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Spherically Symmetric Boundary-Value Problems in One-Speed Transport Theory*

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A Green's function technique is applied to one-velocity neutron problems with spherical symmetry. The main advantage of the approach is that it bypasses the need to construct explicitly the appropriate spherical eigenfunctions. Indeed, these can be directly deduced from this new formulation, if one so desires.

1. INTRODUCTION

The eigenfunction expansion technique has achieved considerable success in dealing with boundary-value problems in one-speed linear transport theory. The method developed by one of the authors,[1] though applied extensively to boundary-value problems with planar boundary conditions,[1–4] is readily adapted to more general geometrical configurations. The essential feature of this technique lies in constructing a complete set of eigenfunctions (normal modes) of the appropriate transport equation, expanding the neutron angular density in terms of the complete set, and finding the expansion coefficients from the boundary conditions. There are some drawbacks in this kind of treatment. They are, among others:

(i) The set of eigenfunctions (for example, the energy-dependent transport equation) may not form a complete set, which means one must construct appropriate additional functions to make the set complete.

(ii) In most cases it is not always easy to prove completeness.

In a recent paper by Case,[5] a fresh approach has been introduced. It draws on analogy with the Green's function technique in dealing with classical boundary-value problems. The advantages of this approach, among others, are:

(i) It incorporates the normal mode expansion technique in the scheme.

(ii) The eigenfunctions arise in a rather natural way, and thus the necessity of proving their completeness (if they form a complete set) is eliminated.

This paper utilizes the new approach to deal with various spherically symmetric boundary-value problems in one-speed transport theory. In particular, we treat albedo, critical, and Milne problems for the interior of a sphere, and the Milne problem for the exterior of a black sphere. In formulating the boundary-value problems for specific cases, we encounter an apparent difficulty in managing the regular integral equations by analytic methods. To circumvent this difficulty, we introduce a reduction operator which permits us to transform these regular integral equations into integral equations with singular kernels, but with the original coefficients. In other words, the reduction operator essentially reduces the spherical eigenfunctions in the integral equations to the planar ones. The resulting singular integral equations are then solved for the coefficients by the conventional method developed for planar problems.

2. CONSTRUCTION OF THE GREEN'S FUNCTION

The time-independent transport equation in the one-speed approximation is

$$(1 + \mathbf{\Omega}\cdot\mathbf{\nabla})\psi(\mathbf{r},\mathbf{\Omega}) = \frac{c}{4\pi}\int d\Omega'\psi(\mathbf{r},\mathbf{\Omega}') + Q(\mathbf{r}, Q), \tag{1}$$

where $\mathbf{\Omega} = \mathbf{v}/v$ is the unit velocity vector, $\mathbf{r}$ is the position vector, $Q(\mathbf{r},\mathbf{\Omega})$ is a given neutron-source function, and c is the average number of secondary neutrons per collision produced by a neutron of velocity $\mathbf{v}$. In this treatment we will assume that c is a known constant.

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In the treatment of the boundary-value problems, the standard way of incorporating the boundary conditions is to convert the differential equation into an integral equation. With this end in mind, consider the Green's function for the above transport equation:

$$(1 + \mathbf{\Omega} \cdot \mathbf{\nabla})G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_0, \mathbf{\Omega}_0) = \frac{c}{4\pi} \int d\Omega' G(\mathbf{r}, \mathbf{\Omega}', \mathbf{r}_0, \mathbf{\Omega}_0) + \delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0), \tag{2}$$

where $\mathbf{\Omega}_0$ is the direction of the monodirectional point source located at $\mathbf{r}_0$ and $\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0)$ is the surface δ function defined in the usual manner; i.e.,

$$\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0) = 0, \quad \mathbf{\Omega} \neq \mathbf{\Omega}_0, \tag{3a}$$

$$\int d\mathbf{\Omega} f(\mathbf{\Omega})\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0) = f(\mathbf{\Omega}_0), \tag{3b}$$

if somewhere in the domain of integration $\mathbf{\Omega}_0 = \mathbf{\Omega}$. By construction, the solution[5] of Eq. (1) is

$$\psi(\mathbf{r}, \mathbf{\Omega}) = \int d\Omega' \, d^3r' G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}', \mathbf{\Omega}')Q(\mathbf{r}', \mathbf{\Omega}') + \int dS \, d\Omega_s G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_s, \mathbf{\Omega}_s)\hat{\mathbf{n}}(r_s) \cdot \mathbf{\Omega}_s \psi(\mathbf{r}_s, \mathbf{\Omega}_s), \tag{3c}$$

where $\mathbf{r}_s$ and $\mathbf{\Omega}_s$ are position and velocity vectors of the neutron at the boundary surface, respectively, and $\hat{\mathbf{n}}(\mathbf{r}_s)$ is the corresponding normal pointing toward the region where the solution of the transport equation is being sought.

Let us now construct the Green's function by taking the Fourier transform of Eq. (2) with respect to $\mathbf{r}$; i.e., let

$$G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_0, \mathbf{\Omega}_0) = \frac{1}{(2\pi)^3} \int d^3k \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_0)]G_k(\mathbf{\Omega}, \mathbf{\Omega}_0). \tag{4}$$

Equation (2) then becomes

$$G_k(\mathbf{\Omega}, \mathbf{\Omega}_0) = \frac{1}{1 + i\mathbf{k} \cdot \mathbf{\Omega}} \frac{c}{4\pi} \int d\Omega' G_k(\mathbf{\Omega}', \mathbf{\Omega}_0) + \frac{\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0)}{1 + i\mathbf{k} \cdot \mathbf{\Omega}}. \tag{5}$$

By integrating both sides of Eq. (5) with respect to $\mathbf{\Omega}$, we obtain

$$\int d\Omega G_k(\mathbf{\Omega}, \mathbf{\Omega}_0) = \frac{1}{\Lambda(k) \cdot (1 + i\mathbf{k} \cdot \mathbf{\Omega})}, \tag{6}$$

where

$$\Lambda(k) = 1 - \frac{c}{4\pi} \int \frac{d\Omega}{1 + i\mathbf{k} \cdot \mathbf{\Omega}} \tag{7}$$

is the familiar dispersion function.

Substituting the integral in Eq. (5) by the expression (6), we get

$$G_k(\mathbf{\Omega} \cdot \mathbf{\Omega}_0) = [\Lambda(1 + i\mathbf{k} \cdot \mathbf{\Omega})(1 + i\mathbf{k} \cdot \mathbf{\Omega}_0)]^{-1} + \delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0)[1 + i\mathbf{k} \cdot \mathbf{\Omega}]^{-1}.$$

The Green's function is then simply given by

$$G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}_0, \mathbf{\Omega}_0) = \frac{1}{(2\pi)^3} \int d^3k \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_0)] \times \left[\frac{c}{4\pi\Lambda} [(1 + i\mathbf{k} \cdot \mathbf{\Omega})(1 + i\mathbf{k} \cdot \mathbf{\Omega}_0)]^{-1} + \frac{\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_0)}{1 + i\mathbf{k} \cdot \mathbf{\Omega}}\right]. \tag{8}$$

This is the fundamental Green's function which will serve to determine the solution of the one-speed transport equation for any given source and any incident distribution. We, therefore, turn to Eq. (3), which represents such a solution, and cast it in a more useable form.

Let us introduce the explicit expression for G [as given by Eq. (8)] in Eq. (3c) and rearrange the terms to obtain

$$\psi(\mathbf{r}, \mathbf{\Omega}) = \psi_q(\mathbf{r}, \mathbf{\Omega}) + \psi_0(\mathbf{r}, \mathbf{\Omega}) + \psi_c(\mathbf{r}, \mathbf{\Omega}), \tag{9}$$

where

$$\psi_q(\mathbf{r}, \mathbf{\Omega}) = \int d\Omega' \, d^3r' G(\mathbf{r}, \mathbf{\Omega}; \mathbf{r}', \mathbf{\Omega}')Q(\mathbf{r}', \mathbf{\Omega}') \tag{10}$$

is a known function,

$$\psi_0(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi^2} \int d^3k \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{1 + i\mathbf{k} \cdot \mathbf{\Omega}} H, \tag{11}$$

$$H = \frac{1}{2\pi} \int dS \, d\Omega_s \hat{\mathbf{n}}(\mathbf{r}_s) \cdot \mathbf{\Omega}_s \psi(\mathbf{r}_s, \mathbf{\Omega}_s)\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}_s)e^{-i\mathbf{k} \cdot \mathbf{r}_s} = \frac{1}{2\pi} \int dS \hat{\mathbf{n}}(\mathbf{r}_s) \cdot \mathbf{\Omega}_s \psi(\mathbf{r}_s, \mathbf{\Omega})e^{-i\mathbf{k} \cdot \mathbf{r}_s}, \tag{12}$$

$$\psi_c(\mathbf{r}, \mathbf{\Omega}) = \frac{c}{8\pi^2} \int d^3k \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{1 + i\mathbf{k} \cdot \mathbf{\Omega}} \frac{T}{\Lambda}, \tag{13}$$

and

$$T = \frac{1}{4\pi^2} \int dS \, d\Omega_s \hat{\mathbf{n}}(\mathbf{r}_s) \cdot \mathbf{\Omega}_s \psi(\mathbf{r}_s, \mathbf{\Omega}_s) \frac{e^{-i\mathbf{k} \cdot \mathbf{r}_s}}{1 + i\mathbf{k} \cdot \mathbf{\Omega}_s}. \tag{14}$$

From Eq. (9) we see that the solution of any boundary-value problem is known, provided we can find the surface distribution $\psi(\mathbf{r}_s, \mathbf{\Omega})$ or, equivalently, the coefficients T and H. Of course, if $\psi(\mathbf{r}_s, \mathbf{\Omega})$ is known *a priori*, then we are done. However, this is not always possible, for, in most instances, we only know either the incident or the outgoing distribution, but not both. It is, therefore, necessary to supplement Eq. (9) with another equation which determines $\psi(\mathbf{r}_s, \mathbf{\Omega})$. This is easily done by passing to the limit [in Eq. (9)] as $\mathbf{r} \to \mathbf{r}_s$ from within the region of interest; i.e.,

$$\psi(\mathbf{r}_s, \mathbf{\Omega}) = \psi_q(\mathbf{r}_s, \mathbf{\Omega}) + \psi_0(\mathbf{r}_s, \mathbf{\Omega}) + \psi_c(\mathbf{r}_s, \mathbf{\Omega}). \quad (15)$$

For further discussion of the equation that determines the surface distribution, see Ref. 5.

The rest of this paper is devoted to the study of the integral equation (9) in conjunction with Eq. (15), subject to various boundary conditions in spherically symmetric problems. Specifically, we shall consider two categories of problems, the interior and the exterior of a sphere.

