

physics, the results strongly support their validity. The only result that deviated from the theoretical prediction was that for the drag on a slow sheet. Here the deviation amounts to about 50%. It appears that the theoretical result may depend on the detailed behavior of the Debye clouds for two colliding particles.

Finally, the model should be able to serve as a useful guide for obtaining theories of nonequilibrium properties, and nonlinear phenomenon.

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## Oscillations in a Relativistic Plasma

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The linear oscillations in a hot plasma which is representable by the relativistic Vlasov equation with the self-consistent fields are investigated. The generalization of Bernstein's method for the relativistic case is used to obtain the formal solution of the linearized problem. Particular attention is given to the case when the system initially is in the relativistic equilibrium state. The dispersion equation is derived and studied for the case when the propagation is along the direction of the unperturbed magnetic field, considering the spatial dispersions explicitly. The asymptotic expansions are developed corresponding to the dispersion relations of the cases studied. It is found that transverse waves propagating along the unperturbed field are Landau damped if  $\nu^2 \geq 1 - \Omega^2/\omega^2$ ,  $\nu$  and  $\Omega$  being the index of refraction and the gyrofrequency, respectively. In the absence of the external field the cutoff frequency, which is found to be the same for both longitudinal and the transverse modes, is shown to be a monotonically decreasing function of the temperature.

### I. INTRODUCTION

IN the recent literature the problems involving the hot ionized gases have increasingly attracted the plasma investigators. The relativistic Vlasov equation together with Maxwell's equations for the self-consistent fields have been used in most of these approaches. Since the correlations are ignored as a whole in this model, the validity and applicability of this representation are somewhat restricted. The extent to which this imposes limitations has not yet been made evident in the literature. However, leaving these questions unanswered, in this paper it will be assumed that this model can properly represent the system under consideration to some extent.

Furthermore, to study the oscillatory phenomena, a linearized theory will be employed with the presence of a constant external magnetic field. As a special case, the unperturbed distribution function will be assumed to be the relativistic equilibrium distribution function. It will be shown that, taking

appropriate limits of the results of the present paper, one can obtain the results of the former analyses in which, in the absence of the external field, the weakly relativistic and ultrarelativistic cases were studied using the approximate forms of the equilibrium distribution corresponding to the respective cases.

The formulation of the mathematical problem will be given in Sec. II. The derivation of the formal solution of the linearized system and the dispersion relation will be sketched subsequently. Particular attention will be given to the case in which the propagation is along the external field.

Sections III, IV, and V will be devoted to the study of the longitudinal, transverse, and the magnetohydrodynamic waves, respectively. A short discussion of the results deduced will be given in Sec. VI.

### II. FORMAL SOLUTION OF THE LINEARIZED PROBLEM

The system of equations by which the plasma is assumed to be represented is given as

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$$u_\mu \frac{\partial f_A}{\partial x_\mu} + \frac{e_A}{m_A c} F_{k\mu} u_\mu \frac{\partial f_A}{\partial u_k} = 0, \quad (A = 1, 2, \dots, N),$$

$$\frac{\partial F_{\nu\mu}}{\partial x_\mu} = \frac{4\pi}{c} J_\nu = \frac{4\pi}{c} \sum e_A \int d^3u \frac{u_\nu}{\gamma} f_A, \quad (1)$$

$$\epsilon_{\sigma\mu\nu\lambda} \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0.$$

The summation convention both for Greek (1 to 4) and Latin (1 to 3) indices are used here. The reduced velocity  $u_\mu$  and the field tensor  $F_{\nu\mu}$  are defined as

$$u_k = v_k \gamma, \quad u_4 = ic\gamma, \\ \gamma = (1 - v^2/c^2)^{-1/2} = (1 + u^2/c^2)^{1/2}, \quad (v^2 = v_i v_i), \quad (2)$$

$$F_{ij} = \epsilon_{ijk} H_k, \quad F_{4i} = -F_{i4} = iE_i.$$

The symbol  $\sum$  represents a summation over all plasma species, and will be used in this sense unless otherwise specified.

The fourth component of the space-time is chosen as  $x_4 = ict$ . The symbols  $\epsilon_{ijk}$  and  $\epsilon_{\sigma\mu\nu\lambda}$  denote the completely antisymmetric (Levi-Civita) tensor densities in 3- and 4-space, respectively. The rest mass and the electric charge of the species of type  $A$  are denoted by  $m_A$  and  $e_A$ , respectively. The suffix  $A$  will be suppressed in what follows whenever there is no ambiguity.

Consider small perturbations about a space-time independent (zero-order) state with the presence of a constant external magnetic field  $\mathbf{H}^{(0)}$ . The general solution to the unperturbed state equation is

$$f^{(0)}(u_k) = g(\mathbf{u} \cdot \mathbf{h}, u^2), \quad (3)$$

$\mathbf{h}$  being the unit vector along  $\mathbf{H}^{(0)}$ , and  $g$  being an arbitrary function of its arguments. The relativistic Maxwell-Boltzmann (Jüttner<sup>1</sup>) distribution (MBJ), which will be given more attention later, satisfies the above condition.

Integrating along the unperturbed world-lines, the first-order equation can be written as<sup>2</sup>

$$f^{(1)}(x_\lambda, u_k) - f^{(1)}\left[x_\lambda - G_{\lambda\mu} \left(\frac{x_4}{u_4}\right) u_\mu, R_{ki} \left(\frac{x_4}{u_4}\right) u_j\right] \\ = -\frac{e}{mc} u_\nu \frac{\partial f^{(0)}}{\partial u_i} \\ \cdot \int_0^{x_4/u_4} F_{k\mu}^{(1)} [x_\lambda - G_{\lambda\rho}(s) u_\rho] R_{\mu\nu}(s) R_{ki}(s) ds, \quad (4)$$

<sup>1</sup> F. Jüttner, Ann. Physik 34, 856 (1911a).

