Electromagnetic and Acoustic Scattering by a Semi-Infinite Body of Revolution

CRAIGE E. SCHENSTED
Willow Run Research Center, University of Michigan, Ann Arbor, Michigan
(Received April 9, 1954)

The first two terms of Kline's asymptotic expansion are obtained for the scattering of a plane wave incident along the axis of a perfectly reflecting semi-infinite body of revolution. When this method is applied to the paraboloid the *exact* electromagnetic solution is obtained in closed form. The accuracy of the method of physical optics is studied by using the asymptotic expansion.

MORRIS Kline has given an asymptotic solution to hyperbolic partial differential equations valid for short wavelengths.[1] This method can be readily applied to the scattering by perfectly reflecting bodies when there are no "shadow" regions present. In this paper the first two terms of this expansion are obtained for the scattering of a plane wave incident along the axis of a perfectly reflecting semi-infinite body of revolution. The first term of this expansion corresponds to geometric-optics, while the second term gives the deviation from geometric-optics when the wavelength is sufficiently small compared to the minimum radius of curvature of the body.

When the body is taken to be a paraboloid, the exact electromagnetic solution is given (in closed form) by the first term of the expansion.

At the end of the paper the asymptotic solution is compared with the result given by the current-distribution method[2] and its acoustic counterpart.

A brief resume of Kline's method, as it applies to the problem being considered, will be given here. The solution is given in terms of the function $S(x,y,z)$ which is defined so that $S(x,y,z)=\text{const}$ is a surface of constant phase in the geometric-optics approximation and so that $(\nabla S)^2=1$. The form of the asymptotic solution (in the electromagnetic and acoustic cases respectively) is

$$\mathbf{E}=\sum_{n=0}^{\infty}\lambda^n\mathbf{E}_n e^{i(kS-\omega t)},\quad U=\sum_{n=0}^{\infty}\lambda^n U_n e^{i(kS-\omega t)}. \quad (1)$$

Here $\mathbf{E}$ is the electric field, U is the velocity potential, λ is the wavelength, $k=2\pi/\lambda$, and $\omega=kc$ where c is the velocity of light. The wave equation then gives the following equations for $\mathbf{E}_n$ and U_n

$$\begin{aligned}\frac{\partial\mathbf{E}_n}{\partial S}+\tfrac{1}{2}\mathbf{E}_n\nabla\cdot\mathbf{s}&=\frac{i}{4\pi}\nabla^2\mathbf{E}_{n-1},\\ \frac{\partial U_n}{\partial S}+\tfrac{1}{2}U_n\nabla\cdot\mathbf{s}&=\frac{i}{4\pi}\nabla^2 U_{n-1},\end{aligned}\quad (2)$$

where $\mathbf{s}=\nabla S$ and $\partial/\partial S\equiv\mathbf{s}\cdot\nabla$ (lower-case, boldface is used to denote a unit vector). For the electromagnetic problem the requirement that the divergence of the electric field vanish gives the additional equation

$$\mathbf{s}\cdot\mathbf{E}_n=\frac{i}{2\pi}\nabla\cdot\mathbf{E}_{n-1}. \quad (3)$$

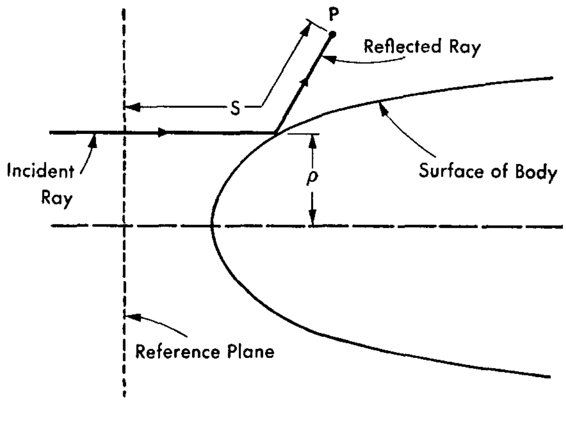
In addition to the above equations, the following boundary conditions must be satisfied on the surface of the scatterer:

$$\mathbf{n}\times\mathbf{E}_n=0,$$
$$U_n=\frac{i}{2\pi}\frac{\mathbf{n}\cdot\nabla U_{n-1}}{\mathbf{n}\cdot\mathbf{s}}, \quad (4)$$

where $\mathbf{n}$ is the unit normal to the surface.