3. SPHERICALLY SYMMETRIC PROBLEMS

By this we mean that the angular density ψ will be a function of r and $\mu = \hat{\mathbf{r}} \cdot \mathbf{\Omega}$, with $\mu > 0$ corresponding to the outgoing neutrons and $\mu < 0$ to the incoming ones. Under the spherical symmetry, the integral equation (9) is considerably simplified by carrying out the appropriate angular integrations. Thus, noting that T, given by Eq. (14), is a function of the magnitude of k, we can write

$$\psi_c(r, \mu) = \frac{c}{8\pi^2} \int_0^\infty dk k^2 \frac{T(k)}{\Lambda(k)} \int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{dt}{1 + ikt} \times \exp(ik\{t\mu + [(1-\mu^2)(1-t^2)]^{\frac{1}{2}} \cos(\varphi - \varphi_k)\}),$$

or

$$\psi_c(r, \mu) = \frac{c}{8\pi} \int_{-\infty}^{\infty} dk k^2 I(k, r, \mu) \frac{T(k)}{\Lambda(k)}, \quad (16)$$

where

$$I(k, r, \mu) = \int_{-1}^{1} \frac{dt}{1 + ikt} e^{iktr\mu} J_0\{kr[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}\}. \quad (17)$$

Similarly,

$$\psi_0(r, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk k^2 \int_{-1}^{1} \frac{dt}{1 + ikt} e^{iktr\mu} \times J_0\{kr[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}\} H(k, t), \quad (18)$$

where

$$H(k, t) = \int_{-1}^{1} d\mu' \mu' \psi(r_s, \mu') e^{-iktr_s\mu'} \times J_0\{kr_s[(1-\mu'^2)(1-t^2)]^{\frac{1}{2}}\} \quad (19)$$

and r_s denotes the radius of the sphere. The equation that determines the angular density is then

$$\psi(r, \mu) = \psi_q(r, \mu) + \psi_0(r, \mu) + \psi_c(r, \mu). \quad (20)$$

Before we express $\psi(r, \mu)$ (for interior and exterior problems) in terms of eigenfunctions of the transport equation, let us examine the analytical properties of $\Lambda(k)$, $I(k, r, u)$, and $T(k)$ in the complex k plane. From Eq. (7) it is clear that Λ is sectionally holomorphic in the k plane with the branch cuts extending from $-i\infty$ to $-i$ and i to $i\infty$. This property is shared by the functions $I(k, r, \mu)$ and $T(k)$. (We shall use this fact in constructing the eigenfunctions of the transport equation.) The zeros of $\Lambda(k)$ are either purely real or purely imaginary, depending on whether $c < 1$ or $c > 1$. It may seem peculiar, at first, that the angular density [or more appropriately, the Green's function given by Eq. (8)] is not uniquely determined when the zeros of Λ are real. However, we will show later (when we deal with the critical problem) that it is *not necessary* to prescribe any one particular recipe for treating the real zeros of Λ; that is, all prescriptions lead to a unique determination of the angular density. Finally, we note that, for complex values of k, the functions $I(k, r, \mu)$, $T(k)$, and $H(k, t)$ diverge at infinity. However, we show in the following sections that these functions can always be written as a sum of two, one of which converges in the upper half k plane and the other in the lower half.

A. Interior Problems

In Eq. (20), let us first consider $\psi_c(r, \mu)$ given by Eq. (16). We wish to express $\psi_c(r, \mu)$ in terms of eigenfunctions. To do that, we need to change the path of integration from the real axis to the contour surrounding the cut, as shown in Fig. 1. Since $r < r_s$, it is necessary to decompose $T(k)$ only. The decomposition is readily obtained by expanding the exponential in Eq. (14) in terms of spherical harmonics. Thus,

$$e^{-i\mathbf{k}\cdot\mathbf{r}_s} = 4\pi \sum_{n,m} i^n j_n(kr_s) Y^*_{nm}(\hat{\mathbf{k}} \cdot \mathbf{\Omega}_s) Y_{nm}(-\hat{\mathbf{r}}_s \cdot \mathbf{\Omega}_s). \quad (21)$$

The expression for $T(k)$ now becomes

$$T(k) = \frac{1}{\pi} \int dS\, d\Omega_s \hat{\mathbf{n}}(\mathbf{r}_s) \cdot \mathbf{\Omega}_s \frac{\psi(\mathbf{r}_s, \mu_s)}{1 + i\mathbf{k} \cdot \mathbf{\Omega}_s} \times \sum_{n,m} i^n j_n(kr_s) Y^*_{nm}(\hat{\mathbf{k}} \cdot \mathbf{\Omega}_s) Y_{nm}(-\hat{\mathbf{r}}_s \cdot \mathbf{\Omega}_s).$$

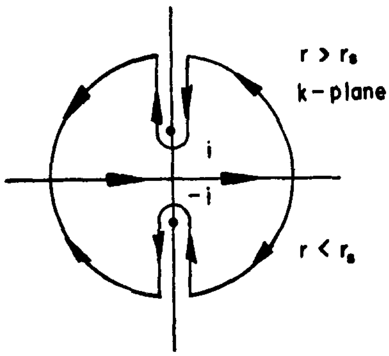

FIG. 1. Contours for $r > r_s$ and $r < r_s$.

Because of the azimuthal symmetry of the surface angular density $\psi(r_s, \mu_s)$, the only term in the sum over m which is nonzero is that for which $m = 0$. Hence,

$$T(k) = \int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s) \times \sum_{n=0}^{\infty} i^n(2n + 1) j_n(kr_s) P_n(-\mu_s) Q_n(k), \quad (22)$$

where $Q_n(k)$ is the Legendre function of the second kind defined by

$$Q_n(k) = \int_{-1}^{1} dt \frac{P_n(t)}{1 + ikt}. \quad (23)$$

Now the spherical Besssel function $j_n(z)$ can be expressed as a sum of Hankel functions of the first and the second kind. In Sommerfeld's notation,[6]

$$2j_n(z) = \zeta_n^{(1)}(z) + \zeta_n^{(2)}(z). \quad (24)$$

Putting (24) into (23), we get the following decomposition for $T(k)$:

$$T(k) = \tfrac{1}{2}[T_1(k) + T_2(k)], \quad (25)$$

where

$$T_{1,2}(k) = \int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s) \zeta^{(1),(2)}(k, r_s, -\mu_s) \quad (26)$$

and

$$\zeta^{(1),(2)}(k, r_s, -\mu_s) = \sum_{n=0}^{\infty} i^n(2n + 1) \zeta_n^{(1),(2)}(kr_s) P_n(-\mu_s) Q_n(k). \quad (27)$$

Let us note that

$$T_2(-k) = T_1(k). \quad (28)$$

The expression (16) for $\psi_c(r, \mu)$ may now be rewritten as a sum of two integrals by inserting the decomposition of $T(k)$ given by Eq. (25). One may, then, readily show that one of the integrals converges in the upper half k plane and the other in the lower half. To simplify the subsequent calculations, we use the relation (28) and write

$$\psi_c(r, \mu) = \frac{c}{8\pi} \int_{-\infty}^{\infty} dk k^2 \frac{T_1(k)}{\Lambda} I(k, r, \mu). \quad (29)$$

In much the same way, the expression for $\psi_0(r, \mu)$ [see Eq. (18)] may be cast in the following form:

$$\psi_0(r, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk k^2 \int_{-1}^{1} \frac{dt}{1 + ikt} e^{iktr\mu} \times J_0\{kr[(1 - \mu^2)(1 - t^2)]^{\frac{1}{2}}\} H_1(k, t), \quad (30)$$

where

$$H_1(k, t) = \int_{-1}^{1} d\mu' \mu' \psi(r_s, \mu') \times \sum_{n=0}^{\infty} i^n(2n + 1) \zeta_n^{(1)}(kr_s) P_n(-\mu') P_n(t). \quad (31)$$

Now we can change the path of integration from the real axis to a contour surrounding the cut in the upper half k plane. For $\psi_c(r, \mu)$, we have

$$\psi_c(r, \mu) = \frac{c}{8\pi} \int_{i}^{i\infty} dk k^2 \left(\frac{I^- T_1^-}{\Lambda^-} - \frac{I^+ T_1^+}{\Lambda^+} \right) + \text{d. c.} \quad (32)$$

where $-$ $(+)$ denotes the boundary value as we approach the cut from the right(left), and $D \cdot C$ is the discrete contribution arising from the zero of Λ. For the present we have assumed that $c < 1$; i.e., the zeros of $\Lambda(k)$ are purely imaginary. Let us simplify the integrand by constructing the eigenfunctions. This we do as follows. First we note that, from Plemelj's formula, we have

$$I^{\pm} = \mathcal{P} I \pm i\pi I_\delta, \quad (33)$$

where

$$\mathcal{P} I = \mathcal{P} \int_{-1}^{1} dt \frac{e^{iktr\mu}}{1 + ikt} J_0\{kr[(1 - \mu^2)(1 - t^2)]^{\frac{1}{2}}\}, \quad (34a)$$

$$I_\delta = \int_{-1}^{1} dt e^{iktr\mu} J_0\{kr[(1 - \mu^2)(1 - t^2)]^{\frac{1}{2}}\} \delta(1 + ikt) = \frac{i}{k} e^{-r\mu} J_0\{r[(1 - \mu^2)(k^2 + 1)]^{\frac{1}{2}}\}, \quad (34b)$$

and k is purely imaginary. Thus,

$$A = (I^- T_1^- / \Lambda^-) - (I^+ T_1^+ / \Lambda^+) = (T_1^- / \Lambda^-)(\mathcal{P} I + i\pi I_\delta) - (T_1^+ / \Lambda^+)(\mathcal{P} I - i\pi I_\delta).$$

To construct the eigenfunctions, we introduce $\lambda(k)$ in A as follows:

$$A = (T_1^- / \Lambda^-)[\mathcal{P} I + \lambda I_\delta - (\lambda - i\pi) I_\delta] - (T_1^+ / \Lambda^+)[\mathcal{P} I + \lambda I_\delta - (\lambda + i\pi) I_\delta].$$

Now choose λ such that

$$\phi(k, r, \mu) = \mathcal{P} I + \lambda I_\delta \quad (35)$$

are spherical eigenfunctions of the transport equation with continuous spectrum. By straightforward calculations, one may readily show that the appropriate λ is

$$\lambda = i\pi(\Lambda^+ + \Lambda^-)/(\Lambda^+ - \Lambda^-). \quad (36)$$

If we choose λ in this way, then A becomes

$$A = [(T_1^- / \Lambda^-) - (T_1^+ / \Lambda^+)] \phi(k, r, \mu) + 2\pi i (T_1^- - T_1^+ / \Lambda^- - \Lambda^+) I_\delta.$$

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[6] A. Sommerfeld, *Lectures on Theoretical Physics, Vol. 6: Partial Differential Equations in Physics* (Academic Press, New York, 1949).

Equation (32) for $\psi_c(r, \mu)$ now becomes

$$\psi_c(r, \mu) = \int_i^{i\infty} \frac{dk}{k} \phi(k, r, \mu)\Gamma_<(k) + \frac{ic}{4}\int_i^{i\infty} dkk^2 \frac{T_1^- - T_1^+}{\Lambda^- - \Lambda^+} I_\delta + \phi(ik_0, r, \mu)\Gamma_<^0(k_0), \quad (37)$$

where

$$\Gamma_<(k) = \frac{k^3 c}{8\pi}\left(\frac{T_1^-}{\Lambda^-} - \frac{T_1^+}{\Lambda^+}\right), \quad (38)$$

and we have written down the explicit expression for the discrete contribution in which

$$\Gamma_<^0(k_0) = -\frac{ic}{4} k_0^2 \frac{T_1(ik_0)}{\Lambda'(ik_0)}, \quad (39)$$

and

$$\phi(ik_0, r, \mu) = \int_{-1}^{1} \frac{dt}{1 - k_0 t} e^{-k_0 r\mu} I_0\{k_0 r[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}\} \quad (40)$$

is the discrete eigenfunction of the transport equation with the eigenvalue ik_0 the zero of $\Lambda(k)$; i.e.,

$$\Lambda(ik_0) = 0, \quad 0 < k_0 < 1.$$

For $\psi_0(r, \mu)$ [see Eq. (30)], we get

$$\psi_0(r, \mu) = \frac{2\pi i}{4\pi}\int_i^{i\infty} dkk^2 \int_{-1}^{1} dt \times e^{iktr\mu} J_0\{kr[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}\} \times \delta(1 + ikt)H(k, t),$$

or

$$\psi_0(r, \mu) = -\tfrac{1}{2}e^{-r\mu}\int_i^{i\infty} dkk J_0\{r[(1-\mu^2)(k^2+1)]^{\frac{1}{2}}\} \times H\left(k, -\frac{1}{ik}\right). \quad (41)$$

Now one may easily show that $\psi_0(r, \mu)$, as given by Eq. (41), is equal to minus the second term on the right-hand side of Eq. (37). With this in mind, Eq. (20), for the angular density $\psi(r, \mu)$, becomes

$$\psi(r, \mu) = \psi_q(r, \mu) + \int_0^1 d\nu \frac{1}{\nu} \Gamma_<\left(\frac{i}{\nu}\right)\phi\left(\frac{i}{\nu}, r, \mu\right) + \phi\left(\frac{i}{\nu_0}, r, \mu\right)\Gamma_<^0\left(\frac{1}{\nu_0}\right), \quad (42)$$

where we have set $k = i/\nu$ and $k_0 = i/\nu_0$.