<sup>2</sup> For details see K. Imre [The University of Michigan Radiation Laboratory Rept. 2756-1-F, (ARL-TR-60-274, Part II) (1961)].

where

$$R_{\mu\nu}(s) = \delta_{\mu\nu} - \sin(\Omega s) b_{\mu\nu} - (1 - \cos \Omega s) b_{\mu\nu}^2, \\ G_{\mu\nu}(s) = \int_0^s R_{\mu\sigma}(s') ds', \quad (5)$$

$$\Omega = \frac{eH^{(0)}}{mc}, \quad b_{\mu\nu} = \frac{F_{\mu\nu}^{(0)}}{H^{(0)}}, \quad b_{\mu\nu}^2 = b_{\mu\sigma} b_{\sigma\nu}.$$

The utilization of the Fourier transform enables one to solve this integral equation formally. Introduce the transform  $\psi^+(k_\lambda)$  of an arbitrary function  $\psi^{(1)}(x_\lambda)$  as

$$\psi^+(k_\lambda) = \int_{(+)} e^{-ik_{\rho\tau} x_\rho} \psi^{(1)}(x_\lambda) d^4x. \quad (6)$$

The symbol  $+$  indicates that the  $x_4$  integration is to be restricted to the positive  $x_4$  range.

Eliminating the field variables, one obtains (suppressing the initial conditions)

$$f^+ = A_i \sum e \int d^3u \frac{u_i}{\gamma} f^+ = A_i J_i^+, \quad (7)$$

where<sup>3</sup>

$$A_i = \frac{4\pi e i}{m p^2 k_\rho^2} u_\nu \frac{\partial f^{(0)}}{\partial u_\alpha} \delta_{\mu k}^{\lambda i} k_\lambda (k_i k_j + k_4^2 \delta_{ij}) \\ \cdot \int_0^\infty e^{-\nu(s)} R_{\mu\nu}(s) R_{k\alpha}(s) ds, \quad (8)$$

$$y(s) = ik_\rho u_\sigma G_{\rho\sigma}(s) = i(\mathbf{k} \cdot \mathbf{u} - ip\gamma s) \\ + i(1/\Omega)(1 - \cos \Omega s) \mathbf{u} \cdot (\mathbf{k} \times \mathbf{h}) \\ - i(1/\Omega)(\Omega s - \sin \Omega s) (\mathbf{u} \times \mathbf{h}) \cdot (\mathbf{k} \times \mathbf{h}), \quad (9)$$

with  $k_\rho^2 \equiv k_\rho k_\rho$  and  $p = -k_4 c$ . The latter can be interpreted as the Laplace transform parameter corresponding to the time variable. Multiplying Eq. (7) by  $e(u_i/\gamma) d^3u$  and integrating over the  $\mathbf{u}$  space, and then summing over all species one obtains

$$(S_{ij} - \nu^{-2} \epsilon_{ij}) J_i^+ = \zeta_j, \quad (10)$$

where  $\nu^2 = -k^2 c^2/p^2$ ,  $S_{ij}$  is the projection tensor [ $S_{ij} = \delta_{ij} - (k_i k_j/k^2)$ ],  $\epsilon_{ij}$  is the dielectric tensor [ $\epsilon_{ij} \equiv \delta_{ij} + (4\pi\sigma_{ij}/p)$ ], and  $\sigma_{ij}$  is the conductivity tensor which is defined as

<sup>3</sup> The Kronecker delta  $\delta_{\nu_1 \nu_2 \dots \nu_n}^{\mu_1 \mu_2 \dots \mu_n}$  may be defined as

$$\delta_{\nu_1 \nu_2 \dots \nu_n}^{\mu_1 \mu_2 \dots \mu_n} \equiv \det (\delta_{\nu_i}^{\mu_j}) = \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \dots & \delta_{\nu_1}^{\mu_n} \\ \vdots & & \vdots \\ \delta_{\nu_n}^{\mu_1} & \dots & \delta_{\nu_n}^{\mu_n} \end{vmatrix}.$$

Also,  $\delta_j^\alpha = 0$  for  $\alpha = 4$  and  $= \delta_{jk}$  for  $\alpha = k(=1, 2, 3)$ .

$$\begin{aligned} \sigma_{\lambda j} &= \delta_{\rho\lambda}^{\nu j} \sigma_{\lambda\rho} k_\nu / k_\lambda, \\ \sigma_{\lambda\rho} &= i \sum \frac{e^2}{mc} \int d^3u \frac{u_\nu u_\lambda}{\gamma} \frac{\partial f^{(0)}}{\partial u_k} \\ &\quad \cdot \int_0^\infty e^{-\nu R_{\rho k} R_{\rho\nu}} ds, \end{aligned} \quad (11)$$

so that  $J_\lambda^+ = \sigma_{\lambda j} E_j^+ = i\sigma_{\lambda\rho} F_{j\rho}^+$ . Here  $\zeta_i$  is an explicit function of the initial perturbations and the unperturbed parameters only, and contains all the terms of that nature which have been suppressed in the previous equations [cf. Eq. (7)].

