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Final Report

THE GENERAL CONSERVATION LAWS OF A DILUTE PLASMA

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ORA Project 02929

under contract with:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
OFFICE OF SPACE SCIENCES
GRANT NO. NSG-22-59
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

July 1963

FOREWORD

This report represents the final report on the NASA Grant NSG-22-59. It is essentially a continuation of the work done on the general study of the physics of a low density ionized gas assuming no collisions. The aim of this study is to familiarize ourselves with the properties of such a dilute plasma in the hope to gain insight into phenomena such as instabilities, discontinuities and the all-important derivation of the "magneto-hydrodynamic" equations from a kinetic theoretical point of view.

The project is directed by Professor R.S.B. Ong of The University of Michigan. The contents of this report will be condensed and submitted for publication shortly.

LIST OF SYMBOLS

$\vec{r} = (r_x, r_y, r_z)$	position vector pertaining to electrons
$\vec{R} = (R_x, R_y, R_z)$	position vector pertaining to ions
$\vec{v} = (v_x, v_y, v_z)$	velocity vector pertaining to electrons
$\vec{V} = (V_x, V_y, V_z)$	velocity vector pertaining to ions
$\phi(\lvert\vec{r}_1 - \vec{r}_2\rvert)$	Coulomb potential between two electrons
$\phi(\lvert\vec{R}_1 - \vec{R}_2\rvert)$	Coulomb potential between two ions
$\phi(\lvert\vec{r} - \vec{R}\rvert)$	Coulomb potential between an electron at position $\vec{r}$ and an ion at position $\vec{R}$
e	electron charge
m	electron mass
M	ion mass
t	time
$\vec{E}(\vec{r})$	induced electric field
$\vec{H}(\vec{r})$	induced magnetic field

1. INTRODUCTION

In the case of Coulomb forces in a dilute plasma the effect of weak interactions is more important than the effect of single collisions. In order to give a good evaluation of these multiple interactions the two-particle distribution function is used. The usual expression for the effect of binary collisions in the kinetic equation is then replaced by an integral containing the various two-particle velocity distribution functions. In the case of a dilute, fully ionized hydrogen plasma the basic equations are of the following forms (for the meaning of the symbols see page v).[1]

$$\frac{\partial}{\partial t} f_1(\vec{r},\vec{v};t) + v_{1\alpha}\frac{\partial f_1}{\partial r_{1\alpha}} - \frac{e}{m}\left[E_\alpha(\vec{r}_1) + \frac{1}{c}(\vec{v}_1 \times \vec{H})\right]\frac{\partial f_1}{\partial v_{1\alpha}}$$
$$= \frac{4\pi e^2}{m}\int d\vec{r}_2 d\vec{v}_2 \frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{r}_2|)\frac{\partial}{\partial v_{1\alpha}} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \qquad (1.1)$$
$$- \frac{4\pi e^2}{m}\int d\vec{R} d\vec{V} \frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{R}|)\frac{\partial}{\partial v_{1\alpha}} f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t)$$

The function $f_1(\vec{r}_1,\vec{v}_1;t)$ is the single electron velocity distribution function defined in such a way that $f_1(\vec{r}_1,\vec{v}_1;t)d\vec{r}_1 d\vec{v}_1$ (where $d\vec{r}_1 = dr_{1x}dr_{1y}dr_{1z}$ and $d\vec{v}_1 = dv_{1x}dv_{1y}dv_{1z}$) gives the probability of finding an electron at a given instant t located within the element $d\vec{r}_1$ of the coordinate space and having velocity vectors with their end points in the element $d\vec{v}_1$ of the velocity space. The subscript 1 under the space coordinate $\vec{r}$ and velocity $\vec{v}$ indicates that a typical electron is being considered. The function $f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$ is defined in such a way that $f_2\, d\vec{r}_1 d\vec{r}_2 d\vec{v}_1 d\vec{v}_2$ gives the probability of finding electron 1 at the instant t located within $d\vec{r}_1 d\vec{v}_1$ together with electron 2 located within $d\vec{r}_2 d\vec{v}_2$ at the same instant. The integration with respect to $d\vec{r}_2$ and $d\vec{v}_2$ in the multiple integral must be extended over the entire coordinate space and velocity space available to electrons. The function $f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)$ is defined in such a way that $f_2\, d\vec{r}d\vec{R}d\vec{v}d\vec{V}$ gives the probability of finding electron 1 at the instant t located within $d\vec{r}_1 d\vec{v}_1$ together with an ion located within $d\vec{R}d\vec{V}$ at the same instant. Note that capital letters refer to ion positions and velocities while lower case letters indicate electron positions and velocities. The integration with respect to $d\vec{R}$ and $d\vec{V}$ must be extended over the entire coordinate space and velocity space available to ions.