An equation that determines the coefficients $\Gamma_<$ and $\Gamma_<^0$ is obtained by letting $r \to r_s$ in Eq. (42). Thus,

$$\psi(r_s, \mu) = \psi_q(r_s, \mu) + \int_0^1 d\nu \frac{1}{\nu} \Gamma_<\left(\frac{i}{\nu}\right)\phi\left(\frac{i}{\nu}, r_s, \mu\right) + \phi\left(\frac{i}{\nu_0}, r_s, \mu\right)\Gamma_<^0\left(\frac{1}{\nu_0}\right). \quad (43)$$

This is a regular integral equation. Its solution is difficult to discuss. Therefore, we shall seek the help of an operator (the reduction operator) which, when applied to Eq. (20), produces an auxiliary equation with the same coefficients as that in Eq. (43), but with a singular kernel. Before we present such an operator, let us first consider the class of exterior problems.

B. Exterior Problems

Now since $r > r_s$, in order to express $\psi_c(r, \mu)$ [see Eq. (16)] in terms of eigenfunctions by the change of path of integration to a contour surrounding the cut in the upper half plane, we need to decompose $I(k, r, \mu)$ [see Eq. (17)] and then follow the same procedure as that in the interior problems. Thus, to decompose $I(k, r, \mu)$, we write

$$e^{iktr\mu}J_0(kr[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}) = \tfrac{1}{2}\sum_{n=0}^{\infty} i^n(2n+1)[\zeta_n^{(1)}(kr) + \zeta_n^{(2)}(kr)]P_n(\mu)P_n(t), \quad (44)$$

where we have used the decomposition (24) of the spherical Bessel function. Then,

$$I(k, r, \mu) = \tfrac{1}{2}[\zeta^{(1)}(k, r, \mu) + \zeta^{(2)}(k, r, \mu)], \quad (45)$$

where $\zeta^{(1),(2)}(k, r, \mu)$ is given by (27) with r_s replaced by r and $-\mu_s$ by μ. Equation (16) for $\psi_c(r, \mu)$ now becomes

$$\psi_c(r, \mu) = \frac{c}{8\pi}\int_{-\infty}^{\infty} dkk^2 \zeta^{(1)}(k, r, \mu)\frac{T(k)}{\Lambda(k)}. \quad (46)$$

The resulting integral equation for $\psi(r, \mu)$ is given by

$$\psi(r, \mu) = \psi_q(r, \mu) + \int_0^1 d\nu(1/\nu)\Gamma_>(i/\nu)Z(i/\nu, r, \mu) + Z(i/\nu_0, r, \mu)\Gamma_>^0(1/\nu_0). \quad (47)$$

An equation that determines the coefficients $\Gamma_>(i/\nu)$ and $\Gamma_>^0(1/\nu_0)$ is

$$\psi(r_s, \mu_s) = \psi_q(r_s, \mu) + \int_0^1 d\nu(1/\nu)\Gamma_>(i/\nu)Z(i/\nu, r_s, \mu) + Z(i/\nu_0, r_s, \mu)\Gamma_>^0(1/\nu_0), \quad (48)$$

where

$$\Gamma_>(i/\nu) = \frac{c}{8\pi i\nu^3}\left[\frac{T^-(i/\nu)}{\Lambda^-(i/\nu)} - \frac{T^+(i/\nu)}{\Lambda^+(i/\nu)}\right], \quad (49)$$

$$Z(i/\nu, r, \mu) = \sum_{n=0}^{\infty} i^n(2n+1)\zeta_n^{(1)}(ir/\nu)P_n(\mu)\left[\mathcal{P}\int_{-1}^{1}\frac{dtP_n(t)\nu}{\nu - t} + \nu P_n(\nu)i\pi\frac{\Lambda^+ + \Lambda^-}{\Lambda^+ - \Lambda^-}\right], \quad (50)$$

and

$$\Gamma_>^0(1/\nu_0) = \frac{-ic}{4\nu_0^2}\frac{T(i/\nu_0)}{\Lambda'(i/\nu_0)}. \quad (51)$$

In Eq. (47), the regular eigenfunctions occur implicitly in the coefficients $\Gamma_>(i/\nu)$ and $\Gamma_>^0(1/\nu_0)$. Thus, from Eq. (22) for $T(k)$, we have

$$T(k) = \int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s) I(k, r_s, -\mu_s).$$

Therefore,

$$\frac{T^-(k)}{\Lambda^-} - \frac{T^+(k)}{\Lambda^+} = \int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s) \left[\frac{I^-(k, r_s, -\mu_s)}{\Lambda^-} - \frac{I^+(k, r_s, -\mu_s)}{\Lambda^+}\right],$$

where k is a purely imaginary number (i.e., $k = i/\nu$). Now, by definition,

$$\begin{aligned}
&[I^-(k, r_s, -\mu_s)/\Lambda^-] - [I^+(k, r_s, -\mu_s)/\Lambda^+] \\
&\quad = (1/\Lambda^-)[\mathcal{P}I(k, r_s, -\mu_s) + i\pi I_\delta(k, r_s, -\mu_s)] \\
&\qquad - (1/\Lambda^+)[\mathcal{P}I(k, r_s, -\mu_s) - i\pi I_\delta(k, r_s, -\mu_s)] \\
&\quad = (1/\Lambda^+\Lambda^-)\{\mathcal{P}I(k, r_s, -\mu_s) \\
&\qquad + i\pi[(\Lambda^+ + \Lambda^-)/(\Lambda^+ - \Lambda^-)]I_\delta(k, r_s, -\mu_s)\} \\
&\quad = (1/\Lambda^+\Lambda^-)\phi(k, r_s, -\mu_s).
\end{aligned}$$

We have already mentioned that $\phi(k, r, \mu)$, as given by Eq. (35), are the regular eigenfunctions of the transport equation with continuous spectrum. Here,

$$\phi(k, r_s, -\mu_s) = \mathcal{P}I(k, r_s, -\mu_s) + i\pi[(\Lambda^+ + \Lambda^-)/(\Lambda^+ - \Lambda^-)]I_\delta(k, r_s, -\mu_s) \quad (52)$$

are the regular eigenfunctions of the adjoint equation with the same continuous spectrum. Thus, explicitly,

$$\Gamma_>\left(\frac{i}{\nu}\right) = \frac{c}{8\pi i\nu^3}\int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s)\frac{1}{\Lambda^+\Lambda^-}\phi\left(\frac{i}{\nu}, r_s, -\mu_s\right). \quad (53)$$

Similarly,

$$\Gamma_>^0\left(\frac{1}{\nu_0}\right) = \frac{-ic}{4\nu_0^2}\frac{1}{\Lambda'(i/\nu_0)} \times \int_{-1}^{1} d\mu_s \mu_s \psi(r_s, \mu_s)\phi\left(\frac{i}{\nu_0}, r_s, -\mu_s\right), \quad (54)$$

where

$$\phi\left(\frac{i}{\nu_0}, r_s, \mu_s\right) = \nu_0\int_{-1}^{1}\frac{dt}{\nu_0 - t} e^{r_s\mu_s/\nu_0} I_0\left\{\frac{r_s}{\nu_0}[(1-\mu_s^2)(1-t^2)]^{\frac{1}{2}}\right\} \quad (55)$$

is the regular discrete eigenfunction of the adjoint equation with the point spectrum ν_0 ($ik_0 = i/\nu_0$ being the zero of Λ).

To solve the integral equation (48) for the coefficients, we shall seek the help of the reduction operator. In the next section, we present such an operator.

4. THE REDUCTION OPERATOR

Let us consider an operator θ, given by

$$\theta \equiv \lim_{r\to r_s}\int_{-\infty}^{\infty} dr' r' K(r - r', \mu) \times \left(1 + \mu\frac{\partial}{\partial r'} + \frac{1-\mu^2}{r'}\frac{\partial}{\partial\mu}\right), \quad (56)$$

where the kernel $K(r - r', \mu)$ is

$$K(r - r', \mu) = (1/\mu)e^{-(r-r')/\mu}\Theta(r - r')\Theta(\mu) - (1/\mu)e^{-(r-r')/\mu}\Theta(r' - r)\Theta(-\mu). \quad (57)$$

Let us write the operator θ, formally, as

$$\theta \equiv \lim_{r\to r_s} KS, \quad (58)$$

where

$$S \equiv 1 + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}$$

is the streaming part of the transport operator.

A. Application of the Operator θ to Interior Problems

Owing to the linearity of the operator θ, its application to Eq. (20) gives us

$$\theta\psi(r, \mu) = \theta\psi_a(r, \mu) + \theta\psi_0(r, \mu) + \theta\psi_c(r, \mu). \quad (59)$$

First, consider the left-hand side of Eq. (59). Since θ is a product of two operators [Eq. (58)], K and S, let us apply S first. Thus,

$$\begin{aligned}
S\psi(r,\mu)\Theta(r_s - r) &= \psi(r,\mu)\Theta(r_s - r) + \mu\Theta(r_s - r)\frac{\partial\psi}{\partial r} \\
&\quad - \mu\psi\delta(r_s - r) + \frac{1-\mu^2}{r}\Theta(r_s - r)\frac{\partial\psi}{\partial\mu}.
\end{aligned}$$

For $\mu < 0$,

$$\theta[\psi(r, \mu)\Theta(r_s - r)] = r_s\psi(r_s, \mu). \quad (60)$$

Next consider $\theta\psi_c$. We have, from Eq. (16),

$$\theta\psi_c(r, \mu) = \frac{c}{8\pi}\int_{-\infty}^{\infty} dk k^2 \frac{T(k)}{\Lambda(k)}[\theta I(k, r, \mu)]. \quad (61)$$

Now,

$$SI(k, r, \mu) = (2\sin kr)/kr.$$

Therefore, for $\mu < 0$,

$$KSI(k, r, \mu) = -\frac{2}{r}\int_{-\infty}^{\infty} dr' r' e^{-(r-r')/\mu}\Theta(r' - r)\frac{\sin kr'}{kr'}$$

or

$$KSI(k, r, \mu) = \frac{i}{k}\left[-\frac{e^{ikr}}{1 + ik\mu} + \frac{e^{-ikr}}{1 - ik\mu}\right]. \quad (62)$$

If we put Eq. (62) into Eq. (60), the result is

$$\theta\psi_c(r,\mu) = \frac{ic}{8\pi}\lim_{r\to r_s}\int_{-\infty}^{\infty} dkk\,\frac{T(k)}{\Lambda(k)} \times\left[-\frac{e^{ikr}}{1+ik\mu}+\frac{e^{-ikr}}{1-ik\mu}\right]. \quad (63)$$

Now, decomposing $T(k)$ into the sum of $T_1(k)$ and $T_2(k)$ as before [see Eq. (25)] and following the same procedure of integration around the cut in the upper half k plane, we obtain

$$\begin{aligned}\theta\psi_c(r,\mu) &= \frac{ic}{8\pi}\int_i^{i\infty} dkk[\phi^0(-k,r_s,\mu)-\phi(k,r_s,\mu)] \times\left(\frac{T_1^-}{\Lambda^-}-\frac{T_1^+}{\Lambda^+}\right)\\ &- \frac{c}{4}\int_i^{i\infty} dkk[e^{-ikr_s}\delta(1-ik\mu) - e^{ikr_s}\delta(1+ik\mu)]\left(\frac{T_1^- - T_1^+}{\Lambda^- - \Lambda^+}\right) + \text{d.c.}\end{aligned} \quad (64)$$

where

$$\begin{aligned}\phi^0(k,r_s,\mu) = e^{ikr_s}\{&\mathcal{T}(1+ik\mu)^{-1}\\ &+ i\pi[(\Lambda^+ + \Lambda^-)/(\Lambda^+ - \Lambda^-)]\delta(1+ik\mu)\}\end{aligned} \quad (65)$$

are the planar eigenfunctions corresponding to the continuous k spectrum. Also, for $\mu < 0$, these eigenfunctions are regular. However, if the sign of k is reversed, then these planar eigenfunctions become irregular.