The dispersion equation is, therefore,

$$\det(S_{ij} - \nu^{-2}\epsilon_{ij}) = 0, \quad (12)$$

whenever  $\zeta_i$  is an analytic function (as will be assumed in this paper). The latter can be written as

$$\epsilon^l(\nu^2 - a_+)(\nu^2 - a_-) = 0, \quad (13)$$

where

$$\begin{aligned} a_\pm &= [\frac{1}{2} \text{tr } \epsilon - \frac{1}{2}(\epsilon^2)^l / \epsilon^l] \\ &\quad \mp \{[\frac{1}{2} \text{tr } \epsilon - \frac{1}{2}(\epsilon^2)^l / \epsilon^l]^2 - \det(\epsilon) / \epsilon^l\}^{\frac{1}{2}}, \\ \epsilon^l &= k_i k_j \epsilon_{ij} / k^2, \\ (\epsilon^2)^l &= k_i k_j \epsilon_{ij} \epsilon_{ij} / k^2, \\ \text{tr } \epsilon &= \epsilon_{jj}. \end{aligned} \quad (14)$$

The two possible modes indicated by  $\mp$  correspond to the ordinary and extraordinary waves.

The general study of Eq. (12) is quite complicated and will not be attempted here. However, when the propagation is along the unperturbed magnetic field, which will be assumed in what follows, the longitudinal and the transverse modes can be completely decoupled; hence the algebra involved is considerably reduced.

Moreover, if the unperturbed state is isotropic in the  $\mathbf{u}$  space, one has further simplifications. In this case Eqs. (8) and (11) read<sup>4,5</sup>

$$f^+ = -\frac{e}{mc} \frac{u_i u_k}{u} \frac{df^{(0)}}{du} F_{i\lambda}^+ \int_0^\infty e^{-\nu R_{ik}} ds, \quad (8')$$

$$\sigma_{ij} = -\sum \frac{e^2}{m} \int d^3u \frac{u_i u_k}{u} \frac{df^{(0)}}{du} \int_0^\infty e^{-\nu R_{ik}} ds, \quad (11')$$

where, with  $\mathbf{k} \times \mathbf{h} = 0$ , one has

$$y(s) = ik_\mu u_\mu s = i(\mathbf{k} \cdot \mathbf{u} - i\nu\gamma)s.$$

Selecting  $\mathbf{k}$  (thus  $\mathbf{h}$ ) along the  $z$  axis, for the sake of simplicity, it is seen that  $\sigma_{11} - \sigma_{22} = \sigma_{12} + \sigma_{21} = \sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$ , and therefore,

$$a_\pm = 1 + (4\pi\sigma'_\pm/p), \quad (15)$$

where  $\sigma'_\pm \equiv \sigma_{11} \mp i\sigma_{21}$ .

In particular if  $f^{(0)}$  is the MBJ distribution, [i.e.,

$$f_{\text{MBJ}} = (n/4\pi c^3)[\beta/K_2(\beta)] \exp(-\beta\gamma), \quad (16)$$

where  $\beta = mc^2/\Theta$ ,  $\Theta$  being the temperature in ergs,  $K_n(z)$  is the modified Bessel function of the second kind, and  $n = \int f_{\text{MBJ}} d^3u$ ], one obtains

$$\begin{aligned} \sigma_{ij} &= \sum \frac{e^2 n}{m} \frac{\beta^2}{K_2(\beta)} \\ &\quad \cdot \int_0^\infty \left[ \frac{K_2(\varpi)}{\varpi^2} R_{ij} - c^2 k_\rho k_\rho \frac{K_3(\varpi)}{\varpi^3} G_{i\rho} G_{\rho j} \right] ds, \end{aligned} \quad (17)$$

where

$$\varpi(s) = (\beta + ps)^2 + 2c^2 k_i k_i M_{ii}(s),$$

$$M_{ii}(s) = \int_0^s G_{ii}(s') ds' = \int_0^s \int_0^{s'} R_{ii}(s'') ds'' ds'$$

The latter form has already been given by Trubnikov.<sup>5</sup> Note that when  $\mathbf{k} \times \mathbf{h} = 0$ , one has

$$\varpi(s) = (\beta + ps)^2 + k^2 c^2 s^2. \quad (18)$$

### III. LONGITUDINAL OSCILLATIONS

The longitudinal oscillations are represented by the first factor of Eq. (13), thus the corresponding dispersion relation reads

$$\epsilon^l \equiv 1 + (4\pi\sigma^l/p) = 0, \quad (19)$$

where

$$\begin{aligned} \sigma^l &\equiv \sigma_{ii} k_i k_i / k^2 \\ &= -\sum \frac{e^2}{m} \int \frac{df^{(0)}}{du} \frac{u_x^2}{u} \int_0^\infty \exp(-ik_\mu u_\mu s) ds d^3u. \end{aligned}$$

Here, we have used the fact that  $\mathbf{k} \times \mathbf{H}^{(0)} = 0$  implies  $b_{ij} k_j = b_{ii}^2 k_i = 0$ , so that  $R_{ij} k_j = k_i$ , as expected, for  $R_{ij}$  is the rotation transformation about  $\mathbf{H}^{(0)}$ .

It is seen that the longitudinal mode is completely independent of the unperturbed field, which is of course expected.

The relation (19), thus, is identical to the one which is derived and studied to some extent in the literature<sup>6</sup> in dealing with the case when  $\mathbf{H}^{(0)}$  is zero. Therefore, only the results will be given here without going into details.

<sup>4</sup> W. E. Drummond and M. N. Rosenbluth, Phys. Fluids 3, 45 (1960).

<sup>5</sup> B. A. Trubnikov, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, edited by M. A. Leontovich, translation editor, J. Turkevich (Pergamon Press, New York, 1960), Vol. 3.

