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"A DISCUSSION OF THEORETICAL ANALYSES OF PROBE-INDUCED THERMAL FLUX PERTURBATIONS"

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I. INTRODUCTION

The problem to be formulated and discussed in some detail herein is quite narrowly specialized, nevertheless it is of sufficient importance and complexity to have warranted considerable experimental and theoretical attention in the past several years. In fact, this attention is still in being because the practical importance remains forceful and the experimental versus theoretical state of affairs apparently remains somewhat controversial. It will be the purpose of this review to attempt a clear statement of the problem and a careful comparison of some of the various calculational procedures that have been developed in the endeavor to solve the problem theoretically.

The over-all task is simply stated, i.e., the interpretation of a counted activity in a probe of some size, shape, and composition in terms of the neutron density that was present in the medium at the point of insertion of the probe prior to its insertion. But only a portion of this task will concern us here. We will assume that somehow the measured activity has been satisfactorily related to a steady state neutron absorption rate in the probe, and will therefore regard this latter quantity as if it were actually the observed quantity. That is, we will regard A as the measured quantity where A is defined by

$$A = \int_{\text{Probe}} d^3r \int_{E=0}^{\infty} dE \int_{\underline{\Omega}} d\Omega \Sigma_a^D(E)\, vN\,(\underline{r},E,\underline{\Omega}) \,. \qquad (1)$$

In this expression, $\Sigma_a^D(E)$ is the energy dependent macroscopic absorption cross-section appropriate to the probe, v is the speed of neutrons of energy E, and $N(\underline{r},E,\underline{\Omega})d^3rdEd\Omega$ is the expected number of neutrons in d^3r about $\underline{r}$ having energies E in dE going in directions $\underline{\Omega}$ in $d\Omega$. We presume that the distribution of nuclei in the probe is uniform so that Σ_a^D depends only upon energy, and that a steady state absorption rate is measured. Evidently all that remains is to determine the density as a function of its arguments and as a functional of the unperturbed density.

Usually, of course, A is measured for the explicit purpose of determining the "thermal" neutron density -- the density of neutrons with energies less than some preassigned E_T not too large compared to the thermal energy of the ambient medium. How this is to be accomplished will not concern us -- we will simply assume that a quantity A_T is measured where A_T is defined by

$$A_T = \int_V d^3r \int_0^{E_T} dE \int_{\underline{\Omega}} d\Omega\, \Sigma_a^D(E)\, vN\,(\underline{r},E,\underline{\Omega}) \quad . \qquad (2)$$

Here we have introduced the symbol V to represent the volume of the probe.

The calculation of the density proceeds via the familiar balance relation

$$\begin{aligned} &v\underline{\Omega}\cdot\underline{\nabla}N(\underline{r},E,\underline{\Omega}) + \Sigma_t(\underline{r},E)\, vN\,(\underline{r},E,\underline{\Omega}) = S(\underline{r},E,\underline{\Omega}) \\ &+ \int_{E'=0}^{\infty} dE' \int_{\underline{\Omega}'} d\Omega'\Sigma_s(\underline{r},E')\, v'N(\underline{r},E',\underline{\Omega}')\, F\,(E',\underline{\Omega}';E,\underline{\Omega}|\underline{r}), \end{aligned} \qquad (3)$$

plus the boundary condition that the solution to Equation (3) shall approach the unperturbed density at points sufficiently removed from the probe. The balance relation (3) is presumably quite adequate for our purposes, i.e., though it is not exact, it is probably as accurate as the experiments with which its solutions are to be compared. Note that the parameters Σ_t, Σ_s and F are explicit functions of position -- if only because of the material inhomogeneities introduced by the insertion of the probe.

An alternative and, in the context of the problem that concerns us here, very useful form of the balance relation (3) is its integral equivalent,

$$
\begin{aligned}
vN(\underline{r},E,\underline{\Omega}) &= vN\,(\underline{r}-s\,\underline{\Sigma},E,\underline{\Omega})\; e\left[-\int_0^s \Sigma_t(\underline{r}-y\underline{\Omega})ey\right] \\
&+ \int_0 S(\underline{r}-y\underline{\Omega},E,\underline{\Omega})\,dy\; e\left[-\int_0^y \Sigma_t(E,\underline{r}-z\underline{\Omega})dz\right] \\
&+ \int_0^s dy \int_0^\infty dE' \int_{\underline{\Omega}'} d\Omega'\Sigma_s(E',\underline{r}-y\underline{\Omega})\; v'N\,(r-y\underline{\Omega},E',\underline{\Omega}') \\
&\quad (x)\; F\,(E',\underline{\Omega}';E,\underline{\Omega})\; e\left[-\int_0^y \Sigma_t(E,\underline{r}-z\underline{\Omega})\,dz\right]
\end{aligned}
\tag{4}
$$

In this expression the space dependence of most of the parameters has been indicated explicitly. As indicated above, such dependence would be expected due to variation of the atomic densities as well as to temperature non-uniformities in otherwise homogeneous media. The analyses of probe-induced perturbations usually proceeds on the tacit assumption that atomic temperature variations are not present; an

assumption -- tacit or otherwise -- that must be distinguished from the further assumption that the mean speed of the "thermal" neutrons is also space independent. The latter assumption will be discussed in some detail later, the former will be explicitly accepted throughout this summary.

II. THE INFINITE SLAB APPROXIMATION TO THE FOIL

Many of the earliest attempts to calculate the relation between a measured activity in a foil and the pre-probe neutron density proceeded on the assumption that, since the radius of the foil was large compared to its thickness, edge effects arising from the finiteness of the radius might be ignored altogether -- at least to first approximation. Thus, the calculational model supposed an infinite slab of the material to be activated to be inserted into the medium in which the neutron density is to be measured. To simplify matters still further, it is supposed that there are no sources of neutrons with energies less than E_T in the slab (i.e., no neutron slowing down in the slab) and that scattering in the slab is negligible compared to absorption -- again, presumably in the sense of a first approximation. In view of these assumptions, the absorption rate in the slab is nicely calculated in terms of the density at the surface employing Equation (4).

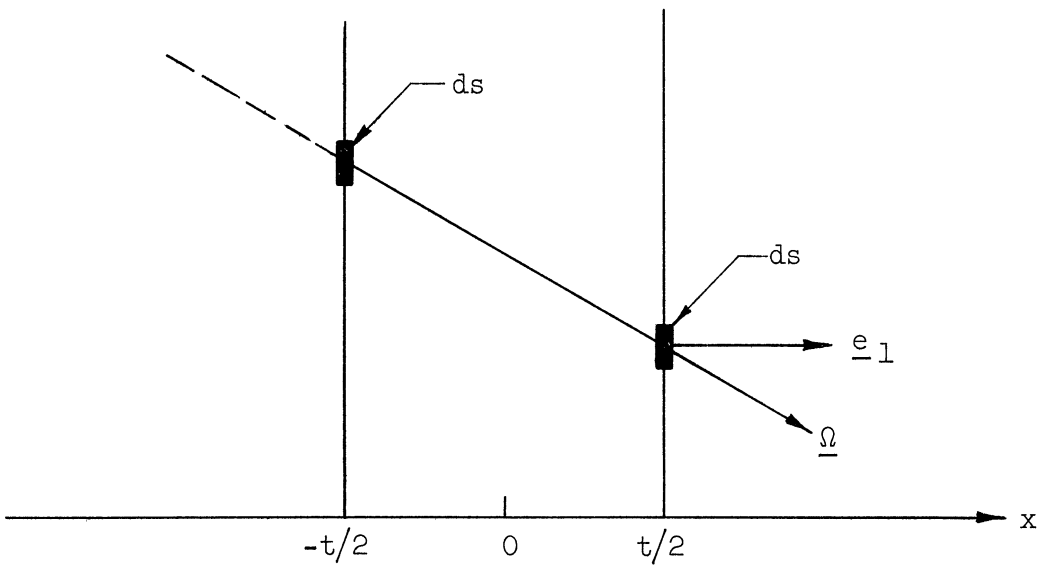

Figure 1.