In Eq. (64), let us introduce the coefficients $\Gamma_<$ and $\Gamma_<^0$ as given by Eqs. (38) and (39), respectively. Thus,

$$\begin{aligned}\theta\psi_c(r,\mu) &= i\int_i^{i\infty}\frac{dk}{k^2} \times[\phi^0(-k,r_s,\mu)-\phi^0(k,r_s,\mu)]\Gamma_<(k)\\ &- \frac{c}{4}\int_i^{i\infty} dkk\,\frac{T_1^- - T_1^+}{\Lambda^- - \Lambda^+} \times[e^{-ikr_s}\delta(1-ik\mu) - e^{ikr_s}\delta(1+ik\mu)]\\ &+ \left(\frac{1}{k_0}\right)[\phi^0(-ik_0,r_s,\mu) - \phi^0(ik_0,r_s,\mu)]\Gamma_<(k_0),\end{aligned} \quad (66)$$

where

$$\phi^0(ik_0,r_s,\mu) = e^{-ik_0r_s}/(1-k_0\mu), \quad 0<k_0<1, \quad (67)$$

is the planar discrete eigenfunction corresponding to the point spectrum k_0.

One may readily show again that the second term on the right-hand side of Eq. (66) is equal to $-\theta\psi_0(r,\mu)$. Consequently, the reduced angular density given by Eq. (59) may now be written as follows:

$$\begin{aligned}r_s\psi(r_s,\mu) = \theta\psi_Q(r,\mu)\int_0^1 d\nu&\left[\phi^0\left(-\frac{i}{\nu},r_s,\mu\right) - \phi^0\left(\frac{i}{\nu},r_s,\mu\right)\right]\Gamma_<\left(\frac{i}{\nu}\right)\\ &+ \nu_0\left[\phi^0\left(-\frac{i}{\nu_0},r_s,\mu\right) - \phi^0\left(\frac{i}{\nu_0},r_s,\mu\right)\right] \times\Gamma_<^0\left(\frac{i}{\nu_0}\right), \quad \mu<0.\end{aligned} \quad (68)$$

Putting the explicit forms of the planar eigenfunctions in Eq. (67), we re-express this equation in the standard notation.[1] Thus,

$$\begin{aligned}&\frac{1}{2}\left[\Lambda^+\left(-\frac{i}{\mu}\right)+\Lambda^-\left(-\frac{i}{\mu}\right)\right]\tilde{\Gamma}_<\left(-\frac{i}{\mu}\right)\\ &+\frac{1}{2\pi i}\left[\Lambda^+\left(-\frac{i}{\mu}\right)-\Lambda^-\left(-\frac{i}{\mu}\right)\right]\mathcal{T}\int_0^1\frac{d\nu}{\nu+\mu}\,\tilde{\Gamma}_<\left(\frac{i}{\nu}\right)\\ &= \left[\Lambda^+\left(-\frac{i}{\nu}\right)-\Lambda^-\left(-\frac{i}{\nu}\right)\right]f(\mu), \quad \mu<0,\end{aligned} \quad (69)$$

where

$$\begin{aligned}f(\mu) = \frac{1}{2\pi i}\Bigg[&\nu_0^2\Gamma_<^0\left(\frac{1}{\nu_0}\right)\left(\frac{e^{-r_s/\nu_0}}{\nu_0-\mu}-\frac{e^{r_s/\nu_0}}{\nu_0+\mu}\right)\\ &+\int_0^1\frac{d\nu}{\nu-\mu}\,e^{-2r_s/\nu}\tilde{\Gamma}_<\left(\frac{i}{\nu}\right)\\ &+ r_s\psi(r_s,\mu) - \theta\psi_Q(r,\mu)\Bigg]\end{aligned} \quad (70)$$

and

$$\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) = \nu e^{r_s/\nu}\Gamma_<\left(\frac{i}{\nu}\right). \quad (71)$$

Equation (69) is the auxiliary equation which can be solved for the coefficients $\tilde{\Gamma}_<$ and $\Gamma_<^0$ by the standard technique developed for planar problems.[1] We note that the existence of such a solution guarantees the completeness of the regular spherical eigenfunctions given by Eqs. (35) and (40) in the sense stated in the following theorem.

Theorem 1: "Any" function $\psi(r,\mu)$ in the domain $-\mu < 1$ and regular at $r = 0$ may be expanded in terms of the regular spherical eigenfunctions $\phi(i/\nu, r,\mu)$ and $\phi^0(i/\nu_0, r,\mu)$, corresponding to the continuous spectrum $0<\nu<1$ and the point spectrum ν_0.

Let us mention that this is the half-range completeness of the eigenfunctions. The proof of the theorem is demonstrated by constructing the coefficients $\Gamma_<$ and $\Gamma_<^0$ from Eq. (69). For further details on this we refer the reader to Ref. 1. Here we shall merely use the results of such a solution in various specific boundary-value problems. Before we do that, however, let us complete our presentation by giving the analog of Eq. (69) for the class of exterior problems.

B. Application of the Operator θ to Exterior Problems

Consider the left-hand side of Eq. (59). For $r > r_s$, we have

$$S\{\psi(r,\mu)\Theta(r-r_s)\} = \psi(r,\mu)\Theta(r-r_s) + \mu\Theta(r-r_s)\frac{\partial\psi}{\partial r} + \mu\psi\delta(r-r_s) + \frac{1-\mu^2}{r}\Theta(r-r_s)\frac{\partial\psi}{\partial\mu}.$$

For $\mu > 0$, we get

$$\theta\{\psi(r,\mu)\Theta(r-r_s)\} = r_s\psi(r_s,\mu). \quad (72)$$

In the right-hand side of Eq. (59) consider $\theta\psi_c$:

$$\theta\psi_c(r,\mu) = \frac{c}{8\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2}\frac{T(k)}{\Lambda(k)}[\theta I(k,r,\mu)].$$

For $\mu > 0$, we have

$$KSI(k,r,\mu) = K\frac{2\sin kr}{kr} = \frac{2}{k\mu}\int_{-\infty}^{r} dr' e^{-(r-r')/\mu}\sin kr' = \frac{1}{ki}\left[\frac{e^{ikr}}{1+ik\mu} - \frac{e^{-ikr}}{1-ik\mu}\right].$$

Therefore,

$$\theta\psi_c(r,\mu) = \frac{c}{8\pi i}\lim_{r\to r_s}\int_{-\infty}^{\infty} dkk\frac{T(k)}{\Lambda} \times\left[\frac{e^{ikr}}{1+ik\mu} - \frac{e^{-ikr}}{1-ik\mu}\right],$$

or

$$\theta\psi_c(r,\mu) = \frac{c}{4\pi i}\lim_{r\to r_s}\int_{-\infty}^{\infty} dkk\frac{T(k)}{\Lambda}\frac{e^{ikr}}{1+ik\mu}. \quad (73)$$

Again, by changing the path of integration to a contour surrounding the cut in the upper half k plane, we cast Eq. (73) into the form given below:

$$\theta\psi_c(r,\mu) = \frac{c}{4\pi i}\int_i^{i\infty} dkk\phi^0(k,r_s,\mu)\left(\frac{T^-}{\Lambda^-} - \frac{T^+}{\Lambda^+}\right) + \frac{c}{2}\int_i^{i\infty} dkke^{ikr_s}\delta(1+ik\mu) \times\frac{T^- - T^+}{\Lambda^- - \Lambda^+} + \text{d. c.} \quad (74)$$

where $\phi^0(k,r_s,\mu)$ is the planar eigenfunction represented by Eq. (65). Now let us introduce the coefficients $\Gamma_>$ and $\Gamma_>^0$ defined by Eqs. (50) and (51), respectively, and put the form of $\theta\psi_c$ given by Eq. (73) into Eq. (59). We obtain

$$r_s\psi(r_s,\mu) = \theta\psi_q(r,\mu) - 2\int_0^1 d\nu\Gamma_>\left(\frac{i}{\nu}\right)\phi^0\left(\frac{i}{\nu},r_s,\mu\right) - 2\nu_0\phi^0\left(\frac{i}{\nu_0},r_s,\mu\right)\Gamma_>^0\left(\frac{i}{\nu_0}\right), \quad (75)$$

where $\phi^0(i/\nu_0,r_s,\mu)$ is the discrete planar eigenfunction given by Eq. (67).

Let us cast Eq. (74) into a more usable form by introducing the explicit expressions for the eigenfunctions. Thus,

$$\frac{1}{2}\left[\Lambda^+\left(\frac{i}{\mu}\right) + \Lambda^-\left(\frac{i}{\mu}\right)\right]\tilde{\Gamma}_>\left(\frac{i}{\mu}\right) + \frac{1}{2\pi i}\left[\Lambda^+\left(\frac{i}{\mu}\right) - \Lambda^-\left(\frac{i}{\mu}\right)\right]\mathcal{P}\int_0^1\frac{d\nu}{\nu-\mu}\tilde{\Gamma}_>\left(\frac{i}{\nu}\right) = (\Lambda^+ - \Lambda^-)g(\mu), \quad \mu > 0, \quad (76)$$

where

$$g(\mu) = \frac{1}{4\pi i}\left[-\nu_0^2\Gamma_>^0\left(\frac{i}{\nu_0}\right)e^{-r_s/\nu_0}/(\nu_0-\mu) + \theta\psi_q(r,\mu) - r_s\psi(r_s,\mu)\right] \quad (77)$$

and

$$\tilde{\Gamma}_>(i/\mu) = \mu e^{-r_s/\mu}\Gamma_>(i/\mu). \quad (78)$$

In the next section, we consider some specific interior and exterior problems.

5. APPLICATIONS

In the integral equation (69), we notice that, except for the terms involving the incident distribution and the distribution due to source(s), the rest of the features are common to all interior problems. This is also true for Eq. (76) for the class of exterior problems. For this reason it is convenient to write down the most general solutions for the corresponding coefficients and treat just the distinguishing part separately for each problem. Thus, for the class of interior problems, the solution of the singular integral equation (69) is[1]

$$\tilde{\Gamma}_>\left(\frac{i}{\mu}\right) = -\frac{1}{2\pi i}\left[\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right]\left\{\nu_0^2\Gamma_>^0\left(\frac{1}{\nu_0}\right) \times\left[e^{r_s/\nu_0}\frac{X(\nu_0)}{\mu+\nu_0} - e^{-r_s/\nu_0}\frac{X(-\nu_0)}{\mu-\nu_0}\right] + \int_0^1 d\nu e^{-2r_s/\nu}\frac{\tilde{\Gamma}_<(i/\nu)}{\nu-\mu}X(-\nu) + \int_0^1 d\mu'\frac{X^+(\mu')-X^-(\mu')}{\mu'+\mu} \times[r_s\psi(r_s,-\mu') - \theta\psi_q(r,-\mu')]\right\}, \quad \mu < 0. \quad (79)$$

The equation that determines $\Gamma^0_<$ is

$$\nu_0^2\Gamma^0_<\left(\frac{1}{\nu_0}\right)[X(\nu_0)e^{-r_s/\nu_0} - X(-\nu_0)e^{-r_s/\nu_0}] - \int_0^1 d\nu X(-\nu)\exp(-2r_s/\nu)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) - \int_0^1 d\mu'[X^+(\mu') - X^-(\mu')] \times [r_s\psi(r_s, -\mu') - \theta\psi_q(r, -\mu')] = 0, \quad (80)$$

where

$$X(z) = \frac{1}{1-z}\exp\left[\frac{1}{\pi}\int_0^1 d\mu \frac{\arg\Lambda^+(\mu)}{\mu - z}\right]. \quad (81)$$

Similarly, for the exterior problems, Eq. (76) has the solution

$$\tilde{\Gamma}_>\left(\frac{i}{\mu}\right) = -\frac{1}{4\pi i}\left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)}\right) \times \left\{-\nu_0^2\Gamma^0_>\left(\frac{1}{\nu_0}\right)e^{-r_s/\nu_0}\frac{X(+\nu_0)}{\mu - \nu_0} + \int_0^1 d\mu' \frac{X^+(\mu') - X^-(\mu')}{\mu' - \mu} \times [\theta\psi(r, \mu') - r_s\psi(r_s, \mu')]\right\}, \quad \mu > 0. \quad (82)$$

The equation that determines $\Gamma^0_>$ is

$$\nu_0^2\Gamma^0_>\left(\frac{1}{\nu_0}\right)X(+\nu_0)e^{-r_s/\nu_0} + \int_0^1 d\mu'[X^+(\mu') - X^-(\mu')] \times [\theta\psi_q(r, \mu') - r_s\psi(r_s, \mu')] = 0. \quad (83)$$

Let us now consider some specific problems and determine the coefficients $\tilde{\Gamma}_{\lessgtr}$ and $\Gamma^0_{\lessgtr}$ explicitly. We treat the interior problems first.