An equation similar to (1.1) applies for the distribution function $F_1(\vec{R},\vec{V};t)$ for single ions. Furthermore Eq. (1.1) is one equation of a hierarchy of equations for the multi-particle distribution functions of increasing order; the analogous Bogoliubov-Born-Green-Kirkwood-Yvon equations for a dilute plasma.[1] Along with Eq. (1.1) we shall be concerned with the equation for $f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$ which has the form:

$$
\begin{aligned}
&\frac{\partial}{\partial t} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) + v_{1\alpha}\frac{\partial}{\partial r_{1\alpha}} f_2 + v_{2\alpha}\frac{\partial}{\partial r_{2\alpha}} f_2 - \frac{e}{m}\Big\{E_\alpha(\vec{r}_1) \\
&\quad + \frac{1}{c}[\vec{v}_1 x \vec{H}]_\alpha\Big\}\frac{\partial f_2}{\partial v_{1\alpha}} - \frac{e}{m}\Big\{E_\alpha(\vec{r}_2) + \frac{1}{c}[\vec{v}_2 x \vec{H}]_\alpha\Big\}\frac{\partial f_2}{\partial v_{2\alpha}} \\
&= \frac{4\pi e^2}{m}\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{r}_2|)\frac{\partial}{\partial v_{1\alpha}} f_2 + \frac{4\pi e^2}{m}\frac{\partial}{\partial r_{2\alpha}}\phi(|\vec{r}_1-\vec{r}_2|)\frac{\partial f_2}{\partial v_{2\alpha}} \\
&\quad + \frac{4\pi e^2}{m}\int d\vec{v}_3 d\vec{r}_3\Big\{\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{r}_3|)\frac{\partial}{\partial v_{1\alpha}} \\
&\quad + \frac{\partial}{\partial r_{2\alpha}}\phi(|\vec{r}_1-\vec{r}_3|)\frac{\partial}{\partial v_{2\alpha}}\Big\} f_3(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{v}_1,\vec{v}_2,\vec{v}_3;t) \\
&\quad - \frac{4\pi e^2}{m}\int d\vec{R} d\vec{V}\Big\{\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{R}|)\frac{\partial}{\partial v_{1\alpha}} \\
&\quad + \frac{\partial}{\partial r_{2\alpha}}\phi(|\vec{r}_2-\vec{R}|)\Big\} f_3(\vec{r}_1,\vec{r}_2,\vec{R},\vec{v}_1,\vec{v}_2,\vec{V};t)
\end{aligned} \tag{1.2}
$$

The three-particle distribution functions f_3 are defined similarly as the definition of f_2 and the integrations with respect to $d\vec{r}_3$ and $d\vec{v}_3$ must be extended over the entire coordinate space and velocity space available to electrons and those with respect to $d\vec{R}$ and $d\vec{V}$ again over the entire coordinate space and velocity space available to ions.

A similar equation also holds for the corresponding two-particle distribution function $F_2(\vec{R}_1,\vec{R}_2,\vec{V}_1,\vec{V}_2;t)$ for the ions. Since this equation is exactly analogous to Eq. (1.2), hence there is no need to write it down. However, we will be concerned with the equation for the electron-ion distribution function $f_2(\vec{r}_1,\vec{R}_1,\vec{v}_1,\vec{V}_1;t)$ which is of the form:

$$\frac{\partial}{\partial t} f_2(\vec{r}_1,\vec{R}_1,\vec{v}_1,\vec{V}_1;t) + v_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} f_2 + V_{1\alpha} \frac{\partial}{\partial R_{1\alpha}} f_2$$

$$- \frac{e}{m} \left\{ E_\alpha(\vec{r}_1) + \frac{1}{c} [\vec{v}_1 x \vec{H}]_\alpha \right\} \frac{\partial f_2}{\partial v_{1\alpha}} + \frac{e}{M} \left\{ E_\alpha(\vec{R}) + \frac{1}{c} [\vec{V}_1 x \vec{H}]_\alpha \right\} \frac{\partial f_2}{\partial V_{1\alpha}}$$

$$= - \frac{4\pi e^2}{m} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial}{\partial v_{1\alpha}} f_2 - \frac{4\pi e^2}{M} \frac{\partial}{\partial R_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial f_2}{\partial V_{1\alpha}}$$

$$- \frac{4\pi e^2}{m} \int d\vec{r}_2 d\vec{v}_2 \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial}{\partial v_{1\alpha}} f_3(\vec{r}_1,\vec{r}_2,\vec{R},\vec{v}_1,\vec{v}_2,\vec{V};t) \tag{1.3}$$