The number of neutrons leaving the slab per sec. through the area ds at $x = t/2$ going in directions $\underline{\Omega}$ in $d\Omega$ having energies E in dE is given by

$$(\underline{e}_1 \cdot \underline{\Omega})\ vN\ (\frac{t}{2}, \underline{\Omega}, E)\ d\Omega dE ds \quad . \tag{5}$$

But according to Equation (4) this quantity is also equal to

$$\mu vN\ (-\frac{t}{2}, \underline{\Omega}, E)\ d\Omega dE ds\ \ e^{-\Sigma_a^D(E)t/\mu} \quad , \tag{6}$$

where we have introduced the symbol $\mu = \underline{e}_1 \cdot \underline{\Omega}$. This equality obtains, of course, because within the slab we have set S and Σ_s^D equal to zero, and have presumed the slab homogenous. The number per second entering through ds at $x = -t/2$ is given by Equation (5) with the density evaluated at $(-t/2)$. Thus, the difference between the number entering and those leaving, which represents the number absorbed along the trajectory indicated in Figure 1, is given by

$$\mu vN\ (-\frac{t}{2}, \underline{\Omega}, E)\ d\Omega dE ds\ \left[1 - e^{-t\Sigma_a^D(E)/\mu}\right] . \tag{7}$$

Thus, the absorption rate per cm^2 resulting from neutrons impinging upon the slab from both sides is,

$$\begin{aligned} A_T/ds = & \int_0^{E_T} dE \int_{\phi=0}^{2\pi} d\phi \int_{\mu=0}^{1} d\mu\mu\ vN\ (-\frac{t}{2}, \mu, E) \left[1 - e^{-t\Sigma_a^D(E)/\mu}\right] \\ & - \int_0^{E_T} dE \int_{\phi=0}^{2\pi} d\phi \int_{\mu=-1}^{0} d\mu\mu\ vN\ (\frac{t}{2}, \mu, E) \left[1 - e^{t\Sigma_a^D(E)/\mu}\right] . \end{aligned} \tag{8}$$

Here, we have employed ϕ to represent an azimuthal angle measured about the x-axis and A_T to represent the absorption rate in the portion of the slab whose volume is tds. If now we assume that the density was uniform before the insertion of the probe, we may assert that $N(-\frac{t}{2},\mu,E) = N(\frac{t}{2},-\mu,E)$ so that

$$A_T/ds = 4\pi \int_0^{E_T}\int_0^1 d\mu\mu\, vN(\tfrac{t}{2},-\mu,E)\left[1 - e^{-t\Sigma_a^D(E)/\mu}\right]. \qquad (9)$$

Evidently, to proceed further it is necessary to calculate the surface density in some self-consistent way. So far, only limited progress has been made with respect to this problem. Nevertheless, the formula (9) is of considerable importance to us here, as it forms the basis for diverse further approximated calculations of the effects presently under consideration.

A. <u>The Diffusion Approximation</u>

A calculation of the surface density employing "one-speed" diffusion theory was presented by Bothe[1] -- if not in detail, at least in essence. The density in formula (9) is expanded in Legendre polynomials according to

$$\begin{aligned} N(\tfrac{t}{2},-\mu,E) &= \sum_\ell \frac{2\ell+1}{4\pi} N_\ell(\tfrac{t}{2},E)\, P_\ell(-\mu) \\ &= \sum_\ell (-1)^\ell \frac{2\ell+1}{2\pi} N_\ell(\tfrac{t}{2},E)\, P_\ell(\mu)\,. \end{aligned} \qquad (10)$$

Introducing the symbol $\bar{A} = A_T/ds$, displaying the energy dependent absorption cross-section of the probe as

$$\Sigma_a^D(E) = a(E)\,\Sigma_{ao}^D \quad , \tag{11}$$

and letting $\tau_o = t\,\Sigma_{ao}^D$ one finds that Equation (9) becomes

$$\bar{A} = \sum_\ell (-1)^\ell (2\ell+1) \int_0^{E_T} dE\, N_\ell(\tfrac{t}{2},E)\, v \int_0^1 d\mu\mu\, P_\ell(\mu) \left[1 - e^{-\tau_o a(E)/\mu} \right] \quad . \tag{12}$$

The quantity Σ_{ao}^D defined in Equation (11) is the energy dependent cross-section evaluated at some convenient energy, E_o. Thus, $a(E)$ is a dimensionless function carrying the energy dependence of $\Sigma_a^D(E)$. In particular, for a one-over-vee cross-section,

$$a(E) = \sqrt{E_o/E} \quad . \tag{13}$$

Finally, we write (12) as

$$\bar{A} = \sum_\ell (-1)^\ell (2\ell + 1) \int_0^{E_T} dE N_\ell(\tfrac{t}{2},E) v \chi_\ell(\tau_o,E), \tag{14}$$

where we have defined

$$\chi_\ell(\tau_o,E) = \int_0^1 d\mu\mu\, P_\ell(\mu) \left[1 - e^{-\tau_o a(E)/\mu} \right] \tag{15}$$

As it is our intention in this section to complete the calculation of $\bar{A}$ by an application of diffusion theory (the P_1- approximation) in the external medium, it is evident that only N_o and N_1 will be accessible to our analysis. Thus, we approximate

$$\bar{A} \simeq \int_0^{E_T} dE\, N_o(\tfrac{t}{2}, E)\, v\chi_o(\tau_o, E) - 3\int_0^{E_T} dE\, N_1(\tfrac{t}{2}, E)\, v\chi_1(\tau_o, E) \,, \tag{16}$$

and turn immediately to the task of estimating N_o and N_1.

The starting point here is, of course, the transport Equation (3). However, the familiar detail of the deduction of the pair of coupled equations which approximately describe the functions $N_o(x,E)$ and $N_1(x,E)$ from the transport equation is hardly worth our attention at this point. This is so on at least two counts: firstly, the P_1-equations thereby deduced are wholly intractible analytically and practically so numerically; and secondly, such attention to detail is apt to suggest that the formula (16) for the absorption rate in the infinite foil is worth more than is warranted. The only significant point that would emerge from such an analysis would be the practical necessity of assuming that the neutron density is separable (even in and in the vicinity of the probe) in the sense that $N(x,E,\mu) = n(x,\mu)M(E)$, where $M(E)$ is the energy spectrum of the density normalized to unity. Given this assumption of separability (no spectrum hadrening?), it follows that

$$N_o(x,E) = n_o(x)\, M\,(E)$$

and

$$N_1(x,E) = n_1(x)\, M\,(E) \quad .$$

Furthermore, in this instance, Equation (16) simplifies to

$$A \simeq n_o(\tfrac{t}{2}) \int_0^{E_T} dE\, v\, M(E)\, X_o(\tau_o,E) - 3n_1(\tfrac{t}{2}) \int_0^{E_T} dE\, v\, M(E)\, X_1(\tau_o,E) \,. \tag{17}$$

Since X_o and X_1 are known to us, and M is to be guessed (a typical guess being the Maxwell-Boltzmann spectrum), all that remains is to calculate the "one-speed" flux, ϕ_o and net current, ϕ_1 at the surface of the slab and relate them to the corresponding densities, n_o and n_1. They are presumably described by the equations,

$$D \frac{d^2\phi_o}{dx^2} - \Sigma_a\phi_o + \Sigma_a\bar{\phi}_o = 0, \quad \text{and}$$
$$\phi_1 = - D \frac{d\phi_o}{dx} \quad \text{for} \quad \frac{t}{2} \leq x < \infty \,, \tag{18}$$

and the boundary conditions,

$$\phi_o(x) \rightarrow \bar{\phi}_o \quad \text{as} \quad x \rightarrow \infty \,,$$
$$\phi_1(\tfrac{t}{2}) = - \bar{A}/2 \,. \tag{19}$$

In Equation (18) we have introduced $\bar{\phi}_o$ to represent the spatially independent flux that existed in the medium prior to the insertion of the probe. The cross-sections appearing in (18) are those for the medium (the cross-sections for the probe will always carry the superscript, D) and must be interpreted as the averages of the energy dependent medium cross-sections taken with respect to the distribution, $v\,M(E)$.

Solutions to the Equations (18) subject to the boundary conditions (19) are readily obtained,

$$\phi_o(x) = \bar{\phi}_o - \frac{\bar{A}L}{2D} e^{-(x - \frac{t}{2})/L} \tag{20}$$

so that at the surface of the slab

$$\phi_o(\tfrac{t}{2}) = \bar{\phi}_o - \frac{\bar{A}L}{2D} \quad . \tag{21}$$

Of course, $\phi_1(\frac{t}{2})$ is given by the boundary condition, (19). As usual, the symbol L appearing in Equations (20) and (21) is the diffusion length, i.e., $L^2 = D/\Sigma_a$.