Along the rays incident on the scatterer, $\mathbf{E}_0$ and U_0 are taken to represent the incident fields; while all the other $\mathbf{E}_n$ and U_n are taken to be zero. The initial values of $\mathbf{E}_n$ and U_n on the reflected rays are obtained from (3) and (4). $\mathbf{E}_n$ and U_n can then be obtained by integrating the ordinary differential equations given by (2). One component of $\mathbf{E}_n$ can be obtained more readily by (3).

The actual determination of $\mathbf{E}_n$ and U_n from the



S is the distance along a ray to any point P from some reference plane.

ρ is the distance from the axis of the body at which the ray hits the body.

φ is a rotational angle about the axis of the body.

FIG. 1. Coordinate system used in calculations.

[1] Morris Kline, Commun. Pure Appl. Math. 4, 225–262 (1951).
[2] Samuel Silver, *Microwave Antenna Theory and Design* (McGraw-Hill Book Company, Inc., New York, 1949) p. 144.

above relations can be very cumbersome. A useful labor-saving device is the introduction of the coordinate system as shown in Fig. 1.

It may be seen that there are two sets of coordinates for each point because both an incident ray and a reflected ray go through each point. These two coordinate systems are used in the calculation of the incident and reflected fields, respectively.

The equation of the body can be taken to be $S=f(\rho)$ in either coordinate system. If an x,y,z coordinate system is introduced with the z-axis along the body axis and the x,y-plane as the reference plane then (for the coordinate system after reflection)

$$
\begin{aligned}
x&=\rho\cos\phi+\frac{2f^{(1)}}{(f^{(1)})^2+1}(S-f)\cos\phi,\\
y&=\rho\sin\phi+\frac{2f^{(1)}}{(f^{(1)})^2+1}(S-f)\sin\phi,\\
z&=f+\frac{(f^{(1)})^2-1}{(f^{(1)})^2+1}(S-f),
\end{aligned}
\tag{5}
$$

where $(f^{(n)})^m$ is the mth power of the nth derivative. The relationship between the two sets of unit vectors is

$$
\begin{aligned}
\mathbf{s}&=\frac{2f^{(1)}}{(f^{(1)})^2+1}\cos\phi\,\mathbf{i}+\frac{2f^{(1)}}{(f^{(1)})^2+1}\sin\phi\,\mathbf{j}+\frac{(f^{(1)})^2-1}{(f^{(1)})^2+1}\mathbf{k},\\
\boldsymbol{\rho}&=-\frac{(f^{(1)})^2-1}{(f^{(1)})^2+1}\cos\phi\,\mathbf{i}-\frac{(f^{(1)})^2-1}{(f^{(1)})^2+1}\sin\phi\,\mathbf{j}+\frac{2f^{(1)}}{(f^{(1)})^2+1}\mathbf{k},\\
\boldsymbol{\phi}&=-\sin\phi\,\mathbf{i}+\cos\phi\,\mathbf{j}.
\end{aligned}
\tag{6}
$$

The differential operators required in the calculation of $\mathbf{E}_n$ and U_n can be expressed in terms of the S, ρ, ϕ, coordinate system in the usual way by making use of (5). It turns out that the coordinate system is orthogonal. The final results are most easily expressed in terms of the metric coefficients

$$
\begin{aligned}
h_S&=1,\\
h_\rho&=1+\frac{2f^{(2)}}{(f^{(1)})^2+1}(S-f),\\
h_\phi&=\rho+\frac{2f^{(1)}}{(f^{(1)})^2+1}(S-f).
\end{aligned}
\tag{7}
$$

For simplicity, only the case in which there is no multiple reflection will be obtained here. There would be no essentially new features introduced by the removal of this restriction, but the results would be much more complicated.

