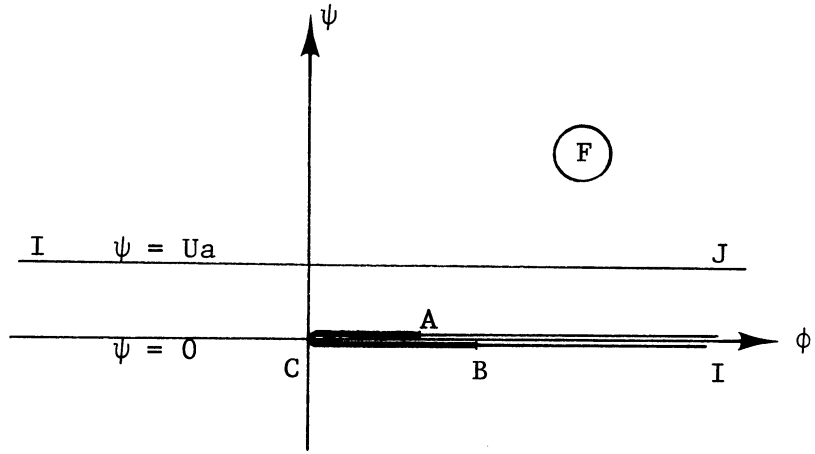

Figure (5-2). Planing Problem in the Plane of the Complex Potential.

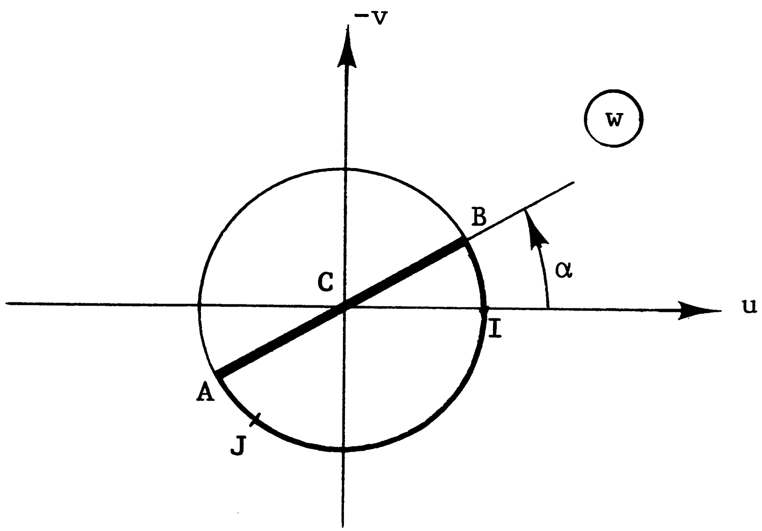

Figure (5-3). Planing Problem in the Plane of the Complex Velocity.

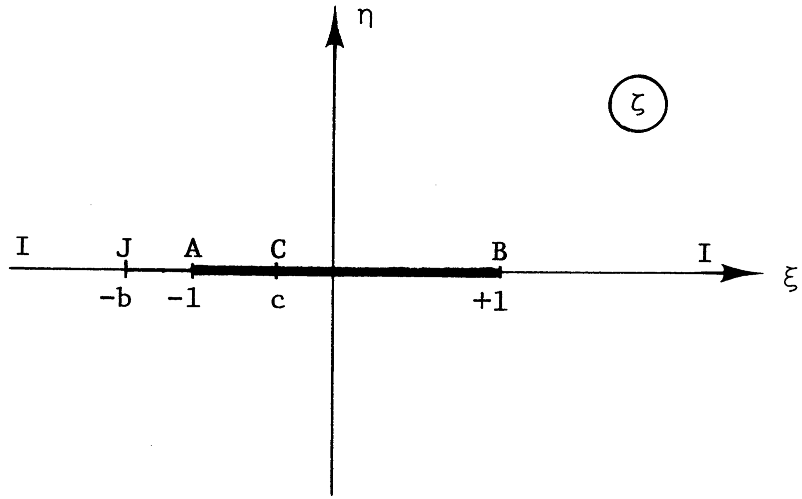

Figure (5-4). Planing Problem in the Auxilliary (ζ) Plane.

$$z(\zeta) = \int_1^{\zeta} \frac{H(\zeta')}{w(z(\zeta'))} \, d\zeta'$$

$$= \frac{a\, e^{-i\alpha}}{\pi(b + c)} \left(-c(\zeta-1) + (1+bc) \log \frac{\zeta+b}{b+1} + i\sqrt{(1-c^2)}\sqrt{(\zeta^2-1)} \right.$$

$$\left. - ib\sqrt{(1-c^2)} \log [\zeta+\sqrt{(\zeta^2-1)}] - i\sqrt{(1-c^2)}\sqrt{(b^2-1)} \log \frac{1+b\zeta-\sqrt{(b^2-1)}\sqrt{(\zeta^2-1)}}{\zeta + b} \right) .$$

So now we have z as a function of ζ, as well as F and w as functions of ζ.

There are three parameters in this solution, a, b, and c, none of which has been determined yet. By letting $|\zeta| \to \infty$, Green came to the conclusion that the flow far away is a uniform stream as required only if:

$$c = -\cos\alpha \quad \text{and} \quad \sqrt{(1-c^2)} = \sin\alpha \ . \tag{5-2}$$

(Both statements are necessary to avoid an ambiguity in sign.) Also, one can use the $z(\zeta)$ formula to evaluate z at the leading edge of the plate:

$$z(-1) = -\ell\, e^{-i\alpha} \ .$$

(Compare Figures (5-1) and (5-4).) This provides a relationship among a, b, and c. But there are no more conditions to be found unless we introduce more information about the physical problem. For example, we could use the solution with unspecified values of a and b, and work out the formula for lift on the plate. (Milne-Thomson gives the formula.) If then we fix the value of lift, we have another condition on a and b. However, this is rather a backwards way of going at the problem. We are most likely to want to solve the entire problem just to find the lift and other interesting physical quantities, and so we have not gained much if we must assume the value of the lift as a given datum.

There is another anomaly in this result: The value of h (See Figure (5-1)) has not been used in any way. In the formula for $z(\zeta)$, let $\zeta = \xi$, with $|\xi|$ very large. Then every value of z computed in this way gives a point on the free surface far away from the planing surface. With a considerable amount of tedious algebra, one can eliminate ξ and express y as a function of x (at least asymptotically, as $|x| \to \infty$). The first term is the most interesting:

$$y \sim -\frac{a\sqrt{(1-c^2)}}{\pi(b + c)} [\log |x| + \text{constant}] \quad .$$

Thus, far away from the planing surface, the free surface apparently drops off logarithmically to $-\infty$. The slope of the free surface approaches zero ($\propto 1/|x|$) and so there is no violation of our assumption that the flow at infinity is simply a streaming motion parallel to the x axis. But obviously the assumption that the trailing edge was located at a height h below the free-surface level at infinity was quite meaningless, and it cannot be enforced in the solution.

There are thus two difficulties: 1) The above solution is not unique (a common difficulty in free-streamline problems); 2) It has unacceptable behavior at infinity.