<sup>6</sup> P. C. Clemmow and A. J. Willson, Proc. Roy. Soc. (London) A237, 117 (1956).

The method of Landau<sup>7</sup> will be used here to calculate the spatial dispersions. Consider the real and imaginary parts of  $(4\pi\sigma^i/p)$  on the imaginary axis of the complex  $p$  plane, to get

$$\operatorname{Re} \left( \frac{4\pi\sigma^i}{p} \right) = - \left( \frac{4\pi}{k} \right)^2 \sum \frac{e^2}{m} \int_0^\infty \gamma u \frac{df^{(0)}}{du} \cdot \left( 1 + \frac{\omega\gamma}{2ku} \ln \left| \frac{\omega\gamma - ku}{\omega\gamma + ku} \right| \right) du, \quad (20)$$

and

$$\operatorname{Im} \left( \frac{4\pi\sigma^i}{p} \right) = -\theta(k^2c^2 - \omega^2) \frac{8\pi^3\omega}{k^3} \cdot \sum \frac{e^2}{m} \int_0^\infty \gamma^2 \frac{df^{(0)}}{du} du, \quad (21)$$

where  $p = -i\omega$  (to ensure that the waves propagate along the positive direction of the  $z$  axis). Here,  $\theta(x)$  is Heaviside's step function, i.e.,

$$\theta(x) = 0 \quad x < 0, \\ = 1 \quad x > 0,$$

and

$$U = (\omega/k)(1 - \omega^2/k^2c^2)^{-\frac{1}{2}}. \quad (22)$$

The step function indicates that there exist purely oscillatory solutions only when the phase speed  $(\omega/k)$  is greater than the speed of light; viz., the index of refraction  $\nu = kc/\omega$  is less than unity.<sup>6</sup>

In the nonrelativistic limit, using  $\gamma^2 \sim 1$ , one obtains the well-known result of Landau<sup>7</sup>

$$\operatorname{Im} \left( \frac{4\pi\sigma^i}{p} \right) \rightarrow \frac{8\pi^3\omega}{k^3} \sum \frac{e^2}{m} f^{(0)} \left( \frac{\omega}{k} \right).$$

In particular let us assume that  $f^{(0)}$  is of the MBJ type as introduced in the previous section. One finds after some manipulations

$$\operatorname{Re} \left( \frac{4\pi\sigma^i}{p} \right) = \sum \frac{\omega_p^2 \beta}{\omega^2 \nu^2} \left[ 1 - \frac{1}{\nu^2 K_2(\beta)} \cdot \int_0^\infty \left( \cosh^2 x + \frac{2}{\beta} \cosh x + \frac{2}{\beta^2} \right) \cdot \frac{\exp(-\beta \cosh x)}{1 - [(1 - \nu^2)/\nu^2] \cosh^2 x} dx \right], \quad (23)$$

and

$$\operatorname{Im} \left( \frac{4\pi\sigma^i}{p} \right) = \theta(\nu^2 - 1) \frac{\pi}{2\nu^3} \cdot \sum \frac{\omega_p^2}{\omega^2} \frac{e^{-\beta}}{\beta K_2(\beta)} (2 + 2\beta\Gamma + \beta^2\Gamma^2), \quad (24)$$

where

$$\omega_p^2 = 4\pi n e^2 / m, \\ \Gamma = (1 + U^2/c^2)^{\frac{1}{2}} = (1 - \nu^{-2})^{-\frac{1}{2}}.$$

Let us note that the  $x$  integration in Eq. (23) can be carried out exactly when  $\nu = 1$ . The result is

$$\operatorname{Re} \left( \frac{4\pi\sigma^i}{p} \right) = - \sum \frac{\omega_p^2}{\omega^2} \left[ \frac{2}{\beta} \frac{K_0(\beta)}{K_2(\beta)} + \frac{K_1(\beta)}{K_2(\beta)} \right]. \quad (25)$$

Alternatively, one may use Eq. (17) to compute  $\sigma^i$  to get

$$1 + (4\pi\sigma^i/p) = 1 + \sum \omega_p^2 \frac{\beta^2}{K_2(\beta)} \cdot \int_0^\infty \left[ \frac{K_3(s)}{s^3} \right] (s\beta + ps^2) ds = 0. \quad (26)$$

Using the above form of the dispersion equation we have developed an asymptotic expansion for the integration involved, in powers of

$$\alpha = (1 - \nu^2)/\beta.$$

The result is (with  $p = -i\omega$ )

$$1 \sim \sum \frac{\omega_p^2}{\omega^2} \sum_{i=0,1,\dots} \frac{K_{i-1}(\beta)}{K_2(\beta)} \left( \alpha^i A_i - \frac{\alpha^{i-1}}{\beta} B_i \right), \quad (27)$$

where

$$A_i = \frac{(-1)^i (2i+2)!}{2^{i+1} (i+1)!}, \quad B_i = \frac{(-1)^i (2i)!}{2^{i-1} (i-1)!} \\ (B_0 = 0).$$

One observes that for  $\nu = 1$  the above expansion becomes identical to Eq. (25). To the first order in  $(1/\beta)$ , Eq. (27) gives the result of Clemmow and Willson.<sup>6</sup>

$$\omega^2 \cong \sum \omega_p^2 [1 + (3/\beta)(\nu^2 - \frac{5}{6})] + O(\beta^{-2}). \quad (28)$$

In deriving the latter, the asymptotic expansion of the Bessel functions involved is used

$$\frac{K_n(x)}{K_m(x)} \sim 1 + \frac{n^2 - m^2}{2x} + \frac{(n^2 - m^2)(n^2 - m^2 - 2)}{2! (2x)^2} + O(x^{-3}).$$

This result is not surprising, since the unperturbed distribution function used here coincides with the one adopted by Clemmow and Willson to the first order in  $1/\beta$ , theirs being of the form  $f^{(0)} \propto \exp(-\frac{1}{2}\beta u^2)$ .

