PATENT NOTICE: When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Qualified requestors may obtain copies of this report from the ASTIA Arlington Hall Station, Arlington 12, Virginia. ASTIA Services for the Department of Defense contractors are available through the "Field of Interest Register" on a "need-to-know" certified by the cognizant military agency of their project or contract.

Copies of WADC Technical Reports and Technical Notes should not be returned to the Wright Air Development Center unless return is required by security considerations, contractual obligations, or notice on a specific document.

RRL-TR-60-274, Part I.
ASTIA

THEORY OF PLASMAS, I

by

R. K. Osborn

March 1960

Report No. 2756-1-T

on

CONTRACT AF 33(616)-5585

Prepared for

AERONAUTICAL RESEARCH LABORATORY
AIR FORCE RESEARCH DIVISION
AIR RESEARCH AND DEVELOPMENT COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

Table of Contents

Abstract

This paper presents a portion of an attempt to develop, in μ-space, a unified theory of plasmas - the main purpose being to localize points of difficulty or obscurity encountered in the deduction of a description of such systems. Within the context of non-relativistic particle dynamics, exact relations describing the particle and photon singlet densities are deduced - the relations for the particle densities being essentially those developed earlier by Brittin[1].

It is then shown that the relations governing the particle densities are crudely reducible to Boltzmann's and/or Vlasov's equations in the sense of certain reasonably clearly stated, but ill-evaluated, approximations. The relation describing the photon singlet density is discussed in considerable detail, and a reduction to the conventional equation of photon transport is accomplished.

---

[1] W.E. Brittin, Phys. Rev., 106, 843, 1957.

Introduction

The purpose of this report is to present the preliminary results of an attempt to develop in a systematic, self-contained way a unified theory of a plasma. In particular, it is an attempt to delineate - in the sense, at least, of indicating sufficiency conditions - the extent to which equations of the Boltzmann type can be expected to be descriptive of the particle and photon distributions in the plasma. Actually, a more accurate characterization of these investigations to date might well be that they represent an attempt to pinpoint the difficulties inherent in the development of such a theory in order that attention can subsequently be focused on relevant aspects of the problem.

For the purpose of delimiting areas of logical difficulty, an axiom-deduction approach has been adopted. Only after strictly deductive analysis from clearly stated axioms has been carried as far as has so far been found feasible, will intuitive assumptions and approximations be introduced in order to bridge the gaps between these tentative deductive termini and the desired balance relations. Accordingly, the material to be presented herein shall be divided into five parts. Part I will incorporate a statement of the axioms and a brief discussion thereof. Parts II and III will present deductive developments of balance relations for the particle and photon distributions respectively, while Parts IV and V shall be devoted to the introduction and discussion of approximations sufficient for the realization of Boltzmann-type equations for the particle and photon densities.

I. The Axioms

The axioms to be invoked here shall be essentially of two kinds and shall be presumed to be sufficient for the complete logical characterization of certain classes of systems of the general type presently under consideration. The first of these shall be for the purpose of specifying the dynamics of the interactions between the "particles" that comprise the plasma, whereas the second shall be for the purpose of introducing statistical concepts into a description of a system characterized by a huge number of degrees of freedom.

The dynamical axiom can be expected to be reasonably firm, at least within certain self-evident limitations such as, for example, nonrelativistic treatment of the particles. However, the statistical axioms enter in a most formal way, and it is in no sense being asserted here that it is either the best or even the right way. It is argued only that the method of axiomatizing the necessary statistical concepts adopted herein is both direct and operationally precise and that some amusing consequences are deducible therefrom.

The dynamical axiom will be stated in the form of an energy density for fields of interacting charged particles and photons, and the Schroedinger equation for the wave function which characterizes the states of such a system. Thus we have,

$$HF = i\hbar \frac{\partial F}{\partial t} \quad , \tag{1}$$

where

$$H = \int_{\underline{x}} \mathcal{H}(\underline{x})\, d^3 x \quad , \tag{2}$$

and

$$\mathcal{H}(\underline{x}) = - \sum_{\sigma} \frac{1}{2m_{\sigma}} \left[(i\hbar \underline{\nabla} - \frac{e_{\sigma}}{c} \underline{A} - \frac{e_{\sigma}}{c} \underline{A}^{e})_j \psi_{\sigma}^{+} \right]$$

$$(x) \left[(i\hbar \underline{\nabla} + \frac{e_{\sigma}}{c} \underline{A} + \frac{e_{\sigma}}{c} \underline{A}^{e})_j \psi_{\sigma} \right] + \left[2\pi c^2 \underline{P}^2 + \frac{1}{8\pi} (\underline{\nabla} \times \underline{A})^2 \right]$$

$$+ \sum_{\sigma} e_{\sigma} \phi \psi_{\sigma}^{+} \psi_{\sigma} + \frac{1}{2} \sum_{\sigma\sigma'} \int e_{\sigma} e_{\sigma'} \frac{\psi_{\sigma}^{+}(\underline{x}) \psi_{\sigma'}^{+}(\underline{x}') \psi_{\sigma}(\underline{x}) \psi_{\sigma'}(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3 x' \; . \tag{3}$$

In equation (3), ψ_{σ} is a wave operator for a field of particles of the σ^{th} kind, $\underline{A}^e$ and ϕ are the vector and scalar potentials of the "external" fields, and $\underline{A}$ and $\underline{P}$ are the "transverse" magnetic and electric operators for the photon field[2]. The external fields are unquantized, whereas the internal field operators satisfy the commutation relations,

$$\left[\psi_{\sigma}(\underline{x}) , \; \psi_{\sigma'}^{+}(\underline{x}') \right] = \delta_{\sigma\sigma'} \delta(\underline{x} - \underline{x}') \quad ,$$

$$\left[A_j(\underline{x}) , \; P_{\ell}(\underline{x}') \right] = i\hbar \; \delta_{j\ell} \; \delta(\underline{x} - \underline{x}')$$

$$- i\hbar \; \nabla_j \; \nabla'_{\ell} \left(\frac{1}{4\pi |\underline{x} - \underline{x}'|} \right) \quad , \tag{4}$$

[2] L.I. Schiff, "Quantum Mechanics", McGraw-Hill Book Company, Inc., 1949.

all other quantities commuting. Actually, the commutation relations indicated for the particle fields are not strictly appropriate for the plasma, since the electron field at least should be characterized by anticommutation rules. However, the subsequent analysis is quite unaffected by the choice of commutation rules except for the issue of the connection between singlet and doublet densities.

The assertion of the transversality of the magnetic and electric operators $\underline{A}$ and $\underline{P}$ is rendered formally precise by the statements

$$(\underline{\nabla} \cdot \underline{A})\, F = (\underline{\nabla} \cdot \underline{P})\, F = 0 \quad . \tag{5}$$

To facilitate subsequent analysis, it is convenient to introduce creation and destruction operators for the particles and photons according to the definitions,

$$\psi_\sigma = \frac{1}{(2\pi)^{3/2}} \int d^3K \, a_\sigma(\underline{K})\, e^{i\underline{K} \cdot \underline{x}} \quad , \tag{6a}$$

$$\underline{A} = \sqrt{\frac{2\pi \hbar c}{(2\pi)^3}} \sum_\lambda \int \frac{d^3k}{\sqrt{k}} \, \underline{\zeta}^{+}_{\lambda}(\underline{k})\, e^{-i\underline{k} \cdot \underline{x}} \quad , \tag{6b}$$

$$\underline{P} = i \sqrt{\frac{\hbar}{8\pi c\, (2\pi)^3}} \sum_\lambda \int \sqrt{k}\, d^3k \, \underline{\zeta}^{-}_{\lambda}(\underline{k})\, e^{-i\underline{k} \cdot \underline{x}} \quad , \tag{6c}$$

where

$$\underline{\zeta}^{\pm}_{\lambda}(\underline{k}) = \alpha^{+}_{\lambda}(\underline{k})\, \underline{\epsilon}_{\lambda}(\underline{k}) \pm \alpha_{\lambda}(-\underline{k})\, \underline{\epsilon}_{\lambda}(-\underline{k}) \quad . \tag{7}$$

In these expressions, a^{+}_{σ} and a_σ are creation and destruction operators for particles of kind σ with momentum $\hbar\, \underline{K}$; and α^{+}_{λ} and α_λ are creation and destruction operators for photons of momentum $\hbar\, \underline{k}$ and polarization λ. The unit vectors $\underline{\epsilon}_\lambda$ (λ = 1,2) are the polarization vectors for the photon field, and are largely determined by the requirements,

$$\underline{\epsilon}_1 \cdot \underline{\epsilon}_1 = \underline{\epsilon}_2 \cdot \underline{\epsilon}_2 = 1 \; ,$$

$$\underline{\epsilon}_1 \cdot \underline{\epsilon}_2 = 0 \; ,$$

$$\underline{\epsilon}_1 \cdot \underline{k} = \underline{\epsilon}_2 \cdot \underline{k} = 0 \; . \tag{8}$$

The commutation rules for these creation and destruction operators are,

$$\left[a_\sigma(\underline{K}) \, , \, a^+_{\sigma'}(\underline{K}') \right] = \delta_{\sigma\sigma'} \, \delta(\underline{K} - \underline{K}') \; ,$$

$$\left[\alpha_\lambda(\underline{k}) \, , \, \alpha^+_{\lambda'}(\underline{k}') \right] = \delta_{\lambda\lambda'} \, \delta(\underline{k} - \underline{k}') \; . \tag{9}$$

In this representation the Hamiltonian is conveniently decomposed into the terms,

$$H = T_P + T_\gamma + H_{P\gamma} + H_{Pe} + H_{P\gamma e} + V_c \; , \tag{10}$$

where

$$T_P = \sum_\sigma \int d^3K \, \frac{\hbar^2 K^2}{2m_\sigma} \, a^+_\sigma(\underline{K}) \, a_\sigma(\underline{K}) \; , \tag{11a}$$

$$T_\gamma = \sum_\lambda \int d^3k \, (\hbar c k) \, \alpha^+_\lambda(\underline{k}) \, \alpha_\lambda(\underline{k}) \; , \tag{11b}$$

$$H_{P\gamma} = -\frac{\hbar}{c} \sqrt{\frac{2\pi\hbar c}{(2\pi)^3}} \sum_{\lambda\sigma} \frac{e_\sigma}{m_\sigma} \int \frac{d^3K d^3K'}{\sqrt{|\underline{K} - \underline{K}'|}} \, a^+_\sigma(\underline{K}') \, a_\sigma(\underline{K}) \underline{K} \cdot \underline{\zeta}^+_\lambda(\underline{K} - \underline{K}')$$

(11c)

$$+ \frac{1}{2c^2} \left(\frac{2\pi\hbar c}{(2\pi)^3} \right) \sum_{\lambda\lambda'\sigma} \frac{e_\sigma^2}{m_\sigma} \int \frac{d^3K d^3K' d^3k}{\sqrt{k|\underline{K} - \underline{K}' - \underline{k}|}} \, a^+_\sigma(\underline{K}') \, a_\sigma(\underline{K}) \, \underline{\zeta}^+_\lambda(\underline{k}) \cdot \underline{\zeta}^+_{\lambda'}(\underline{K} - \underline{K}' - \underline{k}) \; ,$$

$$H_{Pe} = -\frac{\hbar}{2c}\frac{1}{(2\pi)^3}\sum_{\sigma}\frac{e_\sigma}{m_\sigma}\int d^3x d^3K d^3K' \left[\underline{A}^e \cdot (\underline{K}+\underline{K}')\right] \mathcal{a}_\sigma^+(\underline{K}')\, \mathcal{a}_\sigma(\underline{K}) e^{i\underline{x}\cdot(\underline{K}-\underline{K}')}$$

$$+\frac{1}{2c^2}\frac{1}{(2\pi)^3}\sum_{\sigma}\frac{e_\sigma^2}{m_\sigma}\int d^3x d^3K d^3K' (\underline{A}^e)^2\, \mathcal{a}_\sigma^+(\underline{K}')\, \mathcal{a}_\sigma(\underline{K})\, e^{i\underline{x}\cdot(\underline{K}-\underline{K}')}$$

$$+\frac{1}{(2\pi)^3}\sum_{\sigma} e_\sigma \int d^3x d^3K d^3K'\, \phi\, \mathcal{a}_\sigma^+(\underline{K}')\, \mathcal{a}_\sigma(\underline{K})\, e^{i\underline{x}\cdot(\underline{K}-\underline{K}')} \quad , \tag{11d}$$

$$H_{P\gamma e} = \frac{1}{c^2}\sqrt{\frac{2\pi\hbar c}{(2\pi)^3}}\,\frac{1}{(2\pi)^3}\sum_{\lambda,\sigma}\frac{e_\sigma^2}{m_\sigma}\int d^3x\, e^{i\underline{x}\cdot(\underline{K}-\underline{K}'-\underline{k})}$$

$$(x)\ \frac{d^3K d^3K' d^3k}{\sqrt{k}}\ \underline{A}^e \cdot \underline{\xi}_\lambda^+(\underline{k})\, \mathcal{a}_\sigma^+(\underline{K}')\, \mathcal{a}_\sigma(\underline{K}) \quad , \tag{11e}$$

$$V_c = \frac{1}{2(2\pi)^{3/2}}\sum_{\sigma,\sigma'} e_\sigma e_{\sigma'} \int d^3K d^3K' d^3K''\, U(|\underline{K}'-\underline{K}''|)$$

$$(x)\ \mathcal{a}_\sigma^+(\underline{K})\, \mathcal{a}_{\sigma'}^+(\underline{K}')\, \mathcal{a}_\sigma(\underline{K}+\underline{K}'-\underline{K}'')\, \mathcal{a}_{\sigma'}(\underline{K}'') \quad , \tag{11f}$$

where

$$U(|\underline{K}'-\underline{K}''|) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3R}{R}\, e^{i\underline{R}\cdot(\underline{K}'-\underline{K}'')} \quad .$$

The several terms in this expression for the energy are directly interpretable in intuitive terms; i.e., T_P and T_γ being the energies of free particles and photons respectively, $H_{P\gamma}$ and H_{Pe} being the interaction energy of particles with the photons and external electromagnetic fields, $H_{P\gamma e}$ being an energy of interaction between particles, photons, and external fields; and finally V_c being the electrostatic potential energy of the charged particles.