These difficulties were resolved by Rispin (1966) and Wu (1967), who recognized that the solution of Green's problem is part of a near-field (inner) expansion of the complete solution. An inner expansion does not necessarily satisfy the obvious conditions at infinity; it must only match some outer expansion in a proper way. Rispin and Wu produced the appropriate outer expansions and showed that matching does occur. The effects of gravity appear first in the far field, which is hardly surprising, for two reasons: 1) Far away, one expects to find gravity waves as the only disturbance. 2) The divergence of the free-surface shape in Green's solutions is so weak that one might expect the smallest amount of gravity effect to bring the free surface into the region where we expect to find it; thus, the small effect of gravity eventually would have a large consequence, but only far away from the planing surface.

Rispin defines the small parameter:

$$\beta \equiv g\ell/U^2 = 1/F^2 ,$$

where F is the usual Froude number. In the near field, the natural coordinates are used, which means effectively that ℓ is considered to be $O(1)$. Smallness of β is achieved by allowing $g \to 0$ or $U \to \infty$. Rispin treats his small parameter properly by nondimensionalizing everything, so that he then does not have to specify whether $U \to \infty$ or $g \to 0$. Rather than change all variables now, I shall treat g as a small parameter, as in Section 3.2; the results are the same as Rispin's, of course.

In the far field, typical lengths are assumed to be $O(1/\beta)$ in magnitude, or $O(1/g)$, in my loose notation. We could define new coordinates, say,

$$\hat{z} = \beta z \ ; \quad \hat{x} = \beta x \ ; \quad \hat{y} = \beta y ,$$

and consider that $\hat{z} = O(1)$ as $g \to 0$ in the far field, while $z = O(1)$ as $g \to 0$ in the near field. Rather than do this, we shall just keep in mind that such orders of magnitude are to be assumed. Also, we note that $d/dz = O(1)$ in the near field and $d/dz = O(\beta)$ in the far field.

This problem is reversed from the most common kind of stretched-coordinate problem: The inner problem is solved by natural coordinates, and the outer coordinates are compressed. Note, however, that there is no distortion of coordinates between near- and far-fields. There is just a change of scale.

In the far field, the planing surface appears to vanish in the limit, and so the first term in a far-field expansion must represent just the incident uniform stream. That is, if the outer expansion is represented:

$$F(z;\beta) \sim \sum_{n=0}^{N} F_n(z;\beta) , \quad w(z;\beta) \sim \sum_{n=0}^{N} W_n(z;\beta) , \text{ for fixed } \beta z \text{ as } \beta \to 0 ,$$

then clearly we have:

$$F_0(z;\beta) = Uz , \text{ and } W_0(z;\beta) = U .$$

This one-term outer expansion must match the one-term inner expansion, the latter being just Green's solution. This much of the matching procedure is rather obvious, and Green already used this fact to determine the value of c, as given in (5-2).

The next term in the outer expansion is not quite so obvious. In order to facilitate the matching process, Rispin solved the problem in the ζ plane, just as we did above for Green's problem. The free-surface boundary condition on W_1 is not much different from the familiar linearized condition. One can show fairly simply that:

$$Re \left[\frac{dW_1}{d\zeta} + \frac{igA}{U^2} W_1 \right] = 0 \quad \text{on } \eta = 0 ,$$

where $A = a/\pi(b+c)$. (The factor A is just the value of $dz/d\zeta$ far away from the planing surface.) Note that the first term is $O(\beta W_1)$ because of the differentiation, and the second term is the same order because of the g factor. The solution for W_1 must be analytic in the lower half-space and satisfy this condition on $\eta = 0$, $|\xi| > 0$; note the exclusion of the origin, where singularities may occur.

As usual, we try to restrict the singularities to the simplest kind possible. In this case, we would find nothing in the near field to match with if we allowed all kinds of singularities in W_1 . A sufficiently general solution* is the following:

$$W_1(\zeta;\beta) = i\, e^{-igA\zeta/U^2} \int_\infty^\zeta dt\, e^{igAt/U^2} \left[\frac{C_1}{t} + \frac{C_2}{t^2}\right] ,$$

where C_1 and C_2 are real constants yet to be determined (in the matching).

The two-term outer expansion is now:

$$w(z;\beta) \sim U + W_1(\zeta;\beta) ,$$

with W_1 given as above. Its inner expansion to one term is easily found:

$$w(z;\beta) \sim U - \frac{iC_2}{\zeta} .$$

We cannot really say positively that these two terms are the same order of magnitude, but it turns out that they must be if this expression is to match the two-term outer expansion of the one-term inner expansion. The latter is obtained readily from Green's solution for $w(z(\zeta))$ which was given

*Rispin discusses more general solutions, which are needed in constructing higher-order solutions.

in (5-1). It is:

$$w(z;\beta) \sim U + \frac{i\,U \sin\alpha}{\zeta} .$$

Then, obviously, we find that:

$$C_2 = -U \sin\alpha .$$

We cannot determine the other constants, C_1, from the solutions so far obtained. It is necessary to solve for the second term in the inner expansion, and Rispin carries this through. Then, he matches the two-term outer expansion of the two-term inner expansion with the two-term inner expansion of the two-term outer expansion, finding that $C_1 = -aU/\pi$. Thus, C_1 is proportional to the rate at which fluid leaves in the jet; the C_1 term represents a sink, in fact. (The C_2 term represents a vortex.)

Rispin obtains estimates for h as well, but the results are rather complicated, and it would add no perspicuity to the present section to repeat them. The important point in principle is that it is possible now to specify the value of h and not come to a contradiction as a result. The far-field description has effectively provided a height reference, because of the effect of gravity. This effect does not change the first-order inner solution, but it does modify the second-order term. (The velocity magnitude is not constant on the free surface in the second approximation.)

In the second-order term of the inner expansion, there is another interesting phenomenon, namely, the apparent angle of attack changes. This means, physically, that the occurrence of gravity waves modifies the inflow to the planing surface. In the near field, it is still not possible to see the waves that exist far away, but the latter have the effect of making the incident stream appear to be rotated somewhat. It is like a downwash effect (although the physical origin is quite different).

If one were given a planing problem such as we formulated early in this section, with the incident stream and all geometric parameters prescribed, it would be necessary to solve for the parameters a and b. One equation relating these parameters has already been mentioned, namely, the equation relating the length of the plate to these parameters. The other equation comes from the expression (which was not written out here) for h as a function of a and b.

Rispin avoided much tedious algebra by solving the inverse problem. He assumed that a, b, and c were given, then solved to find h. He also had to treat the angle of attack as an unknown quantity, and he found an asymptotic expansion for it. (Note that only two of the basic parameters can be prescribed arbitrarily, unless we are prepared also to let ℓ be an unknown quantity.)

One final comment on Rispin's work must be made. He finds terms of six orders of magnitude: $O(1)$, $O(\beta \log \beta)$, $O(\beta)$, $O(\beta^2 \log^2 \beta)$, $O(\beta^2 \log \beta)$, and $O(\beta^2)$. But he finds also that they cannot be determined one at a time. Rather, they must be taken in groups: a) the $O(1)$ terms, b) the terms linear in β (the logarithm being ignored), and c) the terms involving β^2. This is the same kind of matching procedure that would have been used if he had adopted the working rule that logarithms should be treated as if they were $O(1)$. (See Section 1.2.)

5.2 Flow Around Bluff Body in Free Surface.

A problem related to that of Rispin (1966) and Wu (1967) has been studied by Dagan & Tulin (1969). They have concerned themselves with the flow at the bow of a blunt ship, where any kind of linearization procedure must be completely wrong. In order to handle such a situation, they have adopted essentially the same procedure that the previous authors used, namely, they set up inner- and outer-expansion problems in which the nonlinearity is confined initially to the near field and the effects of gravity are confined initially to the far field. Then, by limiting their study to a two dimensional problem, the nonlinear near-field problem can be solved by the hodograph method, and the far-field problem is a simple variation of a well-studied problem in water-wave theory.