The fluxes and currents appearing in (18), (19), (20), and (21) have been defined as the integrals of the energy dependent fluxes and currents over the energy range from 0 to E_T, i.e.,

$$\phi_o(x) = \int_0^{E_T} dE\ \phi_o(x,E), \tag{22a}$$

$$\phi_1(x) = \int_0^{E_T} dE\ \phi_1(x,E). \tag{22b}$$

Because of the assumed separability of the density under all circumstances, it follows that

$$\begin{aligned} \phi_o(x) &= v_T n_o(x), \\ \phi_1(x) &= v_T n_1(x), \end{aligned} \tag{23}$$

where

$$v_T = \int_0^{E_T} dE\ v\ M(E) \Big/ \int_0^{E_T} dE\ M(E) \ . \tag{24}$$

Thus, the relevant densities in this problem are

$$n_o(\frac{t}{2}) = \bar{n}_o - \frac{\bar{A}L}{2Dv_T} \quad , \tag{25a}$$

$$n_1(\frac{t}{2}) = - \bar{A}/2v_T \quad . \tag{25b}$$

Substituting these results in Equation (17), we obtain as our estimate of the absorption rate per unit area,

$$\bar{A} = \frac{\bar{n}_o v_T \zeta_o}{1 + \frac{L}{2D}\zeta_o - \frac{3}{2}\zeta_1} \quad , \tag{26}$$

where we have defined

$$\zeta_\ell(\tau_o) = \int_0^{E_T} dE \frac{v}{v_T} M(E) \chi_\ell(\tau_o', E) \quad . \tag{27}$$

Considering the approximations involved in its derivation, the result embodied in formula (26) is probably worth no more than brief, qualitative consideration. In the first place, the result is separated nicely, though somewhat deceptively, into a "self-protection" factor, ζ_o, which measures the mean depression of the flux in the probe with respect to the surface flux, and a "flux depression" factor, $[1 + \frac{L}{2D}\zeta_o - \frac{3}{2}\zeta_1]^{-1}$, which measures the depression of the surface flux relative to the original, unperturbed flux. This factorization is deceptive because it is a direct consequence of the approximations that led to it. Strictly speaking, no such separation of "self-protection" and "flux depression" is possible. In the second place, the result is interesting because of the implication of the very different energy averaging of the cross-sections

required in the probe and in the medium when the problem is viewed from the present vantage point. Some of the qualitative significance of this difference has been discussed in detail by Hanna[2] and Tittle[3], and thus will not be dwelt upon here. However, we shall return to this issue qualitatively again in the next section.

B. A Transport Calculation of the Surface Density

A somewhat different method for estimating the absorptions per second in the infinite slab was proposed by Skyrme[4]. Like Bothe, he chose as the starting point for his analysis Equation (9), so that the task that remained to him was the calculation of the surface density. But unlike Bothe (and others[5] who employed diffusion theory or some variant thereof to estimate this density), Skyrme solved a transport problem which presumably provides a more refined estimate of the activity. This section will be devoted to a sketch of some aspects of Skyrme's approach to this problem.

In actuality, Skyrme's analysis did not follow strictly along the lines to be discussed below. Instead it seems that he proceeded somewhat intuitively. Thus, the discussion entered into here will be a paraphrase of the writer's interpretation of Skyrme's ideas into the context of a systematic, successive approximation calculation. The reason for doing this, of course, is to attempt to illuminate some of the approximations that are implicit in Skyrme's analysis. It will be seen that, in the sense of these approximations, the present paraphase yields results somewhat intermediate between those obtained by

Skyrme and those presented by Ritchie and Eldridge[6] in a semi-empirical recasting of Skyrme's formulas.

It is first suggested that one display the surface density as

$$N(\frac{t}{2},-\mu,E) = \frac{1}{4\pi}\ \bar{n}_o M(E) - \eta(\frac{t}{2},-\mu)M(E)\ . \tag{28}$$

Here, as before, separability of the density into a factor dependent only on energy and another dependent upon space and neutron direction of motion is assumed. The quantity $\bar{n}_o$ represents the unperturbed spatial neutron density, which must be taken to be uniform to ensure the validity of Equation (9). It follows, then, that $\eta(\frac{t}{2},-\mu)$ is a density which represents the difference between the unperturbed density before probe insertion and the actual surface density existing when the probe is present. Substituting (28) into (9), one obtains

$$\begin{aligned}
\bar{A} &= \bar{n}_o \int_0^{E_T} dE \int_0^1 d\mu\mu\ v\ M(E) \left[1 - e^{-\tau_o a(E)/\mu}\right] \\
&- 4\pi \int_0^{E_T} dE \int_0^1 \mu d\mu\ v\ \eta(\frac{t}{2},-\mu)M(E) \left[1 - e^{-\tau_o a(E)/\mu}\right] \\
&= \bar{n}_o\ \zeta_o(\tau_o)\ v_T \\
&- \sum_\ell (-1)^\ell (2\ell + 1)\ \eta_\ell(\frac{t}{2})\ v_T \zeta_\ell(\tau_o)\ ,
\end{aligned} \tag{29}$$

where we have again made use of the expansion (10), and the definitions (15),(24), and (27). Evidently, Skyrme's "first" approximation

(representing only the "self-protection" effect) is the same as obtained in Equation (26). The corrections due to "flux depression" in the external medium, which appear in the denominator of (26), are contained in the sum appearing Equation (29) as is obvious since the density depression is η by definition. The estimate of $\bar{A}$ is then approximated by the retention of $\eta_o(\frac{t}{2})$ only in (29), i.e.,

$$\bar{A} \simeq \bar{n}_o v_T \zeta_o - \eta_o(\tfrac{t}{2}) v_T \zeta_o , \qquad (30)$$

so that the remainder of the task is the calculation of $\eta_o(\frac{t}{2})$.

The derivation of an equation for η_o proceeds from the integro-differential form of the balance relation satisfied by the neutron density, Equation (3). The latter equation is integrated over the energy range from 0 to E_T yielding (bearing in mind that there are no true thermal sources),

$$\underline{\Omega} \cdot \underline{\nabla} \int_0^{E_T} dE\, v\, N(\underline{r},\underline{\Omega},E) + \int_0^{E_T} dE\, \Sigma_t(\underline{r},E)\, v\, N(\underline{r},\underline{\Omega},E) = q(\underline{r},\underline{\Omega},E_T)$$

$$+ \int_{E=0}^{E_T} dE \int_{E'=0}^{E_T} dE' \int_{\underline{\Omega}'} d\Omega' \Sigma_s(\underline{r},E') v' N(\underline{r},\underline{\Omega}',E')\, F\,(E',\underline{\Omega}';E,\underline{\Omega}) , \qquad (31)$$

where the slowing-down-density evaluated at E_T entered explicitly through the identification, $q(\underline{r},\underline{\Omega},E_T)$

$$= \int_0^{E_T} dE \int_{E_T}^{\infty} dE' \int_{\underline{\Omega}'} d\Omega' \Sigma_s(\underline{r},E') v' N(\underline{r},E',\underline{\Omega}')\, F\,(E',\underline{\Omega}';E,\underline{\Omega}). \qquad (32)$$

If, now, we again invoke the separability assumption, i.e.,

$$N(\underline{r},\underline{\Omega},E) = n(\underline{r},\underline{\Omega})\, M\,(E),$$

we obtain the equation

$$\underline{\Omega} \cdot \underline{\nabla}\, n(\underline{r},\underline{\Omega})\, v_T + \Sigma_t(\underline{r})\, v_T n(\underline{r},\underline{\Omega}) = q(\underline{r},\underline{\Omega},E_T)$$
$$+ \Sigma_s(\underline{r})\, v_T \int_{\underline{\Omega}'} d\Omega' n(\underline{r},\underline{\Omega}')\, \mathcal{H}(\underline{\Omega}' \cdot \underline{\Omega}) , \tag{33}$$

where v_T was defined in Equation (24), the space dependent cross-sections are defined by

$$\Sigma(\underline{r}) = \int_0^{E_T} dE\; v\, M(E)\, \Sigma\,(\underline{r},E)/v_T \quad , \tag{34}$$

and the thermal scattering frequency is defined by

$$\int_0^{E_T} dE' \int_0^{E_T} dE\; \Sigma_s(\underline{r},E')\, M\,(E')\, F\,(E',\underline{\Omega}';E,\underline{\Omega})$$
$$= \mathcal{H}(\underline{\Omega}' \cdot \underline{\Omega}) \int_0^{E_T} dE'\; \Sigma_s(\underline{r},E')\, v'M\,(E') \tag{35}$$
$$= \Sigma_s(\underline{r})\, v_T\, \mathcal{H}(\underline{\Omega} \cdot \underline{\Omega}').$$