$$- \frac{4\pi e^2}{M} \int d\vec{r}_2 d\vec{v}_2 \frac{\partial}{\partial R_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial}{\partial V_{1\alpha}} f_3(\vec{r}_1,\vec{r}_2,\vec{R},\vec{v}_1,\vec{v}_2,\vec{V};t)$$

$$- \frac{4\pi e^2}{m} \int d\vec{R}_2 d\vec{V}_2 \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial}{\partial v_{1\alpha}} f_3(\vec{r}_1,\vec{R}_1,\vec{R}_2;\vec{v}_1,\vec{V}_1,\vec{V}_2;t)$$

$$- \frac{4\pi e^2}{M} \int d\vec{R}_2 d\vec{V}_2 \frac{\partial}{\partial R_{1\alpha}} \phi(|\vec{r}_1-\vec{R}_1|) \frac{\partial}{\partial V_{1\alpha}} f_3(\vec{r}_1,\vec{R}_1,\vec{R}_2;\vec{v}_1,\vec{V}_1,\vec{V}_2;t)$$

The problem is how to deal with these functions without running into an endless chain of higher order distribution functions. Obviously, expressions must be found giving the f_3's in terms of the f_1's and f_2's for both electrons and ions. This will then yield a bona fide kinetic equation for a dilute plasma which is analogous to the Boltzmann equation for neutral gases. For the general case of a spatially non-homogeneous plasma this leads to a rather complicated kinetic equation. For the special case of a spatially homogeneous plasma we have the less complicated Balescu-Guernsey equation.[2] At any rate, in order to derive the macroscopic magneto-hydrodynamic equations from kinetic theoretical principles two preliminary studies are required.

1. Study of the approach to equilibrium.

2. Derivation of the general conservation laws.

The study of the approach to equilibrium should reveal the character of the relaxation times, which in turn would be very useful in the attempt to seek for approximate solutions to the complicated kinetic equation. The conservation laws are usually obtained by taking moments of the kinetic equation. They are equations giving relations between the various macroscopic variables, e.g., density, temperature, mass average velocity, pressure, etc. Unfortunately they form an open set of equations and in order to close them we need to overcome our ignorance concerning the characteristic relaxation times of a dilute plasma. However, it is important to note that only the complete set of macroscopic (in this case magneto-hydrodynamical) equations are of physical interest. Hence, although we are still far from obtaining the full set of magneto-hydrodynamic equations derived from first principles of physics, yet it is of interest to study the open set of general macroscopic conservation equations. This is usually obtained by taking moments of the kinetic equation. However, it is also possible to obtain these conservation equations directly from the B-B-G-K-Y hierarchy, i.e., from Eqs. (1.1), (1.2), (1.3), and the corresponding ones for the ions. They will clearly exhibit the role which the two-particle distribution functions play in the various macroscopic quantities.

2. DERIVATION OF THE GENERAL CONSERVATION LAWS

We start with Eq. (1.1) of the preceding section. Integrating with respect to $\vec{v}_1$ from $-\infty$ to $+\infty$ and assuming that the distribution vanishes as $\vec{v}_1$ goes to ∞ we obtain the equation:

$$\frac{\partial}{\partial t} n(\vec{r},t) + \frac{\partial}{\partial r_\alpha} \{n(\vec{r},t) u_\alpha(\vec{r},t)\} = 0 \tag{2.1}$$

where

$$n(\vec{r},t) \equiv \int_{-\infty}^{\infty} f_1(\vec{r},\vec{v};t)\, d\vec{v}_1 = \text{the number density.}$$

$$\vec{u}(\vec{r},t) \equiv \frac{1}{n} \int_{-\infty}^{\infty} \vec{v}_1 f_1 d\vec{v}_1 = \text{the macroscopic density.}$$

The terms in (1.1) involving the two-particle distribution function f_2 do not contribute anything, for e.g.,

$$\int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) \frac{\partial}{\partial v_{1\alpha}} f_2 d\vec{v}_1$$

$$= \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) \int_{-\infty}^{\infty} \frac{\partial}{\partial v_{1\alpha}} f_2 d\vec{v}_1$$

$$= \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 \left[\frac{\partial\phi}{\partial r_{11}} \int_{-\infty}^{\infty} \frac{\partial}{\partial v_{11}} f_2 \, dv_{11} dv_{12} dv_{13} + \frac{\partial\phi}{\partial r_{12}} \int_{-\infty}^{\infty} \frac{\partial}{\partial v_{12}} f_2 \, dv_{12} dv_{13} dv_{11} \right.$$

$$\left. + \frac{\partial\phi}{\partial r_{13}} \int_{-\infty}^{\infty} \frac{\partial}{\partial v_{13}} f_2 \, dv_{13} dv_{11} dv_{12} \right] = 0 ,$$

$$\text{since} \int_{-\infty}^{\infty} \frac{\partial f_2}{\partial v_{1i}} dv_{1i} = 0 \qquad (i = 1,2,3)$$