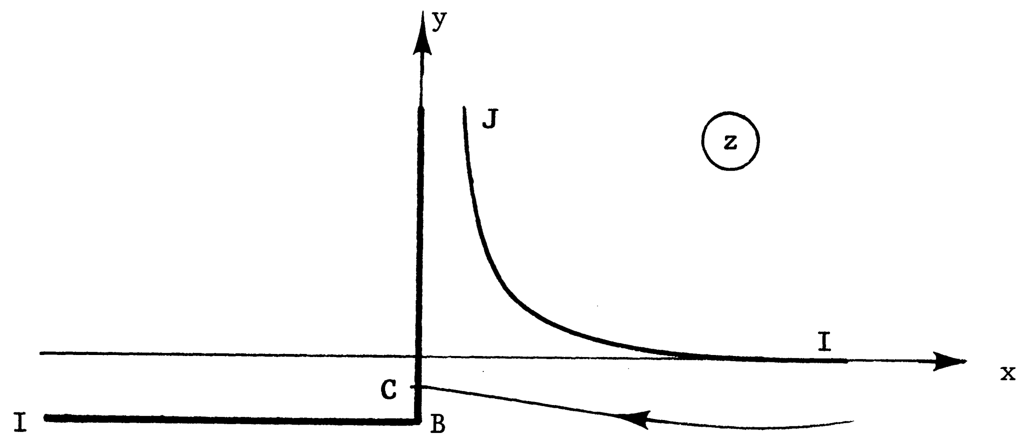

Figure (5-5). Bluff Body in the Free Surface.

The geometry of their problem is shown in Figure (5-5), which is reproduced from their paper. They argue that at very low speed there will be a smooth flow up to and then down under the bow, with a stagnation point at the location of highest free-surface rise, but that that flow becomes unstable as speed increases, until finally a jet forms, as sketched in Figure (5-5). Regardless of whether their description of the flow at very low speed is correct*, this jet model appears to

*Their Section III.2 has some questionable aspects.

be entirely reasonable physically; a barge-like body usually causes a region of froth just ahead of the bow, and this froth is probably caused by such a jet being thrown upward and forward, then dropping downward (which the theory overlooks). Thus it seems appropriate to study the formation of such a free-surface jet by the use of free-streamline theory, and one may expect that the details of the formation of the jet are not terribly sensitive to the effect of gravity.

The body, as shown in Figure (5-5), extends downstream to infinity. (In a sense, the whole problem is part of the inner expansion of a much larger problem, in which the stern of the body would be visible and in which waves would follow the body.) Thus, there is no Kutta condition or equivalent which can effectively cause a circulation type of flow in the fluid region. In Green's problem, for example, the flow at great distances appears to have been caused by a vortex. It is this property that causes the apparent logarithmic deflection of the free surface far away from the body, and it is this property that requires the far-field description (as in Rispin's problem) to contain a logarithmic singularity at the origin. Dagan and Tulin have no such logarithmic solutions.

They find that the jet appears, from far away, to be caused by a singularity of algebraic type. Specifically, the outer expansion of their inner expansion shows the complex velocity behaving like $Z^{-3/2}$, where Z is the complex variable defined in the physical plane, shown in Figure (5-5). Thus, their far-field expansion must exhibit a singularity at the origin of this same type.

This result, if correct, is most interesting, for, as Dagan and Tulin point out, it means that the far-field expression for pressure is not integrable, and so one *must* use the near-field expansion for any force calculation.

Furthermore, it is a disturbing result, because it suggests that many previous attempts to incorporate bow-wave nonlinearities into linear-theory singularities have been futile exercises.

Personally, I am not yet willing to admit that the possibility of having the complex velocity behave like $z^{-1/2}$ is really to be rejected, as Dagan and Tulin claim. Wagner (1932) analyzed the region of the jet and the stagnation point for the flow against a flat plate of infinite extent donwstream, and he showed that this flow, from far away, has the behavior of a flow around the leading edge of an airfoil, that is, the velocity varied with $z^{-1/2}$. Physically it seems rather difficult to imagine that, by curving the body around just behind the stagnation point, one causes such a drastic change in the apparent singularity.

Dagan and Tulin present a figure (their Figure 2) in which they have placed many symbols showing beam/draft ratios of more than a hundred ships, and it is quite evident that most ships have values of this ratio considerably greater than unity. They then use this fact as an alleged justification for claiming that their 2-D model of the bow flow (as in Figure (5-5)) will have some validity in describing the flow around the bow of an actual ship — since most ships are presumably of the "flat" variety. However, this claim is completely misleading. The theory might apply to a scow, but not to a ship. After all, beam/draft ratio is measured amidships, and even ships with the largest block coefficients have entrance angles less than 180°.

Also, it is appropriate to mention again the warning against defining a small parameter precisely and then trying to interpret on some absolute basis whether a particular value of the parameter is "small enough." For example, it is conceivable that a thin-ship analysis would be valid for a ship with beam/draft ratio of 10 , whereas a flat-ship analysis might fail for the same ship. I am not saying that this is likely, but it is *possible*. In one problem, a value of 10 might be "small," whereas in another problem a value of 1/10 might be "not small."

Notwithstanding these objections, the paper by Dagan and Tulin has provided a refreshing change in outlook on the bow-flow problem, and perhaps it will be more fruitful eventually than the usual attempts to place complicated singularities at the bow in the frame-work of linearized theory.

5.3 Submerged Body at Finite Speed

Since the principal difficulty in solving free-surface problems follows from the nonlinear conditions at the free surface, we are always seeking new arguments to justify linearizing the conditions. One possible basis for linearizing is that a body is deeply submerged. Then its effect on the free surface will presumably be small, even if it is not appropriate to linearize the problem in the immediate neighborhood of the body itself.

Such problems were discussed by Wehausen & Laitone (1960), where the previous history may also be found. Tuck (1965b) introduced a more systematic treatment for the case of a circular cylinder. Salvesen (1969) solved the problem for a hydrofoil (with Kutta condition and thus with circulation), and he compared his results with the data from experiments which he conducted. In the earlier studies of such problems, the approach was usually an iterative one in which the body boundary condition was first satisfied, then an additional term was added to the solution so that the free-surface condition would be satisfied; the latter would cause the body boundary condition to be violated, and so another term would have to be added to correct that error, but then there would again be an error in the free-surface condition. And so on. The free-surface condition that was satisfied once during each cycle was generally the conventional linearized condition. Thus, if the procedure converged, one obtained a solution which exactly satisfied the body boundary condition and the linearized free-surface condition. The contribution of Tuck seems to have been in systematizing the procedure in terms of a small parameter varying inversely with depth of the body and in pointing out that a consistent iteration scheme involves using the exact free-surface conditions as a starting point. Then, as the boundary condition on the body is corrected at each stage, so also is the free-surface condition made more and more nearly exact.