The statistical axioms to be employed herein may be rather directly stated. We define singlet particle and photon density operators by,

$$\rho_\sigma = \mathcal{a}^+_\sigma(\underline{K}+\underline{Q})\,\mathcal{a}_\sigma(\underline{K}-\underline{Q}) \quad ,$$

$$\rho_\gamma = \sum_\lambda \alpha^+_\lambda(\underline{k}+\underline{q})\,\alpha_\lambda(\underline{k}-\underline{q}) \quad , \tag{12}$$

and then define the particle and photon singlet densities,

$$f_\sigma(\underline{x},\underline{K},t) = \pi^{-3}\int d^3Q\, e^{-2i\underline{x}\cdot\underline{Q}}(F,\rho_\sigma F)$$

$$= \pi^{-3}\int d^3z\, e^{-2i\underline{K}\cdot\underline{z}}\left(F, \psi^+(\underline{x}-\underline{z})\,\psi(\underline{x}+\underline{z})\,F\right) \quad ,$$

$$\chi_\gamma(\underline{x}, \underline{k}, t) = \pi^{-3}\int d^3q\, e^{-2i\underline{x}\cdot\underline{q}}(F, \rho_\gamma F) \quad . \tag{13}$$

It is intended that the densities f_σ and χ_γ shall have the physical significance expressible as

> $f_\sigma(\underline{x}, \underline{K}, t)\, d^3x\, d^3K$ = the expected number of particles of kind σ to be found in $d^3x\, d^3K$ about $(\underline{x}, \underline{K})$ at time t,

and

> $\chi_\gamma(\underline{x}, \underline{k}, t)\, d^3x\, d^3k$ = the expected number of photons to be found in $d^3x\, d^3k$ about $(\underline{x}, \underline{k})$ at time t.

It will be indicated in the sequel that these densities satisfy - in the sense of certain approximations - the "conventional" transport equations for particles and photons. Thus it is somewhat interesting to note at this point that in no way has the concept of the ensemble been invoked for the purpose of defining these densities.

II. The Density, f_σ

The time rate of change of the singlet density for the particles of the σ^{th} kind is given by,

$$\frac{\partial f_\sigma}{\partial t} = \frac{i\pi^{-3}}{\hbar} \int d^3Q\, e^{-2i\underline{x}\cdot\underline{Q}} \left(F, [H, \rho_\sigma]\, F\right) . \tag{14}$$

It is a straightforward, but tedious, task to evaluate the indicated commutators and show that

$$\frac{\partial f_\sigma}{\partial t} = -\frac{\hbar}{m_\sigma} K_j \frac{\partial f_\sigma}{\partial x_j}$$

$$+ \sum_{\sigma'} \frac{i\pi^{-3}}{\hbar} \quad d^3 z\, d^3 x'\, d^3 K'\, d^3 K''\, e^{-2i\underline{z}\cdot(\underline{K}-\underline{K}'')} \left[\frac{e_\sigma e_{\sigma'}}{|\underline{x}'-\underline{x}+\underline{z}|}\right.$$

$$\left. - \frac{e_\sigma e_{\sigma'}}{|\underline{x}'-\underline{x}-\underline{z}|}\right] f_{\sigma\sigma'}(\underline{x},\underline{K}''; \underline{x}', \underline{K}'; t)$$

$$+ e_\sigma \frac{i\pi^{-3}}{\hbar} \int d^3 z\, d^3 k'\, e^{-2i\underline{z}\cdot(\underline{k}-\underline{k}')} \left[\phi(\underline{x}-\underline{z}) - \phi(\underline{x}+\underline{z})\right] f_\sigma(\underline{x}, \underline{k}', t)$$

$$- \frac{2e_\sigma}{m_\sigma c} K_j \pi^{-3} \int d^3Q\, e^{-2i\underline{x}\cdot\underline{Q}} \left(F, \sin\left\{\frac{\underline{\nabla}_x\cdot\underline{\nabla}_K}{2}\right\} (A_j + A_j^e)\, \rho_\sigma\, F\right)$$

$$+ \frac{e_\sigma}{m_\sigma c} \pi^{-3} \int d^3Q \left[\frac{\partial}{\partial x_j} e^{-2i\underline{x}\cdot\underline{Q}}\right] \left(F, \cos\left\{\frac{\underline{\nabla}_x\cdot\underline{\nabla}_K}{2}\right\} (A_j + A_j^e)\, \rho_\sigma\, F\right)$$

$$+ \frac{e_\sigma^2}{m_\sigma c^2} \frac{\pi^{-3}}{\hbar} \int d^3Q\, e^{-2i\underline{x}\cdot\underline{Q}} \left(F, \sin\left\{\frac{\underline{\nabla}_x\cdot\underline{\nabla}_K}{2}\right\} (\underline{A}+\underline{A}^e)^2 \rho_\sigma\, F\right) . \tag{15}$$

This result is substantially the same as that obtained by Brittin[1]. In fact, equation (15) differs mainly only in the explicit formulation of the densities in terms of the field theoretic formalism and in the introduction of the doublet density, $f_{\sigma\sigma'}$, at this stage of the analysis. This doublet density is defined in terms of a doublet density operator,

$$\rho_{\sigma\sigma'} = \mathcal{a}_{\sigma}^{+}(\underline{K}+\underline{Q})\ \mathcal{a}_{\sigma'}^{+}(\underline{K}'+\underline{Q}')\ \mathcal{a}_{\sigma}(\underline{K}-\underline{Q})\ \mathcal{a}_{\sigma'}(\underline{K}'-\underline{Q}') \quad , \tag{16}$$

and the further definition,

$$f_{\sigma\sigma'}(\underline{x}, \underline{K}, \underline{x}', \underline{K}', t) = \pi^{-6} \int d^3Q\, d^3Q'\, e^{-2i\underline{x}\cdot\underline{Q}-2i\underline{x}'\cdot\underline{Q}'} (F, \rho_{\sigma\sigma'} F) \, . \tag{17}$$

The subsequent discussion of equation (15) herein is novel, however; departing from Brittin in that his concern is for the "magnetohydrodynamic" equations whereas ours is for Boltzmann's equation.

"Equation" (15) as it stands is, of course, not actually an equation at all, but merely a connection between diverse functions. The argument by means of which we attempt to replace this rigorous, but more or less meaningless, relation between obscure quantities by an equation of the Boltzmann type will be reserved for Section IV.

---

[1] W.E. Brittin, Phys. Rev., 106, 843, 1957.

III. The Density, χ_γ

An important aspect of plasma behavior which does not seem to have been given explicit consideration from a fundamental point of view is that of photon transport. It should be noted however, that some of the investigations of Klimontovich[3] and Klimontovich and Temko[4] have been directed quite closely to some of the issues under consideration herein. But most of the attempts to deal with the problem of the interaction of radiation with plasmas have employed the conventional phenomenological electrodynamics of the macroscopic continuum. However, the validity of such an approach is not always intuitively self-evident in the context of the plasma. Furthermore, the macroscopic theory is inherently incapable of delineating the range of physical situations for which it is a suitable description. Consequently, a microscopic description of plasma electrodynamics is desirable, if only for the purpose of estimating the limitations - if any - of the macroscopic theory. For these reasons, among others, we present a detailed - though preliminary - discussion of the relations characterizing photon transport, as well as certain other general considerations of plasma electrodynamics.

---

[3] Iu. L. Klimontovich, Soviet Physics, JETP, 35 , No. 5, 891 , 1959.

[4] Iu. L. Klimontovich and S.V. Temko, Soviet Physics, JETP, 35 , No. 5, 799, 1959.

Recalling equations (13) and (14) we again note that

$$\frac{\partial \chi_\gamma}{\partial t} = \frac{i\pi^{-3}}{\hbar} \int d^3q\, e^{-2i\underline{x}\cdot\underline{q}} \left(F, \left[H, \rho_\gamma\right] F\right) \quad . \tag{18}$$

In this case, as in that of the previous section, it is straightforward but tedious to obtain the following:

$$\frac{\partial \chi_\gamma}{\partial t} = -2c\,\chi_\gamma \left\{\sin\left(\frac{\overleftarrow{\underline{\nabla}}_x \cdot \overrightarrow{\underline{\nabla}}_K}{2}\right)\right\} |\underline{k}| - \frac{i\pi^{-3}}{\hbar}\,\frac{\hbar}{c}\,\sqrt{\frac{2\pi hc}{(2\pi)^3}}$$

$$(x) \int d^3q\, e^{-2i\underline{x}\cdot\underline{q}} \Bigg(F, \Bigg\{ \sum_{\lambda,\sigma} \frac{e_\sigma}{m} \int \frac{d^3K}{\sqrt{|\underline{k}+\underline{q}|}}\, a_\sigma^+(\underline{K}+\underline{k}+\underline{q})\, a_\sigma(\underline{K})\underline{K}\cdot \underline{\epsilon}_\lambda(\underline{k}+\underline{q})\, \alpha_\lambda(\underline{k}-\underline{q})$$

$$- \sum_{\lambda,\sigma} \frac{e_\sigma}{m_\sigma} \int \frac{d^3K}{\sqrt{|\underline{k}-\underline{q}|}}\, a_\sigma^+(\underline{K}-\underline{k}+\underline{q})\, a_\sigma(\underline{K})\underline{K}\cdot \underline{\epsilon}_\lambda(\underline{k}-\underline{q})\, \alpha_\lambda^+(\underline{k}+\underline{q}) \Bigg\} F \Bigg)$$

$$+ \frac{i\pi^{-3}}{\hbar}\,\frac{1}{c^2}\,\frac{2\pi\hbar c}{(2\pi)^3} \int d^3q\, e^{-2i\underline{x}\cdot\underline{q}}$$

$$(x) \Bigg(F \sum_{\lambda\lambda'\sigma} \frac{e_\sigma^2}{m_\sigma} \int \frac{d^3K\, d^3K'\, a_\sigma^+(\underline{K}')\, a_\sigma(\underline{K})\, \underline{\epsilon}_\lambda(\underline{k}+\underline{q})\cdot \underline{\xi}_{\lambda'}^+(\underline{K}-\underline{K}'+\underline{k}+\underline{q})\, \alpha_\lambda(\underline{k}-\underline{q})}{\sqrt{|\underline{k}+\underline{q}|\;|\underline{K}-\underline{K}'+\underline{k}+\underline{q}|}}$$

$$- \sum_{\lambda\lambda'\sigma} \frac{e_\sigma^2}{m_\sigma} \int \frac{d^3K\, d^3K'\, a_\sigma^+(\underline{K}')\, a_\sigma(\underline{K})\, \alpha_\lambda^+(\underline{k}+\underline{q})\, \underline{\epsilon}_\lambda(\underline{k}-\underline{q})\cdot \underline{\xi}_{\lambda'}^+(\underline{K}-\underline{K}'-\underline{k}+\underline{q})}{\sqrt{|\underline{k}-\underline{q}|\;|\underline{K}-\underline{K}'-\underline{k}+\underline{q}|}} \Bigg\} F \Bigg)$$

$$+ \frac{1}{c^2} \sqrt{\frac{2\pi\hbar c}{(2\pi)^3}} \frac{i\pi^{-3}}{\hbar} \sum_{\lambda,\sigma} \frac{e_\sigma^2}{m_\sigma} \int d^3K d^3K' d^3q \, e^{-2i\underline{x}\cdot\underline{q}}$$

$$(x) \left[\frac{\underline{A}^e(\underline{K}-\underline{K}'+\underline{k}+\underline{q}) \cdot \underline{\epsilon}_\lambda(\underline{k}+\underline{q})}{\sqrt{|\underline{k}+\underline{q}|}} \left(F, \, a_\sigma^+(\underline{K}') \, a_\sigma(\underline{K}) \alpha_\lambda(\underline{k}-\underline{q}) \, F\right) \right.$$

$$\left. - \frac{\underline{A}^e(\underline{K}-\underline{K}'+\underline{q}-\underline{k}) \cdot \underline{\epsilon}_\lambda(\underline{k}-\underline{q})}{\sqrt{|\underline{k}-\underline{q}|}} \left(F, \, a_\sigma^+(\underline{K}') \, a_\sigma(\underline{K}) \, \alpha_\lambda^+(\underline{k}+\underline{q}) \, F\right) \right] \tag{19}$$

where we have introduced the Fourier transform,

$$\underline{A}^e(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x \, e^{i\underline{k}\cdot\underline{x}} \, \underline{A}^e(\underline{x}) \quad . \tag{20}$$

This expression is, at best, obscure. However, we shall see in Section V that some interesting comment upon the problem of photon transport is available even at this stage and that a certain amount of intuitive interpretation of the several terms in equation (19) is perhaps already accessible.

Furthermore, in Section VI we accomplish an approximate reduction of equation (19) to the form of a transport equation useful for characterizing the balance of high frequency radiation in the plasma.

IV. A "Boltzmann Equation" for the Particle Singlet Densities

At this point we redirect our attention to the relation (15) and attempt to extract from it a Boltzmann-type equation for the singlet density f_σ. The indicated relation is rigorous, but contentless. In order to specialize it to a significant and useful form, it shall be sufficient (but not yet demonstrably necessary) to introduce certain approximations which, though they may be stated in operationally precise terms, will be ill-evaluated qualitatively and more or less completely unevaluated quantitatively.

Before proceeding to a discussion of these approximations, however, it is convenient to rewrite equation (15) in a slightly different form; i.e.,

$$\frac{\partial f_\sigma}{\partial t} = - \frac{\hbar K_j}{m_\sigma} \frac{\partial f_\sigma}{\partial x_j}$$

$$+ \frac{2}{\hbar} \sum_{\sigma'} \int d^3x' d^3K' \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \sin \left\{ \frac{\overleftarrow{\underline{\nabla}}_x \cdot \overrightarrow{\underline{\nabla}}_K}{2} \right\} f_{\sigma\sigma'} (\underline{x}, \underline{K}, \underline{x}', \underline{K}', t)$$

$$+ \frac{2}{\hbar} e_\sigma \phi \sin \left\{ \frac{\overleftarrow{\underline{\nabla}}_x \cdot \overrightarrow{\underline{\nabla}}_K}{2} \right\} f_\sigma$$

$$- \frac{2e_\sigma}{m_\sigma c} K_j \pi^{-3} \int d^3Q \, e^{-2i\underline{x}\cdot\underline{Q}} \sin \left\{ \frac{\underline{\nabla}_x \cdot \underline{\nabla}_K}{2} \right\} \left(F, \left\{ A_j + A_j^e \right\} \rho_\sigma F \right)$$

$$+ \frac{e_\sigma}{m_\sigma c} \pi^{-3} \int d^3Q \left[\frac{\partial}{\partial x_j} e^{-2i\underline{x}\cdot\underline{Q}} \right] \cos \left\{ \frac{\underline{\nabla}_x \cdot \underline{\nabla}_K}{2} \right\} \left(F, \left\{ A_j + A_j^e \right\} \rho_\sigma F \right)$$

$$+ \frac{e_\sigma^2}{m_\sigma c^2} \frac{\pi^{-3}}{\hbar} \int d^3Q \, e^{-2i \underline{x} \cdot \underline{Q}} \sin\left\{\frac{\underline{\nabla}_x \cdot \underline{\nabla}_K}{2}\right\} \left(F, \left\{A_j + A_j^e\right\}^2 \rho_\sigma F\right) . \quad (21)$$

Equation (21) merely takes explicit cognizance of equations (1 and 2) which imply that the state vector F is not an explicit function of either $\underline{x}$ or $\underline{K}$. It should be further borne in mind that $\underline{A}$ and $\underline{A}^e$ are functions of $\underline{x}$ but not of $\underline{K}$, whereas ρ_σ is a function of $\underline{K}$ but not of $\underline{x}$. (Also that $\underline{A}^e$ is a given vector field specified by charge and current distributions outside the plasma - and hence could be removed from the averaging process also, while $\underline{A}$ is the photon operator defined in equation (6b).)