We now introduce the difference-density defined in (28) into Equation (33) obtaining,

$$- \underline{\Omega} \cdot \underline{\nabla}\, \eta(\underline{r},\underline{\Omega})\, v_T + \frac{1}{4\pi}\, \Sigma_t(\underline{r})\, v_T\, \bar{n}_o - \Sigma_t(\underline{r})\, v_T \eta(\underline{r},\underline{\Omega})$$
$$= q(\underline{r},E_T,\underline{\Omega}) + \frac{1}{4\pi}\, \Sigma_s(\underline{r})\, v_T\, \bar{n}_o \tag{36}$$
$$- \Sigma_s(\underline{r})\, v_T \int_{\underline{\Omega}'} d\Omega'\; \eta(\underline{r},\underline{\Omega}')\, \mathcal{H}(\underline{\Omega}' \cdot \underline{\Omega}).$$

Let us further approximate the equation for η by assuming isotropic scattering everywhere, i.e., $\mathcal{H}(\underline{\Omega}' \cdot \underline{\Omega}) = 1/4\pi$.

It has been assumed that the medium prior to probe insertion was homogeneous, and that the probe itself is homogeneous. It has also been assumed that there is no space-energy coupling, even when the probe is present. Thus, the space dependence of the cross-sections arises solely because the nuclear type and nuclear density are different in the medium and the probe. Hence, we may display

$$\Sigma(\underline{r}) = \Sigma + \delta\,\Sigma\,(\underline{r}) \tag{37}$$

where Σ is the constant cross-section of the medium and $\delta\Sigma(\underline{r})$ is the step function

$$\begin{aligned} \delta\,\Sigma\,(\underline{r}) &= 0, \qquad \underline{r} \text{ not in probe} \\ &= \Sigma^{D} - \Sigma, \qquad \underline{r} \text{ in probe .} \end{aligned} \tag{38}$$

Entering (37) into (36) we get after some rearrangement,

$$\begin{aligned} &- \underline{\Omega} \cdot \underline{\nabla}\,\eta(\underline{r},\underline{\Omega})\,v_T - \Sigma_t v_T \eta(\underline{r},\underline{\Omega}) + \frac{\Sigma_s v_T}{4\pi} \int_{\underline{\Omega}'} d\Omega'\eta\,(\underline{r},\underline{\Omega}') \\ &+ \frac{1}{4\pi}\,\Sigma_t \bar{n}_o\,v_T - q(\underline{r},\underline{\Omega},E_T) - \frac{1}{4\pi}\,\Sigma_s v_T\,\bar{n}_o \\ &= -\frac{1}{4\pi}\,\delta\,\Sigma_t(\underline{r})\,v_T\bar{n}_o + \delta\,\Sigma_t(\underline{r})\,v_T\eta\,(\underline{r},\underline{\Omega}) \\ &+ \frac{1}{4\pi}\,\delta\,\Sigma_s\,v_T\,\bar{n}_o - \frac{\delta\Sigma_s(\underline{r})v_T}{4\pi} \int_{\underline{\Omega}'} d\Omega'\eta\,(\underline{r},\underline{\Omega}') . \end{aligned} \tag{39}$$

If we approximate the slowing down density in (39) by the original

uniform, isotropic slowing down source and note its equality with $\Sigma_a v_T \bar{n}_o$, we obtain an equation for the difference density η of the form

$$\underline{\Omega} \cdot \underline{\nabla} \eta v_T + \Sigma_t \eta v_T = \delta S + \frac{\Sigma_s v_T}{4\pi} \int_{\underline{\Omega}'} d\Omega' \eta' \quad , \tag{40}$$

where δS is zero outside the probe. Inside the probe,

$$\begin{aligned} \delta S = & \frac{1}{4\pi} (\Sigma_a^D - \Sigma_a) v_T \bar{n}_o - (\Sigma_t^D - \Sigma_t) v_T \eta \\ & + \frac{1}{4\pi} (\Sigma_s^D - \Sigma_s) \int_{\underline{\Omega}'} d\Omega' \eta' \quad . \end{aligned} \tag{41}$$

It is now a straight-forward matter to convert the integro-differential equation (40) to an integral equation analogous to Equation (4); which, when integrated over directions of motion of the neutrons, yields the equation for $\eta_o(\underline{r})$,

$$\begin{aligned} \eta_o(\underline{r}) = & \Sigma_a^D \bar{n}_o \int_V d^3r' \frac{e^{-\Sigma_t |\underline{r} - \underline{r}'|}}{4\pi |\underline{r} - \underline{r}'|^2} \\ & - \Sigma_a^D \int_V d^3r' \eta_o(\underline{r}') \frac{e^{-\Sigma_t |\underline{r} - \underline{r}'|}}{4\pi |\underline{r} - \underline{r}'|^2} \\ & + \Sigma_s \int d^3r' \eta_o(\underline{r}') \frac{e^{-\Sigma_t |\underline{r} - \underline{r}'|}}{4\pi |\underline{r} - \underline{r}'|^2} \quad . \end{aligned} \tag{42}$$

The symbol, V, on the integrals in (42) implies that the region of integration is the volume of the probe, the unspecified integral ranges over all space. In deriving (42) explicit use of the boundary

condition, $\eta_o(\underline{r}) \to 0$ as $r \to \infty$ was made. Lastly, the second term on the right-hand side of (42) is ignored, and the equation solved by Skyrme for the $\eta_o(\frac{t}{2})$ required in (30) is

$$\eta_o(\underline{r}) = \Sigma_a^D \bar{n}_o \int_V d^3r' \frac{e^{-\Sigma_t|\underline{r} - \underline{r}'|}}{4\pi\,|\underline{r} - \underline{r}'|^2}$$

$$+ \Sigma_s \int d^3r'\, \eta_o(\underline{r}') \frac{e^{-\Sigma_t|\underline{r} - \underline{r}'|}}{4\pi\,|\underline{r} - \underline{r}'|^2} \qquad (43)$$

Actually, the solution to (43) was obtained in the limit that the volume distribution of "sources" represented by the term proportional to $\bar{n}_o$ was replaced by a surface distribution. From Skyrme's analysis, one finds,

$$\eta_o(\tfrac{t}{2}) = \bar{n}_o\, \tau\, (D_1 - D_1') \quad , \qquad (44)$$

where the quantities D_1 and D_1' have been discussed by Skyrme and by Ritchie and Eldridge. Inserting this result into (30) one finds

$$\bar{A} = \bar{n}_o\, v_T\, [\zeta_o - \zeta_o \tau\, (D_1 - D_1')] \quad . \qquad (45)$$

This result differs from that presented by Skyrme, and at least two reasons for the difference are not hard to find. In the first place, the energy averaging within the probe has here been carried out quite differently than was proposed by Skyrme. He employed an effective "one-speed" calculational model ab initio in which presumably all cross-sections had been "flux" averaged, i.e., averaged

with respect to the function, $v\,M(E)$ in the present notation. Under such circumstances, one finds, instead of Equation (29),

$$\bar{A}' = \bar{n}_o\, v_T\, X_o(\tau) - \sum_{\ell} (-1)^{\ell}(2\ell + 1)\, \eta_{\ell}\, (\tfrac{t}{2})\, v_T X_{\ell}(\tau) \quad , \tag{29'}$$

where $X_{\ell}(\tau)$ is defined analogously to Equation (15), i.e.,

$$X_{\ell}(\tau) = \int_0^1 d\mu\mu\, P_{\ell}(\mu)[1 - e^{-\tau/\mu}] \tag{15'}$$

and τ has been defined by

$$\begin{aligned} \tau &= t < \Sigma_a^D > \\ &= \frac{t}{v_T} \int_0^{E_T} dE\; v\, M(E)\, \Sigma_a^D(E) \; . \end{aligned} \tag{46}$$

Note that τ as defined by Equation (46) is equal to $t\Sigma_a(v_T)$ for a $(1/v)$ cross-section. Thus, in such a case, τ and τ_o introduced earlier would be the same if Σ_{ao}^D is the absorption cross-section of the probe evaluated at $v = v_T$.