Tuck concluded, in fact, that it was more important to include nonlinear free-surface effects than to improve the satisfaction of the body boundary condition if one were most interested in certain free-surface phenomena, *e.g.*, predicting wave resistance and near-surface lift. Salvesen agreed with this conclusion only on the condition that the body speed be not too large. At fairly high speed, his results indicated that precision in satisfying the body boundary condition was just as important as precision in satisfying the free-surface condition. Figure (5-6) is taken from Salvesen's paper; it shows the theoretical wave resistance of a particular body as a function of (depth) Froude number, the resistance being calculated by three different approximations: 1) linearized free-surface theory, 2) theory in which the free-surface condition is satisfied to second order, and 3) theory in which both the free-surface condition and the body boundary condition are satisfied to second order. The differences are quite apparent.

The figure is a very interesting one. The difference between the linear-theory curve and either of the other two curves is presumably a second-order quantity, and yet that difference is — in one case — of the same order of magnitude numerically as the linear-theory curve itself. The problem is worth further discussion.

Salvesen defines his small parameter as follows:

$$\varepsilon \equiv t/b ,$$

where t is the thickness (or some other characteristic dimension) of the body, and b is the submergence of the body below the undisturbed free-surface level. It is *not* assumed that the body is "thin" in any sense; it could be

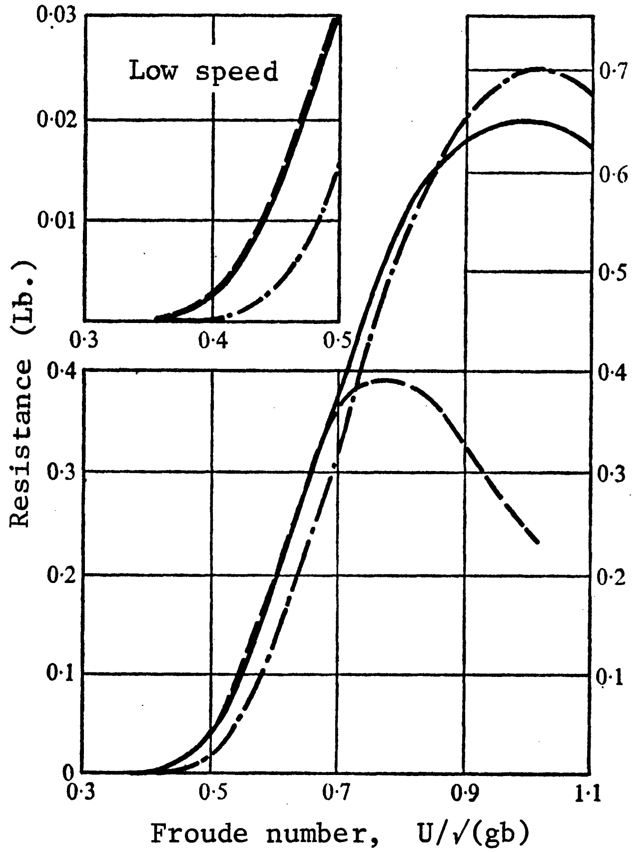

Figure (5-6). Theoretical Wave-Resistance Curves for $\varepsilon = t/b = 0.30$.

—— - ——, first-order theory;
————, inconsistent second-order theory (neglecting body-correction effects);
— — — —, consistent second-order theory.

(From Salvesen (1969))

a circular cylinder (Tuck's problem), for example. Salvesen's calculations and experiments were carried out for a rather fat, wing-shaped body with a sharp trailing edge. The body was symmetrical about the horizontal plane at depth b. If the free surface had not been present, there would have been no lift on the body.

A complex velocity potential, $F(z) = \phi(x,y) + i\psi(x,y)$, can be defined for the problem, with $z = x+iy$ measured from an origin located in the body at a depth b below the undisturbed free surface. Salvesen expands the complex potential in a series which he groups in two alternate ways:

$$F(z) = [Uz + F_{b_0}] + [F_{f_1} + F_{b_1}] + \ldots \qquad (5\text{-}3)$$

$$= Uz + [F_{b_0} + F_{f_1}] + [F_{b_1} + F_{f_2}] + \ldots \qquad (5\text{-}4)$$

These terms are defined in terms of the iteration scheme already mentioned. The grouping in (5-3) is to be used near the body, and the grouping in (5-4) applies far away from the body; in particular, the latter applies on and near the free surface. Salvesen points out that this distinction means that: a) near the body, we are considering the zero-order flow to be that flow which would occur in the presence of the body and the absence of the free surface, and b) near the free-surface, the basic flow is just the uniform incident stream. Thus, in (5-3), we must determine F_{b_0} so that $[Uz + F_{b_0}]$ satisfies the kinematic boundary condition on the body and so that $|\nabla F_{b_0}| \to 0$ as $|z| \to \infty$ (in any direction).

Next, Salvesen assumes that F_{b_0} is $O(\varepsilon)$ far away from the body. The two terms so far obtained do not satisfy a free-surface condition, and so F_{f_1} must be determined so that, when it is added to the first two terms, the sum satisfies the appropriate free-surface condition, which is:

$$Re\{F_{b_0}' + F_{f_1}' + i\kappa F_{b_0} + i\kappa F_{f_1}\} = 0 \quad \text{on} \quad y = b \qquad (5\text{-}5)$$

where $\kappa = g/U^2$. Since F_{b_0} is assumed to be $O(\varepsilon)$ near the surface, then the same should be true for F_{f_1} .

Now the three terms in the series do not satisfy the body condition, and so F_{b_1} is determined so that, when it is added to the first three terms, the sum satisfies the condition properly. Then F_{b_1} is assumed to be $O(\varepsilon^2)$ near the free surface, and a new function F_{f_2} is found to provide a further correction needed near the free surface.

It is in this last step that the Tuck-Salvesen approach differs from the previous treatments of such problems. If F_{b_1} is really $O(\varepsilon^2)$, then the free-surface condition ought to be satisfied to that order of magnitude. It can be shown that this implies the following condition on F_{f_2} :

$$Re\{F_{b_1}' + F_{f_2}' + i\kappa F_{b_1} + i\kappa F_{f_2}\}$$

$$= \eta_1 \, Im\{F_{b_0}'' + F_{f_1}'' + i\kappa F_{b_0}' + i\kappa F_{f_1}'\}$$

$$- (1/2U) \, |F_{b_0}' + F_{f_1}'|^2 \; . \qquad (5\text{-}6)$$

The right-hand side of this equation takes account of the nonlinearity of the free-surface conditions, since obviously it involves just the potential function from the previous cycle of the iteration. η_1 is the free-surface elevation from the previous approximation; it is given by:

$$\eta_1(x) = -(U/g) \, Re\{F_{b_0}' + F_{f_1}'\} ,$$

with the right-hand side evaluated on $y = b$. One might try to cut corners in (5-6) in either of two ways, namely, 1) ignore the right-hand side by setting it equal to zero,

2) drop the terms involving F_{b_1} on the left-hand side. The first is equivalent to retaining just a linear free-surface condition. The second is equivalent to neglecting the effect of the second-order body correction at the free surface; this is the "inconsistent" second-order theory to which Figure (5-6) refers.

Apparently, Salvesen did not prove one important step in his development, namely, his claim that F_{b_0} is $O(1)$ near the body and $O(\varepsilon)$ far away from the body. In fact, with his definition of $\varepsilon = t/b$, it appears that the statement is wrong. The potential F_{b_0} represents just a thickness effect, since it is the solution of the problem of a symmetrical body in a uniform stream. Although the body can be replaced by a distribution of sources, the disturbance will appear from far away to have been caused by a dipole, and so it must have the form: $F_{b_0} \sim C/z$. If the body were a circular cylinder, we could evaluate C : $C = Ut^2$, where t is the radius of the cylinder. The complex fluid velocity on the free surface caused by the body is, in the first approximation, $-C/z^2 = O(\varepsilon^2)$, since $z = x+ib$ on the level of the undisturbed free surface. This conclusion contradicts Salvesen's assumption that the free-surface disturbance is $O(\varepsilon)$, but perhaps it does not matter. At this point, the results would presumably be just the same if he had defined: $\varepsilon \equiv (t/b)^{1/2}$. (The argument above for a circular cylinder agrees with Tuck's conclusions.)