The calculations outlined above have been carried through for the incident fields $\mathbf{E}_i=\mathbf{i}e^{i(kS-\omega t)}$ and $U_i=e^{i(kS-\omega t)}$. The results are

$$
\begin{aligned}
\mathbf{E}_0&=\left(\frac{\rho}{h_\rho h_\phi}\right)^{\frac{1}{2}}(-\cos\phi\boldsymbol{\rho}+\sin\phi\boldsymbol{\phi}),\\
U_0&=\left(\frac{\rho}{h_\rho h_\phi}\right)^{\frac{1}{2}},\\
\mathbf{E}_1&=\frac{i}{2\pi}\left(\frac{\rho}{h_\rho h_\phi}\right)^{\frac{1}{2}}\Biggl\{\Biggl[\biggl(\frac{1}{8\rho^2}-\frac{f^{(1)}f^{(3)}}{h_\rho((f^{(1)})^2+1)}\biggr)\\
&\quad\times\frac{1}{2f^{(2)}}(1-h_\rho^{-1})+\frac{2f^{(3)}+\rho f^{(4)}}{16\rho(f^{(2)})^2}(1-h_\rho^{-1})^2\\
&\quad-\frac{1}{16\rho f^{(1)}}-\frac{5(f^{(3)})^2}{48(f^{(2)})^3}(1-h_\rho^{-1})^3\Biggr]((f^{(1)})^2+1)\\
&\quad\times(\cos\phi\boldsymbol{\rho}-\sin\phi\boldsymbol{\phi})+\Biggl[\frac{f^{(2)}}{2}\biggl(1+\frac{(f^{(1)})^2-1}{(f^{(1)})^2+1}\frac{1}{h_\rho}\biggr)\\
&\quad-\frac{f^{(1)}}{2\rho}(1-h_\rho^{-1})-\frac{(5(f^{(1)2})+1)(3(f^{(1)})^2-1)}{16f^{(1)}((f^{(1)})^2+1)h_\phi}\Biggr]\\
&\quad\times\cos\phi\boldsymbol{\rho}+\Biggl[\frac{f^{(2)}}{2}(1-h_\rho^{-1})-\frac{f^{(1)}}{2\rho}(1+h_\rho^{-1})\\
&\quad+\frac{15(f^{(1)})^2-1}{16f^{(1)}h_\phi}\Biggr]\sin\phi\boldsymbol{\phi}+\Biggl[\frac{f^{(3)}}{2f^{(2)}h_\rho}(1-h_\rho^{-1})\\
&\quad-\biggl(\frac{1}{2\rho}+\frac{f^{(1)}f^{(2)}}{(f^{(1)})^2+1}\biggr)\frac{1}{h_\rho}+\frac{3(f^{(1)})^2+1}{2((f^{(1)})^2+1)h_\phi}\Biggr]\cos\phi\mathbf{s}\Biggr\},\\
U_1&=-\frac{i}{2\pi}\left(\frac{\rho}{h_\rho h_\phi}\right)^{\frac{1}{2}}\Biggl\{\biggl(\frac{(f^{(1)})^2+1}{16\rho^2f^{(2)}}-\frac{f^{(1)}f^{(3)}}{2f^{(2)}h_\rho}\biggr)(1-h_\rho^{-1})\\
&\quad+\frac{f^{(1)}+\rho f^{(2)}}{2\rho}(1+h_\rho^{-1})+\Biggl[\frac{1}{f^{(1)}h_\phi}-\frac{1}{\rho f^{(1)}}\\
&\quad+\frac{2f^{(3)}+\rho f^{(4)}}{\rho(f^{(2)})^2}(1-h_\rho^{-1})^2\\
&\qquad-\frac{5(f^{(3)})^2}{3(f^{(2)})^3}(1-h_\rho^{-1})^3\Biggr]\frac{(f^{(1)})^2+1}{16}\Biggr\}.
\end{aligned}
\tag{8}
$$