A. The Albedo Problem

The albedo problem involves the determination of neutron angular density everywhere inside the source-free sphere $[\psi_q(r, \mu) \equiv 0]$ with an incident distribution given by

$$\psi(r_s, \mu) = (r_s^2)^{-1}\delta(\mu - \mu_0), \quad \mu < 0, \quad \mu_0 < 0. \quad (84)$$

Under this boundary condition, Eq. (79) for $\tilde{\Gamma}_<$ becomes

$$\tilde{\Gamma}\left(\frac{i}{\mu}\right) = -\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right)\left\{\nu_0^2\Gamma^0_<\left(\frac{1}{\nu_0}\right) \times \left[\exp\left(\frac{r_s}{\nu_0}\right) - \exp\left(\frac{-r_s}{\nu_0}\right)\frac{X(-\nu_0)}{\mu - \nu_0}\right] + \int_0^1 d\nu \exp(-2r_s/\nu)\frac{\tilde{\Gamma}_<(i/\nu)}{\nu - \mu}X(-\nu) + \frac{1}{r_s}\frac{X^+(-\mu_0) - X^-(-\mu_0)}{\mu - \mu_0}\right\}. \quad (85)$$

For $\Gamma^0_<$, we have

$$\nu_0^2\Gamma^0_<\left(\frac{1}{\nu_0}\right)\left[X(\nu_0)\exp\left(\frac{r_s}{\nu_0}\right) - X(-\nu_0)\exp\left(-\frac{r_s}{\nu_0}\right)\right] - \int_0^1 d\nu X(-V)\exp\left(\frac{-2r_s}{\nu}\right)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) - \frac{1}{r_s}[X^+(-\mu_0) - X^-(-\mu_0)] = 0. \quad (86)$$

Equations (85) and (86) are well suited for asymptotic expansions of the angular density. Thus, for a large sphere, one may neglect the integral term in Eq. (85) involving $\exp(-2r_s/\nu)$, and solve the equation by iteration. In particular, in the zeroth approximation, we have

$$\tilde{\Gamma}_<\left(\frac{i}{\mu}\right) \cong -\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right) \times \left\{\nu_0^2\Gamma^0_<\left(\frac{1}{\nu_0}\right)\left[\exp\left(\frac{r_s}{\nu_0}\right)\frac{X(\nu_0)}{\mu + \nu_0} - \exp\left(\frac{-r_s}{\nu_0}\right)\frac{X(-\nu_0)}{\mu - \nu_0}\right] + \frac{1}{r_s}\frac{X^+(-\mu_0) - X^-(-\mu_0)}{\mu - \mu_0}\right\} = 0 \quad (87)$$

and

$$\nu_0^2\Gamma^0_<\left(\frac{1}{\nu_0}\right) \doteq \frac{[X^+(-\mu_0) - X^-(-\mu_0)]}{r_s[X(\nu_0)\exp(r_s/\nu_0) - X(-\nu_0)\exp(-r_s/\nu_0)]}. \quad (88)$$

Now, by eliminating $\nu_0^2\Gamma^0_<(1/\nu_0)$ from Eq. (87), we obtain the explicit expression for $\tilde{\Gamma}_<$. The angular density is then readily obtained from Eq. (42).

B. Milne Problem for the Interior of a Sphere

The Milne problem involves the determination of neutron angular density everywhere inside the sphere with a source at the center and zero incident distribution. Thus, the boundary condition is

$$\psi(r_s, \mu) = 0, \quad \mu < 0. \quad (89)$$

We assume an isotropic source; i.e.,

$$Q(\mathbf{r}, \mathbf{\Omega}) = \delta(\mathbf{r}). \quad (90)$$

Putting this expression for Q into Eq. (10), we obtain

$$\psi_Q(r,\mu) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} \frac{dkk^2}{\Lambda} I(k,r,\mu), \tag{91}$$

where $I(k, r, \mu)$ is given by Eq. (17).

As before, we may express ψ_Q in terms of boundary values about the branch cut in the upper half k plane by first decomposing $I(k, r, \mu)$ as shown in Eq. (45). To avoid repetition of calculations, we merely state the answer. Thus,

$$\psi_Q(r,\mu) = -\frac{c}{8\pi}\int_i^{i\infty} \frac{dkk}{\Lambda^+\Lambda^-} Z(k,r,\mu) - \frac{ik_0^2}{4\pi\Lambda'(ik_0)}\zeta^{(1)}(ik_0,r,\mu), \tag{92}$$

where $Z(k, r, \mu)$ and $S^{(1)}(ik_0, r, \mu)$ are given by Eqs. (50) and (27), respectively. Put $k = i/\nu$ and $k_0 = 1/\nu_0$ to get

$$\psi_Q(r,\mu) = \frac{c}{8\pi}\int_0^1 \frac{d\nu}{\nu^3\Lambda^+\Lambda^-} Z\left(\frac{i}{\nu}, r, \mu\right) + \frac{1}{4\pi^2 i\nu_0\Lambda'}\zeta^{(1)}\left(\frac{i}{\nu_0}, r, \mu\right). \tag{93}$$

After the application of the reduction operator to Eq. (91) for $\mu < 0$, we get

$$\theta\psi_Q(r,\mu) = \frac{c}{4\pi}\int_0^1 \frac{d\nu}{\nu\Lambda^+\Lambda^-}\frac{e^{-r_s/\nu}}{\nu-\mu} + \frac{1}{2\pi i\Lambda'}\frac{e^{-r_s/\nu_0}}{\nu_0-\mu}, \quad \mu < 0. \tag{94}$$

Now let us subject Eq. (79) to the boundary condition (92) and insert the expression (94) for $\theta\psi_Q$. The result is

$$\begin{aligned}\tilde{\Gamma}_<\left(\frac{i}{\mu}\right) = &-\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right) \\ &\times \left\{\nu_0^2\Gamma_<^0\left(\frac{1}{\nu_0}\right)\left[\exp\left(\frac{r_s}{\nu_0}\right)\frac{X(\nu_0)}{\mu+\nu_0} - \exp\left(\frac{-r_s}{\nu_0}\right)\frac{X(-\nu_0)}{\mu-\nu_0}\right]\right. \\ &- \frac{1}{2\pi i\Lambda'}\exp\left(\frac{-r_s}{\nu_0}\right)\frac{X(-\nu_0)}{\mu-\nu_0} \\ &+ \int_0^1 d\nu\exp\left(\frac{-2r_s}{\nu}\right)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right)\frac{X(-\nu)}{\nu-\mu} \\ &\left.+ \frac{c}{4\pi}\int_0^1 \frac{d\nu}{\nu\Lambda^+\Lambda^-}\exp\left(\frac{-r_s}{\nu}\right)\frac{X(-\nu)}{\nu-\mu}\right\}, \\ &\mu < 0. \end{aligned} \tag{95}$$

From Eq. (79), we get

$$\begin{aligned}&\nu_0^2\Gamma_<^0\left(\frac{1}{\nu_0}\right)\left[\exp\left(\frac{r_s}{\nu_0}\right)X(\nu_0) - \exp\left(\frac{-r_s}{\nu_0}\right)X(-\nu_0)\right] \\ &- \frac{1}{2\pi i\Lambda'}\exp\left(\frac{-r_s}{\nu_0}\right)X(-\nu_0) \\ &- \int_0^1 d\nu\exp\left(\frac{-2r_s}{\nu}\right)X(-\nu)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) \\ &- \frac{c}{4\pi}\int_0^1 \frac{d\nu}{\nu\Lambda^+\Lambda^-}\exp\left(\frac{-r_s}{\nu}\right)X(-\nu) = 0. \end{aligned} \tag{96}$$

For a large sphere, we may neglect the integrals in Eqs. (95) and (96) involving $\exp(-2r_s/\nu)$. Thus, in the zeroth approximation, Eq. (96) becomes

$$\begin{aligned}\nu_0^2\Gamma_<^0\left(\frac{1}{\nu_0}\right) \simeq &\frac{1}{2\pi i\Lambda'}\exp\left(\frac{-2r_s}{\nu_0}\right)\frac{X(-\nu_0)}{X(\nu_0)} \\ &+ \frac{c}{4\pi}\exp\left(\frac{-r_s}{\nu_0}\right)\int_0^1 \frac{d\nu}{\nu\Lambda^+\Lambda^-} \\ &\times \exp\left(\frac{-r_s}{\nu}\right)\frac{X(-\nu)}{X(\nu_0)}, \end{aligned} \tag{97}$$

while Eq. (95) becomes

$$\begin{aligned}\tilde{\Gamma}\left(\frac{i}{\mu}\right) \simeq &-\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right) \\ &\times \left[\frac{1}{\pi i\Lambda'}\exp\left(\frac{-r_s}{\nu_0}\right)\frac{\nu_0 X(-\nu_0)}{(\mu+\nu_0)(\mu-\nu_0)}\right. \\ &+ \frac{c}{4\pi(\mu+\nu_0)}\int_0^1 \frac{d\nu'}{\nu'\Lambda^+\Lambda^-} \\ &\left.\times \exp\left(\frac{-r_s}{\nu'}\right)\frac{X(-\nu')}{\nu'-\mu}(\nu'+\nu_0)\right]. \end{aligned} \tag{98}$$

These are precisely the coefficients which occur in the half-space Milne problem,[1] as expected. The first order approximation of $\Gamma_<^0$ and $\tilde{\Gamma}_<$ may now be readily obtained by computing the integrals previously neglected in Eqs. (96) and (95) from the first iterations (97) and (98). Thus,

$$\nu_0^2\tilde{\Gamma}_<\left(\frac{1}{\nu_0}\right) \cong \frac{1}{2\pi i\Lambda'}\exp\left(\frac{-2r_s}{\nu_0}\right)\frac{X(-\nu_0)}{X(\nu_0)} + \frac{c}{4\pi}\exp\left(\frac{-r_s}{\nu_0}\right)\int_0^1\frac{d\nu}{\nu\Lambda^+\Lambda^-}\exp\left(\frac{-r_s}{\nu}\right)\frac{X(-\nu)}{X(\nu_0)} - \frac{1}{2\pi i}\int_0^1 d\nu\exp\left(\frac{-r_s}{\nu}\right)X(-\nu)\left(\frac{1}{X^+(\nu)}-\frac{1}{X^-(\nu)}\right) \times\left[\frac{1}{\pi i\Lambda'}\exp\left(\frac{-r_s}{\nu_0}\right)\frac{\nu_0X(-\nu_0)}{(\nu_0-\nu)(\nu_0+\nu)} - \frac{c}{4\pi(\nu_0-\nu)}\int_0^1\frac{d\nu'}{\nu'\Lambda^+\Lambda^-} \times\exp\left(\frac{-r_s}{\nu'}\right)\frac{X(-\nu')}{\nu'+\nu}(\nu'+\nu_0)\right] \quad (99)$$

and

$$\tilde{\Gamma}_<\left(\frac{i}{\mu}\right) \cong \frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)}-\frac{1}{X^-(-\mu)}\right) \times\left\{\frac{1}{\pi i\Lambda'}\exp\left(\frac{-r_s}{\nu_0}\right)\frac{\nu_0X(-\nu_0)}{(\nu_0-\nu)(\nu_0+\nu)} - \frac{c}{4\pi(\nu_0-\nu)}\int_0^1\frac{d\nu'}{\nu'\Lambda^+\Lambda^-} \times\exp\left(\frac{-r_s}{\nu'}\right)\frac{X(-\nu')}{\nu'+\nu}(\nu'+\nu_0) - \frac{1}{2\pi i}\int_0^1\frac{d\nu}{\nu-\mu} \times\exp\left(\frac{-2r_s}{\nu}\right)X(-\nu)\left(\frac{1}{X^+(\nu)}-\frac{1}{X^-(\nu)}\right) \times\left[\frac{1}{\pi i\Lambda'}\exp\left(\frac{r_s}{\nu_0}\right)\frac{\nu_0X(-\nu_0)}{(\nu_0-\nu)(\nu_0+\nu)} - \frac{c}{4\pi(\nu_0-\nu)}\int_0^1\frac{d\nu'}{\nu'\Lambda^+\Lambda^-} \times\exp\left(\frac{-r_s}{\nu'}\right)\frac{X(-\nu')}{\nu'+\nu}(\nu'+\nu_0)\right]\right\}. \quad (100)$$