The subscript 1 in n and $\vec{u}$ is not needed anymore and to conform with the usual notation for the macroscopic quantities it is therefore omitted. Equation (2.1) yields the equation for the conservation of mass:

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \frac{\partial}{\partial r_\alpha}[\rho(\vec{r},t)u_\alpha(\vec{r},t)] = 0 \tag{2.2}$$

where

$$\rho(\vec{r},t) \equiv mn(\vec{r},t) = \text{electron mass density.}$$

Equation (1.1) is now multiplied by v_{1i} and integrated with respect to $\vec{v}_1$ from $-\infty$ to $+\infty$. We then obtain:

$$\begin{aligned} m\frac{\partial}{\partial t}(nu_i) &+ m\frac{\partial}{\partial r_\alpha}(nu_iu_\alpha) + \frac{\partial}{\partial r_\alpha}P^{\kappa}_{i\alpha}(\vec{r},t) + en\{E_i + \frac{1}{c}[\vec{u}x\vec{H}]_i\} \\ &= 4\pi e^2\int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 d\vec{r}_2 v_{1i}\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{r}_2|)\frac{\partial}{\partial v_{1\alpha}}f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \\ &- 4\pi e^2\int_{-\infty}^{\infty} d\vec{v}_1 d\vec{V} d\vec{R} v_{1i}\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{R}|)\frac{\partial}{\partial v_{1\alpha}}f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t) \end{aligned} \tag{2.3}$$

with

$$P^{\kappa}_{ij}(\vec{r},t) \equiv \int_{-\infty}^{\infty} d\vec{v}\, U_iU_jf_1(\vec{r},\vec{v};t) \tag{2.4}$$

In (2.3) we have put $v_i = u_i+U_i$; $\vec{U}$ = thermal velocity. $P^{\kappa}_{ij}(\vec{r},t)$ is the familiar expression of the pressure tensor due to the kinetic motion of the molecules. Using (2.1) Eq. (2.3) may be written in the form:

$$\begin{aligned} \rho\frac{Du_i}{Dt} &= -\frac{\partial}{\partial r_\alpha}P^{\kappa}_{i\alpha} - en\{E_i + \frac{1}{c}[\vec{u}x\vec{H}]_i\} \\ &+ 4\pi e^2\int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 d\vec{r}_2 v_{1i}\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{r}_2|)\frac{\partial}{\partial v_{1\alpha}}f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \\ &- 4\pi e^2\int_{-\infty}^{\infty} d\vec{v}_1 d\vec{V} d\vec{R} v_{1i}\frac{\partial}{\partial r_{1\alpha}}\phi(|\vec{r}_1-\vec{R}|)\frac{\partial}{\partial v_{1\alpha}}f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t) \end{aligned} \tag{2.5}$$

with

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_\alpha\frac{\partial}{\partial r_\alpha}$$

Now it can be shown that

$$4\pi e^2 \int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 d\vec{r}_2 v_{1i} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) \frac{\partial}{\partial v_{1\alpha}} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) = - \frac{\partial P^{\phi_1}_{i\alpha}}{\partial r_\alpha} \quad (2.6)$$

where $P^{\phi_1}_{ij}(\vec{r},t)$ is defined by

$$P^{\phi_1}_{ij}(\vec{r},t) = 2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i \rho_j}{\rho^2} \frac{d}{d\rho} \phi(|\vec{r}_1-\vec{r}_2|) \int_0^{\rho} d\lambda \int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \quad (2.7)$$

Also,

$$-4\pi e^2 \int_{-\infty}^{\infty} d\vec{v}_1 d\vec{V} d\vec{R} v_{1i} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{R}|) \frac{\partial}{\partial v_{1\alpha}} f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t) = - \frac{\partial P^{\phi_2}_{i\alpha}}{\partial r_\alpha} \quad (2.8)$$

where $P^{\phi_2}_{ij}(\vec{r},t)$ is defined by

$$P^{\phi_2}_{ij}(\vec{r},t) = -2\pi e^2 \int_0^{\infty} d\vec{\rho}\,' \frac{\rho_i' \rho_j'}{\rho'^2} \frac{d}{d\rho'} \phi(|\vec{r}_1-\vec{R}|) \int_0^{\rho'} d\lambda' \int_{-\infty}^{\infty} d\vec{v}_1 d\vec{V} f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t) \quad (2.9)$$

with $\vec{\rho} \equiv \vec{r}_2-\vec{r}_1$ and λ is defined such that

$$\begin{cases} \vec{r}_1 = \vec{r} + (\lambda-\rho) \dfrac{\vec{\rho}}{\rho} \\ \vec{r}_2 = \vec{r} + \lambda \dfrac{\vec{\rho}}{\rho} \end{cases}$$