The first approximation that we shall introduce shall consist in the replacement of the quantum by a classical description of the singlet density. Such a description should be generated simply by setting $\hbar$ equal to zero. Actually, such an approximation should not be expected to be serious so long as only non-degenerate plasmas are to be described; i.e., systems for which the number of readily available particle states greatly exceeds the number of particles.

Recalling that $\underline{p} = \hbar \underline{K}$, we obtain, after expanding the explicit dependence of the relation (21) upon $\hbar$ in a power series in $\hbar$ and retain only those terms not explicitly dependent upon $\hbar$,

$$\frac{\partial f_\sigma}{\partial t} + \frac{p_j}{m_\sigma} \frac{\partial f_\sigma}{\partial x_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3p' \left[\frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial}{\partial p_j} f_{\sigma\sigma'}(\underline{x}, \underline{p}, \underline{x}', \underline{p}', t) + e_\sigma \frac{\partial \phi}{\partial x_j} \frac{\partial f_\sigma}{\partial p_j}$$

$$- \frac{e_\sigma}{m_\sigma c} p_j \pi^{-3} \int d^3Q \, e^{-2i\underline{x}\cdot\underline{Q}} \frac{\partial^2}{\partial x_\ell \partial p_\ell} \left(F, \left\{A_j + A_j^e\right\} \rho_\sigma F\right)$$

$$+ \frac{e_\sigma}{m_\sigma c} \pi^{-3} \int d^3Q \left[\frac{\partial}{\partial x_j} e^{-2i\underline{x}\cdot\underline{Q}} \right] \left(F, \left\{A_j + A_j^e\right\} \rho_\sigma F\right)$$

$$+ \frac{e_\sigma^2}{2m_\sigma c^2} \pi^{-3} \int d^3Q \, e^{-2i\underline{x}\cdot\underline{Q}} \frac{\partial^2}{\partial x_\ell \partial p_\ell} \left(F, \left\{A_j + A_j^e\right\}^2 \rho_\sigma F\right) . \qquad (22)$$

It is not quite obvious that our present approach to the classical limit is consistent, since we have ignored an implicit dependence upon $\hbar$ in the quantum expectation values. However, in the sense of subsequent approximations it will be seen that this neglect is apparently justified.

At this point, we introduce a fundamental approximation which is hardly intuitively transparent and which has so far been almost completely unevaluated quantitatively. Consider, for example, the expectation value

$$\left(F, \left\{A_j + A_j^e\right\} \rho_\sigma F\right) = A_j^e (F, \rho_\sigma F) + (F, A_j \rho_\sigma F) , \qquad (23)$$

recalling that $\underline{A}^e$ is an externally determined vector potential. Devoting our attention to the second term in (23), we observe that it is the average of a product of two operators, one exclusively in the photon space and the other exclusively in the particle space. The average of such a product would become the product of averages if the state vector were decomposable into a simple product of photon and particle state vectors - a circumstance that would obtain in the event of no interaction between the particles and photons. Thus in actuality such a decomposition is not to be expected, but it is perhaps reasonable to assert that the approximation,

$$(F, A_j \rho_\sigma F) = (F, A_j F)(F, \rho_\sigma F) + \Big[(F, A_j \rho_\sigma F) - (F, A_j F)(F, \rho_\sigma F)\Big]$$

$$\cong (F, A_j F)(F, \rho_\sigma F) \quad , \tag{24}$$

is quantitatively justified in certain interesting classes of cases. It should be noted that a first order estimate of the quantitative significance of this approximation is readily accessible in the context of conventional perturbation analysis of the states characterizing interacting photons and particles. However, such an investigation will not be entered into in this section.

It is further to be noted that this decomposition of averages of products into products of averages will, for the moment, be reserved for those instances in which at least one of the factors is in the photon space. Specifically, it will not be extrapolated at this stage in the development of the argument to the case in which both factors are in particle spaces as is implicit in the definition of $f_{\sigma\sigma'}$.

We now observe that, in the sense of this approximation, equation (23) could as well be written as

$$\left(F, \left\{A_j + A_j^e\right\} \rho_\sigma F\right) \cong \left(F, \left\{A_j + A_j^e\right\} F\right) (F, \rho_\sigma F) \quad , \tag{25}$$

so that equation (22) becomes,

$$\frac{\partial f_\sigma}{\partial t} + \frac{p_j}{m_\sigma} \frac{\partial f_\sigma}{\partial x_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3p' \left[\frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial}{\partial p_j} f_{\sigma\sigma'}(\underline{x}, \underline{p}, \underline{x}', \underline{p}', t) + e_\sigma \frac{\partial \phi}{\partial x_j} \frac{\partial f_\sigma}{\partial p_j}$$

$$- \frac{e_\sigma}{m_\sigma c} p_j \frac{\partial (F, \{A_j + A_j^e\} F)}{\partial x_\ell} \frac{\partial f_\sigma}{\partial p_\ell} + \frac{e_\sigma}{m_\sigma c} (F, \{A_j + A_j^e\} F) \frac{\partial f_\sigma}{\partial x_j}$$

$$+ \frac{e_\sigma^2}{2 m_\sigma c^2} \frac{\partial (F, \{A_j + A_j^e\}^2 F)}{\partial x_\ell} \frac{\partial f_\sigma}{\partial p_\ell} \quad , \tag{26}$$

recalling the definition of f_σ; i.e., equation (13). It is convenient to introduce the symbol

$$R_j = (F, \{A_j + A_j^e\} F) \cdot = A_j^e + (F, A_j F) \quad , \tag{27}$$

and rewrite (26) compactly as

$$\frac{\partial f_\sigma}{\partial t} + \frac{p_j}{m_\sigma} \frac{\partial f_\sigma}{\partial x_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3p' \left[\frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial f_{\sigma\sigma'}}{\partial p_j} + e_\sigma \frac{\partial \phi}{\partial x_j} \frac{\partial f_\sigma}{\partial p_j}$$

$$- \frac{e_\sigma}{m_\sigma c} p_j \frac{\partial R_j}{\partial x_\ell} \frac{\partial f_\sigma}{\partial p_\ell} + \frac{e_\sigma}{m_\sigma c} R_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma^2}{2 m_\sigma c^2} \frac{\partial R_j^2}{\partial x_\ell} \frac{\partial f_\sigma}{\partial p_\ell} . \tag{28}$$

Since it is particle velocities which are observable and not particle momenta, it is desirable to transform coordinates according to

$$(\underline{x}, \underline{p}) \longrightarrow (\underline{x}, \underline{v}) \quad , \qquad p = m\underline{v} - \frac{e}{c} \underline{R} \quad , \tag{29}$$

in which case the densities transform according to the definition

$$f_\sigma(\underline{x}, \underline{v}, t) = J \left| \left(\frac{\underline{x}, \underline{p}}{\underline{x}, \underline{v}} \right) \right| f_\sigma(\underline{x}, m_\sigma \underline{v} - \frac{e_\sigma}{c} \underline{R}, t) . \tag{30}$$

Since $J = m_\sigma^3$ and is thus a constant, and since equation (28) is homogeneous, with respect to this Jacobian, we may ignore it henceforth. In terms of these new independent variables, equation (28) becomes,

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} - \frac{e_\sigma}{m_\sigma c} \frac{\partial R_j}{\partial t} \frac{\partial f_\sigma}{\partial v_j} - \frac{e_\sigma}{m_\sigma} \frac{\partial \phi}{\partial x_j} \frac{\partial f}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} \left[\underline{v} \times (\nabla \times \underline{R}) \right]_j \frac{\partial f_\sigma}{\partial v_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j} . \tag{31}$$

Recalling the definition of $\underline{R}$, we observe that equation (31) may be rewritten as,

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma}{m_\sigma} E_j^e \frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} (\underline{v} \times \underline{H}^e)_j \frac{\partial f_\sigma}{\partial v_j}$$

$$- \frac{e_\sigma}{m_\sigma c} \frac{\partial (F, A_j F)}{\partial t} \frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} \left[\underline{v} \times \left\{ \underline{\nabla} \times (F, \underline{A} F) \right\} \right]_j \frac{\partial f_\sigma}{\partial v_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j} , \qquad (32)$$

where we have introduced the symbols $\underline{E}^e$ and $\underline{H}^e$ to represent the external electric and magnetic fields respectively.

At this point it is necessary to recall the meaning of the expectation values $(F, \underline{A}F)$ and investigate the significance of their space and time derivatives (Note that the presence of these quantities implies that the description of the particle densities at this stage is still not completely classical.)

Define the operators,

$$\underline{E}^T = - \frac{1}{c} \overset{o}{\underline{A}} = \frac{i}{\hbar c} \left[H, \underline{A} \right] = - 4\pi c \, \underline{P}$$

and

$$\underline{B} = \underline{\nabla} \times \underline{A} . \qquad (33)$$

Further define the expectation values of these operators to be,

$$\underline{\mathcal{E}}^T = (F, \underline{E}^T F) ,$$

$$\underline{\mathcal{B}} = (F, \underline{B} F) . \qquad (34)$$

It is then a straightforward matter to show that these averages (note - not "ensemble" averages) satisfy the equations[5],

$$\underline{\nabla} \times \underline{\mathcal{E}}^T = -\frac{1}{c}\frac{\partial}{\partial t}\underline{\mathcal{B}} ,$$

$$\underline{\nabla} \cdot \underline{\mathcal{B}} = 0 ,$$

$$\underline{\nabla} \cdot \underline{\mathcal{E}}^T = 0 ,$$

$$\frac{1}{c}\frac{\partial}{\partial t}\underline{\mathcal{E}}^T = \underline{\nabla} \times \underline{\mathcal{B}} - \frac{4\pi i}{\hbar}\left(F, \left[H_{p\gamma}, \underline{P}\right] F\right) . \tag{35}$$

These are, of course, the appropriate subset of Maxwell's equations for the description of the magnetic and transverse electric fields in the presence of an irrotational current density given by

$$\underline{J}^T = \frac{ic}{\hbar}\left(F, \left[H_{p\gamma}, \underline{P}\right] F\right) . \tag{36}$$

In order to complete these equations we must adjoin the description of the longitudinal part of the electric field. This is accomplished by defining the longitudinal electric field operator,

$$\underline{E}^L = \sum_{\sigma} e_{\sigma} \int_{\underline{x}'} \frac{(\underline{x}-\underline{x}')\,\psi_{\sigma}^{+}(\underline{x}')\,\psi_{\sigma}(\underline{x}')}{|\underline{x}-\underline{x}'|^3}\, d^3x'$$

$$= -\sum_{\sigma} e_{\sigma} \underline{\nabla} \int_{\underline{x}'} \frac{\psi_{\sigma}^{+}(\underline{x}')\,\psi_{\sigma}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, d^3x' , \tag{37}$$

---

[5] See for example, G. Wentzel, "Quantum Theory of Fields", Interscience Publ. Inc., New York, 1949.

and its expectation value

$$\underline{\mathcal{E}}^L \equiv (F, \underline{E}^L F) \; ; \tag{38}$$

which, we note, satisfies the equation,

$$\underline{\nabla} \cdot \underline{\mathcal{E}}^L = 4\pi \sum_\sigma e_\sigma \; (F, \psi_\sigma^+ \; \psi_\sigma \; F)$$

$$= 4\pi \sum_\sigma e_\sigma \int_{\underline{K}} f_\sigma(\underline{x}, \underline{K}, t) \, d^3K \; . \tag{39}$$

This equation is merely Poisson's equation for the longitudinal component of the observable electric field. If we now calculate the time derivative of $\underline{\mathcal{E}}^L$ and add it to $\underline{\mathcal{E}}^T$, we find, for $\underline{\mathcal{E}} = \underline{\mathcal{E}}^T + \underline{\mathcal{E}}^L$, the equation

$$\frac{1}{c} \frac{\partial}{\partial t} \underline{\mathcal{E}} = \underline{\nabla} \times \underline{\mathcal{B}} - \frac{4\pi}{c} \underline{J}^T - \frac{4\pi}{c} \underline{J}^L \quad , \tag{40}$$

where the longitudinal component of the current density is given by

$$\underline{J}^L = - \frac{i}{4\pi \hbar} \; (F, [\, H, \underline{E}^L \,] \; F) \; . \tag{41}$$

Finally, if we compute $\underline{J}$ in accordance with equations (36 and 41), we may summarize the macroscopic electromagnetic equations for the plasma implied by the axioms employed herein as:

$$\underline{\nabla} \times \underline{\mathcal{E}} + \frac{1}{c} \frac{\partial}{\partial t} \underline{\mathcal{B}} = 0 \; ,$$

$$\underline{\nabla} \cdot \underline{\mathcal{B}} = 0$$

$$\underline{\nabla} \cdot \underline{\mathcal{E}} = 4\pi \sum_\sigma e_\sigma \int_{\underline{v}} f_\sigma \; (\underline{x}, \underline{v}, t) \, d^3v$$

$$\underline{\nabla} \times \underline{\mathcal{B}} - \frac{1}{c}\frac{\partial}{\partial t}\underline{\mathcal{E}} = \frac{4\pi}{c}\sum_{\sigma} e_\sigma \int_{\underline{V}} \underline{v}\, f_\sigma(\underline{x}, \underline{v}, t)\, d^3v$$

$$+ \frac{4\pi}{c}\sum_{\sigma} \frac{e_\sigma^2}{m_\sigma c}\left[(F, \underline{A}F)(F, \psi_\sigma^+ \psi_\sigma F) - (F, \underline{A}\, \psi_\sigma^+ \psi_\sigma F)\right]. \quad (42)$$

This result is somewhat curious because of the inclusion in the expression for the current of the correlation term. The presence of this term is quite possibly a direct consequence of the unconventional formulation of the statistical axiom as employed herein. If this should be the case, then perhaps a careful re-investigation of the question of how to introduce statistical concepts into this problem is in order. However, it should be noted that, in the sense of approximations already explicitly invoked, this correlation term is actually ignorable. Thus though there may be something of importance to be scrutinized here, this issue will be tentatively bypassed.