When the first free-surface correction is found, namely, F_{f_1}, its effect in the neighborhood of the body is not diminished by an order of magnitude, since at least one part of F_{f_1} involves an exponential decay with depth, the exponent being $\kappa(y-b)$. Near the body, $y \simeq 0$, and so the exponential-decay factor is $e^{-\kappa b}$, and it has been assumed that κb is $O(1)$. (See Salvesen's paper.)

Since F_{f_1} is $O(\varepsilon^2)$ near the body, the order of magnitude of the next correction term, F_{b_1} , must be the same. This time, however, the nature of the body disturbance is quite different from a dipole disturbance. The effective incident flow corresponding to F_{f_1} is not a uniform stream, and so the presence of a sharp trailing edge on the body requires that a Kutta condition be imposed, and then a circulation flow occurs. From far away, it appears that F_{b_1} is caused by a combination of a vortex and a dipole. If the strengths of the two apparent singularities were comparable, the vortex behavior would dominate the dipole behavior far away, and the induced velocity would diminish in proportion to $1/z$, rather than $1/z^2$, which was the case for the dipole. Thus, F_{b_1} would be $O(\varepsilon^3)$ near the free surface. In the absence of a sharp trailing edge which can cause the formation of a vortex flow, the corresponding F_{b_1} would be $O(\varepsilon^4)$. This matter remains to be resolved.

There are other interesting aspects to this problem. One relates to the interpretation of the small parameter, $\varepsilon = t/b$. In defining such a dimensionless perturbation parameter, one normally assumes that the smallness of ε can be realized physically either by letting t be extremely small or by letting b be very large. In the present problem, this choice is not really available to us. The reason is that there is another length scale in the problem, namely, $1/\kappa = U^2/g$, and this length scale appears generally in combination with the dimension b . It has been assumed that $\kappa b = O(1)$ as $\varepsilon \to 0$. Therefore, if we want to consider the problem of a body which is more and more deeply submerged, ($b \to \infty$), then we must also restrict our attention to higher and higher speeds. This is awkward.

Finally, one more important aspect must be mentioned. The relation between wave number, κ , and forward speed, U , namely, $\kappa = g/U^2$, is based on linearized free-surface theory. In general, if one seeks to find the nature of non-linear waves which can propagate without change of form, the wavelength of those waves is not related to their speed in this simple fashion. To be sure, the relationship is approximately correct if the waves are not terribly big in amplitude, and so one might expect that the wavelength or the wavenumber can be expressed as an asymptotic series in ε

$$\kappa \sim \kappa_0 + \kappa_1 + \kappa_2 + \dots ,$$

with $\kappa_0 = g/U^2$. This can indeed be done, but it turns out to be much more convenient to assume that κ is precisely given and then to find the value of forward speed that corresponds to that wave number. Thus, one expands the forward speed, U , into an asymptotic expansion:

$$U \sim u_0 + u_1 + u_2 + \dots .$$

This procedure is discussed by Wehausen & Laitone (1960), and Salvesen uses it in his hydrofoil problem. I was able to omit mention of it in writing Equations (5-5) and (5-6) because it turns out that $u_1 = 0$, and so the effect of this speed shift (or period shift) does not enter the problem until the third approximation is being sought. However, this is a classic example of the kind of expansion described in Section 1.3. If one did not allow for a variation in either κ or U , the third approximation would not be valid at infinity, and so one would have great difficulty in predicting wave resistance, since that quantity depends explicitly on the wave height at infinity.

Figure (5-7) is taken from Salvesen (1969). It shows very clearly the change in wavelength that arises in the third-order solution. In fact, it appears in this case that the change of wavelength is practically the *only* third-order effect. This figure also speaks well for Salvesen's experimental technique!

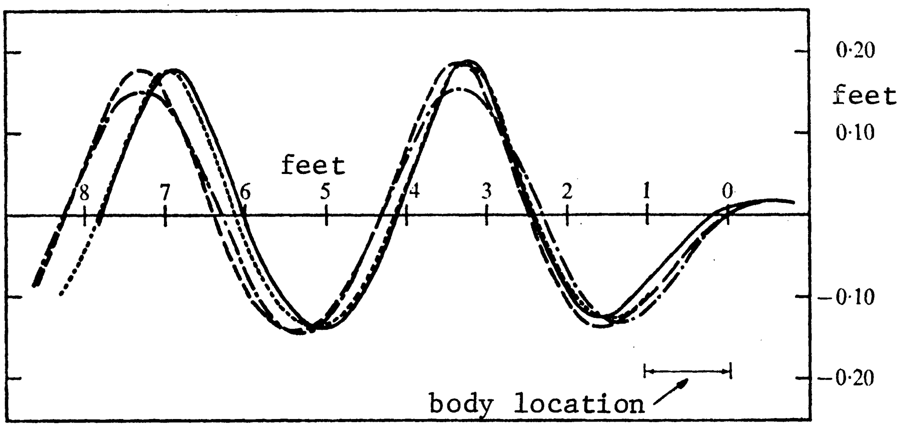

Figure (5-7). Third-Order Effect on Wavelength.

—— - —— , first-order theory;
— — — — , second-order theory;
---------- , third-order theory;
———— , experiment.

Froude number = 0.71 ;
$\varepsilon = t/b = 0.30$.

(From Salvesen (1969))

5.4 Submerged Body at Low Speed

Salvesen (1969) computed the wave height behind a hydrofoil up to the third approximation, as already mentioned in Section 5.3. Although his third approximation is not really consistent, he gives what appear to be sufficient arguments to demonstrate that the consistent result would not be much different from the results presented in his paper. Figure (5-8), from Salvesen (1969), presents the wave-height computations in a way that shows the relative importance of the

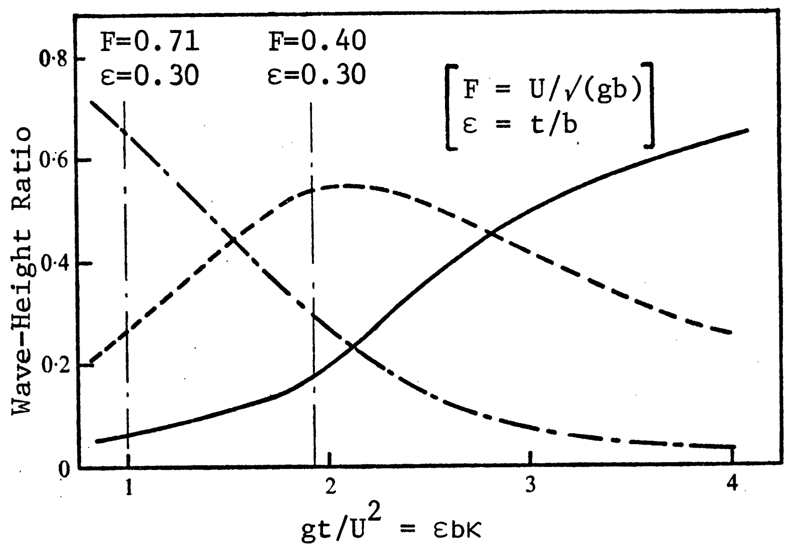

Figure (5-8). First-, Second-, and Third-Order Wave Heights at Low Speeds.