Furthermore $\vec{\rho}\,' \equiv \vec{R}-\vec{r}_1$ and λ' is defined such that

$$\begin{cases} \vec{r}_1 = \vec{r} + (\lambda'-\rho) \dfrac{\vec{\rho}}{\rho} \\ \vec{R} = \vec{r} + \lambda \dfrac{\vec{\rho}}{\rho} \end{cases}$$

$P^{\phi_1}_{ij}(\vec{r},t)$ is the stress tensor due to the potential between pairs of electrons and $P^{\phi_2}_{ij}(\vec{r},t)$ is the stress tensor due to the potential between pairs of electrons and ions.

Thus Eq. (2.5) becomes:

$$\rho \frac{Du_i}{Dt} = - \frac{\partial}{\partial r_\alpha} P_{i\alpha} - en\{E_i + \frac{1}{c} [\vec{u}x\vec{H}]_i\} \tag{2.10}$$

where

$$P_{ij}(\vec{r},t) = P^{\kappa}_{ij}(\vec{r},t) + P^{\phi_1}_{ij}(\vec{r},t) + P^{\phi_2}_{ij}(\vec{r},t)$$

Hence $P_{ij}(\vec{r},t)$ is the total stress tensor and it is symmetric.

<u>Proof of Equation (2.6)</u>: (See S. T. Choh and G. E. Uhlenbeck, "The Kinetic Theory of Phenomena in Dense Gases," U.M.R.I. Report, Feb. 1958).

We define the pair density distribution:

$$n_2(\vec{r}_1,\vec{r}_2;t) = \int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$$

$n_2(\vec{r}_1,\vec{r}_2;t)$ is a symmetric function of the two points $\vec{r}_1$ and $\vec{r}_2$.

$$\int_{-\infty}^{\infty} d\vec{v}_1 d\vec{v}_2 d\vec{r}_2 v_{1i} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) \frac{\partial}{\partial v_{1\alpha}} f_2 = \int_0^{\infty} d\vec{\rho} \frac{\rho_i}{\rho} \frac{d\phi}{d\rho} n_2(\vec{r}_1,\vec{r}_2;t)$$

where $\vec{\rho} \equiv \vec{r}_2-\vec{r}_1$. Now:

$$\frac{\partial P^{\phi_1}_{i\alpha}}{\partial r_\alpha} = 2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i\rho_\alpha}{\rho^2} \frac{d\phi}{d\rho} \int_0^{\rho} d\lambda \frac{\partial}{\partial r_\alpha} n_2(\vec{r} + (\lambda-\rho) \frac{\vec{\rho}}{\rho}, r + \lambda \frac{\vec{\rho}}{\rho}; t)$$

But

$$\frac{\partial n_2}{\partial r_i} = \frac{\rho}{\rho_i} \frac{\partial n_2}{\partial \lambda}$$

Therefore,

$$\frac{\partial P^{\phi_1}_{i\alpha}}{\partial r_\alpha} = 2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i}{\rho} \frac{d\phi}{d\rho} \{n_2(\vec{r},\vec{r}+\vec{\rho};t) - n_2(\vec{r}-\vec{\rho},\vec{r};t)\}$$

$$= 4\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i}{\rho} \frac{d\phi}{d\rho} n_2(\vec{r}_1,\vec{r}_2;t)$$

because of symmetry of n_2. Q.E.D.

Equation (2.8) can be derived in a similar manner. Equation (2.10) is the equation for the conservation of momentum. To get the equation for the transport of the kinetic energy density we multiply (1.1) by $\frac{1}{2} m v_1^2$ and integrate over $\vec{v}_1$. After rearranging terms and making use of the equation (2.10) scalarly multiplied by $\vec{u}$ we obtain:

$$\rho \left(\frac{D}{Dt} \frac{\epsilon^{\kappa}}{\rho}\right) + \frac{\partial}{\partial r_\alpha} q_\alpha^{\kappa} + P_{\alpha\beta} D_{\alpha\beta} - \frac{\partial}{\partial r_\alpha} (u_\alpha P_{\alpha\beta}^{\phi})$$

$$= 4\pi e^2 \int_{-\infty}^{\infty} d\vec{R} d\vec{V} d\vec{v}_1 v_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1 - \vec{R}|) f_2(\vec{r}_1, \vec{R}, \vec{v}_1, \vec{V}; t) \qquad (2.11)$$

$$- 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 v_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1 - \vec{r}_2|) f_2(\vec{r}_1, \vec{r}_2, \vec{v}_1, \vec{v}_2; t)$$

where

$$q_i^{\kappa}(\vec{r}, t) \equiv \frac{m}{2} \int U^2 U_i f_1(\vec{r}, \vec{v}; t) d\vec{v}$$ = the kinetic part of the heat current density.