We now rewrite equation (32) taking account of the definitions (34), obtaining

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma}{m_\sigma}(E_j^e + \mathcal{E}_j^T)\frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c}\left[v \times (\underline{H}^e + \underline{\mathcal{B}})\right]_j \frac{\partial f_\sigma}{\partial v_j}$$

$$= \sum_{\sigma'} \int d^3x' d^3v' \left[\frac{1}{m_\sigma}\frac{\partial}{\partial x_j}\frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|}\right]\frac{\partial f_{\sigma\sigma'}}{\partial v_j}. \quad (43)$$

In order to complete the argument converting this relation to an equation, we need to relate $f_{\sigma\sigma'}$ to some functional of f_σ and $f_{\sigma'}$. At least two (crudely intuitive) methods of accomplishing this recommend themselves. The first of these is quite direct but presumably somewhat unsophisticated. It consists of simply exhibiting

$$f_{\sigma\sigma'} = f_\sigma f_{\sigma'} + (f_{\sigma\sigma'} - f_\sigma f_{\sigma'}) \quad , \tag{44}$$

and then asserting that the correlation term is in some sense ignorable. Accepting this approach, one finds that the Coulomb term becomes

$$\frac{\partial f_\sigma}{\partial v_j} \frac{e_\sigma}{m_\sigma} \frac{\partial}{\partial x_j} \sum_{\sigma'} \int d^3x' d^3v' \frac{e'_\sigma}{|\underline{x} - \underline{x}'|} f_{\sigma'} = - \frac{e_\sigma}{m_\sigma} \mathcal{E}^L_j \frac{\partial f_\sigma}{\partial x_j} \quad , \tag{45}$$

in accordance with equation (37). Inserting this result in equation (43) and defining the total fields,

$$\begin{aligned} \underline{H} &= \underline{H}^e + \underline{\mathcal{B}} \quad , \\ \underline{E} &= \underline{E}^e + \underline{\mathcal{E}}^T + \underline{\mathcal{E}}^L \\ &= \underline{E}^e + \underline{\mathcal{E}} \quad , \end{aligned} \tag{46}$$

we obtain the Vlasov[6] equations:

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma}{m_\sigma} E_j \frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} (\underline{v} \times \underline{H})_j \frac{\partial f_\sigma}{\partial v_j} = 0 \quad . \tag{47}$$

These equations have been extensively employed in plasma studies - particularly in investigations of plasma oscillations.[7]

---

[6] A. Vlasov, Journ. Exper. and Theor. Phys., 8, 291, 1938.

[7] I.B. Bernstein, Phys. Rev., 109, 10, 1958.

The second approach alluded to above perhaps yields a slightly better approximation to the relation (43). However, whether it is a better or worse approximation has yet to be established deductively from "first principles"; nor is it at all apparent that there is sufficient experimental evidence bearing on this issue to provide a resolution. Thus the following argument is presented mainly for the purpose of obtaining a specific form for the description of the plasma.

In this instance we first break up the Coulomb integral in such a way that part of the electrostatic interaction between particles is characterized by a self-consistent longitudinal electric field and the remainder (the high momentum transfer interactions) by binary collisions. It is first assumed that the density variations throughout the system are neither so great nor so irregular that the introduction of a meaningful average inter-particle spacing, ℓ, is precluded. Then the term describing Coulomb interactions in equation (43) is exhibited as a sum of two terms; i.e.,

$$\sum_{\sigma'} \int_{|\underline{x}-\underline{x}'|>\ell} d^3x'd^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x}-\underline{x}'|}\right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j}$$

$$+\sum_{\sigma'} \int_{|\underline{x}-\underline{x}'|<\ell} d^3x'd^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x}-\underline{x}'|}\right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j} . \tag{48}$$

The first term in the expression (48) is dealt with in precisely the same terms that lead to equation (45); thus we may rewrite (48) as

$$- \frac{e_\sigma}{m_\sigma} \tilde{\mathcal{E}}^L_j \frac{\partial f_\sigma}{\partial x_j} + \sum_{\sigma'} \int_{|\underline{x}-\underline{x}'|<\ell} d^3x' d^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x}-\underline{x}'|} \right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j} \quad , \tag{49}$$

where we have introduced the symbol $\tilde{\underline{\mathcal{E}}}^L$ to represent the longitudinal electric field at the point $\underline{x}$ at time t due to the charge distribution throughout the plasma, excepting a region of volume $\sim \ell^3$ centered at the point $\underline{x}$.

The second term in (49) may be argued to be approximated in some sense by the conventional collision representation of high momentum transfer interactions, provided the differential cross section is appropriately truncated in angle so as to correspond to maximum distance of closest approach somewhat less than ℓ. The argument is practically purely intuitive, though plausible. It will not be reproduced here[+] inasmuch as - in simplest terms - it consists of nothing more than a somewhat fuzzy adaptation of the argument deemed appropriate to the development of the binary collision characterization of interactions in gases comprised of particles which interact only via short-range forces.[8] The principal unique feature of the argument when employed in the present context is the seeming necessity for the introduction of a true Coulomb cross section truncated in angle, rather than a cross section defined by a shielded Coulomb potential.

---

8 R. Eisenschitz, Statistical Theory of Irreversible Processes, Oxford Univ. Press, 1958.

+ See Appendix A

If it is accepted, and if the total magnetic and electric fields are defined as in (46) - except for the replacement of $\underline{\mathcal{E}}^L$ by $\tilde{\underline{\mathcal{E}}}^L$ - then our description of the plasma becomes,

$$\frac{\partial f_\sigma}{\partial t} + v_j \frac{\partial f_\sigma}{\partial x_j} + \frac{e_\sigma}{m_\sigma} E_j \frac{\partial f_\sigma}{\partial v_j} + \frac{e_\sigma}{m_\sigma c} (\underline{v} \times \underline{H})_j \frac{\partial f_\sigma}{\partial v_j}$$

$$= \sum_{\sigma'} \int d^3 v_1 \, d\Omega \, | \underline{v} - \underline{v}_1 | \, \sigma_{\sigma\sigma'} \left[f_\sigma(\underline{v}') \, f_{\sigma'}(\underline{v}_1') - f_\sigma(\underline{v}) \, f_{\sigma'}(\underline{v}_1) \right] . \qquad (50)$$

These equations have been extensively employed in investigations of plasma behavior[9]. Evidently they are essentially Vlasov's equations (eq. 47) corrected by the addition of the binary "collision" terms. That equation (50) is - in some sense - a "better" description of the distribution of particles of the σ^{th} kind in the plasma than equation (47) is at least suggested by the observation that, although collisions may be an unimportant contribution to the characterization of system states far from equilibrium (in which case equation (50) reduces to equation (47)), they become all-important in the description of the equilibrium state itself.

A curious aspect of the present development of the subject - merely mentioned in passing earlier - should be emphasized here. The singlet densities have herein been defined without recourse to an ensemble concept. This is, at

---

9 L. Sptizer, Jr., "Physics of Fully Ionized Gases", Interscience Publ. Inc., New York, 1956.

least formally, at variance with the usual attempts to deal fundamentally with irreversible statistical processes. Furthermore, if one attempts a comparable deduction of equation (50) employing strictly classical concepts ab initio one finds that, indeed, some sort of an ensemble concept is seemingly necessary in order to give the densities, f_σ , their conventional statistical significance. It would appear probable therefore that the formulation of the statistical axiom provides a focal point for further investigation. See Section VII for further discussion of this point.

V. Further Discussion of Photon Transport.

In order to facilitate some further investigation of the implications of equation (19) for the description of photon distributions in plasmas, it is convenient to rewrite that equation in a slightly more immediately interpretable notation; i.e.,

$$\frac{\partial \chi_\gamma}{\partial t} + 2c\,\chi_\gamma \left\{ \sin\left(\frac{\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_k}{2} \right) \right\} |k|$$

$$= -\frac{i\pi^{-3}}{c} \sqrt{\frac{2\pi\hbar c}{(2\pi)^3}} \int \frac{d^3K d^3Q}{\sqrt{Q}} \sum_{\sigma,\lambda} \frac{e_\sigma}{m_\sigma}$$

$$(x)\Big[e^{-2i\underline{x}\cdot(\underline{Q}-\underline{k})} \left\{ \underline{K}\cdot \epsilon_\lambda(\underline{Q}) \right\} \left(F, \rho_\sigma(\underline{K}+\underline{Q},\underline{K})\,\alpha_\lambda(2\underline{k}-\underline{Q})\, F \right)$$

$$- e^{-2i\underline{x}\cdot(\underline{Q}+\underline{k})} \left\{ \underline{K}\cdot \epsilon_\lambda(-\underline{Q}) \right\} \left(F, \rho_\sigma(\underline{K}+\underline{Q},\underline{K})\,\alpha_\lambda^{+}(2\underline{k}+\underline{Q})\, F \right) \Big]$$

$$+ \frac{i\pi^{-3}}{(2\pi)^2 c} \int d^3q\, d^3K\, d^3Q\, e^{-2i\underline{x}\cdot\underline{q}} \sum_{\sigma,\lambda,\lambda'} \frac{e_\sigma^2}{m_\sigma}$$

$$(x)\Bigg[\frac{\epsilon_\lambda(\underline{k}+\underline{q})\cdot \epsilon_{\lambda'}(\underline{k}+\underline{q}-\underline{Q})}{\sqrt{|\underline{k}+\underline{q}|\,|\underline{k}+\underline{q}-\underline{Q}|}} \left(F, \rho_\sigma(\underline{K}+\underline{Q},\underline{K})\,\alpha_{\lambda'}^{+}(\underline{k}+\underline{q}-\underline{Q})\,\alpha_\lambda(\underline{k}-\underline{q})\, F \right)$$

$$- \frac{\epsilon_\lambda(\underline{k}-\underline{q})\cdot \epsilon_{\lambda'}(\underline{k}+\underline{Q}-\underline{q})}{\sqrt{|\underline{k}-\underline{q}|\,|\underline{q}-\underline{k}-\underline{Q}|}} \left(F, \rho_\sigma(\underline{K}+\underline{Q},\underline{K})\,\alpha_\lambda^{+}(\underline{k}+\underline{q})\,\alpha_{\lambda'}(\underline{k}+\underline{Q}-\underline{q})\, F \right) \Bigg]$$

$$+ \frac{i\pi^{-3}}{(2\pi)^2 c} \int d^3q\, d^3K\, d^3Q\, e^{-2i\underline{x}\cdot\underline{q}} \sum_{\sigma,\lambda,\lambda'} \frac{e_\sigma^2}{m_\sigma}$$

$$(x)\left[\frac{\underline{\epsilon}_{\lambda}(\underline{k}+\underline{q})\cdot\underline{\epsilon}_{\lambda'}(\underline{Q}-\underline{k}-\underline{q})}{\sqrt{|\underline{k}+\underline{q}||\underline{k}+\underline{q}-\underline{Q}|}}\left(F,\rho_{\sigma}(\underline{K}+\underline{Q},\underline{K})\,\alpha_{\lambda'}(\underline{Q}-\underline{k}-\underline{q})\,\alpha_{\lambda}(\underline{k}-\underline{q})\,F\right)\right.$$

$$\left.-\frac{\underline{\epsilon}_{\lambda}(\underline{k}-\underline{q})\cdot\underline{\epsilon}_{\lambda'}(\underline{q}-\underline{k}-\underline{Q})}{\sqrt{|\underline{k}-\underline{q}|\,|\underline{q}-\underline{k}-\underline{Q}|}}\left(F,\rho_{\sigma}(\underline{K}+\underline{Q},\underline{K})\,\alpha^{+}_{\lambda}(\underline{k}+\underline{q})\alpha^{+}_{\lambda'}(\underline{q}-\underline{k}-\underline{Q})\,F\right)\right]$$

$$+\frac{i\pi^{-3}\sqrt{2\pi\hbar c}}{c^{2}\hbar(2\pi)^{3}}\int d^{3}q\,d^{3}K\,d^{3}Q\,e^{-2i\underline{x}\cdot\underline{q}}\sum_{\sigma,\lambda}\frac{e_{\sigma}^{2}}{m_{\sigma}}$$

$$(x)\left[\frac{\underline{A}^{e}(\underline{k}+\underline{q}-\underline{Q})\cdot\underline{\epsilon}_{\lambda}(\underline{k}+\underline{q})}{\sqrt{|\underline{k}+\underline{q}|}}\left(F,\rho_{\sigma}(\underline{K}+\underline{Q},\underline{K})\,\alpha_{\lambda}(\underline{k}-\underline{q})\,F\right)\right.$$

$$\left.-\frac{\underline{A}^{e}(\underline{q}-\underline{k}-\underline{Q})\cdot\underline{\epsilon}_{\lambda}(\underline{k}-\underline{q})}{\sqrt{|\underline{k}-\underline{q}|}}\left(F,\rho_{\sigma}(\underline{K}+\underline{Q},\underline{K})\,\alpha^{+}_{\lambda}(\underline{k}+\underline{q})\,F\right)\right],\qquad(51)$$

where we have introduced the notation,

$$\rho_{\sigma}(\underline{K}+\underline{Q},\underline{K}-\underline{Q}) = \mathcal{a}^{+}_{\sigma}(\underline{K}+\underline{Q})\,\mathcal{a}_{\sigma}(\underline{K}-\underline{Q})\ .$$

Though some intuitively satisfying interpretation of this expression is accessible in its present form, it is actually more pertinent at this stage (the reduction of this relation to something resembling the conventional photon transport equation has yet to be accomplished) to view it piecewise in different physical contexts. Specifically, we examine this relation solely for its information about transport per se, and then give some attention to the interaction terms in the infinite, homogeneous medium.