Silin<sup>8</sup> studied the ultrarelativistic case assuming  $f^{(0)} \propto \exp(-\beta u/c)$ . His result for the longitudinal

<sup>7</sup> J. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

<sup>8</sup> V. P. Silin, J. Exptl. Theoret. Phys. 38, 1577 (1960) [Soviet Phys.—JETP 11, 1136 (1960)].

case may be deduced from Eq. (26), by approximating the Bessel function  $K_3(\omega) \approx 8/\omega^3$ , ( $\beta \ll 1$ ). Carrying out the integration, one finds in this limit

$$\nu^2 = \sum (\omega_p^2/\omega^2)\beta[(1/\nu) \tanh^{-1} \nu - 1]. \quad (29)$$

For  $|\nu| \ll 1$ , approximating successively one obtains

$$\frac{2}{3}\nu^2 \cong 1 - \frac{1}{3} \sum (\omega_p^2/\omega^2)\beta,$$

which indicates that no longitudinal waves can propagate in an ultrarelativistic plasma with frequencies less than the cutoff frequency  $\omega_0 = (\frac{1}{3} \sum \omega_p^2\beta)^{\frac{1}{2}}$ . Further discussion will be given later for the cutoff frequencies.

It is seen from Eq. (29) that as the phase speed approaches the speed of light, the frequency grows up infinitely. However, this is not the case, since from Eq. (25) one can compute the frequency for this limit without making the approximation for the Bessel functions

$$\omega^2 = \sum \omega_p^2 \left[ \frac{2 K_0(\beta)}{\beta K_2(\beta)} + \frac{K_1(\beta)}{K_2(\beta)} \right], \quad (\nu = 1).$$

This observation indicates the limitation of Silin's result in the vicinity of  $\nu = 1$ . One may write for  $\nu = 1$  and  $\beta \ll 1$

$$\omega^2 \approx \sum \omega_p^2 \beta (\frac{1}{2} - \ln \beta).$$

IV. TRANSVERSE OSCILLATIONS

The two possible modes of the transverse oscillations are given by

$$\nu^2 = a_* = 1 + (4\pi\sigma_*^i/p).$$

Proceeding as before, consider the real and imaginary parts of

$$(4\pi\sigma_*^i/p) \equiv (4\pi/p)(\sigma_{11} \mp i\sigma_{21})$$

on the imaginary axis of the complex  $p$  plane. After performing the angular integrations in the  $\mathbf{u}$  space, one obtains

$$\begin{aligned} \text{Re} \left( \frac{4\pi\sigma_*^i}{p} \right) &= \frac{8\pi^2}{\omega k} \sum \frac{e^2}{m} \int_0^\infty u^2 \frac{df^{(0)}}{du} \left\{ \frac{\omega\gamma \mp \Omega}{ku} \right. \\ &\quad \left. + \frac{1}{2} \left[ 1 - \left( \frac{\omega\gamma \mp \Omega}{ku} \right)^2 \right] \ln \left| \frac{\omega\gamma \mp \Omega + ku}{\omega\gamma \mp \Omega - ku} \right| \right\} du, \quad (30) \end{aligned}$$

and

$$\begin{aligned} \text{Im} \left( \frac{4\pi\sigma_*^i}{p} \right) &= -\frac{4\pi^3}{\omega k} \sum \frac{e^2}{m} \theta(k^2c^2 - \omega^2 + \Omega^2) \\ &\quad \cdot \int_{U_1}^{U_2} u^2 \frac{df^{(0)}}{du} \left[ 1 - \left( \frac{\omega\gamma \mp \Omega}{ku} \right)^2 \right] du. \quad (31) \end{aligned}$$

The upper and lower limits of the integral of the latter are given as

$$U_1 = c \left| \frac{|\nu\Omega/\omega| - (\nu^2 - 1 + \Omega^2/\omega^2)^{\frac{1}{2}}}{\nu^2 - 1} \right|,$$

$$U_2 = c \frac{|\nu\Omega/\omega| + (\nu^2 - 1 + \Omega^2/\omega^2)^{\frac{1}{2}}}{1 - \nu^2}$$

(extraordinary mode, for  $\nu < 1$ ), (32)

=  $\infty$  (ordinary mode, and extraordinary mode for  $\nu \geq 1$ ).

The step function in Eq. (31) indicates that the transverse waves are Landau damped unless

$$\nu^2 < 1 - \Omega^2/\omega^2. \quad (33)$$

In the absence of the unperturbed field the above result becomes identical to the one obtained for the longitudinal waves [cf., Eq. (37)].