$$\epsilon^{\kappa}(\vec{r}, t) \equiv \frac{m}{2} \int_{-\infty}^{\infty} d\vec{v}_1 U_1^2 f_1(\vec{r}_1, \vec{v}_1; t)$$ = thermal energy density.

$$D_{ij}(\vec{r}, t) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$ = deformation tensor. (2.12)

$$P_{ij}^{\phi}(\vec{r}, t) = P_{ij}^{\phi_1}(\vec{r}, t) + P_{ij}^{\phi_2}(\vec{r}, t).$$

Next we derive the transport equation for the potential energy density. For this we need to use Eqs. (1.2) and (1.3) of the B-B-G-K-Y hierarchy as described in Section 1. Multiplying Eq. (1.2) by $4\pi e^2 \phi(|\vec{r}_1 - \vec{r}_2|)$ and integrating over $\vec{r}_2, \vec{v}_1$, and $\vec{v}_2$ we obtain:

$$\frac{\partial}{\partial t}\epsilon^{\phi} + \frac{\partial}{\partial r_\alpha}(u_\alpha \epsilon^{\phi}) + \frac{\partial}{\partial r_\alpha} q_\alpha^{\phi}$$

$$= 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_1 d\vec{v}_2 v_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$$

$$+ 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_1 d\vec{v}_2 v_{2\alpha} \frac{\partial}{\partial r_{2\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \qquad (2.13)$$

where

$$\epsilon^{\phi}(\vec{r},t) = 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_1 d\vec{v}_2 \phi(|\vec{r}_1-\vec{r}_2|) f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$$

= the potential energy density resulting from the potential between pairs of electrons.

$$q_i^{\phi}(\vec{r},t) \equiv 4\pi e^2 \int d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 \phi(|\vec{r}_1-\vec{r}_2|) U_{1i} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)$$

= the electron-electron potential part of the heat current density.

Equation (2.13) is the transport equation for the potential energy density resulting from the potential between pairs of electrons. In a similarly way we obtain the transport equation for the potential energy density resulting from the potential between single electrons and single ions. For this we multiply Eq. (1.3) from Section 1 by $-4\pi e^2 \phi(|\vec{r}-\vec{R}|)$ and integrate over $\vec{v}$, $\vec{R}$, $\vec{V}$. The result is:

$$\frac{\partial}{\partial t}\epsilon^{\Phi} + \frac{\partial}{\partial r_\alpha}(u_\alpha \epsilon^{\Phi}) + \frac{\partial}{\partial r_\alpha} q_\alpha^{\Phi}$$

$$= -4\pi e^2 \int_{-\infty}^{\infty} d\vec{R} d\vec{V} d\vec{v} v_\alpha \frac{\partial}{\partial r_\alpha} \phi(|\vec{r}-\vec{R}|) f_2(\vec{r},\vec{R},\vec{v},\vec{V};t) \qquad (2.14)$$

$$- 4\pi e^2 \int_{-\infty}^{\infty} d\vec{R} d\vec{V} d\vec{v} V_\alpha \frac{\partial}{\partial R_\alpha} \phi(|\vec{r}-\vec{R}|) f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)$$

where

$$\epsilon^{\Phi}(\vec{r},t) = -4\pi e^2 \int_{-\infty}^{\infty} d\vec{R}d\vec{V}d\vec{v}\ \phi(|\vec{r}-\vec{R}|)\, f_2(\vec{r},\vec{v},\vec{R},\vec{V};t)$$

= the potential energy density resulting from the potential between pairs of electrons and ions.

$$q_i^{\Phi}(\vec{r},t) = -4\pi e^2 \int_{-\infty}^{\infty} d\vec{R}d\vec{V}d\vec{v}U_i\ \phi(|\vec{r}-\vec{R}|)\, f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)$$

= the electron-ion potential part of the heat current density.