The term describing photon transport in equation (51) is, of course,

$$2c\, \chi_\gamma \left\{ \sin\left(\frac{\underline{\nabla}_x \cdot \underline{\nabla}_k}{2} \right) \right\} |\underline{k}| \quad . \tag{52}$$

The "classical limit" of this expression is,

$$c\, \underline{\Omega} \cdot \underline{\nabla}\, \chi_\gamma \quad , \tag{53}$$

where $\underline{\Omega} = \underline{k}/|\underline{k}|$. This is the conventional form of the transport term appearing in the usual, phenomenological formulations of the description of the photon density, χ_γ. It is curious to observe that, in some sense, photon transport is a non-classical process, whereas particle transport - characterized by the term,

$$v_j \frac{\partial f}{\partial x_j} \quad -$$

is described the same way regardless of whether the process is examined in classical or quantum mechanical formalisms. However, it is doubtful that the quantum corrections to the description of photon transport (terms proportional to non-zero powers of $\hbar$) are more than a curiosity; hence the left-hand-side of equation (51) may, most probably, be rewritten quite adequately as,

$$\frac{\partial \chi_\gamma}{\partial t} + c\, \underline{\Omega} \cdot \underline{\nabla}\, \chi_\gamma \quad . \tag{54}$$

The observation incorporated in equation (54) suggests that the terms on the right-hand-side of equation (51) should be - in some approximation - interpretable in terms of rates of gain and loss of photons in the volume element $d^3x\, d^3k$ due to photoelectric emission and absorption, compton scattering, pair production, and bremsstrahlung. If so, such an interpretation will be as observable in the infinite, homogeneous medium as in the finite system. Furthermore, some simplification of equation (51) is to be expected in the space-independent situation. Hence we proceed to an examination of the effect of photon-particle interactions upon the photon distribution in the context of the equation obtained from (51) by integration over all space; i.e.,

$$\frac{\partial \chi_\gamma}{\partial t} = -\frac{i}{c}\sqrt{\frac{2\pi\hbar c}{(2\pi)^3}}\int \frac{d^3K}{\sqrt{k}}\sum_{\sigma,\lambda}\frac{e_\sigma}{m_\sigma}$$

$$(x)\left[\left\{\underline{K}\cdot\underline{\epsilon}_\lambda(\underline{k})\right\}(F,\rho_\sigma(\underline{K}+\underline{k},\underline{K})\,\alpha_\lambda(\underline{k})\,F) - \left\{\underline{K}\cdot\underline{\epsilon}_\lambda(\underline{k})\right\}(F,\rho_\sigma(\underline{K}-\underline{k},\underline{K})\alpha_\lambda^+(\underline{k})F)\right]$$

$$+\frac{i}{(2\pi)^2 c}\int d^3K\,d^3Q\sum_{\sigma,\lambda,\lambda'}\frac{e_\sigma^2}{m_\sigma}$$

$$(x)\left[\frac{\underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}-\underline{Q})}{\sqrt{k\,|\underline{k}-\underline{Q}|}}(F,\rho_\sigma(\underline{K}+\underline{Q},\underline{K})\alpha_{\lambda'}^+(\underline{k}-\underline{Q})\alpha_\lambda(\underline{k})\,F)\right.$$

$$\left.-\frac{\underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{k}+\underline{Q})}{\sqrt{k\,|\underline{k}+\underline{Q}|}}(F,\rho_\sigma(\underline{K}+\underline{Q},\underline{K})\alpha_\lambda^+(\underline{k})\,\alpha_{\lambda'}(\underline{k}+\underline{Q})\,F)\right]$$

$$+\frac{i}{(2\pi)^2 c}\int d^3K\,d^3Q\sum_{\sigma,\lambda,\lambda'}\frac{e_\sigma^2}{m_\sigma}$$

$$(x)\left[\frac{\underline{\epsilon}_\lambda(\underline{k})\cdot\underline{\epsilon}_{\lambda'}(\underline{Q}-\underline{k})}{\sqrt{k\,|\underline{k}-\underline{Q}|}}(F,\rho_\sigma(\underline{K}+\underline{Q},\underline{K})\alpha_{\lambda'}(\underline{Q}-\underline{k})\alpha_\lambda(\underline{k})\,F)\right.$$

$$- \frac{\underline{\epsilon}_{\lambda}(\underline{k}) \cdot \underline{\epsilon}_{\lambda'}(-\underline{Q} - k)}{\sqrt{k \, | \underline{k} + \underline{Q} |}} \left(F, \rho_{\sigma} (\underline{K}+\underline{Q}, \underline{K}) \, \alpha^{+}_{\lambda}(\underline{k}) \, \alpha^{+}_{\lambda'}(-\underline{k} - \underline{Q}) \, F \right) \Bigg]$$

$$+ \frac{i \sqrt{2\pi\hbar c}}{c^2 \hbar (2\pi)^3} \int d^3K d^3Q \sum_{\sigma, \lambda} \frac{e_{\sigma}^2}{m_{\sigma}}$$

$$(x) \Bigg[\frac{\underline{A}^e(\underline{k} - \underline{Q}) \cdot \underline{\epsilon}_{\lambda}(\underline{k})}{\sqrt{k}} \left(F, \rho_{\sigma} (\underline{K} + \underline{Q}, \underline{K}) \, \alpha_{\lambda}(\underline{k}) \, F \right)$$

$$- \frac{\underline{A}^e(-\underline{k} - \underline{Q}) \cdot \underline{\epsilon}_{\lambda}(\underline{k})}{\sqrt{k}} \left(F, \rho_{\sigma} (\underline{K} + \underline{Q}, \underline{K}) \, \alpha^{+}_{\lambda}(\underline{k}) \, F \right) \Bigg] \, . \qquad (55)$$

Ultimately it is necessary to reduce this relation to an equation for χ_{γ}. However, at this stage, it is feasible only to indicate that qualitatively the meaning of the interaction terms in this expression is more or less what it is expected to be.

The first pair of interaction terms - linear in the creation or destruction operators for photons - describe (to first order) transitions in which particles gain or lose momentum in the amount $\hbar \underline{k}$ while a photon of the same momentum is correspondingly being destroyed or created. Thus, since the component particle states comprising F include bound states and continuum states for interacting particles, it is to be expected that these terms describe the effect upon the photon density, χ_{γ}, of general particle transitions accompanied by the emission or absorption of a photon.

The second pair of interaction terms are bilinear in the creation and destruction operators for photons. Thus these characterize processes in which photons gain or lose momentum while particles lose or gain momentum correspondingly. Hence they should reduce - in the sense of certain approximations - to the conventional formulae for the effect of compton scattering upon the description of the photon density.

The third pair of terms are bilinear in the creation or destruction operators for photons. However, they are also (as are all of the others) bilinear in the creation and destruction operators for the particles. Thus they describe processes in which a particle undergoes a transition accompanied by the emission or absorption of two photons. Such two-photon events may be expected to be of higher order (in the sense of less probable) than the processes described by the other terms in equation (55), and consequently will not appear in the lowest order formulation of the balance relation for the photon density. Finally the last two terms describe particle transitions in the presence of external fields accompanied by the emission or absorption of photons.

It is to be noted that this balance relation provides no accounting for the effects of pair production and annihilation. This, however, is to be expected as a consequence of the nonrelativistic treatment of the particle dynamics.

VI. Development of a "Transport Equation" for Photons

In previous sections (I, III), a distribution function for photons in a plasma was defined and a relation presumably characterizing it was deduced. That relation was, however, inspected in only the most cursory and qualitative terms, Section V. It is our purpose in this section to explore some of the implications of that balance equation in considerable detail.

For this purpose, we recall the definition of a photon density adopted above, equation (13), which we write here as

$$\chi(\underline{x}, \underline{k}, t) = \frac{8}{V}\sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} (F, \rho F), \tag{56}$$

where again

$$\rho \equiv \sum_{\lambda} \alpha_{\lambda}^{+}(\underline{k} + \underline{q})\, \alpha_{\lambda}(\underline{k} - \underline{q}), \tag{57}$$

and where

$$\pi^{-3}\int d^3q \longrightarrow \frac{8}{V}\sum_{\underline{q}}, \text{ i.e.,}$$

we have enclosed our system in a large but finite cube of volume, V. We will develop the present argument for a plasma experiencing no external fields, reserving for a subsequent report the details of the analysis of the radiation balance in the presence of a temporally constant, spatially uniform magnetic field. Consequently the Hamiltonian (10) reduces to

$$H = T_P + T_\gamma + H_{P\gamma} + V_c, \tag{58}$$

where the several terms are exhibited explicitly in (11a), (11b), (11c) and (11f) respectively.

It is convenient to exhibit χ in the interaction representation. Recall that the state vector, F, was presumed to satisfy

$$HF = i\hbar \frac{\partial F}{\partial t} . \tag{59}$$

Defining the new state vector G by

$$F = UG \tag{60}$$

where

$$U = e^{-i(T_P + T_\gamma)t/\hbar} , \tag{61}$$

we have for the distribution function

$$\chi = \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} (G, \rho_t G), \tag{62}$$

where

$$\rho_t = U^+ \rho U. \tag{63}$$

In these terms, we have for the time rate of change of χ

$$\frac{\partial \chi}{\partial t} = \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} \left(G, \frac{i}{\hbar}\left[T_\gamma, \rho_t\right] G\right) + \frac{8}{V} \sum_{\underline{q}} e^{-2i\underline{x}\cdot\underline{q}} \left[\left(\frac{\partial G}{\partial t}, \rho_t G\right) + \left(G, \rho_t \frac{\partial G}{\partial t}\right)\right]. \tag{64}$$

The first term yields the conventional description of transport (53). If then we integrate the relation (64) over a sufficiently large element of volume, we find for the rate of change of photons of momentum $\underline{k}$,

$$\frac{\partial \chi(\underline{k}, t)}{\partial t} \cong -c \int \hat{\underline{n}} \cdot \underline{\Omega}\, \chi(\underline{x}_s, \underline{k}, t)\, ds + \left[\left(\frac{\partial G}{\partial t}, \rho_t G\right) + \left(G, \rho_t \frac{\partial G}{\partial t}\right)\right], \tag{65}$$

where

$$\chi(\underline{k}, t) = \int \chi(\underline{x}, \underline{k}, t)\, d^3 x,$$

$\hat{\underline{n}}$ is an outgoing unit normal to the surface bounding the volume of integration, and $\underline{x}_s$ is a position vector to the point on the surface at which $\hat{\underline{n}}$ is located. We also note that now

$$\rho_t = U^+ \sum_\lambda \alpha_\lambda^+(\underline{k})\, \alpha_\lambda(\underline{k})\, U. \tag{66}$$

The contribution to the rate of change of photons in the volume element of interest may now be estimated by a calculation proceeding along more or less conventional lines[10]. Nevertheless, it is convenient to proceed somewhat

[10] R.C. Tolman, The Principles of Statistical Mechanics, Oxford Univ. Press, 1938.

deviously for the purpose of clarifying a certain amount of notation, hence the details of an argument leading to such an estimate will at least be sketched.

Expand G in the eigenstates of free photons and particles,

$$G = \sum_{n\eta} b_{n\eta}(t) \, | n\eta \rangle \, , \tag{67}$$

where

$$T_P | n \rangle = E_n | n \rangle \, ,$$

$$T_\gamma | \eta \rangle = \epsilon_\eta | \eta \rangle \, ,$$

$$| n\eta \rangle = | n \rangle | \eta \rangle \, ,$$

and

$$\langle n'\eta' | n\eta \rangle = \delta_{nn'} \, \delta_{\eta\eta'} \, .$$

The symbol $|n\rangle$ is shorthand for the symbol $| n_{\sigma K_1}, n_{\sigma K_2} \ldots, n_{\sigma' K_1}, n_{\sigma' K_2} \ldots \rangle$ which is a system state in which there are $n_{\sigma K_1}$ particles of the σth kind having momentum $\hbar \underline{K}_1$ present, $n_{\sigma K_2}$ particles of the .... etc. The interaction terms in the relation (65) then become

$$\left.\frac{\delta \chi}{\delta t}\right)_{Int} = \sum_{n\eta n'\eta'} (\dot{b}^*_{n\eta} b_{n'\eta'} + b^*_{n\eta} \dot{b}_{n'\eta'}) \langle n\eta | \rho_t | n'\eta' \rangle$$

$$= \sum_{n\eta\lambda} (\dot{b}^*_{n\eta} b_{n\eta} + b^*_{n\eta} \dot{b}_{n\eta}) \, \eta_{\lambda k} \, . \tag{68}$$

From (67) it is seen that

$$\dot{b}_{n\eta} = \langle n\eta \mid \dot{G}\rangle = -\frac{i}{\hbar} \sum_{n'\eta'} \langle n\eta \mid H_{P\gamma_t} + V_{c_t} \mid n'\eta'\rangle b_{n'\eta'}, \tag{69}$$

so that it follows that

$$b_{n\eta}(t+s) = b_{n\eta}(t) + \sum_{n'\eta'} \int_t^{t+s} v_{n\eta n'\eta'}(t')\, b_{n'\eta'}(t')\, dt' \tag{70}$$

where we have defined

$$v_{n\eta n'\eta'} = -\frac{i}{\hbar} \langle n\eta \mid H_{P\gamma_t} + V_{c_t} \mid n'\eta'\rangle .$$

Iterating, it is readily shown that

$$b_{n\eta}(t+s) = b_{n\eta}(t) + \sum_{n'\eta'} \sum_{j=1}^{\infty} Q^{(j)}_{n\eta n'\eta'}\, b_{n'\eta'}(t), \tag{71}$$

where

$$Q^{(j)}_{n\eta n'\eta'} = \int_{s_1=0}^{s} \cdots\cdots \int_{s_j=0}^{s_{j-1}} ds_1 \ldots ds_j\, v_{n\eta n_1\eta_1}(t+s_1) \ldots\ldots v_{n_{j-1}\eta_{j-1}n'\eta'}(t+s_j) \tag{72}$$