In the case when  $f^{(0)}$  is the MBJ distribution, the  $u$  integration in Eq. (31) can be carried out, giving

$$\begin{aligned} \text{Im} \left( \frac{4\pi\sigma_*^i}{p} \right) &= \frac{\pi}{2\nu^3} \sum \frac{\omega_p^2}{\omega^2} \frac{e^{-\beta\gamma}}{\beta K_2(\beta)} \\ &\quad \cdot \left\{ \nu^2 - 1 + \beta \left[ \gamma(\nu^2 - 1) \mp \frac{\Omega}{\omega} \right] \right. \\ &\quad \left. + \frac{\beta^2}{2} \left[ \gamma^2(\nu^2 - 1) \mp 2\gamma \frac{\Omega}{\omega} - \nu^2 - \frac{\Omega^2}{\omega^2} \right] \right\} \Big|_{U_1}^{U_2}. \quad (34) \end{aligned}$$

For  $\nu > 1$  one gets

$$\begin{aligned} \text{Im} \left( \frac{4\pi\sigma_*^i}{p} \right) &= \frac{\pi}{2\nu^3} \sum \frac{\omega_p^2}{\omega^2} [\beta K_2(\beta)]^{-1} \\ &\quad \cdot \exp \left[ -\beta \frac{\nu(\nu^2 - 1 + \Omega^2/\omega^2)^{\frac{1}{2}} - |\Omega/\omega|}{\nu^2 - 1} \right] \\ &\quad \cdot \left\{ \nu^2 - 1 + \beta \left[ \nu \left( \nu^2 - 1 + \frac{\Omega^2}{\omega^2} \right)^{\frac{1}{2}} - 2 \left| \frac{\Omega}{\omega} \right| \right] \theta \left( \pm \frac{\Omega}{\omega} \right) \right. \\ &\quad \left. - 2\beta^2 \frac{\nu |\Omega/\omega| (\nu^2 - 1 + \Omega^2/\omega^2)^{\frac{1}{2}} - \Omega^2/\omega^2}{\nu^2 - 1} \theta \left( \pm \frac{\Omega}{\omega} \right) \right\}. \quad (35) \end{aligned}$$

When the unperturbed field is absent the results are somewhat simpler, and can be obtained by setting  $\Omega = 0$  in Eqs. (30) and (31).

$$\begin{aligned} \text{Re} \left( \frac{4\pi\sigma_*^i}{p} \right) &= \frac{8\pi^2}{\omega k} \sum \frac{e^2}{m} \int_0^\infty u^2 \frac{df^{(0)}}{du} \\ &\quad \cdot \left[ \frac{\omega\gamma}{ku} - \left( 1 - \frac{\omega^2\gamma^2}{k^2u^2} \right)^{\frac{1}{2}} \ln \left| \frac{\omega\gamma - ku}{\omega\gamma + ku} \right| \right] du, \quad (36) \end{aligned}$$

$$\begin{aligned} \text{Im} \left( \frac{4\pi\sigma_*^i}{p} \right) &= -\theta(k^2c^2 - \omega^2) \frac{4\pi^3}{\omega k} \\ &\quad \cdot \int_U^\infty u^2 \left( 1 - \frac{\omega^2\gamma^2}{k^2u^2} \right) \frac{df^{(0)}}{du} du. \quad (37) \end{aligned}$$

And again for the MBJ distribution

$$\text{Im} \left( \frac{4\pi\sigma'}{p} \right) = \frac{\pi}{2\nu} \sum \frac{\omega_p^2}{\omega^2} \frac{1 + \beta\Gamma}{\beta K_2(\beta)} \frac{e^{-\beta\Gamma}}{\Gamma^2} \theta(\nu^2 - 1), \quad (38)$$

where  $\Gamma = (k/\omega)U$ .

In the nonrelativistic limit, i.e.,  $\beta \gg 1$ , and for  $\nu > 1$ , one gets

$$\text{Im} \left( \frac{4\pi\sigma'}{p} \right) \cong \frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{\nu} \sum \frac{\omega_p^2}{\omega^2} \beta^{\frac{1}{2}} \Gamma^{-1} e^{-\beta(\Gamma-1)},$$

and for  $\nu \gg 1$

$$\approx \left( \frac{\pi}{2} \right)^{\frac{1}{2}} \sum \frac{\omega_p^2}{\omega^2} \frac{\beta^{\frac{1}{2}}}{\nu} e^{-\beta/2\nu^2}. \quad (39)$$

Approximating successively, the latter leads to the Landau damping decrement corresponding to the transverse waves.<sup>9</sup>

The ultrarelativistic limit again can be considered in a similar manner; one gets Silin's result<sup>8</sup> to the first order in  $\beta$ , for  $\nu^2 > 1$ ,

$$\text{Im} \left( \frac{4\pi\sigma'}{p} \right) = \frac{\pi}{2\nu} \sum \left( \frac{\omega_p}{\omega} \right)^2 \frac{\beta}{\Gamma^2} + O(\beta^3). \quad (40)$$

The study of the real part may be performed more conveniently, as is done in the previous case, by the use of the form given in Eq. (17). One finds after some manipulations the following dispersion equation:

$$1 + \frac{p}{k^2 c^2 + p^2} \sum \omega_p^2 \frac{\beta^2}{K_2(\beta)} \cdot \int_0^\infty \frac{K_2(\varpi)}{\varpi^2} e^{\pi i \Omega s} ds = 0, \quad (41)$$

where  $\varpi(s)$  is as given in Eq. (18). It can be shown that the latter goes to the well-known relation

$$\nu^2 \equiv \frac{k^2 c^2}{\omega^2} = 1 - \sum \frac{(\omega_p/\omega)^2}{1 \mp \Omega/\omega}$$

in the nonrelativistic limit, i.e.,  $\beta \rightarrow \infty$ .