—— - ——, $H_1/(H_1+H_2+H_3)$;
— — — —, $H_2/(H_1+H_2+H_3)$;
————, $H_3/(H_1+H_2+H_3)$.

(From Salvesen (1969))

first-, second-, and third-order terms. Let the wave amplitude be expressed by the series:

$$H \sim H_1 + H_2 + H_3 \ , \quad \text{where} \quad H_{n+1} = o(H_n) \quad \text{as} \quad t \to 0 \ .$$

(t is the thickness of the foil, as in the last section.) Then the figure shows the three ratios, $H_n/(H_1+H_2+H_3)$, for $n = 1,2,3$; that is, each curve shows the relative contribution to the wave height of one of the first three terms in the wave-height expansion. As speed decreases (toward the right-hand side of the figure), the second-order part comes to dominate the linear-theory part, and then the third-order part dominates the first two. It seems quite likely that the fourth-order term would take over if the graph were extended, then the fifth-, sixth-, ... order terms.

Salvesen's analysis is based on the condition that t (or, more properly, t/b, where b is the body depth) is very small; the Froude number is simply a parameter unrelated to t, which is equivalent to saying that Froude number $\equiv U/\sqrt{(gb)}$ is $O(1)$ as $t/b \to 0$. Perhaps it is not surprising if Salvesen's expansion is not uniformly valid with respect to Froude number. That is all that Figure (5-8) really says.

The reason for its nonuniformity has already been mentioned: In the expansion of the solution near the free surface, it has been assumed that the lowest-order approximation is just the uniform-stream term, Ux; all other terms in the expansion of the potential must be very small compared to this term. And this is nonsense if we consider the limit process $U \to 0$. Of course, we might have been lucky: It could have turned out that the velocity perturbation approached zero more rapidly than U. But it does not. And so we have here a genuine singular perturbation problem.

Let us consider a sequence of steady-motion experiments, each lasting for an infinite length of time. We arrange the sequence of experiments according to decreasing values of body speed, U, and we suppose that all conditions except forward speed are identical in all experiments. We shall discuss what happens when " $U \to 0$ ", and we shall understand by the limit operation that we are passing through the sequence of experiments toward the limit case in which there is no forward speed at all. In each experiment, U is a constant.*

*This is the same point that I belabored in the last paragraph of Section 1.2. Again, I apologize to those to whom it is obvious.

As $U \to 0$, we certainly expect all fluid motion to vanish. But we would like to know to what extent the velocity field vanishes in proportion to U (that is, what part is $O(U)$), what part vanishes more rapidly than U (that is, what part is $o(U)$), and what part, if any, vanishes less rapidly than U .

In an infinite fluid, the velocity everywhere is exactly proportional to U . Far away, the velocity approaches zero; it drops off like $1/r$ if there is a circulation around the body, and it drops off like $1/r^2$ if there is no circulation. But in both cases the constant of proportionality is $O(U)$. No matter how distant our point of observation is from the body, the velocity is $O(U)$ as $U \to 0$.

At very low speed, one expects that gravity will force the free surface to remain plane. The constant-pressure condition will be violated to the extent that the magnitude of the fluid velocity on that plane is not quite constant, but the error in satisfying the dynamic condition will be proportional to the *square* of the fluid velocity magnitude. The kinematic condition will be satisfied in a trivial manner. Accordingly, it seems quite reasonable to assume that the free-surface disturbance is $O(U^2)$ as $U \to 0$, and so the velocity potential in the first approximation is the same as if the free surface were replaced by a rigid wall. Let the rigid-wall velocity potential be denoted by $\phi_0(x,y)$. Clearly, it is true that:

$$\phi_0(x,y) = O(U) \quad .$$

This follows by the same arguments as those used in the preceding paragraph. The more important problem is to determine the order of magnitude of $[\phi(x,y) - \phi_0(x,y)]$, where $\phi(x,y)$

is the exact velocity potential for the case of the body moving at speed U under the free surface.

In order to be specific now, let $\phi_0(x,y)$ be the velocity potential in two dimensions which satisfies the conditions:

$$\frac{\partial \phi_0}{\partial n} = 0 \text{ , on body; } \quad |\phi_0 - Ux| \to 0 \text{ , as } x \to -\infty \text{ ; } \quad \frac{\partial \phi_0}{\partial y} = 0 \text{ on } y = 0 \text{ .}$$

The body is at rest in our reference frame.

The rigid-wall solution satisfies all conditions of the free-surface problem except the dynamic condition on the free surface. The latter could be used to define the free-surface shape. Thus, if the free-surface disturbance is expressed by:

$$\eta(x) \sim \eta_0(x) + \eta_1(x) + \ldots \text{ ,}$$

the dynamic free-surface boundary condition says that:

$$\eta(x) \sim \eta_0(x) \equiv \frac{1}{2g} [U^2 - \phi_{0_x}^2(x,0)] \quad . \qquad (5\text{-}7)$$

Of course, the kinematic condition is now violated, but an additional velocity field which is $O(U^2)$ can correct that. And so it appears plausible that:

$$\phi(x,y) - \phi_0(x,y) = o(U) \quad . \qquad (5\text{-}8)$$

One point should be noticed about this conclusion. The limit process " $U \to 0$ " implies that Froude number goes to zero. Nothing has been said about the length scale used in defining Froude number, but it does not matter so long as all dimensions are fixed. The submergence and the body dimensions may be quite comparable, for example. Thus, we are not considering t/b as small, in the sense that Salvesen did. However, *both* t and b are supposed to be large compared with the length U^2/g ; we imply this if we state that *all dimensions must be fixed as* $U \to 0$.

It would be wrong to take $\phi_0(x,y)$ as the potential for the flow around the body in an infinite fluid (without either free surface or a rigid-wall substitute). The body can be quite near to the free surface in Salvesen's sense, and so the effect of its image cannot be neglected. Furthermore, at least part of the effect of the image is $O(U)$, even if the body is very far away from the free surface, and such an effect must be included in the first term of the approximation which is supposed to be valid as $U \to 0$.

The next problem is to find $[\phi(x,y) - \phi_0(x,y)]$. We consider two possible approaches in the following subsections.