If now we interpret $b_{n\eta}(t+s) - b_{n\eta}(t) = s\dot{b}_{n\eta}$ for sufficiently small s, and then substitute (71) into (68) we find that

$$\left.\frac{\delta\chi}{\delta t}\right)_{Int} = \sum_{n\eta\lambda} \frac{1}{s} \left[\sum_{j=1}^{\infty} Q^{(j)}_{n\eta n'\eta'}\, b^*_{n\eta}(t)\, b_{n'\eta'}(t) + \sum_{j=1}^{\infty} Q^{(j)*}_{n\eta n'\eta'}\, b^*_{n'\eta'}(t)\, b_{n\eta}(t) \right.$$
$$\left. + \sum_{j,k} Q^{(j)*}_{n\eta n'\eta'}\, Q^{(k)}_{n\eta n''\eta''}\, b^*_{n'\eta'}(t)\, b_{n''\eta''}(t) \right] \eta_{\lambda k} \tag{73}$$

At this point we digress briefly to recall the physical significance of the quantities, $b_{n\eta}$. For this purpose it is convenient to reconsider the expectation of ρ in the present context, i.e.,

$$(F, \rho F) = \sum_{n\eta n'\eta'} C^*_{n\eta}(t)\, C_{n'\eta'}(t) \langle n\eta | \rho | n'\eta' \rangle , \tag{74}$$

where

$$\begin{aligned} C_{n\eta} &= \langle n\eta | F \rangle = \langle n\eta | U G \rangle \\ &= e^{-i(E_n + \epsilon_\eta)t/\hbar}\, b_{n\eta} . \end{aligned} \tag{75}$$

The quantities $|C_{n\eta}|^2 = |b_{n\eta}|^2$ are interpretable as the probability of finding the system at time t in the state characterized by the sets of occupation numbers $|n\rangle\ |\eta\rangle$. Furthermore, it is convenient to introduce the concept of the density matrix by the definition

$$D_{n\eta,\, n'\eta'} = C^*_{n'\eta'} C_{n\eta}. \tag{76}$$

Thus it follows that

$$b^*_{n'\eta'} b_{n\eta} = e^{i(E_n + \epsilon_\eta - E_{n'} - \epsilon_{\eta'})t/\hbar} D_{n\eta,\, n'\eta'} , \tag{77}$$

and we observe that the quantities $|b_{n\eta}|^2$ are the diagonal elements of the system density matrix (and have the probabilistic interpretation indicated above), whereas the quantities $b^*_{n\eta} b_{n'\eta'}$ are proportional to the off-diagonal elements of same. (Or, equivalently, these quantities are the elements of the density matrix of the system in the interaction representation). If now we assert that the off-diagonal elements of these density matrices are ignorable in comparison with the diagonal elements (without present regard for the extent of the validity of such an assertion), equation (73) may be approximated by

$$\left.\frac{\delta \chi}{\delta t}\right)_{Int} \cong \sum_{n\eta\lambda} \eta_{\lambda k} \left[\frac{1}{s} \sum_j \left(Q^{(j)}_{n\eta n\eta} + Q^{(j)*}_{n\eta n\eta} \right) D_{n\eta n\eta} \right.$$

$$\left. + \frac{1}{s} \sum_{j,k} Q^{(j)*}_{n\eta n'\eta'} \, Q^{(k)}_{n\eta n'\eta'} \, D_{n'\eta' n'\eta'} \right] . \qquad (78)$$

We note here in passing that

$$\begin{aligned} \chi(\underline{k}, t) &= (F, \rho F) \\ &= \sum_{n\eta n'\eta'} C^*_{n\eta} C_{n'\eta'} \langle n\eta | \sum_\lambda \alpha^+_\lambda(\underline{k}) \, \alpha_\lambda(\underline{k}) | n\eta \rangle \\ &= \sum_{n\eta\lambda} D_{n\eta,\, n\eta} \, \eta_{\lambda k} . \end{aligned} \qquad (79)$$

To proceed further, it is necessary to obtain explicit representations of the Q's. To calculate all of them would seem to be a most formidable task, but in the instance of the interactions of photons with charged particles such an extreme may be presumed to be unnecessary. In fact, for our purposes herein it is quite legitimate to expect that significant results will be forthcoming from an analysis restricted to a lowest order calculation of the effects on the rate of change of χ due to one-photon processes only. Since, in the present discussion, we are ignoring external fields and have effectively treated the interactions between particles as "perturbations", it is desirable to take as our physical model for the following analysis a fully ionized, high temperature plasma. In such an instance, the only one-photon processes are Compton scattering (in this case Thompson scattering because of the non-relativistic treatment of the particle dynamics) which is calculable in the sense of first order perturbation theory, and those second order processes in which a photon is either emitted or absorbed by a particle (most probably an electron) which either previously or subsequently is "scattered" by another particle. Under these circumstances we may take for V_c and $H_{P\gamma}$ (see equations 11c and 11f),

$$H_{P\gamma} \equiv H^{(1)} + H^{(2)}$$

$$= -\frac{\hbar}{c}\sqrt{\frac{2\pi\hbar c}{V}} \sum_{\lambda\sigma\, KK'k} \frac{e_\sigma}{m_\sigma} \frac{a^+_\sigma(\underline{K})\, a_\sigma(\underline{K}')\, \underline{K}'\cdot \underline{\zeta}^+_\lambda(\underline{k})\, \delta(\underline{k}-\underline{K}'+\underline{K})}{\sqrt{k}},$$

$$+ \frac{1}{2c^2}\left(\frac{2\pi\hbar c}{V}\right) \sum_{\lambda\lambda'\sigma KK'kk'} \frac{e_\sigma^2}{m_\sigma} \frac{a^+_\sigma(\underline{K}')\, a_\sigma(\underline{K})\, \underline{\zeta}^+_\lambda(\underline{k})\cdot \underline{\zeta}^+_{\lambda'}(\underline{k}')\, \delta(\underline{k}+\underline{K}-\underline{k}'-\underline{K}')}{\sqrt{kk'}},$$

$$V_c \equiv V^c$$

$$= \frac{1}{2V} \sum_{\sigma\sigma' KK'K''K'''} e_\sigma e_{\sigma'} U(|\underline{K}-\underline{K}''|)\, a^+_{\sigma'}(\underline{K})\, a^+_\sigma(\underline{K}')\, a_{\sigma'}(\underline{K}'')\, a_\sigma(\underline{K}''')$$

$$(x)\quad \delta(\underline{K}+\underline{K}'-\underline{K}''-\underline{K}'''), \qquad (80)$$

where

$$U(|\underline{K}-\underline{K}''|) = \int \frac{d^3R}{R}\, e^{i\underline{R}\cdot(\underline{K}''-\underline{K})} . \qquad (81)$$

The symbol δ is here to be interpreted as a Kronecker delta, and V is the volume of the cube at whose walls the particle and photon field operators are presumed to obey periodic boundary conditions - all other symbols represent quantities previously defined above.

In view of the above remarks, the relevant transition probabilities which contribute to the rate equation (78) are

$$Q^{(1)}_{n\eta n\eta} + Q^{(1)*}_{n\eta n\eta} = 0,$$

$$Q^{(1)*}_{n'\eta' n\eta}\, Q^{(1)}_{n'\eta' n\eta} = \frac{4}{\hbar^2}\left|H^I_{n'\eta' n\eta}\right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n\eta}-\omega_{n'\eta'}}{2} s}{(\omega_{n\eta}-\omega_{n'\eta'})^2},$$

$$Q^{(2)}_{n\eta n\eta} + Q^{(2)*}_{n\eta n\eta} = -\frac{4}{\hbar^2}\sum_{m\alpha}\left|H^I_{n\eta m\alpha}\right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n\eta}-\omega_{m\alpha}}{2} s}{(\omega_{n\eta}-\omega_{m\alpha})^2},$$

$$Q^{(3)}_{n\eta n\eta} + Q^{(3)*}_{n\eta n\eta} = 0,$$

$$Q^{(1)*}_{n'\eta' n\eta}\, Q^{(2)}_{n'\eta' n\eta} + Q^{(2)*}_{n'\eta' n\eta}\, Q^{(1)}_{n'\eta' n\eta}$$

$$+ Q^{(1)*}_{n'\eta' n\eta}\, Q^{(3)}_{n'\eta' n\eta} + Q^{(3)*}_{n'\eta' n\eta}\, Q^{(1)}_{n'\eta' n\eta} = 0,$$

$$Q^{(4)*}_{n\eta n\eta} + Q^{(4)}_{n\eta n\eta}$$

$$= - \frac{4}{\hbar^2} \sum_{m\alpha} \left| \sum_{r\beta} \frac{H^{II}_{n\eta r\beta} \; H^{II}_{r\beta m\alpha}}{\hbar(\omega_{n\eta} - \omega_{r\beta})} \right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{(\omega_{n\eta} - \omega_{m\alpha})^2},$$

$$Q^{(2)*}_{n'\eta' n\eta} \; Q^{(2)}_{n'\eta' n\eta}$$

$$= \frac{4}{\hbar^2} \left| \sum_{m\alpha} \frac{H^{II}_{n\eta m\alpha} \; H^{II}_{m\alpha n'\eta'}}{\hbar(\omega_{n\eta} - \omega_{m\alpha})} \right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n'\eta'} - \omega_{n\eta}}{2} s}{(\omega_{n'\eta'} - \omega_{n\eta})^2} . \qquad (82)$$

Thus, to this approximation, equation (78) becomes

$$\left.\frac{\delta\chi}{\delta t}\right)_{Int} = \sum_{n\eta\lambda} \eta_{\lambda k} \Bigg[\sum_{m\alpha} W^{(1)}_{n\eta,\, m\alpha} \Big\{ D_{m\alpha\, ,\, m\alpha}$$

$$- D_{n\eta,\, n\eta} \Big\} + \sum_{m\alpha} W^{(2)}_{n\eta,\, m\alpha} \Big\{ D_{m\alpha\, ,\, m\alpha}$$

$$- D_{n\eta,\, n\eta} \Big\} \Bigg], \qquad (83)$$

where we have introduced the notation

$$W^{(1)}_{n\eta,\, m\alpha} = \frac{4}{\hbar^2} \left| H^{I}_{n\eta,\, m\alpha} \right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{s\,(\omega_{n\eta} - \omega_{m\alpha})^2},$$

$$W^{(2)}_{n\eta,\, m\alpha} = \frac{4}{\hbar^2} \left| \sum_{\ell\sigma} \frac{H^{II}_{n\eta,\ell\sigma} \; H^{II}_{\ell\sigma,\, m\alpha}}{\hbar(\omega_{n\eta} - \omega_{\ell\sigma})} \right|^2 \frac{\mathrm{Sin}^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{s\,(\omega_{n\eta} - \omega_{m\alpha})^2},$$

$H^{II} = H^{(1)} + V^{c}$, $H^{I} = H^{(2)}$, and

$$\omega_{n\eta} = (E_n + \epsilon_\eta)/\hbar. \tag{84}$$

The peaked functions,

$$\frac{\mathrm{Sin}^2 \frac{\omega_{n\eta} - \omega_{m\alpha}}{2} s}{s\,(\omega_{n\eta} - \omega_{m\alpha})^2},$$

essentially guarantee conservation of energy between the initial and final states; and for all practical purposes may be replaced by

$$\frac{\pi}{2}\,\delta\,(\omega_{n\eta} - \omega_{m\alpha}),$$

where this δ is to be interpreted as the Dirac delta function. The matrix elements themselves conserve momentum. It is to be noted that, if H^I is Hermitian, the transition probabilities per unit time, W, are symmetric.

It is convenient at this point to introduce a somewhat more suggestive notation for the diagonal elements of the density matrix. We have already observed that the element $D_{n\eta,\,n\eta}(t)$ is interpretable as the probability of finding the system in the state characterized by the occupation numbers $|\,n\eta\rangle$ at time t; thus we introduce $P(n\eta,\,t) = D_{n\eta,n\eta}(t)$. It is also convenient in some of the subsequent manipulation to introduce the notation,

$$\bar{\eta}_{\lambda k} = \sum_{n\eta} \eta_{\lambda k}\; P(n\eta, t) \tag{85}$$

for the expected number of photons to be found in the region of interest at time t. Then, of course,

$$\chi = \sum_{\lambda} \bar{\eta}_{\lambda k} \; . \tag{86}$$

The calculation of the indicated transition probabilities is straight-forward, and one finds that $\bar{\eta}_{\lambda k}$ satisfies the equation

$$\frac{\partial \bar{\eta}_{\lambda k}}{\partial t} = \sum_{\sigma K K' \lambda' k'} S^{\sigma K, \lambda k}_{\sigma K', \lambda' k'} \overline{\eta_{\lambda' k'} (\eta_{\lambda k} + 1) \, n_{\sigma K'} (n_{\sigma K} + 1)}$$

$$- \sum_{\sigma K K' \lambda' k'} S^{\sigma K', \lambda' k'}_{\sigma K, \lambda k} \overline{\eta_{\lambda k} (\eta_{\lambda' k'} + 1) \, n_{\sigma K} (n_{\sigma K'} + 1)}$$

$$+ \sum_{\sigma \sigma' K K_1 K_2 K_3} T_e(\lambda k)^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1} \overline{(\eta_{\lambda k} + 1) \, n_{\sigma K} \, n_{\sigma' K_1} (n_{\sigma K_2} + 1)(n_{\sigma' K_3} + 1)}$$

$$- \sum_{\sigma \sigma' K K_1 K_2 K_3} T_a(\lambda k)^{\sigma K, \sigma' K_1}_{\sigma K_2, \sigma' K_3} \overline{\eta_{\lambda k} (n_{\sigma K} + 1)(n_{\sigma' K_1} + 1) \, n_{\sigma K_2} \, n_{\sigma' K_3}} \; . \tag{87}$$