A particularly simple result is obtained on applying these formulas to the paraboloid $f=\rho^2/2R$ (R is the radius of curvature at the nose of the paraboloid). In this case

$$
\begin{aligned}
\mathbf{E}_0&=\frac{\rho^2+R^2}{2R(S+R/2)}(-\cos\phi\boldsymbol{\rho}+\sin\phi\boldsymbol{\phi}),\\
U_0&=\frac{\rho^2+R^2}{2R(S+R/2)},\\
\mathbf{E}_1&=0,\\
U_1&=-\frac{i}{2\pi R}\frac{\rho^2+R^2}{2R(S+R/2)}\left[1+\frac{\rho^2+R^2}{2R(S+R/2)}\right].
\end{aligned}
\tag{9}
$$

Since $\mathbf{E}_1\equiv 0$, it follows from (2), (3), and (4) that $\mathbf{E}_n\equiv 0$ $(n\neq 0)$. Thus, in this case, the asymptotic expansion contains only a single term. Since the field given by this one term satisfies the wave equation, the divergence condition, the boundary condition, and the radiation condition,* the exact solution to the scattering problem has been obtained in closed form.†

For the acoustic problem the exact solution has been obtained in closed form by Horton and Karal.[3] The scattered field is

$$U_S=e^{i(kS-\omega t)}\frac{e^{-ikQ}[-si(kQ)+iCi(kQ)]}{2/kR-e^{-ikR}[-si(kR)+iCi(kR)]}, \tag{10}$$

where $Q\equiv R^2(R+2S)/(\rho^2+R^2)$ and si and Ci are the sine and cosine integrals

$$si(x)=-\int_x^\infty \frac{\sin t}{t}dt;\quad Ci(x)=-\int_x^\infty \frac{\cos t}{t}dt.$$

It is easily verified that the two terms of Eq. (8) are just the first two terms of the asymptotic expansion of (10).

If $\beta=2\cos^{-1}R/(\rho^2+R^2)^{\frac{1}{2}}$ (the angle between the axis and the direction to the field point) then the differential scattering cross section is

$$\sigma_V=\frac{R^2}{4\cos^4(\beta/2)} \tag{11}$$

in the vector (electromagnetic) case and

$$\sigma_S=\frac{R^2}{4\cos^4(\beta/2)}\frac{1}{g(kR)}, \tag{12}$$

where

$$g(x)=(2+x\cos x\, si(x)-x\sin x\, Ci(x))^2 + (x\sin x\, si(x)+x\cos x\, Ci(x))^2$$

in the scalar (acoustic) case. The factor $g(x)$ varies monotonically from $\frac{1}{4}$ for $x=0$ to 1 for $x=\infty$.

It is interesting to compare the asymptotic expansion given in (8) with the physical-optics (current-distribution) approximation. The physical-optics method is based on the following exact expression for the scattered fields[4]

$$\mathbf{E}_S=\frac{1}{4\pi}\int[ik(\mathbf{n}\times\mathbf{H})-(\mathbf{n}\cdot\mathbf{E})\nabla]\frac{e^{ikr}}{r}dS,$$
$$U_S=\frac{1}{4\pi}\int U\mathbf{n}\cdot\nabla\frac{e^{ikr}}{r}dS, \tag{13}$$

where the integration is carried out over the scattering surface and r is the distance between the integration and field points. The physical optics method consists of replacing the fields inside the integrals by their geometric-optics approximation. In the electromagnetic case this corresponds to using the geometric-optics approximations for the current and charge distributions on the scattering surface. If the incident fields are those assumed above, then the physical-optics approximations for the back-scattered fields at large distances from the body are