C. The Critical Problem

We mentioned earlier that, when the zeros of the dispersion function are real ($c > 1$), the Green's function [Eq. (5)] is not uniquely determined. The angular density, however, is still uniquely determined regardless of the manner we choose to treat the singularities. To illustrate this point, let us consider the critical problem. Assuming no volume sources, the integral equation (20) becomes

$$\psi(r,\mu) = \psi_0(r,\mu) + \psi_c(r,\mu). \quad (101)$$

The dispersion function occurs only in ψ_c, which is given by Eq. (29). We rewrite this equation as follows:

$$\psi_c(r,\mu) = \frac{c}{8\pi}\int_{-\infty}^{\infty} dkk^2T_1(k)I(k,r,\mu) \times\left[\mathcal{P}\frac{1}{\Lambda} + \lambda_1\delta(k-k_1) + \lambda_2\delta(k-k_2)\right], \quad (102)$$

where $\mathcal{P}$ implies the Cauchy principal value, λ_1 and λ_2 are some arbitrary functions of k, and k_1, k_2 are the real zeros of $\Lambda(k)$. Since $\Lambda(-k) = \Lambda(k)$, $-k_2 = k_1 = k_0 > 0$. Also from Eq. (28), we have $T_1(-k) = T_2(k)$. With this in mind, let us re-express Eq. (101) in the following form:

$$\psi_c(r,\mu) = \frac{c}{8\pi}\int_{-\infty}^{\infty} dkk^2\frac{T_1(k)}{\Lambda}I(k,r,\mu) + \phi(k_0,r,\mu)\Gamma^c(k_0), \quad (103)$$

where we have omitted writing the principal value symbol and

$$\Gamma^c(k_0) = \frac{c}{8\pi}k_0^2[\lambda_1T_1(k_0) + \lambda_2T_2(k_0)] \quad (104)$$

and

$$\phi(k_0,r,\mu) = \int_{-1}^{1}\frac{dt}{1+ik_0t}e^{ik_0tr\mu}J_0(k_0r[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}) \quad (105)$$

is the discrete regular eigenfunction which is of oscillatory type in contrast to the eigenfunctions constructed previously.

The calculation of the integral in Eq. (103) may now be carried out in exact analogy with the interior problems for $c < 1$. Here we merely state the final result. Thus,

$$\psi(r,\mu) = \int_0^1\frac{d\nu}{\nu}\Gamma_<\left(\frac{i}{\nu}\right)\phi\left(\frac{i}{\nu},r,\mu\right) + \phi(k_0,r,\mu)\Gamma^c(k_0), \quad (106)$$

where the coefficient $\Gamma_<$ is given by Eq. (38), and, as before, $\phi(i/\nu, r, \mu)$ [see Eq. (35)] are the regular eigenfunctions corresponding to the continuous spectrum.

The implication of our previous statement as to the uniqueness of the angular density should now be obvious. In particular, we see from Eq. (106) that the coefficients $\Gamma_<$ and Γ^c are determined *uniquely by the* *boundary conditions*, and the angular density, therefore, *does not* depend on how we set up these coefficients. Equation (104) illustrates such an arbitrariness in Γ^c.

The steps involved in obtaining the auxiliary equation for the coefficients are exactly those involved in the interior problems for $c < 1$. Thus, Eq. (69) represents that equation with $\theta\psi_q \equiv 0$, Γ^c replacing $\Gamma^0_<$, and the discrete eigenfunctions by their oscillatory counterpart. In fact, from Eq. (79), which represents the most general solution of Eq. (69), we have

$$\tilde{\Gamma}_<\left(\frac{i}{\mu}\right) = -\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right) \times \left\{\nu_0^2\Gamma^c\left(\frac{1}{\nu_0}\right)\left[\exp\left(\frac{ir_s}{\nu_0}\right)\frac{X(-i\nu_0)}{\mu - i\nu_0} - \exp\left(\frac{-ir_s}{\nu_0}\right)\frac{X(i\nu_0)}{\mu + i\nu_0}\right] + \int_0^1 d\nu \exp\left(\frac{-2r_s}{\nu}\right)\frac{\tilde{\Gamma}_<(i/\nu)}{\nu - \mu}X(-\nu) + \int_0^1 d\mu' \frac{X^+(\mu') - X^-(\mu')}{\mu' + \mu} r_s\psi(r_s, \mu)\right\}. \tag{107}$$

The equation that determines Γ^c is

$$\nu_0^2\Gamma^c\left(\frac{1}{\nu_0}\right)\left[X(-i\nu_0)\exp\left(\frac{-ir_s}{\nu_0}\right) - X(i\nu_0)\exp\left(\frac{ir_s}{\nu_0}\right)\right] - \int_0^1 d\nu X(-\nu)\exp\left(\frac{-2r_s}{\nu}\right)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) - \int_0^1 d\mu'[X^+(\mu') - X^-(\mu')]r_s\psi(r_s, -\mu') = 0. \tag{108}$$

The boundary condition for the critical problem is

$$\psi(r_s, \mu) = 0, \quad \mu < 0. \tag{109}$$

Inserting this boundary condition into Eq. (107), we get

$$\tilde{\Gamma}_<\left(\frac{i}{\mu}\right) = -\frac{1}{2\pi i}\left(\frac{1}{X^+(-\mu)} - \frac{1}{X^-(-\mu)}\right) \times \left\{\nu_0^2\Gamma^c\left(\frac{1}{\nu_0}\right)\left[\exp\left(\frac{ir_s}{\nu_0}\right)\frac{X(-i\nu_0)}{\mu - i\nu_0} - \exp\left(\frac{-ir_s}{\nu_0}\right)\frac{X(i\nu_0)}{\mu + i\nu_0}\right] + \int_0^1 d\nu \exp\left(\frac{-2r_s}{\nu}\right)\frac{\tilde{\Gamma}_<(i/\nu)}{\nu - \mu}X(-\nu)\right\}, \tag{110}$$

and, from Eq. (107), we get

$$\nu_0^2\Gamma^c\left(\frac{1}{\nu_0}\right)\left[X(-i\nu_0)\exp\left(\frac{ir_s}{\nu_0}\right) - X(i\nu_0)\exp\left(\frac{-ir_s}{\nu_0}\right)\right] - \int_0^1 d\nu X(-\nu)\exp\left(\frac{-2r_s}{\nu}\right)\tilde{\Gamma}_<\left(\frac{i}{\nu}\right) = 0. \tag{111}$$

For a large sphere, if we neglect the integral in Eq. (111) involving $\exp(-2r_s/\nu_0)$, we get

$$\exp(-ir_s/\nu)X(+i\nu_0) - X(-i\nu_0)\exp(ir_s/\nu_0) = 0, \tag{112}$$

which merely states that the asymptotic density is to vanish at the extrapolated end point. This problem has been extensively treated for planar geometry by the normal mode expansion technique.[1,4]

As a final application of the Green's function technique, let us consider the Milne problem for the exterior of a black sphere.

D. Milne Problem for the Exterior of a Black Sphere

The problem under consideration involves the determination of the neutron angular density outside a purely absorbing sphere (black sphere). Far away from the sphere, there is a source which supplies the neutrons. Since the black sphere implies zero emergent distribution, the appropriate boundary condition is

$$\psi(r_s, \mu) = 0, \quad \mu > 0. \tag{113}$$

In calculating the angular density $\psi_q(r, \mu)$ in Eq. (47), let us assume that a spherically symmetric source is located at some distance R outside the black sphere. Thus, let

$$Q(r, \Omega) = (q_0/R^2)\delta(r - R), \quad R > r_s. \tag{114}$$

Putting this source function into Eq. (10), we get

$$\psi_q(r, \mu) = \frac{q_0}{4\pi Ri}\int_{-\infty}^{\infty}\frac{dkk}{\Lambda}e^{ikR}I(k, r, \mu), \tag{115}$$

where $I(k, r, \mu)$ is given by Eq. (17).

Let us split Eq. (115) into two parts as follows:

$$\psi_q(r, \mu) = \frac{q_0}{4\pi iR}\left[\int_{-\infty}^{\infty}\frac{dkk}{\Lambda}e^{ikR}I(k, r, \mu)\Theta(R - r) + \int_{-\infty}^{\infty}\frac{dkk}{\Lambda}e^{ikR}I(k, r, \mu)\Theta(r - R)\right]. \tag{116}$$

Now, if we push the source to infinity (i.e., let $R \to \infty$), we see that the second integral in Eq. (116) will make no contribution. Furthermore, in the same limit, the modes with continuous spectrum must also disappear. Thus, in Eq. (110), if we choose

$$q_0 = R\, e^{k_0 R} i[k_0\Lambda'(k_0)]^{-1}, \tag{117}$$

where $\Lambda(ik_0) = 0$, and let $R \to \infty$, we get

$$\psi_q(r, \mu) = -\phi(ik_0, r, \mu). \tag{118}$$

The application of the reduction operator for $\mu > 0$ to Eq. (116) gives us, in the limit $R \to \infty$ and the same choice of q_0 as given in Eq. (117),

$$\theta\psi_q(r,\mu) = \frac{1}{2k_0}\left(\frac{e^{-k_0 r_s}}{1-k_0\mu} - \frac{e^{k_0 r_s}}{1+k_0\mu}\right), \quad \mu > 0. \tag{119}$$

The coefficients $\tilde{\Gamma}_>$ and $\Gamma^0_>$ given by Eqs. (82) and (83), respectively, which solve the integral equation (47), may now be readily obtained. Thus, inserting the boundary condition (113) and Eq. (119) into Eq. (82), we get

$$\tilde{\Gamma}_>\left(\frac{i}{\mu}\right) = -\frac{1}{2\pi i}\left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)}\right) \times \left\{\frac{\nu_0^2 X(\nu_0)}{\mu-\nu_0} e^{-r_s/\nu_0}\left[1-\Gamma_>\left(\frac{1}{\nu_0}\right)\right] + \frac{\nu_0^2 X(-\nu_0)}{\mu+\nu_0} e^{r_s/\nu_0}\right\}. \tag{120}$$

The equation that determines $\Gamma^0_>$ is similarly obtained from Eq. (82). Thus,

$$\Gamma^0_>(1/\nu_0) = 1 + [X(-\nu_0)/X(\nu_0)] \exp(2r_s/\nu_0). \tag{121}$$

One usually writes

$$X(-\nu_0)/X(\nu_0) = -\exp(-2r_0/\nu_0), \tag{122}$$

where r_0 is the so called extrapolation distance which determines the distance where the asymptotic neutron density vanishes.

APPENDIX

In the main body of this paper, we have dealt with the media with regeneration property. In the course of the treatment we encountered certain complicated looking functions such as $\zeta^{(1),(2)}$, defined by the infinite series (27), which are hard to relate to any classically known functions. In this section we consider the Green's function for media without regeneration property (i.e., $c = 0$). (For a geometrical interpretation of the Green's function for purely absorbing media, see Ref. 1.) Here also we encounter similar type of functions which do not seem to have classical analogs, but their certain integrals are related to Dirac's delta function. Consequently, they give rise to some interesting mathematical identities and completeness relations (half and full range). For the planar geometry, the completeness relations are rather trivial. However, for the sake of comparing the hierarchy of complexity, we also present these trivial completeness relations.