The appearance of factors like $(n_{\sigma K} + 1)$ in the first term for example is a direct consequence of the assumption that the particles were bosons. If instead they had been presumed to be fermions (had imposed anti-commutation instead of commutation rules upon the relevant creation and destruction operators), the indicated factors would have appeared as $(1 - n_{\sigma K})$ with the restriction that $n_{\sigma K} = 0$ or 1 only. If further in fact we may assume that the final states for the particles are sufficiently improbably occupied, then we may replace such factors by unity - an assumption leading to Boltzmann statistics for the particles. Given the physical context in which the present analysis is being performed, we incorporate the latter assumption in the remainder of this discussion. The transition probabilities, S and T, are given explicitly by

$$S_{\sigma K', \lambda' k'}^{\sigma K, \lambda k} = \left(\frac{e_\sigma^2}{m_\sigma c^2}\right)^2 \frac{16\pi^2}{V^2} \frac{c^2}{k k'} \left| \underline{\epsilon}_\lambda(\underline{k}) \cdot \underline{\epsilon}_{\lambda'}(\underline{k}') \right|^2$$

$$(x) \; \delta(\underline{k}' + \underline{K}' - \underline{k} - \underline{K}) \frac{\mathrm{Sin}^2 \frac{\omega_f - \omega_i}{2} s}{s(\omega_f - \omega_i)^2}$$

$$= S_{\sigma K, \lambda k}^{\sigma K', \lambda' k'} , \qquad (88a)$$

and

$$T_e(\lambda k)_{\sigma K, \sigma' K_1}^{\sigma K_2, \sigma' K_3} = \frac{2\pi}{V^3} \left(\frac{e_\sigma^2}{m_\sigma c^2}\right)^2 \left(\frac{e_{\sigma'}^2}{\hbar c}\right) \frac{\hbar^2 c^4}{k} \left| U(|\underline{K}_1 - \underline{K}_3|) \right|^2$$

$$(x) \left| \frac{(n_{\sigma k + K_2} + 1) \; \underline{K}_2 \cdot \underline{\epsilon}_\lambda(\underline{k})}{\frac{\hbar^2 K^2}{2 m_\sigma} + \frac{\hbar^2 K_1^2}{2 m_{\sigma'}} - \frac{\hbar^2 K_3^2}{2 m_{\sigma'}} - \frac{\hbar^2 (\underline{k} + \underline{K}_2)^2}{2 m_\sigma}} \right.$$

$$+ \frac{(n_{\sigma\, K-k} + 1) \;\; \underline{K} \cdot \underline{\epsilon}_{\lambda}(\underline{k})}{\frac{\hbar^2 K_3^2}{2m_{\sigma'}} + \frac{\hbar^2 K_2^2}{2m_{\sigma}} - \frac{\hbar^2 K_1^2}{2m_{\sigma'}} - \frac{\hbar^2 (\underline{K} - \underline{k})^2}{2m_{\sigma}}} \Bigg|^2$$

$$(x) \quad \delta(\underline{K} + \underline{K}_1 - \underline{K}_2 - \underline{K}_3 - \underline{k}) \; \frac{\mathrm{Sin}^2 \frac{(\omega_f - \omega_i)}{2} \, s}{s(\omega_f - \omega_i)^2}$$

$$= T_a(\lambda k)^{\sigma K, \sigma' K_1}_{\sigma K_2, \sigma' K_3} \quad . \tag{88b}$$

It should be noted that, since T_e (and T_a) are inversely proportional to the square of the mass of the radiating particle (the $\sigma^{\underline{th}}$ particle in equation (88b)), we are here identifying the radiating particle as an electron and ignoring the additional terms corresponding to radiation by the ions.

If now we replace averages of products by products of averages (without regard at this point for the question as to the validity of such a replacement) in equation (87), we obtain

$$\frac{\partial \bar{\eta}_{\lambda k}}{\partial t} = \sum_{\sigma K K' \lambda' k'} S^{\sigma K, \lambda k}_{\sigma K', \lambda' k'} \left[\bar{\eta}_{\lambda' k'} (\bar{\eta}_{\lambda k} + 1) \bar{n}_{\sigma K'} - \bar{\eta}_{\lambda k} (\bar{\eta}_{\lambda' k'} + 1) \bar{n}_{\sigma K} \right]$$

$$+ \sum_{\sigma\sigma' K K_1 K_2 K_3} T_e(\lambda k)^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1} \left[(\bar{\eta}_{\lambda k} + 1) \bar{n}_{\sigma K} \bar{n}_{\sigma' K_1} - \bar{\eta}_{\lambda k} \bar{n}_{\sigma K_2} \bar{n}_{\sigma' K_3} \right] . \tag{89}$$

It is of some interest at this point to observe that the implications of equation (89) for the equilibrium distributions of particles and photons in a box with perfectly reflecting walls (so that the thorny issues associated with transport may still be held in abeyance) are as anticipated. The requirement that $\eta_{k\lambda}$ be stationary in time will certainly be met if the scattering and the emission and absorption terms separately vanish. Consider first the scattering term. A sufficient condition for its vanishing is that

$$\frac{\bar{\eta}_{\lambda'k'}}{(\bar{\eta}_{\lambda'k'}+1)}\,\bar{n}_{\sigma K'} = \frac{\bar{\eta}_{\lambda k}}{\bar{\eta}_{\lambda k}+1}\,\bar{n}_{\sigma K}\,. \tag{90}$$

Since the scattering process conserves energy, equation (90) is satisfied if we take

$$\bar{\eta}_{\lambda k} = \frac{1}{e^{\hbar c k/\kappa T}-1}\,,$$

and

$$\bar{n}_{\sigma K} \;\alpha\; e^{-\hbar^2 K^2/2m_\sigma \kappa T}\,, \tag{91}$$

where κ is Boltzmann's constant and $\bar{n}_{\sigma K}$ is subject to the normalization,

$$\sum_K \bar{n}_{\sigma K} = N_\sigma\,. \tag{92}$$

But these are just the Planck and Boltzmann distributions for photons and particles respectively which are expected to obtain in the equilibrium systems.

It remains to be shown that the emission and absorption processes also balance for these distributions. To see that indeed they do, rewrite the bracketed factor in the summand of the second term in equation (89) as

$$\left(\bar{\eta}_{\lambda k}+1\right)\left[\bar{n}_{\sigma K}\,\bar{n}_{\sigma' K_1}-\frac{\bar{\eta}_{\lambda k}}{\bar{\eta}_{\lambda k}+1}\,\bar{n}_{\sigma K_2}\,\bar{n}_{\sigma' K_3}\right]$$

$$\alpha\,(\bar{\eta}_{\lambda k}+1)\left[e^{\left[-\frac{\hbar^2 K^2}{2m_\sigma}+\frac{\hbar^2 K_1^2}{2m_{\sigma'}}\right]/\mathcal{K}T}-e^{-\left(\frac{\hbar^2 K_2^2}{2m_\sigma}+\frac{\hbar^2 K_3^2}{2m_{\sigma'}}+\hbar c k\right)/\mathcal{K}T}\right], \qquad (93)$$

where the relations (91) have been explicitly employed. But the exponentials in equation (93) are effectively the same by virtue of the energy conservation condition contained in T_e; hence emission and absorption balance in thermal equilibrium.

In order to obtain the equation for χ, it is convenient to introduce an assumption of random polarization of photons. In such an event,

$$\bar{\eta}_{\lambda k}=\frac{1}{2}\,\bar{\eta}_k=\frac{1}{2}\,\chi\,, \qquad (94)$$

and, after summing equation (89) over λ, we obtain finally,

$$\left.\frac{\delta \chi}{\delta t}\right)_{Int} = \sum_{\sigma KK'k'} \bar{S}^{\sigma K,\, k}_{\sigma K',\, k'} \left[\chi(\underline{k}') \left\{ \chi(\underline{k}) + 2 \right\} \bar{n}_{\sigma K'} - \chi(\underline{k}) \left\{ \chi(\underline{k}') + 2 \right\} \bar{n}_{\sigma K} \right]$$

$$+ \sum_{\sigma\sigma' KK_1K_2K_3} \bar{T}_e(k)^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1} \left[\left\{ \chi(\underline{k}) + 2 \right\} \bar{n}_{\sigma K}\, \bar{n}_{\sigma' K_1} - \chi(\underline{k})\, \bar{n}_{\sigma K_2}\, \bar{n}_{\sigma' K_3} \right], \qquad (95)$$

where we have defined

$$\bar{S}^{\sigma K,\, k}_{\sigma K',\, k'} = \frac{1}{4} \sum_{\lambda\lambda'} S^{\sigma K,\, \lambda k}_{\sigma K',\, \lambda' k'},$$

and

$$\bar{T}_e(k)^{\sigma K_2,\, \sigma' K_3}_{\sigma K,\, \sigma' K_1} = \frac{1}{2} \sum_{\lambda} T_e(\lambda k)^{\sigma K_2,\, \sigma' K_3}_{\sigma K,\, \sigma' K_1}, \qquad (96)$$

i.e., transition probabilities averaged over photon polarization.

Recalling equations (65) and (68), we exhibit a photon transport equation for χ as

$$\frac{\partial \chi}{\partial t} \cong - c \int_{\delta V} \underline{\Omega} \cdot \underline{\nabla} \chi \, d^3 x$$

$$+ \sum_{\sigma K K' k'} \bar{S}^{\sigma K, k}_{\sigma K', k'} \left[\chi' (\chi + 2) \bar{n}_{\sigma K'} - \chi (\chi' + 2) \bar{n}_{\sigma K} \right]$$

$$+ \sum_{\sigma \sigma' K K_1 K_2 K_3} \bar{T}_e(k)^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1} \left[(\chi + 2) \bar{n}_{\sigma K} \bar{n}_{\sigma' K_1} - \chi \bar{n}_{\sigma K_2} \bar{n}_{\sigma' K_3} \right] . \qquad (97)$$

If now it is argued that the element of volume δV is sufficiently large compared to photon wavelengths so as to render the discussion of $\delta \chi / \delta t)_{Int}$ meaningful, but at the same time small compared to distances over which χ is expected to vary appreciably, then it might be legitimate in equation (97) to replace

$$\int_{\delta V} \underline{\Omega} \cdot \nabla \chi \, d^3 x \longrightarrow \underline{\Omega} \cdot \underline{\nabla} \chi , \qquad (98)$$

where, of course, χ has the interpretation as the expected number of photons in δV having momentum $\underline{k}$ at time t. Under such circumstances, equation (97) assumes a form somewhat similar to the one conventionally adopted for the description of photon transport.[11]

[11] L.L. Foldy, Phys. Rev., 81, 395, (1951).

Clearly, the argument leading to the replacement of equation (64) by equation (65) leads to an effective identification of δV with V, i.e., the element of volume over which χ has been spatially "coarse-grained" has been taken to be essentially the same as the volume of quantization, V. Such a procedure bears formal resemblance to Ono's[12] cell method of approach to a similar problem. It contains the clear restriction that the radiation density must vary negligibly over spatial volumes whose least linear dimension is very large compared to the wavelengths of the radiation considered.

Making explicit use of this identification, we redefine the χ of equation (97) as $\chi = V\bar{\chi}$ ($\bar{\chi}$ now being a density in configuration space) and define $\bar{n}_{\sigma K} = V\,\bar{f}_\sigma$. Then equation (97) (recalling (98)) becomes

$$\frac{\partial \bar{\chi}}{\partial t} \cong -c\,\underline{\Omega}\cdot\underline{\nabla}\,\bar{\chi}$$

$$+\sum_{\sigma KK'k'} V\bar{S}^{\sigma K, k}_{\sigma K', k'}\left[\bar{\chi}'\,(\chi+2)\,\bar{f}'_\sigma - \bar{\chi}\,(\chi'+2)\,\bar{f}_\sigma\right]$$

$$+\sum_{\sigma\sigma' KK_1K_2K_3} V\,\bar{T}_e(k)^{\sigma K_2,\,\sigma' K_3}_{\sigma K,\,\sigma' K_1}\left[(\chi+2)\,\bar{f}_\sigma(\underline{K})\,\bar{f}_{\sigma'}(\underline{K}_1)\right.$$

$$\left. -\,\chi\,\bar{f}_\sigma(\underline{K}_2)\,\bar{f}_{\sigma'}(\underline{K}_3)\right]. \qquad (99)$$

Finally if we argue for the photons as for the particles that the final states for

---

[12] S. Ono, Prog. Theor. Phys., (Japan) 12, 113 (1954).

all transitions are sufficiently improbably occupied that we may replace $\chi + 2 \rightarrow 2$, we obtain

$$
\begin{aligned}
\frac{\partial \bar{\chi}}{\partial t} &\cong -c\,\underline{\underline{\Omega}} \cdot \underline{\nabla}\, \bar{\chi} \\
&+ \sum_{\sigma K K' k'} 2V\, \bar{S}^{\sigma K, k}_{\sigma K', k'} \left[\bar{\chi}'\, \bar{f}'_{\sigma} - \bar{\chi}\, \bar{f}_{\sigma} \right] \\
&+ \sum_{\sigma\sigma' K K_1 K_2 K_3} 2V\, \bar{T}_e(k)^{\sigma K_2, \sigma' K_3}_{\sigma K, \sigma' K_1}\, \bar{f}_{\sigma}(\underline{K})\, \bar{f}_{\sigma'}(\underline{K}_1) \\
&- \sum_{\sigma\sigma' K K_1 K_2 K_3} V^2\, \bar{T}_a(k)^{\sigma K, \sigma' K_1}_{\sigma K_2, \sigma' K_3}\, \bar{\chi}\, \bar{f}_{\sigma}(\underline{K}_2)\, \bar{f}_{\sigma'}(\underline{K}_3),
\end{aligned}
\tag{100}
$$

where we have reverted to the notation of equation (87) for the characterization of the absorption term.