5.41 *A Sequence of Neumann Problems.* As above, let there be a velocity potential, $\phi(x,y)$, which provides the solution of the exact problem:

$$g\eta(x) + \frac{1}{2}[\phi_x^2 + \phi_y^2] - \frac{1}{2}U^2 = 0 , \quad \text{on } y = \eta(x) ; \qquad (5\text{-}9)$$

$$\phi_x \eta_x - \phi_y = 0 , \quad \text{on } y = \eta(x) ; \qquad (5\text{-}10)$$

$$\frac{\partial\phi}{\partial n} = 0 , \quad \text{on the body;} \qquad (5\text{-}11)$$

$$\phi(x,y) - Ux \to 0 , \quad \text{as } x \to -\infty ; \qquad (5\text{-}12)$$

The rigid-wall potential, $\phi_0(x,y)$, satisfies (5-11) and (5-12) too, but it does not satisfy the free-surface conditions, of course; instead, we have:

$$\frac{\partial \phi_0}{\partial y} = 0 , \qquad \text{on} \quad y = 0 . \tag{5-13}$$

Now we introduce one more potential function, the difference between the above two potentials:

$$\Phi(x,y) \equiv \phi(x,y) - \phi_0(x,y) . \tag{5-14}$$

It must satisfy the body boundary condition, of course, and it vanishes far upstream. On the free surface, which we now define as:

$$y = \eta(x) = \eta_0(x) + H(x) , \tag{5-15}$$

where $\eta_0(x)$ is defined as in (5-7), the new potential satisfies the two conditions:

$$\begin{aligned} 0 = \; & gH(x) - \frac{1}{2}\phi_{0x}^2(x,0) \\ & + \frac{1}{2}[\phi_{0x}^2 + \phi_{0y}^2 + 2\phi_{0x}\Phi_x + 2\phi_{0y}\Phi_y + \Phi_x^2 + \Phi_y^2]\big|_{y=\eta(x)} ; \end{aligned} \tag{5-16}$$

$$0 = [\eta_0'(x) + H'(x)][\phi_{0x} + \Phi_x]\big|_{y=\eta(x)} - [\phi_{0y} + \Phi_y]\big|_{y=\eta(x)} . \tag{5-17}$$

These conditions are still exact. An obvious approach to solving for $\Phi(x,y)$ and $H(x)$ is to re-express these conditions on $y = \eta(x)$ as conditions on $y = 0$. Here I shall assume that this can be done in the usual way.* Then it follows from the exact conditions that the following are the appropriate simplifications:

$$0 \simeq gH(x) + \phi_{0x}\Phi_x , \qquad \text{on} \quad y = 0 ; \tag{5-18}$$

$$\Phi_y \simeq \eta_0'(x)\phi_{0x} - \eta_0(x)\phi_{0yy} , \qquad \text{on} \quad y = 0 . \tag{5-19}$$

The second condition is a Neumann condition; the right-hand

*This is the crucial point which distinguishes this section from the next section.

side is known, and the condition is prescribed on a known, fixed surface. In fact, (5-19) is satisfied by the real part of:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ds\ p(s)}{s - z} ,$$

where

$$z = x + iy ,$$

$$p(x) = \eta_0(x)\phi_{0_x}(x,0) . \qquad (5\text{-}20)$$

This follows from the Plemelj formula. (See, *e.g.*, Muskhelishvili (1953).) The function $p(x)$ can be interpreted in terms of the fluid velocity which is needed to correct the flow field because of the error incurred by taking the free surface at $y = \eta_0(x)$ while using the potential function $\phi_0(x,y)$ to prescribe the velocity field. This is the same correction which was discussed above in connection with (5-8). Now we may observe that, since $\eta_0 = O(U^2)$ and $\phi_0(x,y) = O(U)$, it follows that $p(x) = O(U^3)$. Thus also:

$$|\nabla\Phi| = O(U^3) \quad \text{as} \quad U \to 0 . \qquad (5\text{-}21)$$

This is certainly a much stronger conclusion than (5-8)!

The integral expression given above is not the solution of the Φ problem, even in the first approximation, since it does not satisfy the body boundary condition. However, since the existence of Φ arises from a defect of ϕ_0 in meeting the free-surface conditions, it is difficult to imagine that the above estimate of the order of magnitude of Φ is not correct.

Numerical procedures could readily be worked out for solving problems of the above type. In fact, all that is needed is one algorithm which handles the problem of a given distribution of the normal velocity component on a surface in the presence of a plane rigid wall. The integral part of the solution given above would lead to a non-zero normal velocity component on the body, and this would have to be offset by a flow which does not change the condition at the plane $y = 0$. Presumably, all higher-order approximations would be solutions of problems which are identical in form to this one.

A variation on this approach has been discussed several times by Professor L. Landweber, although he has not published the work. He points out that the usual linearized free-surface condition,

$$\frac{\partial^2\phi}{\partial x^2} + \kappa \frac{\partial\phi}{\partial z} = 0 , \quad \text{on} \quad z = 0 , \quad \text{where} \quad \kappa = g/U^2 ,$$

becomes the rigid-wall condition when $U \to 0$, and so one might try an iteration scheme in which ϕ is expanded in a series, $\phi \sim \Sigma\phi_n$, and the terms are obtained as the solutions in an iteration scheme:

$$\frac{\partial\phi_1}{\partial y} = 0 , \quad \frac{\partial\phi_n}{\partial y} = -\frac{1}{\kappa}\frac{\partial^2\phi_{n-1}}{\partial x^2} \quad (\text{for} \quad n > 1) \quad \text{on} \quad y = 0 .$$

In order to test the scheme, Professor Landweber proposed trying to obtain the potential function for a Havelock source in this way; this obviates the need to satisfy a body boundary condition, and the known potential for the source can be expanded in a series in terms of $1/\kappa$.

Neither of the above schemes appears very promising to me. Salvesen's findings about the singular low-speed behavior seem to condemn any approach which overlooks the peculiar nature of the free-surface problem at low speeds. The next section should make clear why I am pessimistic about these approaches. It should be obvious even now that the wave-like nature of the problems has been lost, but the difficulty is more serious than that.

5.42 *A Dual-Scale Expansion.* According to linearized wave theory, the wave-like nature of a free surface disturbance loses its identity exponentially with depth. A disturbance created at the free surface is attenuated rapidly with depth, and a disturbance created at some depth causes a free-surface disturbance which decreases with the depth of the cause. The depth effect is essentially proportional to $e^{\kappa y}$, where, as above, $\kappa = g/U^2$ and y is measured as positive in the upward direction.

As U approaches zero, this depth-attenuation factor approaches zero for any fixed y. In other words, the free-surface effects are restricted to a thin layer which approaches zero thickness as $U \to 0$. We might say that the free-surface is separated from the main body of the fluid by this "boundary layer" in which there is a rapid transition from conditions at the surface to conditions inside the bulk of the fluid. From our experience with viscous boundary layers, we should expect the occurrence of large derivatives in this region and also some difficulty in satisfying boundary conditions on a face of the boundary layer.

In a viscous boundary layer, of course, the derivatives are much greater in one direction than in another, and this fact allows us to stretch coordinates anisotropically and apply the limit processes of the method of matched asymptotic expansions. In the free-surface boundary layer, however, this does not appear to be a possible approach. From the linear theory, we expect that there will be a wave motion with wavelengths which are $O(U^2/g)$. Thus, derivatives will be large in at least two directions inside the boundary layer — in the direction normal to the layer and in one direction parallel to the layer.

When I tried to solve this problem two years ago (See Ogilvie (1968)), I did not apply very systematic procedures. Rather, I simply assumed that the first approximation to Φ , as defined in (5-14), would have certain properties, namely,

$$\Phi(x,y) = O(U^5) \; ; \quad \Phi_x(x,y) \; , \quad \Phi_y(x,y) = O(U^3) \; ;$$

also, the surface deflection function would be given by (5-15), with:

$$H(x) = O(U^4) \; ; \quad H'(x) = O(U^2) \; .$$

The order of magnitude of Φ was chosen just so that the velocity components would be $O(U^3)$, and I assumed that differentiation changes a quantity by $1/U^2$ in order of magnitude. The arguments leading up to (5-21) contributed heavily to the conjecture about velocity components, and the $1/U^2$ effect of differentiation was chosen just because the free-surface characteristic length is U^2/g . It is important to note that the rigid-wall potential, ϕ_0 , is still part of the solution, and these statements about orders of magnitude and differentiation do not apply to it. In fact, I assume that ϕ_0 is completely known, and so it is not necessary to conjecture about the effects of differentiation.