The following asymptotic expansions<sup>10</sup> of Eq. (41) have been derived with respect to the parameter  $\alpha = (1 - \nu^2)/\beta$ , for the cases when  $\omega/\Omega$  is greater or less than unity:

<sup>9</sup> Shortly it will be seen that in the absence of the external field one has  $\nu < 1$  [cf. Eq. (44)]. Hence, in this case, there is no Landau damping [cf. Eq. (33)]. However, in some particular problems due to additional effects the total refractive index may exceed unity (e.g., this is the case for plasmas in the proximity of strong dielectrics). Klimontovich and Silin [cf. *Plasma Physics*, edited by J. E. Drummond (McGraw-Hill Book Company, Inc., New York 1961), p. 45] derived a damping factor for such problems using a different method. Equation (39) leads to their result.

<sup>10</sup> Because of the divergent character of the involved series, extra care should be exercised in dealing with the asymptotic expansions given in this paper.

$$\nu^2 \sim 1 + \sum \left( \frac{\omega_p}{\omega} \right)^2 \cdot \sum_{i,j=0,1,\dots} \alpha^i C_i^j \left( \mp \frac{\Omega}{\omega} \right)^{i+2j+1} \frac{K_{2+i+j}(\beta)}{K_2(\beta)}, \quad (42)$$

$$\nu^2 \sim 1 - \sum \left( \frac{\omega_p}{\omega} \right)^2 \cdot \sum_{i,j=0,1,\dots} \alpha^i C_i^j \left( \mp \frac{\Omega}{\omega} \right)^j \frac{K_{i+j-1}(\beta)}{K_2(\beta)},$$

where  $C_i^j = (-1)^i (j+2i)! / (2^i i! j!)$ .

If  $\mathbf{H}^{(0)} = 0$ , setting  $\Omega = 0$  in Eq. (41), one gets

$$k^2 c^2 + p^2 + p \sum \omega_p^2 \frac{\beta^2}{K_2(\beta)} \int_0^\infty \frac{K_2(\varpi)}{\varpi} ds = 0, \quad (43)$$

and the asymptotic expansion becomes

$$\nu^2 \sim 1 - \sum \frac{\omega_p^2}{\omega^2} \sum_{i=0,1,\dots} \frac{(-1)^i (2i)!}{2^i i!} \alpha^i \frac{K_{i-1}(\beta)}{K_2(\beta)}. \quad (44)$$

Keeping the terms up to order  $1/\beta$  in Eq. (42) one obtains

$$\nu^2 = \frac{k^2 c^2}{\omega^2} = \frac{1 - \sum \frac{\omega_p^2}{\omega^2} \frac{1}{1 \pm \Omega/\omega} \left( 1 - \frac{5}{2\beta} \frac{1}{1 \pm \Omega/\omega} \right)}{1 + \sum \frac{\omega_p^2}{\omega^2} \frac{1}{\beta} \left( \frac{1}{1 \pm \Omega/\omega} \right)^3}, \quad (45)$$

which in the absence of the unperturbed field, i.e.,  $\Omega = 0$ , reduces to Buneman's result.<sup>11</sup>

In order to obtain Silin's result for the transverse waves in the ultrarelativistic gas, one may again replace the Bessel function by its small argument expansion in the integrand of the above dispersion relation, to get

$$\nu^2 \cong 1 - \sum \frac{\omega_p^2}{\omega^2} \frac{\beta}{2\nu^2} \left( 1 - \frac{\nu^2 - 1}{\nu} \tanh^{-1} \nu \right). \quad (46)$$

For  $\nu \ll 1$ , by successive approximation one obtains<sup>12</sup>

$$(6/5)\nu^2 = 1 - \frac{2}{3} \sum \beta(\omega_p/\omega)^2.$$

## V. MAGNETOHYDRODYNAMIC WAVES

For simplicity, consider a binary, initially neutral plasma in which the electric charge of the ions is equal to the absolute value of the electronic charge. As customary in the study of the magnetohydrodynamic waves, assume that  $|\omega/\Omega|$  is much less than unity, so that the ion dynamic is particularly

<sup>11</sup> O. Buneman, *Phys. Rev.* **112**, 1504 (1958).

<sup>12</sup> Our formula differs from the one given by Silin by a factor of 2/3.

important. Then, keeping the terms only up to order  $\omega^2/\Omega^2$  in the first of Eqs. (42), one obtains

$$v^2 \cong 1 + \frac{4\pi c^2}{H_0^2} \left[ \rho_+ \frac{K_3(\beta_+)}{K_2(\beta_+)} + \rho_- \frac{K_3(\beta_-)}{K_2(\beta_-)} \right], \quad (47)$$

where the subscripts + and - now denote the ions and electrons, respectively, and  $\rho_{\pm} = nm_{\pm}$ .

Alternatively, the latter may be written as

$$v^2 = 1 + (4\pi/H_0^2)(w_0 + p_0),$$

$w_0$  and  $p_0$  being the zero-order total energy density and pressure, respectively:

$$\begin{aligned} w_0 &= \sum_{+,-} \int mc^2 \gamma f_{MBJ} d^3u \\ &= c^2 \sum_{+,-} \rho \frac{K_3(\beta)}{K_2(\beta)} - p_0, \\ p_0 &= n(\Theta_+ + \Theta_-). \end{aligned}$$

This form may be compared with the result obtained by Harris<sup>13</sup> using the relativistic magnetohydrodynamics formalism.

Using the asymptotic expansion for the Bessel function in Eq. (47), one obtains to the first order in  $1/\beta$

$$v^2 \sim 1 + (c/a)^2 [1 - (5/2)(p_0/c^2)],$$

where  $a$  denotes the Alfvén speed

$$a = H_0 [4\pi(\rho_+ + \rho_-)]^{-1/2}. \quad (48)$$

VI. DISCUSSION AND CONCLUSIONS

In this paper we have attempted the study of the linear oscillations in a hot plasma. It was assumed that the system under consideration can be represented by the relativistic Vlasov equation coupled with Maxwell's field equations for the self-consistent fields.