As a last step in the reduction of equation (100) to familiar form, we return to the continuum in momentum space. Defining

$$
\chi(\underline{x}, \underline{k}, t)\, dk\, d\Omega(\underline{k}) = \sum_{\underline{k} \in d^3 k} \bar{\chi}, \quad \text{and}
$$

$$
f_{\sigma}(\underline{x}, \underline{K}, t)\, d^3K = \sum_{\underline{K} \in d^3 K} \bar{f}_{\sigma}, \tag{101}
$$

so that the photon and particle densities are interpretable as asserted in Section I, we find after considerable manipulation that the rate equation becomes

$$\frac{\partial \chi}{\partial t} \cong - c\,\underline{\Omega} \cdot \underline{\nabla}\,\chi$$

$$+ \sum_{\sigma} \int_{KK'k'} d^3K d^3K' dk' d\Omega' \, c\sigma_T \left\{ \hbar c \, \delta(\underline{k} + \underline{K} - \underline{k'} - \underline{K'}) \, \delta(\hbar ck \right.$$

$$\left. + \frac{\hbar^2 K^2}{2m_\sigma} - \hbar c k' - \frac{\hbar^2 K'^2}{2m_\sigma} \right\} \left[\chi' f'_\sigma - \chi f_\sigma \right]$$

$$+ \sum_{\sigma\sigma'} \int_{KK_1K_2K_3} d^3K d^3K_1 d^3K_2 d^3K_3 \, f_\sigma(\underline{K}) \, f_{\sigma'}(\underline{K}_1) \, \frac{k}{4\pi^2} \, \phi_{\sigma\sigma'}$$

$$-\chi \sum_{\sigma\sigma'} \int_{KK_1K_2K_3} d^3K d^3K_1 d^3K_2 d^3K_3 \, f_\sigma(\underline{K}_2) \, f_{\sigma'}(\underline{K}_3) \, \frac{\pi}{k} \, \phi_{\sigma\sigma'} . \qquad (102)$$

In this expression, we have introduced the Thompson cross-section,

$$\sigma_T = \frac{1}{2} \left(\frac{e_\sigma^2}{m_\sigma c^2} \right) \left[1 + (\hat{\underline{k}} \cdot \hat{\underline{k}}')^2 \right] , \qquad (103)$$

after taking explicit cognizance of the fact that in the non-relativistic limit the scattering of photons produces a negligible change in the frequency of the photons, i.e., $k' \sim k$. The quantity $\phi_{\sigma\sigma'}$ is defined by

$$\phi_{\sigma\sigma'} = \left(\frac{e_\sigma^2}{m_\sigma c^2}\right)^2 \left(\frac{e_{\sigma'}^2}{\hbar c}\right) \frac{\hbar^3 c^4}{|\underline{K}_1 - \underline{K}_3|^4}$$

$$(x) \sum_\lambda \left[\frac{\underline{K}_2 \cdot \underline{\epsilon}_\lambda(\underline{k})}{T_\sigma(K) + T_{\sigma'}(K_1) - T_{\sigma'}(K_3) - T_\sigma(\underline{k} + \underline{K}_2)} \right.$$

$$\left. + \frac{\underline{K} \cdot \underline{\epsilon}_\lambda(\underline{k})}{T_\sigma(K_2) + T_{\sigma'}(K_3) - T_{\sigma'}(K_1) - T_\sigma(\underline{K} - \underline{k})} \right]^2$$

$$(x) \quad \delta(\underline{K} + \underline{K}_1 - \underline{K}_2 - \underline{K}_3 - \underline{k})$$

$$(x) \quad \delta\left[T_\sigma(K) + T_{\sigma'}(K_1) - T_\sigma(K_2) - T_{\sigma'}(K_3) - \hbar c k\right] , \qquad (104)$$

employing the notation,

$$T_\sigma(K) = \frac{\hbar^2 K^2}{2m_\sigma} . \qquad (105)$$

Further discussion of the effects of bremsstrahlung and "inverse" bremsstrahlung upon the rate of change of χ (the next to the last and the last terms of equation (102) respectively) will be postponed to a latter section.

VII. Discussion

In this report we have attempted the development of a description of a plasma from a unified point of view adequate to the characterization of rate processes involving the interaction of particles with particles and particles with photons. It was our sole intention to delineate a set of sufficient conditions in terms of which a formally exact description of a system of particles interacting via electromagnetic forces might be reduced to conventional transport type equations. Clearly then, it is not suggested that the rate equations ultimately presented in Sections IV and VI have been derived. Rather, they have largely been merely argued - in such a fashion, however, that it is hoped that subsequent investigation may be focused upon those points of difficulty at which deduction gave way to intuition. It is certainly not precluded that such subsequent investigation might lead to the conclusion that the conventional rate equations are actually inappropriate for the description of the plasma.

The points of difficulty emerge at essentially two different levels. Most fundamentally, they arise in the axiomatic characterization of the system. It is felt that the dynamical axiom employed herein (within the limits of non-relativistic particle dynamics) should be reasonably secure. However, the so-called statistical axiom is hardly so readily defended, and perhaps some further comment is warranted here.

It has been noted above from time to time that the "statistical" axiom, equation 13, presently adopted is somewhat unconventional (though hardly original[13, 14]). This axiom is somewhat unconventional in that it does not introduce the usual statistical concept of the ensemble at all. It is statistical solely in the sense that all quantum expectation values have statistical interpretations. However, it was pointed out that an implication of equation (32) (the equation describing the particle singlet density in the classical limit) was that this "pure state" quantum expectation value corresponded, in some sense, to a classical ensemble average. The discussion of the essentially identical inference drawn by van Kampen[15] is interesting in this connection.

Another aspect of the present choice of statistical axiom that requires some clarification is that the densities f and χ as defined by equation (13) are not necessarily positive. This fact is in contradiction to the intended interpretation of these quantities. It is surmised, however, that this contradiction is more apparent than real. In fact it is felt that the possibility of negative values of the densities is simply a necessary reflection of the impossibility

---

[13] T. Nishiyama and K. Husimi, Prog. Theor. Phys. (Japan), 5, 909, (1950).

[14] See further in this connection the application of this statistical axiom to the development of rate equations for neutral gases by S. Ono, Proceedings of the International Symposium on Transport Processes in Statistical Mechanics, Interscience Publ. Inc., New York, 1958, p. 229.

[15] N. G. van Kampen, p. 239 of Reference 14.

of the simultaneous specification of positions and momenta of particles with arbitrary accuracy. In order to avoid even the appearance of contradiction, however, it is sufficient merely to qualify the statement of interpretation with the suggestion that the densities have the meaning assigned to them for the purpose of ultimately computing averages. In this sense then these densities are not to be generally regarded as observables, but rather as appropriate weight functions for the calculation of averages which are presumably observable.

The second class of difficulties encountered in the present development are somewhat less fundamental in character than those associated with the choice of axioms, but are nonetheless important and perhaps only slightly less awkward to cope with. These, of course, are the problems posed by the requirement of proceeding logically from an acceptable but formidable description of a system to solutions in specific cases which may be usefully interpreted.

In the present report we have adopted equations (19) and (21) as the exact but intractible descriptions of the photons and particles respectively. Actually, we should add here as an alternative description of the electromagnetic fields in the plasma - the set of equations, (42). As the burden of the report was devoted to pointing up the difficulties of this second class, not much further comment is warranted here. However, one observation does seem to be worthwhile. As just noted, equations (19) and (42) have been presented as equivalent, alternative descriptions of the electromagnetic properties of the plasma. But neither is in general wholly useful, since both are too complicated. In particular the transport description of electromagnetic phenomena as provided by equation (19) is hardly impressive as a practical tool. But the discussion of Section VI leading to the reduction of (19) to (102) suggests a range of physical situations in which the characterization of electromagnetic phenomena in terms of the creation, destruction, scattering, and flow of photons would be expected to be uniquely useful. As suggested there, this range is first of all delineated by the requirement that the photon distribution function vary inappreciably over distances large compared to the wavelengths of radiation under consideration. However, there is a more stringent restriction arising from the practical necessity of limiting the photon transport concept to those situations in which the only significant contributions to the rate of change of the photon density are provided by one-photon processes. This limitation imposes the restriction that the wavelengths of the radiation under consideration be small compared to the mean interparticle spacing - or, equivalently, the radiation emission times which are of the order of the inverse of the natural frequencies of the emitting systems must be small compared to the time required for the photon to travel a distance corresponding to the mean interparticle spacing. Thus the photon transport equation provides a useful tool for investigating the balance of high energy (kev or greater) radiation in a system, whereas lower energy electromagnetic phenomena apparently require the field equations (42), to which must be adjoined equations like (43), (47) or (50) to provide a description of the charge and current densities in the system.

Appendix A

With due regard for the remarks in Section IV concerning the replacement of the integral terms in equation (49) by the conventional binary collision contributions as exhibited in equation (50), we present here for completeness an argument in terms of which this replacement may be more or less intuitively accomplished. For the sake of simplicity we devote our attention to a representative, "truncated range", Coulomb interaction integral, i.e.,

$$\int_{|\underline{x}-\underline{x}'|<\ell} d^3x'\, d^3v' \left[\frac{1}{m_\sigma} \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x} - \underline{x}'|} \right] \frac{\partial f_{\sigma\sigma'}}{\partial v_j} \, . \qquad \text{(A-1)}$$

We first note that this integral may be rewritten as

$$\int_{|\underline{x}-\underline{x}'|<\ell} d^3x'\, d^3v' \left[\frac{1}{m_\sigma} \left\{ \frac{\partial}{\partial x_j} \frac{e_\sigma e_{\sigma'}}{|\underline{x}-\underline{x}'|} \right\} \frac{\partial f_{\sigma\sigma'}}{\partial v_j} \right.$$

$$\left. + \frac{1}{m_{\sigma'}} \left\{ \frac{\partial}{\partial x_j'} \frac{e_\sigma e_{\sigma'}}{|\underline{x}-\underline{x}'|} \right\} \frac{\partial f_{\sigma\sigma'}}{\partial v_j'} \right] , \qquad \text{(A-2)}$$

since the additional term is zero as is readily seen by partial integration with respect to $\underline{v}'$.[16] We now observe that if the range of integration over $\underline{x}'$ is

[16] This particular aspect of this argument plus considerable general discussion of the statistical description of systems in terms similar to those employed herein was presented by M. Schonberg, Nuovo Cimento, 10, 419, 697, 1499 (1953).

sufficiently restricted, i.e., ℓ is required to be sufficiently small with respect to the mean interparticle spacing, then the following identifications and/or approximations may be presumed somewhat justified:

1) The explicit dependence of $f_{\sigma\sigma'}$ upon $\underline{x}'$ and t may be expected to be such that, throughout the range of the integration (and for temporal displacements corresponding to times required for particles to traverse the range of the integration), we may approximate

$$f_{\sigma\sigma'}(\underline{x}, \underline{v}, \underline{x}', \underline{v}', t + \Delta t)$$

$$\cong f_{\sigma\sigma'}(\underline{x}, \underline{v}, \underline{x}, \underline{v}', t). \qquad \text{(A-3)}$$

2) Since the volume of integration is small compared to the average volume per particle, it will be assumed that the probability of finding more than two particles at a time within this volume is negligible (the binary collision assumption), and hence may identify $(\underline{x}, \underline{v})$ as $(\underline{x}_1, \underline{v}_1)$ and $(\underline{x}', \underline{v}')$ as $(\underline{x}_2, \underline{v}_2)$ - the coordinates of particles (1) and (2) respectively.

3) Throughout the interaction of these two closely associated particles it will be presumed that the forces they exert upon each other are sufficiently greater than the resultant of all other forces acting upon either of them that they may be regarded as decoupled from their environment, hence

$$\frac{1}{m_1} \frac{\partial}{\partial x_{1j}} \frac{e_1 e_2}{|\underline{x}_1 - \underline{x}_2|} = - a_{12j},$$

$$\frac{1}{m_2} \frac{\partial}{\partial x_{2j}} \frac{e_1 e_2}{|\underline{x}_1 - \underline{x}_2|} = - a_{21j}, \qquad \text{(A-4)}$$

where $a_{12\,j}$ and $a_{21\,j}$ are the j^{th} components of the accelerations experienced by particles (1) and (2) due to the forces exerted upon them by particles (2) and (1) respectively.

Given these assumptions, it is seen that the expression (A-2) becomes

$$-\int\limits_{|\underline{x}_1-\underline{x}_2|<\ell} d^3x_2\, d^3v_2 \left[a_{12\,j} \frac{\partial f_{12}(\underline{x}_1, \underline{v}_1, \underline{x}_1, \underline{v}_2, t)}{\partial v_{1\,j}} + a_{21\,j} \frac{\partial f_{12}(\underline{x}_1, \underline{v}_1, \underline{x}_1, \underline{v}_2, t)}{\partial v_{2\,j}} \right] . \tag{A-5}$$

We may further interpret

$$a_{12\,j} \frac{\partial f_{12}(\underline{x}_1, \underline{v}_1, \underline{x}_1, \underline{v}_2, t)}{\partial v_{1\,j}} + a_{21\,j} \frac{\partial f_{12}(\underline{x}_1, \underline{v}_1, \underline{x}_1, \underline{v}_2, t)}{\partial v_{2\,j}}$$

$$\cong -\left[f_{12}(\underline{x}_1, \underline{v}_1 - \underline{a}_{12}\tau, \underline{x}_1, \underline{v}_2 - \underline{a}_{21}\tau, t) - f_{12}(\underline{x}_1, \underline{v}_1, \underline{x}_1, \underline{v}_2, t) \right] \frac{1}{\tau} , \tag{A-6}$$

where τ is here interpreted as a "collision time". Clearly we may interpret $(\underline{v}_1 . \underline{v}_2)$ and $(\underline{v}_1 - \underline{a}_{12}\tau,\ \underline{v}_2 - \underline{a}_{21}\tau)$ as the post- and pre-collision velocities of particles (1) and (2) respectively. Furthermore we note that the integrand is no longer a function of the space variable of integration. In order to perform this integration, transform to the center of mass coordinate system so that the variables of integration become the components of the relative displacement, $\underline{R}$. Then, transforming to cylindrical coordinates with the axis along the asymptote of the incident trajectory, the space-integral becomes

$$\int\limits_{|\underline{x}_1-\underline{x}_2|<\ell} d^3x_2 = \int\limits_{R<\ell} d^3R \cong \int_{z=-\ell/2}^{+\ell/2} dz \int_{s=0}^{\ell} ds \int_{\phi=0}^{2\pi} d\phi = \ell \int\limits_{\underline{\Omega}} \sigma_{12}\, d\Omega , \qquad \text{(A-7)}$$

where $\sigma_{12} d\Omega$ is the differential cross section for the Coulomb interaction. Note, however, that the restricted range for the impact parameters (s) implies a corresponding restriction to large scattering angles in the integration over $\underline{\Omega}$. Thus the cross section is defined as a truncated (in angle from below) Coulomb cross section rather than as if generated by a shielded Coulomb potential.

Thus now the expression (A-6) becomes (after an obvious modification of notation),

$$\int d^3v_1\, d\Omega\, \frac{\ell}{\tau}\, \sigma_{12} \left[f_{12}(\underline{x}, \underline{v}', \underline{x}, \underline{v}_1', t) - f_{12}(\underline{x}, \underline{v}, \underline{x}, \underline{v}_1, t) \right] . \qquad \text{(A-8)}$$

Finally we note that, in accordance with the identifications above,

$$\frac{\ell}{\tau} = |\underline{v} - \underline{v}_1| = v_r , \qquad \text{(A-9)}$$

where v_r is the relative speed of the interacting particles. Thus now if we revert to the general notation of (A-2) and approximate

$$f_{\sigma\sigma'}(\underline{x}, \underline{v}, \underline{x}, \underline{v}_1, t) \simeq f_\sigma(\underline{x}, \underline{v}, t)\; f_{\sigma'}(\underline{x}, \underline{v}_1, t) , \qquad \text{(A-10)}$$

we see that we obtain the collision terms as exhibited in equation (50).