In the second place, instead of Equation (30) as an approximate representation of the absorption rate per cm^2, Skyrme writes

$$\bar{A}' \simeq \bar{n}_o v_T X_o(\tau) - \eta_o(\tfrac{t}{2})\, v_T \tau \quad ; \tag{30'}$$

whereas, in accordance with (29'), we would expect

$$\bar{A}' \simeq \bar{n}_o v_T X_o(\tau) - \eta_o(\tfrac{t}{2})\, v_T X_o(\tau) \; . \tag{30''}$$

Noting that for very small τ,

$$X(\tau) \backsim \tau \quad , \tag{47}$$

it is seen that (30') and (30") are essentially the same for very thin probes. Then, given the result presented in Equation (44) for $\eta_o(\frac{t}{2})$ we would find by the "one-speed" analysis

$$\bar{A}' \simeq \bar{n}_o v_T [X_o(\tau) - X_o(\tau)\tau\,(D_1 - D_1')] \; . \tag{45'}$$

If now one replaces $X_o(\tau)$ by τ in the second term of (45'), Skyrme's result is obtained, i.e.,

$$\bar{A}' \simeq \bar{n}_o v_T \, [X_o(\tau) - \tau^2(D_1 - D_1')] \quad . \tag{45"}$$

It is interesting at this point to compare a semi-empirical recasting of Skyrme's formula (45") proposed by Ritchie and Eldridge[6] with (45") and (45'). The latter investigators suggested the formula,

$$\bar{A}" \simeq \frac{\bar{n}_o v_T X_o(\tau)}{1 + X_o(\tau)(D_1 - D_1')} \tag{45"'}$$

If the denominator in (45"') is expanded in a power series and only the first two terms retained, one finds

$$\bar{A}" \simeq \bar{n}_o v_T \, [X_o(\tau) - X_o^2(\tau)(D_1 - D_1')] \quad . \tag{45""}$$

Thus, it is seen that the result presented herein based upon a successive approximation calculation in the context of "one-speed" transport theory is somewhat intermediate between Skyrme's original formula and the recast version of Skyrme's formula proposed by Ritchie and Eldridge.

C. A Transport Calculation of the Absorption Rate per cm^2 in the Infinite Foil

One of the prime defects of the previous approach to the calculation of the activation of the infinite slab is the explicit necessity (at least in principle) for the determination of all of the angular moments of the neutron density at the surface of the slab (recall Equation (14)). This is a direct consequence of representing the absorptions per cm^2 per second in terms of the angular density at the surface -- a procedure that recommends itself in the present context (a non-scattering, sourceless slab immersed in an initially isotropic flux) most naturally because of its seeming simplicity. But actually, from Equation (2), it is seen that quite a different approach characterized by a simplicity of its own is also readily available. That is, the activation sought depends only on the zeroth angular moment of the density throughout the interior of the slab. Thus, apparently this particular problem may be attacked in alternate ways: either a determination of the complete angular density at one point (the surface of the slab) or the calculation of the zeroth moment of the angular density at all points in the slab. In the previous sections (and in calculations not discussed herein[2]) the difficulty of the one-point-full-angular-density approach is made manifest. Hence, an investigation of the alternative procedure would seem to be in order -- and this is what Ritchie and Eldridge carried through.

As this analysis has been discussed in detail in the recent literature and is relatively uncluttered by intuitively based assumptions,

only a few comments will be devoted to it here. In fact, these comments will be restricted to the establishment of the connection between Ritchie and Eldridge's starting equations and Equation (3), which is herein taken to be fundamental in the sense that it presumably describes the problem of probe activation in full generality.

Thus, the starting point for the present analysis is Equation (3), which is promptly converted to Equation (33) in accordance with the arguments of the previous section. Hence, here again, space-energy separability of the density described by Equation (3) is assumed; but in this instance (in contrast to the treatments of the previous sections) all cross-sections, including those of the probe, have been averaged over the distribution, $v\,M(E)$. This follows because Equation (33) is here presumed to hold throughout the interior of the probe as well as in the "external" medium. Finally, the integro-differential Equation (33) is converted to a "one-speed" analogue of Equation (4) which reads,

$$\begin{aligned} v_T n(\underline{r},\underline{\Omega}) &= v_T n(\underline{r}-s\,\underline{\Omega},\underline{\Omega})\, e[-\int_0^s \Sigma_t(\underline{r}-y\underline{\Omega})\,dy] \\ &+ \int_0^s dy\, q(\underline{r}-y\underline{\Omega},\underline{\Omega},E_T)\, e[-\int_0^y \Sigma_t(\underline{r}-z\underline{\Omega})\,dz] \\ &+ \int_0^s dy \int_{\underline{\Omega}'} d\Omega'\Sigma_s(\underline{r}-y\underline{\Omega})\, v_T n(\underline{r}-y\underline{\Omega},\underline{\Omega}') \\ &\qquad (x)\, \mathcal{P}(\underline{\Omega}'\cdot\underline{\Omega})\, e[-\int_0^y \Sigma_t(\underline{r}-z\underline{\Omega})\,dz]. \end{aligned} \tag{48}$$

This equation is then specialized to the case at hand by setting $s = \infty$ and requiring the vanishing of the first term on the right-and side; defining $\phi(\underline{r}) = \int_{\underline{\Omega}} v_T n(\underline{r},\underline{\Omega})\, d\Omega$ and integrating both sides of the equation over Ω; setting $q(\underline{r}-y\underline{\Omega},\underline{\Omega},E_T)$ equal to $\Sigma_a\bar{\phi}_o/4\pi$ outside the probe and equal to zero within the probe; and setting $\Sigma_s(\underline{r}-y\underline{\Omega})\,\mathcal{P}(\underline{\Omega}' \cdot \underline{\Omega}) = \Sigma_s/4\pi$ outside the probe and equal to zero within the probe. The equation then reads,

$$\phi(\underline{r}) = \int_{\underline{r}' \text{ not in probe}} d^3r'[\Sigma_a\bar{\phi}_o + \Sigma_s\phi(\underline{r}')]\,\frac{e[-\int_0^R \Sigma_t(\underline{r}' + y\hat{\underline{R}})dy]}{4\pi R^2}. \qquad (49)$$

The quantity $\Sigma_a\bar{\phi}_o$ is the absorption rate per cm^3 in the external medium when the probe is absent and therefore $\bar{\phi}_o$ is the uniform unperturbed flux, and $\hat{\underline{R}}$ is the unit vector $\underline{R}/R$, $\underline{R} = \underline{r} - \underline{r}'$. Taking advantage of the simplifications inherent in the slab geometry it is now a straight-forward matter to set up the pair of equations for ϕ within and without the probe which Ritchie and Eldridge investigated by variational methods.

III. THE FINITE FOIL

A) The previous discussion has dealt exclusively with the infinite slab approximation to the actual foil. Such complete disregard of edge effects can hardly be justified in all cases. Furthermore, not all useful probes are foils, and the infinite slab analysis would be expected to be even less realistic in such instances. Consequently, some interest attaches to investigations of flux density perturbations induced by finite probes.

Most investigators who have examined these problems at all have given some attention to shape and finite size effects. However, in nearly every instance, the quantitative significance of these effects has been estimated by semi-intuitive calculations grafted onto the infinite probe analysis. Furthermore, the grafting procedures themselves appear to be semi-intuitive in the majority of instances, so that the prospect for developing a consistent, successive approximation analysis in the context of which the various estimates of shape and size effects can be qualitatively compared does not seem bright. Thus, for this reason, if for no other, we devote our attention in this section to a unified treatment of the problem which is potentially suitable for dealing quite accurately with these matters. However, by virtue of its generality, the treatment is quite intractible analytically, so that both qualitative and quantitative results must be sought numerically. Consequently,

the task of comparing the results obtained from such a treatment with the above mentioned semi-intuitive analysis is tedious at best, and to date has been only fragmentarily carried through.

There are reasons other than simply the investigation of shape and size effects that promote interest in a unified treatment of the flux perturbation problem, however. At least five restrictions built into the analyses discussed in the preceding sections should be removed before comparison with experimental results can be made with confidence. These restrictions are:

1) The restriction to "non-scattering" probes. It is not a priori obvious that experiments employing, say, gold and copper probes can be safely analyzed on the assumption that scattering in the probe is ignorable.