Equation (2.14) is the transport equation for the potential energy density resulting from the potential between pairs of electrons and ions. The transport equation for the total electron potential energy is obtained by adding (2.13) and (2.14). After rearranging terms and making use of the equation for the conservation of mass this may be written as:

$$\begin{aligned}
&\rho \frac{D}{Dt}\left(\frac{\epsilon^p}{\rho}\right) + \frac{\partial}{\partial r_\alpha} q_\alpha^p \\
&= 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 v_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} \phi(|\vec{r}_1-\vec{r}_2|)\, f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \\
&+ 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 v_{2\alpha} \frac{\partial}{\partial r_{2\alpha}} \phi(|\vec{r}_1-\vec{r}_2|)\, f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \qquad (2.15) \\
&- 4\pi e^2 \int_{-\infty}^{\infty} d\vec{R}d\vec{V}d\vec{v} v_\alpha \frac{\partial}{\partial r_\alpha} \phi(|\vec{r}-\vec{R}|)\, f_2(\vec{r},\vec{R},\vec{v},\vec{V};t) \\
&- 4\pi e^2 \int_{-\infty}^{\infty} d\vec{R}d\vec{V}d\vec{v} V_\alpha \frac{\partial}{\partial R_\alpha} \phi(|\vec{r}-\vec{R}|)\, f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)
\end{aligned}$$

Combining (2.11) and (2.15) we obtain the transport equation for the total electron energy density. This may be written as:

$$\rho \frac{D}{Dt}\left(\frac{\epsilon}{\rho}\right) + \frac{\partial}{\partial r_\alpha} q_\alpha = - P_{\alpha\beta} D_{\alpha\beta} + \frac{\partial}{\partial r_\alpha}(u_\beta P^p_{\alpha\beta})$$

$$+ 4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 v_{2\alpha} \frac{\partial}{\partial r_{2\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) \qquad (2.16)$$

$$- 4\pi e^2 \int_{-\infty}^{\infty} d\vec{R} d\vec{V} d\vec{v} V_\alpha \frac{\partial}{\partial R_\alpha} \phi(|\vec{r}-\vec{R}|) f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)$$

where

$$\epsilon(\vec{r},t) = \epsilon^{\kappa}(\vec{r},t) + \epsilon^{p}(\vec{r},t)$$

$$= \text{the electron energy density.}$$

$$q_i(\vec{r},t) = q_i^{\kappa}(\vec{r},t) + q_i^{p}(\vec{r},t)$$

$$= \text{the electron heat current density.}$$

Now,

$$4\pi e^2 \int_{-\infty}^{\infty} d\vec{r}_2 d\vec{v}_2 d\vec{v}_1 v_{2\alpha} \frac{\partial}{\partial r_{2\alpha}} \phi(|\vec{r}_1-\vec{r}_2|) f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t) = \frac{\partial J_\alpha}{\partial r_\alpha}$$

if we define

$$J_i(\vec{r},t) \equiv -2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i \rho_\alpha}{\rho^2} \frac{d\phi}{d\rho} \int_0^{\rho} d\lambda \int_{-\infty}^{\infty} v_{2\alpha} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)\, d\vec{v}_1 d\vec{v}_2$$

Moreover,

$$J_i = -2\pi e^2 u_\alpha \int_0^{\infty} d\vec{\rho} \frac{\rho_i \rho_\alpha}{\rho^2} \frac{d\phi}{d\rho} \int_0^{\rho} d\lambda \int_{-\infty}^{\infty} f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)\, d\vec{v}_1 d\vec{v}_2$$

$$-2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i \rho_\alpha}{\rho^2} \frac{d\phi}{d\rho} \int_0^{\rho} d\lambda \int_{-\infty}^{\infty} U_\alpha f_2(\vec{r}_1,\vec{r}_2,\vec{v}_1,\vec{v}_2;t)\, d\vec{v}_1 d\vec{v}_2$$

$$\therefore \quad \frac{\partial J_\alpha}{\partial r_\alpha} = - \frac{\partial}{\partial r_\alpha}(u_\beta P^{\phi_1}_{\alpha\beta}) - \frac{\partial}{\partial r_\alpha} \mathcal{J}_\alpha \qquad (2.17)$$

where

$$\mathcal{J}_i(\vec{r},t) \equiv 2\pi e^2 \int_0^{\infty} d\vec{\rho} \frac{\rho_i \rho_\alpha}{\rho^2} \frac{d\phi}{d\rho} \int_0^{\rho} d\lambda \int_{-\infty}^{\infty} U_\alpha f_2 d\vec{v}_1 d\vec{v}_2 \quad .$$

Similarly,

$$-4\pi e^2 \int_{-\infty}^{\infty} d\vec{R}\,d\vec{V}\,d\vec{v}\,V_\alpha \frac{\partial}{\partial R_\alpha}\phi(|\vec{r}-\vec{R}|)\, f_2(\vec{r},\vec{R},\vec{v},\vec{V};t) \quad = \quad \frac{\partial K_\alpha}{\partial r_\alpha}$$

if we define

$$\mathcal{K}_i(\vec{r},t) \quad \equiv \quad -2\pi e^2 \int_0^{\infty} d\vec{\rho}\,' \,\frac{\rho_i'\rho_\alpha'}{\rho'^2}\frac{d\phi}{d\rho'} \int_0^{\rho'} d\lambda' \int_{-\infty}^{\infty} V_\alpha f_2(\vec{r}_1,\vec{R},\vec{v}_1,\vec{V};t)\, d\vec{v}\, d\vec{V}$$