The derivation of the formal solution of the linearized problem and the dispersion equation is outlined.

Giving particular attention to the spatial dispersion phenomena (in the sense of Landau), we have examined the dispersion equation for the case in which the direction of propagation is parallel to the unperturbed magnetic field. We have developed asymptotic expansions for the cases studied when the unperturbed distribution function is the relativistic equilibrium distribution function.

The longitudinal waves were found to be Landau damped when the index of refraction is greater

<sup>13</sup> E. G. Harris, Phys. Rev. 108, 1358 (1957).

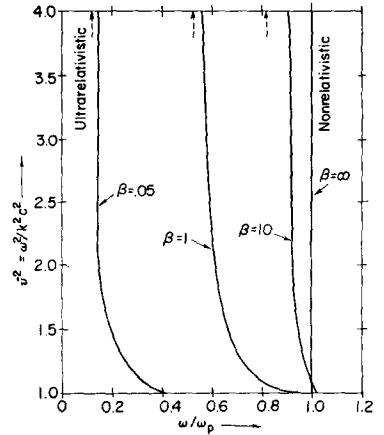


FIG. 1. Longitudinal oscillations. Squared phase velocity vs frequency.

than unity only; this is in agreement with the former investigations.<sup>6,11</sup> It was shown that unattenuated,<sup>14</sup> circularly polarized, transverse waves propagating along the unperturbed field can exist provided  $\omega^2 > k^2 c^2 + \Omega_A^2$ , where the suffix  $A$  represents the species which has the largest  $|e_A/m_A|$  ratio.

Here, a further discussion of this result perhaps is in order. Considering a simple electron gas and ignoring the terms of order  $1/\beta^2$  (or higher), after some simplifications, the above condition may be written as follows [cf., Eq. (45)]:

$$x^2 - \lambda(1 - 1/\beta)x + \lambda/2\beta < 0,$$

where

$$\lambda \equiv \omega_p^2/\Omega^2, \quad x \equiv 1 \pm |\Omega/\omega|.$$

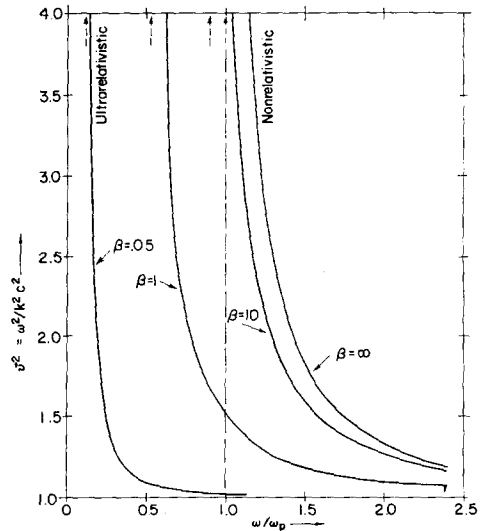


FIG. 2. Transverse oscillations. Squared phase velocity vs frequency.

<sup>14</sup> Here, it should be pointed out that the correlations, which are ignored completely in this work, can provide an additional damping mechanism which is different from the one discussed above.

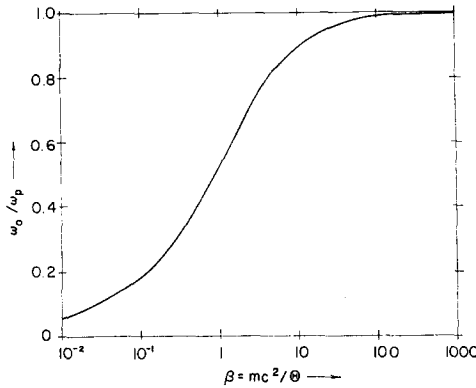


Fig. 3. Cutoff frequency vs inverse temperature.

The ordinary and extraordinary modes are represented by the signs + and -, respectively. The detailed study of the above quadratic form is straightforward, but somewhat lengthy; thus it will not be attempted here.<sup>2</sup> We shall note, however, some of the properties implied.

The frequency band in which the waves under consideration are not Landau damped is determined by the region lying between the roots of the above quadratic form. The resonance frequency, viz.,  $\omega = \Omega$ , lies outside this band. Moreover, it can be seen that whenever the quantity  $\lambda - 3/2\beta$  is less than (or greater than, respectively) unity, the entire

ordinary (or extraordinary) mode is evanescent.

The magnetohydrodynamic waves may be examined in a similar manner. One finds that there is no undamped frequency range in this case.

The results obtained for the case when  $H^{(0)} = 0$  are further illustrated by the following sketches. Assuming  $f^{(0)}$  is the MBJ distribution, the longitudinal and transverse waves in a simple electron gas are considered for different values of the parameter  $\beta$  in Figs. 1 and 2, respectively.

Finally, we considered the cutoff frequencies  $\omega_0$  which are to be determined for both longitudinal and transverse modes by

$$\begin{aligned} \left(\frac{\omega_0}{\omega_p}\right)^2 &= \frac{\beta^2}{K_2(\beta)} \int_{\beta}^{\infty} \frac{K_2(x)}{x^2} dx \\ &\approx 1 - 5/2\beta + 55/8\beta^2 \quad \text{for } \beta \gg 1, \\ &\approx \frac{1}{3}\beta \quad \text{for } \beta \ll 1. \end{aligned}$$

Figure 3 shows  $\omega_0/\omega_p$  vs  $\beta$ .

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