2) The restriction to isotropic scattering in the laboratory coordinate system in the external medium. Though this restriction is approximately or intuitively removable from many of the preceding analyses by simply replacing the total cross-section of the medium by an appropriate transport cross-section, it is nevertheless desirable to have available a consistent and systematic analysis of the effect of anisotropic scattering.

3) The restriction inherent in the assumption that the probe is sourceless and in no way disturbs the source in the surrounding medium.

4) The restriction to initially isotropic neutron densities.

5) The restriction following from the assumption that the energy dependence of the low energy neutron density is everywhere separable from its space and angular dependence.

All of these restrictions may be relaxed in the unified (full-blown-transport and hence hereafter referred to as FBT) treatment; and so it is for this reason, as well as the desire to deal with shape and size effects, that we concentrate in this section on the formulation of such a calculation up to the point suitable for numerical analysis.

Of course, brute strength numerical integration of the transport Equation (3) would constitute FBT (as would a complete Monte Carlo analysis of the problem[8]). However, if careful advantage is taken of the fact that the probe is generally "small", it is possible to so phrase the integration of (3) that the rate of calculation is somewhat increased and the diverse effects referred to above enter additively (and hence may be investigated separately) to first order in the sense of a certain successive approximation procedure.

A clue to the present approach to FBT is discernible in Skyrme's "difference-density" calculation of the surface density to obtain a first estimate of the effect of flux depression discussed in Section II-B above. The approach was somewhat explicityly formulated and partially pressed by Fitch and Drummond[9]. It was considerably further pursued by Dalton and Osborn[10], and yet still further by Dalton[11] subsequently.

The starting point is comprised in Equations (2) and (3) which are herein taken to be essentially exact and complete. For convenience, however, these equations are now rewritten in terms of a slightly modified symbolism, i.e.,

$$A_T = \int_V d^3r \int_0^{E_T} dE \int_{\underline{\Omega}} d\Omega \, \Sigma_a^D(E) \, \Phi \, (\underline{r},E,\underline{\Omega}) \tag{50}$$

and

$$\underline{\Omega} \cdot \underline{\nabla} \, \Phi(\underline{r},E,\underline{\Omega}) + \Sigma_t(\underline{r},E) \, \Phi \, (\underline{r},E,\underline{\Omega}) = S(\underline{r},E,\underline{\Omega})$$
$$+ \int_0^{\infty} dE' \int_{\underline{\Omega}'} d\Omega' \, \Sigma_s(E',\underline{\Omega}';E,\underline{\Omega}|\underline{r}) \, \Phi \, (\underline{r},E',\underline{\Omega}') \, . \tag{51}$$

In these equations we have introduced the "flux", $\Phi = vN$, and the "differential macroscopic scattering cross-section,"

$$\Sigma_s(E',\underline{\Omega}';E,\underline{\Omega}|\underline{r}) = \Sigma_s(\underline{r},E') \, F \, (E',\underline{\Omega}';E,\underline{\Omega}|\underline{r}). \tag{52}$$

It should be borne in mind that here we are still assuming that the velocity distribution of the <u>nuclei</u> is everywhere the same in a given medium, so that the spatial variation of these cross-sections is given solely by inhomogeneities within the system. Thus, in particular within the probe Σ_a^D is independent of position. If now we introduce flux averaged cross-sections according to

$$\overline{\Sigma}(\underline{r}) = \int_0^{E_T} dE \, \Sigma(\underline{r},E) \, \Phi \, (\underline{r},E,\underline{\Omega})/\phi(\underline{r},\underline{\Omega}), \tag{53a}$$

$$\overline{\Sigma}_s(\underline{\Omega}' \cdot \underline{\Omega}|\underline{r})$$
$$= \int_0^{E_T} dE \int_0^{E_T} dE' \Sigma_s(E',\underline{\Omega}';E,\underline{\Omega}|\underline{r}) \, \Phi \, (\underline{r},E',\underline{\Omega}')/\phi(\underline{r},\underline{\Omega}') \tag{53b}$$

where, as before,

$$\phi(\underline{r},\underline{\Omega}) = \int_0^{E_T} dE\ \Phi(\underline{r},E,\underline{\Omega}) \tag{54}$$

and integrate Equation (51) over the energy range from 0 to E_T we find

$$A_T = \int_V d^3r \int_{\underline{\Omega}} d\Omega\ \overline{\Sigma}_a^D(\underline{r})\ \Phi\ (\underline{r},\underline{\Omega}), \tag{55}$$

and

$$\begin{aligned} \underline{\Omega} \cdot \underline{\nabla}\ \phi(\underline{r},\underline{\Omega}) + \overline{\Sigma}_t(\underline{r})\ \phi\ (\underline{r},\underline{\Omega}) &= q(\underline{r},\underline{\Omega},E_T) \\ &+ \int_{\Omega'} d\Omega'\ \overline{\Sigma}_s(\underline{\Omega}' \cdot\ \underline{\Omega}|\underline{r})\ \phi\ (\underline{r},\underline{\Omega}') \end{aligned} \tag{56}$$

The slowing down density appearing in Equation (56) is as defined in Equation (32). These equations are, of course, still exact if one interprets the flux-averaged parameters as having both space and angular dependence. Their dependence upon the direction of motion of the neutrons has been notationally suppressed.

At this point we shall assume that the <u>unperturbed</u>, energy-dependent flux, $\overline{\Phi}(\underline{r},E,\underline{\Omega})$ is separable, i.e., $\overline{\Phi}(\underline{r},E,\underline{\Omega}) = vM(E)\overline{\phi}(\underline{r},\underline{\Omega})$. Then $\overline{\phi}$ satisfies the equation,

$$\begin{aligned} \underline{\Omega} \cdot\ \underline{\nabla}\ \overline{\phi}(\underline{r},\underline{\Omega}) + \Sigma_t\overline{\phi}\ (\underline{r},\underline{\Omega}) &= \overline{q}(\underline{r},\underline{\Omega},E_T) \\ &+ \int_{\underline{\Omega}'} d\Omega'\ \Sigma_s(\underline{\Omega}' \cdot\ \underline{\Omega})\ \overline{\phi}\ (\underline{r},\underline{\Omega}'), \end{aligned} \tag{57}$$

where Σ_t and Σ_s are space-independent medium parameters defined analogously to (34) and (35). In the event that the initial flux is

independent of position and neutron direction of motion, $\Sigma_a\bar{\phi} = \bar{q}$. Let the actual flux ϕ which satisfies (56) be related to the unperturbed flux by $\phi = \bar{\phi} - \psi$. Evidently ψ is the difference-flux. Writing $\bar{\Sigma}(\underline{r}) = \Sigma + [\bar{\Sigma}(\underline{r}) - \Sigma]$ and keeping in mind (57) we find the equation for ψ,

$$\begin{aligned}
\underline{\Omega} \cdot \underline{\nabla}\, \psi\, (\underline{r},\underline{\Omega}) + \Sigma_t\psi(\underline{r},\underline{\Omega}) \;=\; & [(\bar{\Sigma}_t(\underline{r}) - \Sigma_t)\, \bar{\phi}\, (\underline{r},\underline{\Omega}) \\
- \; (\bar{\Sigma}_t(\underline{r}) - \Sigma_t)\, \psi\, (\underline{r},\underline{\Omega}) - \{q(\underline{r},\underline{\Omega},E_T) - \bar{q}(\underline{r},\underline{\Omega},E_T)\} & \qquad (58) \\
- \int_{\underline{\Omega}'} d\Omega'\, \{\bar{\Sigma}_s(\underline{\Omega}'\cdot\, \underline{\Omega}|\underline{r}) - \Sigma\, (\underline{\Omega}'\cdot\, \underline{\Omega}\}\, \bar{\phi}\, (\underline{r},\underline{\Omega}') + \int d\Omega'\, \{\bar{\Sigma}_s(\underline{\Omega}'\cdot\, \underline{\Omega}|\underline{r}) & \\
- \; \Sigma_s(\underline{\Omega}'\cdot\, \underline{\Omega})\}\, \psi\, (\underline{r},\underline{\Omega}')] + \int_{\underline{\Omega}'} d\Omega'\, \Sigma_s(\underline{\Omega}'\cdot\, \underline{\Omega})\, \psi\, (\underline{r},\underline{\Omega}').
\end{aligned}$$

Note that, if there were no spectral shift induced by the probe in the external medium, $\bar{\Sigma}$ would equal Σ outside the probe; and if the probe does not alter the slowing down density in the external medium, $\bar{q} = q$ outside the probe. In such an event, the bracketed term on the right-hand side of Equation (58) would vanish outside the probe. In a given experiment, however, this may not be true. Thus, for the time being, we merely label the bracketed term by the function $\mathscr{S}(\underline{r},\underline{\Omega})$ and display Equation (59) as

$$\begin{aligned}
\underline{\Omega} \cdot\, \underline{\nabla}\, \psi(\underline{r},\underline{\Omega}) \;+\; \Sigma_t\psi(\underline{r},\underline{\Omega}) \;=\; & \mathscr{S}(\underline{r},\underline{\Omega}) \\
+ \int_{\underline{\Omega}'} \Sigma_s(\underline{\Omega}'\cdot\, \underline{\Omega})\, \psi\, (\underline{r},\underline{\Omega}')\, d\Omega' \;. & \qquad (59)
\end{aligned}$$

Presumably, the difference-flux vanishes at points $\underline{r}$ far from the probe.