Derivation of Identities

Let us begin with the Green's function (for $c = 0$) in the form of a Fourier integral

$$G(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_0,\boldsymbol{\Omega}_0) = \frac{1}{(2\pi)^3}\delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0)\int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{1+i\mathbf{k}\cdot\boldsymbol{\Omega}}, \tag{A1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. The integration may be carried out in a straightforward manner by first resolving $\mathbf{k}$ and $\mathbf{R}$ as follows:

$$\mathbf{k} = \boldsymbol{\Omega}\mathbf{k}\cdot\boldsymbol{\Omega} + \mathbf{k}_\perp, \quad \text{such that} \quad \mathbf{k}_\perp\cdot\boldsymbol{\Omega} = 0,$$

and

$$\mathbf{R} = \boldsymbol{\Omega}\mathbf{R}\cdot\boldsymbol{\Omega} + \mathbf{R}_\perp, \quad \text{such that} \quad \mathbf{R}_\perp\cdot\boldsymbol{\Omega} = 0.$$

Then we can write

$$\mathbf{k}\cdot\mathbf{R} = \mathbf{k}\cdot\boldsymbol{\Omega}\mathbf{R}\cdot\boldsymbol{\Omega} + \mathbf{k}_\perp\cdot\mathbf{R}_\perp.$$

Equation (1) then becomes

$$G = \delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0)\frac{1}{2\pi} \times \int_{-\infty}^{\infty} \frac{d(\mathbf{k}\cdot\boldsymbol{\Omega})\exp(i\mathbf{k}\cdot\boldsymbol{\Omega}\mathbf{R}\cdot\boldsymbol{\Omega})}{1+i\mathbf{k}\cdot\boldsymbol{\Omega}}\cdot\frac{1}{(2\pi)^2} \times \int d^2k_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{R}_\perp).$$

Separate parts of the integrals are

$$\frac{1}{(2\pi)^2}\int d^2k_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{R}_\perp) = \delta(\mathbf{R}_\perp)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d(\mathbf{k}\cdot\boldsymbol{\Omega})}{1+i\mathbf{k}\cdot\boldsymbol{\Omega}} e^{i\mathbf{k}\cdot\boldsymbol{\Omega}\mathbf{R}\cdot\boldsymbol{\Omega}} = e^{-\mathbf{R}\cdot\boldsymbol{\Omega}}\Theta(\mathbf{R}\cdot\boldsymbol{\Omega}),$$

where

$$\Theta(\mathbf{R}\cdot\boldsymbol{\Omega}) = 1, \quad \mathbf{R}\cdot\boldsymbol{\Omega} > 0, \qquad = 0, \quad \mathbf{R}\cdot\boldsymbol{\Omega} < 0. \tag{A2}$$

The expression for the Green's function now may be written as

$$G = \exp(-\mathbf{R}\cdot\boldsymbol{\Omega})\delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0)\delta(\mathbf{R}_\perp)\Theta(\mathbf{R}\cdot\boldsymbol{\Omega}). \tag{A3}$$

This equation holds for any arbitrary values of r and r_0.

Let us consider Eq. (1) again and carry out the integration in a manner parallel to the treatment of interior and exterior problems. Define I as

$$I = \frac{1}{(2\pi)^3}\int \frac{d^3k \exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)]}{1+i\mathbf{k}\cdot\boldsymbol{\Omega}}, \tag{A4}$$

and let $I_<$ denote this integral when $r < r_0$, and $I_>$ when $r > r_0$. First consider $r < r_0$. Expanding

$\exp(-i\mathbf{k}\cdot\mathbf{r}_0)$ in terms of spherical harmonics and using the cosine formula to express $\hat{\mathbf{k}}\cdot\mathbf{r} = t$ and $\hat{\mathbf{r}}\cdot\boldsymbol{\Omega} = \mu$, we get

$$I_< = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n e^{\frac{1}{2}im\pi} Y_{nm}(-\hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega}) \times \int_{-\infty}^{\infty} dk k^2 \zeta_n^{(2)}(kr_0) \int_{-1}^{1} \frac{dt}{1+ikt} Y_{nm}^*(t,\varphi) \cdot e^{iktr\mu} J_m\{kr[(1-\mu^2)(1-t^2)]^{\frac{1}{2}}\}. \tag{A5}$$

(Here we have used the decomposition of the spherical Bessel function in terms of spherical Hankel functions $\zeta_n^{(1)}$ and $\zeta_n^{(2)}$.) Now the distortion of the path of integration (with respect to k) to the path surrounding the branch cut in the lower half k plane yields

$$I_< = -e^{-r\mu}\int_{-1}^{0}\frac{d\nu}{\nu^3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n e^{-\frac{1}{2}im\pi}\zeta_n^{(2)}\left(i\frac{r_0}{\nu}\right) \times Y_{nm}(-\hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega})Y_{nm}^*(\nu,\varphi) \times I_m\{r[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\}. \tag{A6}$$

Since $G = I\delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0)$, the comparison of Eq. (6) with Eq. (3) gives us the first identity.

Identity 1:

$$-\exp(-r\mu)\int_{-1}^{0}\frac{d\nu}{\nu^3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n \exp\left(-im\frac{\pi}{2}\right)\zeta_n^{(2)}\left(i\frac{r_0}{\nu}\right) \times Y_{nm}(-\hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega})Y_{nm}^*(\nu,\varphi) \times I_m\{r[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} = \exp(-\mathbf{R}\cdot\boldsymbol{\Omega})\delta(\mathbf{R}_\perp)\Theta(\mathbf{R}\cdot\boldsymbol{\Omega}),$$
$$r < r_0, \quad \mu \equiv \hat{\mathbf{r}}\cdot\boldsymbol{\Omega}.$$

Similarly, for $r > r_0$, we have the second.

Identity 2:

$$\exp(r_0\mu_0)\int_{0}^{1}\frac{d\nu}{\nu^3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n e^{\frac{1}{2}im\pi}\zeta_n^{(1)}\left(i\frac{r}{\nu}\right) \times Y_{nm}(\hat{\mathbf{r}}\cdot\boldsymbol{\Omega})Y_{nm}^*(\nu,\varphi_0) \times I_m(r_0[(1-\mu_s^2)(1/\nu^2-1)]^{\frac{1}{2}}) = \exp(-\mathbf{R}\cdot\boldsymbol{\Omega})\delta(\mathbf{R}_\perp)\Theta(\mathbf{R}\cdot\boldsymbol{\Omega}),$$
$$r > r_0, \quad \mu_s \equiv \hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega}.$$

We remark here that Identities 1 and 2 so far are not restricted to any geometry. Also note that the Green's functions $G_<$ $(r < r_0)$ and $G_>$ $(r > r_0)$ are related to $I_<$ and $I_>$, respectively, by

$$G_{\lessgtr} = \delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0)I_{\lessgtr}. \tag{A7}$$

Now if we express the delta function in Eq. (A7) in the form

$$\delta(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0) = \delta(\mu_s-\mu_0)\delta(\varphi_\Omega-\varphi_{\Omega_0}), \tag{A8}$$

where $\mu_s = \hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega}$, $\mu_0 = \hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega}_0$, and in Eq. (A3) write $\delta(\mathbf{R}_\perp)$ in cylindrical coordinate system, i.e.,

$$\delta(\mathbf{R}_\perp) = r^{-1}(1-\mu^2)^{-\frac{1}{2}} \times \delta[r(1-\mu^2)^{\frac{1}{2}} - r_0(1-\mu_s^2)^{\frac{1}{2}}]\delta(\varphi-\varphi_s), \tag{A9}$$

and integrate Eq. (A7) over all angles except $\mu = \hat{\mathbf{r}}\cdot\boldsymbol{\Omega}$ and $\mu_0 = \hat{\mathbf{r}}_0\cdot\boldsymbol{\Omega}_0$, we get Identities 3 and 4 corresponding to the spherical geometry.

Identity 3:

$$-\tfrac{1}{2}\exp(-r\mu)\int_{-1}^{0}\frac{d\nu}{\nu^3}S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) \times I_0\{r[(1-\mu^2)(1/\nu^2-1)]^{\frac{1}{2}}\} = \exp[-(r\mu - r_0\mu_0)] \times \frac{\delta[r(1-\mu^2)^{\frac{1}{2}} - r_0(1-\mu_0^2)^{\frac{1}{2}}]}{r(1-\mu^2)^{\frac{1}{2}}}\Theta(r\mu - r_0\mu_0),$$
$$r < r_0;$$

Identity 4:

$$\tfrac{1}{2}\exp(r_0\mu_0)\int_{0}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir}{\nu}, \mu, \nu\right) \times I_0\{r_0((1-\mu_0^2)[(1/\nu^2)-1])^{\frac{1}{2}}\} = \exp[-(r\mu - r_0\mu_0)] \times \frac{\delta[r(1-\mu^2)^{\frac{1}{2}} - r_0(1-\mu_0^2)^{\frac{1}{2}}]}{r(1-\mu^2)^{\frac{1}{2}}}\Theta(r\mu - r_0\mu_0),$$
$$r > r_0,$$

where

$$S^{(1),(2)}\left(\frac{ir}{\nu}, \mu, \nu\right) = \sum_{n=0}^{\infty} i^n(2n+1)\zeta_n^{(1),(2)}\left(\frac{ir}{\nu}\right)P_n(\mu)P_n(\nu). \tag{A10}$$

A more convenient form of these identities is obtained if we use the formula

$$\delta(f(x)) = \sum_n \frac{\delta(x-x_n)}{|f'(x_n)|}, \quad f(x_n) = 0,$$

to re-express the right-hand sides. This gives us:

Identity 3′:

$$-\tfrac{1}{2}\exp(-r\mu)\int_{-1}^{0}\frac{d\nu}{\nu^3}S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) \times I_0\{r[(1-\mu^2)(1/\nu^2-1)]^{\frac{1}{2}}\} = \exp[-(r\mu - r_0\mu_0)]\frac{1}{r^2|\mu|} \times [\delta(\mu-\mu_1) + \delta(\mu+\mu_1)]\Theta(r\mu - r_0\mu_0),$$
$$r < r_0;$$

Identity 4′:

$$\tfrac{1}{2}\exp(r_0\mu_0)\int_0^1 \frac{d\nu}{\nu^3} S^{(1)}\left(\frac{ir}{\nu}, \mu, \nu\right) \times I_0\{r_0[(1-\mu_0^2)(1/\nu^2-1)]^{\frac{1}{2}}\}$$
$$= \exp[-(r\mu - r_0\mu_0)](r^2|\mu|)^{-1} \times [\delta(\mu-\mu_1)+\delta(\mu+\mu_1)]\Theta(r\mu - r_0\mu_0), \quad r > r_0,$$

where

$$\mu_1 = [1-(r_0^2/r^2)(1-\mu_0^2)]^{\frac{1}{2}}.$$

Completeness Theorems

Let us note that the left-hand and right-hand sides of identities (3′) and (4′) are two representations of $G_<$ and $G_>$, respectively. The half-range completeness theorems follow from the limits of $G_<$ and $G_>$ as r approaches r_0. Specializing to various particular values of μ and μ_0, let us first obtain a number of useful results. Consider $G_<$ first.