In terms of the general approach of the multiple-scale expansion method, I have assumed that an approximation to the solution can be represented as the sum of two functions.

The first depends only on the length scale appropriate to the body geometry. The second function depends primarily on lengths measured on a scale appropriate to U^2/g , but it also depends on the first function and thus on lengths typical of the body. However, it seems to be possible to keep clear when differentiations are being carried out with respect to each of the length scales.

Physically, the situation may be described in the following way: If U is small enough, the body extends over a distance of many wavelengths of the surface disturbance. The initial disturbance is caused by the body, of course; this is the "rigid-wall" motion, and its dimensions are characteristic of the body. It causes a free-surface disturbance, with the result that waves are created. But these waves are very, very short, whereas the initial disturbance from the body appears to be just a slight nonuniformity in flow conditions when viewed on the scale comparable to the wavelength. The method is, in fact, quite similar to classical methods such as the W-K-B method.

When the assumptions listed above are actually applied, we find that the approximate free-surface conditions given in (5-18) and (5-19) must be replaced by the following:

$$gH(x) + \phi_{0_x}(x,0)\ \Phi_x(x,\eta_0(x)) \simeq 0 \ ; \tag{5-22}$$

$$\Phi_y(x,\eta_0(x)) - \phi_{0_x}(x,0)\ H'(x) \simeq p'(x) \ ;$$

the function $p(x)$ is the same that was given in (5-20). Note that Φ in both conditions here is to be evaluated on $y = \eta_0(x)$, rather than on $y = 0$. The reason is the same that was given in Section 3.2 in the near-field problem: If we tried in the usual way to expand $\Phi(x,\eta_0)$, say as follows:

$$\Phi(x,\eta_0) = \Phi(x,0) + \eta_0\Phi_y(x,0) + \frac{1}{2}\eta_0^2\Phi_{yy}(x,0) + \ldots ,$$

we would find that every term on the right-hand side is the same order of magnitude according to my assumptions. In particular, $\eta_0 = O(U^2)$, and, symbolically, we have: $\partial/\partial y = O(1/U^2)$. So this expansion procedure is not useful.

The two conditions above can be combined consistently into the following:

$$\Phi_y(x,\eta_0(x)) + \frac{1}{g}\phi_{0_x}^2(x,0)\ \Phi_{xx}(x,\eta_0(x)) \simeq p'(x) . \quad (5\text{-}23)$$

This is remarkably similar to the free-surface condition for another problem. In the ordinary linearized theory of gravity waves, suppose that a pressure distribution, $p(x)$, is travelling at a speed U . The free-surface condition would be:

$$\Phi_y(x,0) + \frac{U^2}{g}\Phi_{xx}(x,0) = p'(x) ,$$

if $\Phi(x,y)$ were the potential function for the problem. Replace U by $\phi_{0_x}(x,0)$, the "local stream speed," and evaluate the condition on $y = \eta_0(x)$; then this condition transforms into the condition found for $\Phi(x,y)$ in the low-speed problem. Thus, on a "local" scale (in which a typical length is U^2/g), the free-surface condition is just a very ordinary condition; one cannot see that the stream velocity changes slightly along the free surface, because the change occurs on a scale in which a typical measurement would be a body dimension; the change is very gradual. Also, the level of the undisturbed free surface appears to change gradually, as given by (5-7); this change also cannot be detected on the "local" scale.

It is now clear that the two length scales are quite distinct. We cannot separate the fluid-filled region into distinct parts in each of which only one length scale needs to be considered. Rather, the gradual changes which appear on the body-size scale appear to modify the short-length wave motion in the manner of a modulation.

In trying to find a potential function which satisfies (5-23), I made a nonconformal mapping: $x' = x$, $y' = y - \eta_0(x)$. Then Φ satisfies a complicated partial differential equation in terms of x' and y' , but the terms in the equation can be arranged according to their dependence on U , and it is found that the leading-order terms are simply the terms in the Laplacian, that is,

$$\Phi_{x'x'} + \Phi_{y'y'} \sim 0 \quad ;$$

all other terms are higher order. In this new coordinate system, the free-surface condition, (5-23), is transformed too, but again the leading-order terms are just the same after the transformation (but expressed as functions of x' and y'). Furthermore, the boundary condition is then to be applied on $y' = 0$. Let us now drop the primes on the new variables, for convenience. Then the problem is as follows: Find a velocity potential, $\Phi(x,y)$, which satisfies the Laplace equation in two dimensions and the free-surface condition:

$$\Phi_y(x,0) + \frac{1}{g}\, \phi_{0_x}^2(x,0)\, \Phi_{xx}(x,0) = p'(x) \quad ,$$

where

$$p(x) = \eta_0(x)\, \phi_{0_x}(x,0) \quad .$$

In addition, the potential must satisfy a body boundary condition; this has not been carefully formulated yet, and, in any case, the only solution that has been produced so far is one that satisfies the free-surface condition but not a body condition. There may be some good justification (or rationalization) for proceeding this way, but it is really an open question.

With such restrictions and reservations expressed, we can write down a "solution" of the above problem. Define:

$$\phi_0 \equiv \mathcal{Re}\{f_0(z)\} \quad ; \quad \Phi(x,y) \equiv \mathcal{Re}\{F(z)\} \quad ;$$

$$k(z) \equiv g\left[\frac{df_0}{dz}\right]^{-2} .$$

Note that:

$$f_0'(x) = \phi_{0x}(x,0) \quad ; \quad k(x) = g[\phi_{0x}(x,0)]^{-2} .$$

Then the solution is given by:

$$F'(z) = -\frac{1}{\pi i}\int_{-\infty}^{\infty} ds\ p'(s) \int_{-\infty}^{z} \frac{d\zeta\ k(\zeta)}{s-\zeta} \exp\left[-i\int_{\zeta}^{z} du\ k(u)\right] .$$

The ζ integral is a contour integral starting at $x = -\infty$, located entirely in the lower half-space. It should pass above the location of the singularity in $k(z)$. This solution represents no disturbance at the upstream infinity, as one would expect.

Far downstream, this solution can be approximated:

$$F'(z) \simeq 2i\ e^{-i\kappa z}\int_{-\infty}^{\infty} ds\ p''(s) \exp\left[i\kappa s - i\int_{s}^{z} du\ [k(u) - \kappa]\right] ,$$

where $\kappa = g/U^2$. Then, from (5-22), we obtain the wave shape far aft of the body:

$$H(x) \simeq - \frac{2U}{g} \int_{-\infty}^{\infty} ds \; p''(s) \; \sin \left[\kappa(x-s) + K(s) \right] \quad ,$$

where

$$K(s) = \int_{s}^{\infty} du \; [k(u) - \kappa] \quad .$$

Calculation of the wave resistance is then very simple in principle. (In practice, it is a very tedious calculation.) Note that the expression for the wave shape downstream does not require knowledge of $F'(z)$ (or $\Phi(x,y)$), that is, the surface disturbance far away is a real wave, but its shape and size depend only on the solution of the rigid-wall problem. This is not true of the wave disturbance in the vicinity of the body.

It would be very useful, I am sure, to formulate this problem carefully by the method of multi-scale expansions. The approach described by Ogilvie (1968) is very heuristic and leaves much to be desired.

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