It is convenient to convert the integro-differential Equation (50) to an integral equation by defining the Green's function $G(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}')$ by

$$[\underline{\Omega}\cdot\underline{\nabla} + \Sigma_t - \int_{\underline{\Omega}''} d\underline{\Omega}''\, \Sigma_s(\underline{\Omega}''\cdot\underline{\Omega})]\, G\,(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}') = \delta(\underline{r}'-\underline{r})\,\delta\,(\underline{\Omega}'-\underline{\Omega}), \tag{60}$$

and then noting that

$$\psi(\underline{r},\underline{\Omega}) = \int_{\underline{r}',\underline{\Omega}'} G(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}')\,\mathscr{S}(\underline{r}',\underline{\Omega}')\, d^3r'd\Omega' \,. \tag{61}$$

The general problem formulated in Equation (58) or (61) must proceed from here mainly by numerical analysis. Many such calculations have been carried through, mainly for the purpose of illustrating size and shape effects, the importance of scattering in the probe, anisotropic scattering in the external medium, and the significance of gradients in the originally unperturbed distributions. In fact, only two of the potentially important effects implicit in Equations (58) and (61) have remained largely uninvestigated so far. These are spectral shift effects -- now incorporated into the space and angular dependence of the cross-sections -- and source perturbation effects implicit in the slowing down density, $q(\underline{r},\underline{\Omega},E_T)$. In the next section, we give a little attention to the former of these problems.

A. An Estimate of the Spectral Shift Effect

Of course, the present problem would be quite straight forward (from the numerical point of view) if the quantities $\bar{\Sigma}$ were known as functions of position and neutron direction of motion. But, as is evident from Equation (53), such knowledge is only going to be available when the flux $\Phi(\underline{r},E,\underline{\Omega})$ is known as a function of all of its arguments, and this requires appropriate solution of Equation (51), which so far seems a little out of reach. However, it is perhaps reasonable to expect that the flux hardening effect should be rather small, so that a more or less qualitative estimate might be useful and instructive. Specifically, we shall attempt a guess for the cross-sections and then by first order perturbation methods, calculate a correction to the absorption rates, predicted by the solutions obtained from (61) when spectral shifting is explicitly ignored.

We begin our discussion with Equation (56), and start by assuming that all total cross-sections in the probe and the surrounding medium vary as $(1/v)$ in the thermal energy range. This assumption is actually less far-fetched than it might seem at first sight. In the first place, the $(1/v)$-assumption is usually a good first approximation for most absorption reactions. In the second place, the error introduced by assuming a $(1/v)$ scattering law in the probe is probably a small error accruing to what is expected to be a small correction, and hence is not anticipated to be of much importance. In the third place, if we take water as our

"external" medium, the assumption of a $(1/v)$ scattering law is actually quite good.

With this assumption, we find from (53a) that

$$\bar{\Sigma}(\underline{r},\underline{\Omega}) = \Sigma(\underline{r})\, v_o / \langle v \rangle \quad , \tag{62}$$

where $\Sigma(\underline{r})$ is the cross-section appropriate to the composition of the medium at $\underline{r}$ evaluated at speed v_o, and $\langle v \rangle$ is the space and angle dependent mean speed of neutrons with energies less than E_T. For convenience, we shall take v_o to be the mean speed of Maxwellian distributed neutrons (not 2200 meters per second) so that in the limit of vanishing spectral shifting, $v_o / \langle v \rangle \to 1$. The situation is not so clear for the differential cross-sections, so we will merely assume that

$$\bar{\Sigma}_s(\underline{\Omega}' \cdot \underline{\Omega} | \underline{r}) = \Sigma_s(\underline{\Omega}' \cdot \underline{\Omega} | \underline{r})\, v_o / \langle v \rangle \quad . \tag{63}$$

This assumption is consistent with Equation (62) only if $\langle v \rangle$ is presumed independent of the direction of motion of the neutrons, thus from here on we shall guess that $\langle v \rangle$ is essentially dependent only upon position. (Such would be the case if the neutron speed distribution was a local Maxwellian with a space dependent temperature, for example). Then, recalling that $\phi(\underline{r},\underline{\Omega}) = \langle v \rangle\, n(\underline{r},\underline{\Omega})$, we find that Equation (56) becomes,

$$\begin{aligned} \underline{\Omega} \cdot \underline{\nabla}\, n(\underline{r},\underline{\Omega}) \langle v \rangle + v_o \Sigma_t(\underline{r})\, n(\underline{r},\underline{\Omega}) &= q(\underline{r},\underline{\Omega},E_T) \\ &+ \int_{\underline{\Omega}'} d\Omega' v_o \Sigma_s(\underline{\Omega}' \cdot \underline{\Omega} | \underline{r})\, n(\underline{r},\underline{\Omega}') \; . \end{aligned} \tag{64}$$

We note again that the flux $v_o n$ and the cross-sections $\Sigma(\underline{r})$ appearing in (64) are precisely the fluxes and cross-sections appearing in the space-energy separable problems discussed here and elsewhere. Thus, to the extent that (62) and (63) have some meaning for a real system, it is seen that flux hardening effects enter only via the function, $< v >/v_o$. This function presumably increases as we approach and enter the probe and decreases to unity at points far from the probe. We replace it by

$$< v >/v_o = 1 + g, \tag{65}$$

and regard g as a small quantity henceforth. Now, introducing

$$v_o n(\underline{r},\underline{\Omega}) = \chi(\underline{r},\underline{\Omega}) + f(\underline{r},\underline{\Omega}) \quad , \tag{66}$$

treating f as small in the same sense as g (it is the correction to the flux due to spectral hardening, and hence vanishes as g vanishes), entering (65) and (66) into (64) and retaining only terms linear (at most) in small quantities we find the pair of equations,

$$\begin{aligned} \underline{\Omega} \cdot \underline{\nabla} \chi(\underline{r},\underline{\Omega}) + \Sigma_t(\underline{r})\, \chi\, (\underline{r},\underline{\Omega}) &= q(\underline{r},\underline{\Omega},E_T) \\ &+ \int_{\underline{\Omega}'} d\underline{\Omega}'\, \Sigma_s(\underline{\Omega}' \cdot \underline{\Omega}|\underline{r})\, \chi\, (\underline{r},\underline{\Omega}'), \end{aligned} \tag{67a}$$

and

$$\begin{aligned} \underline{\Omega} \cdot \underline{\nabla} f(\underline{r},\underline{\Omega}) + \Sigma_t(\underline{r})\, f\, (\underline{r},\underline{\Omega}) &= - \underline{\Omega} \cdot \underline{\nabla} \chi(\underline{r},\underline{\Omega})\, g\, (\underline{r}) \\ &+ \int_{\underline{\Omega}'} d\Omega' \Sigma_s(\underline{\Omega}' \cdot \underline{\Omega}\, \underline{r})\, f\, (\underline{r},\underline{\Omega}') \quad . \end{aligned} \tag{67b}$$

Along with this equation, we have for the absorption rate

$$A_T = \int_V d^3r \int_0^{E_T} dE \int_{\underline{\Omega}} d\Omega \, \Sigma_a^D \, v_o \, N(\underline{r},E,\underline{\Omega})$$

$$= \int_V d^3r \int_{\underline{\Omega}} d\Omega \, \Sigma_a^D \, v_o \, n(\underline{r},\underline{\Omega}) \tag{68}$$

$$= \Sigma_a^D \int_V d^3r \int_{\underline{\Omega}} d\Omega \, \chi(\underline{r},\underline{\Omega}) + \Sigma_a^D \int_V d^3r \int_{\underline{\Omega}} d\Omega \, f(\underline{r},\underline{\Omega}) \, ,$$

where the term involving χ represents the contribution from the separable fraction of the flux and the term involving f is a correction due to the spectral shift induced by the probe. In particular, it was the former contribution that was calculated by, say, Ritchie and Eldridge[6] for the infinite slab and Dalton and Osborn[10] in the finite foils and wires.