Again, letting $V_i = v_i + \mathcal{V}_i$, we have

$$\mathcal{K}_i \quad = \quad -2\pi e^2 v_\alpha \int_0^{\infty} d\vec{\rho}\,' \,\frac{\rho_i'\rho_\alpha'}{\rho'^2}\frac{d\phi}{d\rho'} \int_0^{\rho'} d\lambda' \int_{-\infty}^{\infty} f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)\, d\vec{v}\, d\vec{V}$$

$$- \, 2\pi e^2 \int_0^{\infty} d\vec{\rho}\,' \,\frac{\rho_i'\rho_\alpha'}{\rho'^2}\frac{d\phi}{d\rho'} \int_0^{\rho'} d\lambda' \int_{-\infty}^{\infty} \mathcal{V}_\alpha f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)\, d\vec{v}\, d\vec{V}$$

$$\therefore \quad \frac{\partial \mathcal{K}_\alpha}{\partial r_\alpha} \quad = \quad - \frac{\partial}{\partial r_\alpha}(v_\beta P^{\phi_2}_{\alpha\beta}) \; - \; \frac{\partial \mathcal{H}_\alpha}{\partial r_\alpha} \qquad (2.18)$$

where

$$\mathcal{H}_i(\vec{r},t) \equiv 2\pi e^2 \int_0^{\infty} d\vec{\rho}\,' \,\frac{\rho_i'\rho_\alpha'}{\rho'^2}\frac{d\phi}{d\rho'} \int_0^{\rho'} d\lambda' \int_{-\infty}^{\infty} \mathcal{V}_\alpha f_2(\vec{r},\vec{R},\vec{v},\vec{V};t)\, d\vec{v}\, d\vec{V} \, .$$

Hence using (2.17) and (2.18) we can write (2.16) in the form:

$$\rho \frac{D}{Dt}\left(\frac{\epsilon}{\rho}\right) + \frac{\partial}{\partial r_\alpha} q_\alpha \quad = \quad -P_{\alpha\beta}D_{\alpha\beta} - \frac{\partial}{\partial r_\alpha}\{(u_\beta - v_\beta)P^{\phi_2}_{\alpha\beta}\} - \frac{\partial}{\partial r_\alpha}(\mathcal{J}_\alpha + \mathcal{H}_\alpha) \qquad (2.19)$$

Hence, if we define

$$Q_i(\vec{r},t) \quad \equiv \quad q_i(\vec{r},t) + \quad \ldots \quad ,t)$$

= the heat current density for the electrons.

Then the transport equation for the total electron density may finally be written in the form:

$$\rho \frac{D}{Dt}\left(\frac{\epsilon}{\rho}\right) + \frac{\partial}{\partial r_\alpha} Q_\alpha = - P_{\alpha\beta} D_{\alpha\beta} + (u_\beta - v_\beta) \frac{\partial}{\partial r_\alpha} P^{\phi_2}_{\alpha\beta}. \qquad (2.20)$$

Note that the heat current density vector $Q_i(\vec{r},t)$ depends only on the thermal velocities of the electrons and ions.

In a completely similar way we can obtain the conservation laws analogous to Eqs. (2.2), (2.10), and (2.20) for the ions. Hence, in summary, the conservation laws for electrons and ions may be written in the form:

$$\frac{\partial \rho^\pm}{\partial t} + \frac{\partial}{\partial r_\alpha} (\rho^\pm u^\pm_\alpha) = 0 \qquad (2.21)$$

$$\rho^\pm \frac{Du^\pm_i}{Dt} = - \frac{\partial P^\pm_{i\alpha}}{\partial r_\alpha} \pm en^\pm \{E_i + \frac{1}{c} [\vec{u}^\pm x \vec{H}]_i\} \qquad (2.22)$$

$$\rho^\pm \frac{D}{Dt}\left(\frac{\epsilon^\pm}{\rho^\pm}\right) + \frac{\partial}{\partial r_\alpha} Q^\pm_\alpha = -P^\pm_{\alpha\beta} D^\pm_{\alpha\beta} \pm (u^-_\beta - u^+_\beta) \frac{\partial}{\partial r_\alpha} (P^{\phi_2}_{\alpha\beta})^\pm . \qquad (2.23)$$

where the superscript plus refers to the ions and minus to the electrons.

By summing up over the electrons and ions in (2.21), (2.22), and (2.23) we can write the conservation laws for a dilute plasma considered as a mixture of electrons and ions. We would then obtain equations analogous to those derived by Chandrasekhar for the simpler case where the collisionless Landau-Vlasov equations were used as the starting point.[4]

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