$$\mathbf{E}_S\approx \mathbf{i}\frac{i}{\lambda}\frac{e^{i(kS-\omega t)}}{S}\int_0^\infty e^{2ikz}A^{(1)}(z)dz,$$
$$U_S\approx -\frac{i}{\lambda}\frac{e^{i(kS-\omega t)}}{S}\int_0^\infty e^{2ikz}A^{(1)}(z)dz, \tag{14}$$

where $A(z)$ is the cross-sectional area of the body. If $A(z)$ is given by $A(z)=\pi\rho^2=\pi a_1 z+\pi a_2 z^2+\cdots$ then (14) takes the form

$$\mathbf{E}_S\approx -\mathbf{i}\frac{e^{i(kS-\omega t)}}{S}\left(\frac{a_1}{4}+i\lambda\frac{a_2}{8\pi}+\cdots\right),$$
$$U_S\approx\frac{e^{i(kS-\omega t)}}{S}\left(\frac{a_1}{4}+i\lambda\frac{a_2}{8\pi}+\cdots\right). \tag{15}$$

The corresponding expressions obtained from (8) are

$$\mathbf{E}_S\approx -\mathbf{i}\frac{e^{i(kS-\omega t)}}{S}\left(\frac{a_1}{4}+i\lambda\frac{a_2}{8\pi}+\cdots\right),$$
$$U_S\approx\frac{e^{i(kS-\omega t)}}{S}\left(\frac{a_1}{4}+i\lambda\frac{a_2-2}{8\pi}+\cdots\right). \tag{16}$$

From (15) and (16) it can be seen that, asymptotically, physical optics represents an improvement on geometric-optics in the electromagnetic case unless $a_2=0$. In the acoustic case physical optics gives an improvement on geometric-optics unless $|a_2-2|\leq 2$, but the improvement is not as great as it is in the electromagnetic case.

The above analysis lends support to the conjecture put forth by Siegel[5] that physical optics would give the exact electromagnetic back-scattering cross section for the type of problem under consideration in this paper.

* Since the paraboloid is of infinite dimensions the usual radiation condition must be modified. Peters and Stoker [Commun. Pure Appl. Math. 7, 565 (1954)] have suggested that the field be required to have a decomposition $\mathbf{E}=\mathbf{A}+\mathbf{B}$ where $\mathbf{A}$ is specified in some definite way (they suggest letting $\mathbf{A}$ be the geometric-optics solution) and $\mathbf{B}$ is to satisfy the radiation condition. For the scattering of a plane wave incident along the axis of a perfectly conducting paraboloid we get $\mathbf{B}\equiv 0$. M. Lelchter (private communication) has shown that the uniqueness proof of Peters and Stoker can be extended so as to include this problem.

† By requiring that $\mathbf{E}_1$ or U_1 is zero on the scattering surface, it is found that geometric-optics can only give the exact solution in case (for a plane wave incident along the axis of a body of revolution) $f^{(1)}/\rho[f^{(1)}-\rho f^{(2)}/(f^{(1)})^2+1]=0$ in the electromagnetic case or $f^{(1)}+\rho f^{(2)}/\rho=0$ in the acoustic case. The general solutions of these two equations are $f=A+B\rho^2$ and $f=A+B\ln\rho$ where A and B are arbitrary constants. $B=0$ gives an infinite plane in both cases. The general surface in the electromagnetic case is a paraboloid. The general surface in the acoustic case allows multiple reflections and the validity of geometric optics has not been tested for it.

[3] C. W. Horton and F. C. Karal, J. Acoust. Soc. Am. 22, 855 (1950).

[4] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 460–466.

[5] K. M. Siegel, *Far Zone Assumptions Put to Use, Symposium on Microwave Optics* (Eaton Electronics Laboratory, McGill University, Montreal, Canada, June, 1953).