The right-hand side representation of $G_<$ is

$$G_< = \exp[-(r\mu - r_0\mu_0)]\frac{1}{r^2\mu} \times [\delta(\mu+\mu_1)+\delta(\mu-\mu_1)]\Theta(r\mu - r_0\mu_0). \tag{A11}$$

1. $\mu_0 > 0, \quad \mu > 0$

The argument of the Θ function is positive if $\mu > (r_0/r)\mu_0$. Now the first δ function makes no contribution, since its argument cannot vanish. The second δ function can contribute if

$$\mu = \mu_1 = \{(r_0^2/r^2)\mu_0^2 - [(r_0^2/r^2)-1]\}^{\frac{1}{2}} < (r_0/r)\mu_0, \quad (r_0/r) > 1.$$

But then $\Theta(r\mu - r_0\mu_0) = 0$. Hence,

$$G_< = 0, \quad \mu_0 > 0, \quad \text{and} \quad \mu > 0. \tag{A12a}$$

2. $\mu_0 > 0, \quad \mu < 0$

For these values of μ and μ_0, $\Theta\,(r\mu - r_0\mu_0) = 0$. Hence, again

$$G_< = 0, \quad \mu_0 > 0, \quad \mu < 0. \tag{A12b}$$

3. $\mu_0 < 0, \quad \mu > 0$

For this $\Theta = 1$, $\delta(\mu+\mu_1)$ makes no contribution. Therefore, only $\delta(\mu - \mu_1)$ may contribute. Hence,

$$G_< = e^{-(r\mu - r_0\mu_0)}(r^2|\mu|)^{-1}\delta(\mu-\mu_1), \quad \mu_0 < 0, \quad \mu > 0. \tag{A12c}$$

4. $\mu_0 < 0, \quad \mu < 0$

Then $r\mu - r_0\mu_0 = r_0|\mu_0| - r|\mu| > 0$ if

$$|\mu| < (r_0/r)|\mu_0|.$$

Clearly $\delta(\mu-\mu_1)$ makes no contribution. Hence, the possible contribution may come from $\delta(\mu+\mu_1)$. This is easily seen from putting $\mu + \mu_1$ equal to zero:

$$\mu + \mu_1 = 0 = -|\mu| + \mu_1$$

or

$$|\mu| = \{(r_0^2/r^2)\mu_0^2 - [(r_0^2/r^2)-1]\}^{\frac{1}{2}} < (r_0/r)|\mu_0|.$$

The last inequality shows that $\Theta(r\mu - r_0\mu_0) = 1$ is satisfied. Hence,

$$G_< = e^{-(r\mu - r_0\mu_0)}(r^2|\mu|)^{-1}\delta(\mu+\mu_1), \quad \mu_0 < 0, \quad \mu < 0. \tag{A12d}$$

By exactly the same argument, one may show that $G_>$, in the right-hand side representation (see identity 4′), can be written as

$$G_> = e^{-(r\mu - r_0\mu_0)}(r^2|\mu|)^{-1}\delta(\mu-\mu_1), \quad \mu_0 > 0, \quad \mu > 0, \tag{A13a}$$

$$G_> = 0, \quad \mu_0 > 0, \quad \mu < 0, \tag{A13b}$$

$$G_> = [e^{-(r\mu - r_0\mu_0)}/r^2|\mu|]\delta(\mu-\mu_1), \quad \mu_0 < 0, \quad \mu > 0, \tag{A13c}$$

$$G_> = 0, \quad \mu_0 < 0, \quad \mu < 0. \tag{A13d}$$

Now in Eqs. (A12a)–(A12d) and (A13a)–(A13d) let $r \to r_0$. Denoting this limit of $G_{\lessgtr}$ by $G_{\mp}$, we obtain the following set of results:

$$G_- = 0, \quad \mu_0 > 0, \quad \mu > 0, \tag{A14a}$$

$$G_- = 0, \quad \mu_0 > 0, \quad \mu < 0, \tag{A14b}$$

$$G_- = (e^{-2r_0\mu}/r_0^2\mu)\delta(\mu+\mu_0), \quad \mu_0 < 0, \quad \mu > 0, \tag{A14c}$$

$$G_- = -(r_0^2\mu)^{-1}\delta(\mu-\mu_0), \quad \mu_0 < 0, \quad \mu < 0, \tag{A14d}$$

$$G_+ = (r_0^2\mu)^{-1}\delta(\mu-\mu_0), \quad \mu_0 > 0, \quad \mu > 0, \tag{A15a}$$

$$G_+ = 0, \quad \mu_0 > 0, \quad \mu < 0, \tag{A15b}$$

$$G_+ = (e^{-2r_0\mu}/r_0^2\mu)\delta(\mu+\mu_0), \quad \mu_0 < 0, \quad \mu > 0, \tag{A15c}$$

$$G_+ = 0, \quad \mu_0 < 0, \quad \mu < 0. \tag{A15d}$$

Let us remark here that the purpose of tabulating $G_\pm$ for various sets of values of μ, μ_0 (instead of using Heavyside theta function) is that only those representations of $G_\pm$ give rise to completeness relations which correspond to the same sign of μ, μ_0 – for instance, Eqs. (A14a), (A14d), (A15a), and (A15d). The rest give rise to mere identities.

In the left-hand side representations of $G_\mp$ we have from identities (A3′) and (A4′),

$$G_- = -\tfrac{1}{2}e^{-r_0\mu}\int_{-1}^{0}\frac{d\nu}{\nu^3}S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) \times I_0\{r_0[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\}, \quad \text{(A16)}$$

$$G_+ = \tfrac{1}{2}e^{r_0\mu_0}\int_{0}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right) \times I_0\{r_0[(1-\mu_0^2)(1/\nu^2-1)]^{\frac{1}{2}}\}. \quad \text{(A17)}$$

From Eqs. (A14d) and (A15a) we may now conclude our half-range completeness theorems.

Theorem 1 (Half-Range Completeness): For $\mu_0 < 0$ and $\mu < 0$,

$$\frac{r_0^2}{2}e^{-r_0\mu}\int_{-1}^{0}\frac{d\nu}{\nu^3}S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) \times I_0\{r_0[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} = \delta(\mu-\mu_0).$$

Theorem 2 (Half-Range Completeness): For $\mu_0 > 0$ and $\mu > 0$,

$$\tfrac{1}{2}r_0^2\mu e^{r_0\mu_0}\int_{0}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right) \times I_0\{r_0[(1-\mu_0^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} = \delta(\mu-\mu_0).$$

The full-range completeness theorems may now be readily obtained by taking the appropriate differences of G_+ and G_-. Thus, subtracting (A16) from (A17), we have (in the left-hand side representation)

$$G_+ - G_- = \tfrac{1}{2}e^{r_0\mu_0}\int_{0}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right) \times I_0\{r_0[(1-\mu_0^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} + \tfrac{1}{2}e^{-r_0\mu}\int_{-1}^{0}\frac{d\nu}{\nu^3}S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) \times I_0\{r_0[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\}. \quad \text{(A18)}$$

This equation can be cast into a more symmetric form by means of the following relations:

$$e^{-r\mu}I_0\{r_0[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} = \frac{1}{2}\left[S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right) + S^{(2)}\left(\frac{ir_0}{\nu}, \mu, \nu\right)\right], \quad \text{(A19)}$$

$$e^{r\mu}I_0\{r_0[(1-\mu^2)(\nu^{-2}-1)]^{\frac{1}{2}}\} = \frac{1}{2}\left[S^{(1)}\left(\frac{ir_0}{\nu}, -\mu, \nu\right) + S^{(2)}\left(\frac{ir_0}{\nu}, -\mu, \nu\right)\right] \quad \text{(A20)}$$

$$S^{(1)}\left(-\frac{ir_0}{\nu}, \mu, -\nu\right) = S^{(2)}\left(\frac{ir_0}{\nu}, \mu, \nu\right). \quad \text{(A21)}$$

We may now rewrite Eq. (A8) in the form

$$G_+ - G_- = \frac{1}{4}\int_{-1}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right)S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right). \quad \text{(A22)}$$

In the right-hand side representations of G_+ and G_- we have, from (A15a), (A14a) and (A15d), and (A14d),

$$G_+ - G_- = (r_0^2\mu)^{-1}\delta(\mu-\mu_0), \quad \mu_0 > 0, \ \mu > 0, \quad \mu_0 < 0, \ \mu < 0. \quad \text{(A23)}$$

The full-range completeness theorems may now be readily concluded from Eqs. (A22) and (A23).

Theorem 3 (Full-Range Completeness): For any μ, μ_0,

$$\frac{r_0^2}{4}\int_{-1}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, \mu, \nu\right)S^{(2)}\left(\frac{ir_0}{\nu}, -\mu_0, \nu\right) = \frac{1}{\mu}\delta(\mu-\mu_0).$$

Theorem 4 (Full-Range Completeness): For any μ, μ_0,

$$-\tfrac{1}{4}r_0^2\int_{-1}^{1}\frac{d\nu}{\nu^3}S^{(1)}\left(\frac{ir_0}{\nu}, -\mu, \nu\right)S^{(2)}\left(\frac{ir_0}{\nu}, \mu_0, \nu\right) = \frac{1}{\mu}\delta(\mu-\mu_0).$$

We note that Theorem 1 is adjoint to Theorem 2 in the sense that they imply each other under the reflection of μ and μ_0. In particular, this equivalence also exists under the interchange of μ and μ_0. We may, therefore, combine the two theorems into a symmetric form.

Theorem 5 (Full-Range Completeness—Symmetric form): For any μ, μ_0,

$$\tfrac{1}{8}r_0^2\int_{-1}^{1}\frac{d\nu}{\nu^3}\Big[S^{(1)}\Big(\frac{ir_0}{\nu},\mu,\nu\Big)S^{(2)}\Big(\frac{ir_0}{\nu},-\mu_0,\nu\Big) - S^{(1)}\Big(\frac{ir_0}{\nu},-\mu,\nu\Big)S^{(2)}\Big(\frac{ir_0}{\nu},\mu_0,\nu\Big)\Big] = \frac{1}{\mu}\,\delta(\mu-\mu_0).$$

For the sake of comparison we present the corresponding trivial theorems for the planar geometry:

$$-\mu\int_{-1}^{0}\frac{d\nu}{\nu}\,\delta(\nu-\mu_0)\delta(\nu-\mu) = \delta(\mu-\mu_0),\qquad \mu_0<0,\quad \mu<0.$$

$$\mu\int_{0}^{1}\frac{d\nu}{\nu}\,\delta(\nu-\mu_0)\delta(\nu-\mu) = \delta(\mu-\mu_0),\qquad \mu_0>0,\quad \mu>0.$$

$$\mu\int_{-1}^{1}\frac{d\nu}{\nu}\,\delta(\nu-\mu_0)\delta(\nu-\mu_0) = \delta(\mu-\mu_0),\qquad \text{for any}\quad \mu,\mu_0.$$

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On the Theory of Quantum Corrections to the Equations of State and to the Particle-Distribution Functions

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The first-order quantum corrections to the equations of state of an almost-classical N-particle system are calculated to all orders in the particle density by expanding the normalized Wigner distribution function in powers of $\hbar^2$. In this way one avoids the expansion of the partition function, which has the unsatisfactory property that the correction terms diverge in the thermodynamic limit. Similarly, the first-order quantum correction to the pair distribution function is derived.

1. INTRODUCTION

A well-known concept in the quantum statistical treatment of an N-particle system is the Wigner distribution function (WDF[1]) $f(\mathbf{r}, \mathbf{p})$, where $\mathbf{r}$ and $\mathbf{p}$ are the $3N$-dimensional position and momentum vectors. It is defined in such a way that the quantum statistical ensemble average of an arbitrary operator A is given by

$$\langle A\rangle_{\rm qu} = \mathrm{Tr}\,\rho A/\mathrm{Tr}\,\rho = \int a(\mathbf{r},\mathbf{p})f(\mathbf{r},\mathbf{p})\,d\mathbf{r}\,d\mathbf{p}\Big/\int f(\mathbf{r},\mathbf{p})\,d\mathbf{r}\,d\mathbf{p}. \quad (1)$$

Here ρ is the density operator and $a(\mathbf{r}, \mathbf{p})$ represents the classical quantity corresponding to the quantum mechanical operator A. If, in particular, the correspondence between a and A is established according to Weyl's rule (cf., e.g., Ref. 2), one finds for $f(\mathbf{r}, \mathbf{p})$ the expression

$$f(\mathbf{r},\mathbf{p}) = (\pi\hbar)^{-3N}\int \rho(\mathbf{r}-\mathbf{y},\mathbf{r}+\mathbf{y})\exp(2i\hbar^{-1}\mathbf{p}\cdot\mathbf{y})\,d\mathbf{y}, \quad (2)$$

where $\rho(\mathbf{r}, \mathbf{r}')$ is the density operator in coordinate representation.[3] It will be obvious that ρ and $f(\mathbf{r}, \mathbf{p})$ may be multiplied by an arbitrary temperature-dependent factor.

In the case of statistical equilibrium described by a canonical ensemble, the (unnormalized) density operator is given by

$$\rho = \exp(-\beta H),\quad \beta = (kT)^{-1}. \quad (3)$$

This operator satisfies the so-called Bloch equation

$$\frac{\partial\rho}{\partial\beta} = -H\rho. \quad (4)$$

[1] E. Wigner, Phys. Rev. 40, 749 (1932).
[2] K. Schram and B. R. A. Nijboer, Physica 25, 733 (1959).

[3] In fact, the correspondence (2) and Weyl's rule are equivalent, as several authors showed independently; see, e.g., Ref. 2, Ref. 5, and Boris Leaf, J. Math. Phys. 9, 65 (1968).