In Equation (67b) for the function f, the cross-sections are still space dependent due to compositional inhomogeneities resulting from the presence of the probe. However, for a first order estimate of the "hardening" effect, we may neglect this space dependence, i.e., neglect the presence of the probe entirely so far as cross-sections are concerned. Furthermore, since g, χ, and $g\chi$ are certainly continuous, and probably slowly varying, functions of their arguments, it follows that a P_1-approximation to (67b) is apt to be acceptable for present purposes. Also, we shall assume that the initial, unperturbed flux was uniform and isotropic. Then, $\chi = \frac{1}{4\pi} \overline{\phi}_o - \psi$ where ψ is a difference flux, and consequently the

term $g\psi$ is again small to second order. Finally, we also assume that the spectral shift function, g, depends primarily only on position. Thus, we consider the equation,

$$\underline{\Omega} \cdot \underline{\nabla} f(\underline{r},\underline{\Omega}) + \Sigma_t f(\underline{r},\underline{\Omega}) = - \overline{\phi}_o \underline{\Omega} \cdot \underline{\nabla} g(\underline{r}) + \int_{\underline{\Omega}'} d\Omega' \, \Sigma_s(\underline{\Omega}' \cdot \underline{\Omega}) \, f(\underline{r},\underline{\Omega}') \tag{69}$$

In accordance with the above remarks, the P_1-approximation to this equation is

$$D\nabla^2 f_o - \Sigma_a f_o = - \overline{\phi}_o D \nabla^2 g \quad , \tag{70}$$

where

$$f_o(\underline{r}) = \int_{\underline{\Omega}} d\Omega f(\underline{r},\underline{\Omega}) \quad , \quad D = 1/3 \, [\Sigma_t - \overline{\mu}_L \Sigma_s] \quad . \tag{71}$$

To obtain a qualitative estimate of the effect of hardening on the absorption rate, we return to the problem of the infinite slab -- discussed in Section II with disregard for this effect. In such a case, Equation (71) reduces to

$$D \frac{d^2 f_o}{dx^2} - \Sigma_a f_o = - \overline{\phi}_o D \frac{d^2 g}{dx^2} \quad , \tag{72}$$

and the solution to this inhomogeneous equation which vanishes at infinity is

$$f_o(x) = \frac{\overline{\phi}_o}{2L} \int_{-\infty}^{\infty} dx' g(x') \, e^{-|x-x'|/L} - \overline{\phi}_o g(x) \; . \tag{73}$$

To proceed further, it is necessary to guess a reasonable analytical representation for the hardening function, g, defined in Equation (65). If we could assume that the hardened spectrum was everywhere Maxwellian with space-dependent temperature, then

$$g = \frac{\langle v \rangle}{v_o} - 1 = \sqrt{\frac{T}{T_o}} - 1, \tag{74}$$

where $T = T(\underline{r})$ is the actual temperature with probe present and T_o is the temperature characterizing the spectrum before probe insertion. Introducing

$$\Delta T = T - T_o \quad , \tag{75}$$

and assuming that $\Delta T/T_o \ll 1$, we find that (74) may be written as

$$g \simeq \Delta T/2T_o \quad . \tag{76}$$

This suggests that g is most likely small and hence, that most any guess which is qualitatively reasonable will be useful here. Thus, we choose

$$\begin{aligned} g(x) &= a(1 - |x| / x_o), \; - x_o \leq x \leq x_o \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{77}$$

This is a triangle with apex at the origin (also located at the center of the slab) of height, a, and base, $2x_o$. One would presume that $x_o \geq t/2$ i.e., that the hardening extends at least to the boundaries of the slab, and most probably considerably beyond them. Furthermore, the height, a, must vanish as the thickness of the slab vanishes, since then there must be no hardening.

With the choice, (77), for g, the hardening correction, f_o is readily calculated according to (73). Inserting the result into Equation (68) one finds for the hardening correction to the absorption rate per unit area

$$\bar{A}_h \simeq -2a\bar{\phi}_o\tau\left[\frac{L}{2x_o} - \frac{t}{8L} + \frac{t^2}{48Lx_o} - e^{-x_o/L}\left(\frac{L}{2x_o} + \frac{t}{8x_o} + \frac{t^2}{48Lx_o}\right)\right] \tag{78}$$

for sufficiently small t/L. Here, L is the diffusion length of the external medium defined in the usual way. For $x_o = t/2$ (78) reduces to

$$\bar{A}_h \simeq -\frac{1}{2}\, a v_T \bar{n}_o \tau \; ; \tag{79}$$

and for $x_o/L \gg 1$, it becomes

$$\bar{A}_h \simeq -a\frac{L}{x_o} v_T \bar{n}_o \tau \; . \tag{80}$$

This correction is to be compared with something like Equation (45''') which provides an estimate of the first term in Equation (68). Evidently the comparison depends rather sensitively upon the parameter, a which is intuitively given by $\Delta T(0)/2T_o$. Writing $a = \alpha\tau$ in order to guarantee that the hardening effect vanishes as the probe thickness vanishes, it is seen that the correction enters to order τ^2. But a reasonable estimate of the magnitude of the coefficient of τ^2 in this correction term seems difficult to come by. Since the absorption rate is not a linear function of thickness, the present argument merely suggests that hardening effects may not be ignorable in the theoretical interpretation of experimental results.

IV. SUMMARY

Admittedly, this review is incomplete in at least three respects. Firstly, it is not a comprehensive survey of all of the calculational variations of the themes discussed above that have appeared in the literature. For example, the whole-hearted but semi-intuitive application of diffusion theory proposed by Corinaldesi[12] has received no explicit consideration here; and neither has the application of Yvon's double-P_ℓ methods to this problem explored by Bengston[13]. But it was stated in the beginning that such incompleteness was to be anticipated, since we proposed to sketch only those computations which differed fundamentally. However, as we saw, there appear to be only two approaches which so differ -- one which relates the absorption rate to the full angular density at the surface of the probe, and the other which exploits the relationship between the absorption rate and the zeroth angular moment of the density at all points within the probe. As noted, for practical purposes, the former approach is almost necessarily restricted to the infinite slab approximation to a foil in which scattering is ignorable inserted in a medium in which the neutron density is initially istropic and uniform in space. Then, the burden of all such calculations rests on the calculation of the angular density at the surface of the probe and on estimations of finite size corrections. The method is therefore practically intrinsically approximate, and the task of estimating the qualitative and quantitative

significance of the approximation seems overwhelmingly complicated and subtle. Conversely, the latter approaches rest on more or less straightforward attempts to solve the neutron transport equation in inhomogeneous media, and are therefore essentially unrestricted and exact. At the same time, they are almost wholly intractible analytically, and thus do not lend themselves well to qualitative exposition.

Secondly, the theoretical aspects of the problem of relating the observed activity to a steady state "thermal" absorption rate have been completely ignored. A host of questions arise here -- such as self-absorption of the induced activity and the interpretation of cadmium-ratios, particularly when the ratios are not large, to name a couple -- but these were not deemed within the province of the present discussion. In a practical sense, they are probably separable aspects (separable from that aspect dealt with herein) of the problem and may be meaningfully discussed independently.

And thirdly, the various attempts to so devise the experiments as to partially or completely circumvent the problems considered here have not been given attention. Examples are Meister's[7] activations in voids - thus eliminating the effects of flux depression in the external medium, and Randall and Walker's[14] suggested attempts to activate materials which effectively match the microscopic and macroscopic properties of the medium in which the density is to be measured, thus eliminating the probe perturbation effect completely. Though these experimental procedures have great importance in principle in connection with the thorny issues of making reliable comparison between theory and experiment, they are not likely to have wide applicability to the many practical situations in which thermal density measurements are necessary.

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