

## Bremsstrahlung of Slow Electrons in Neutral Gases and Free-Free Absorption of Microwaves

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Bremsstrahlung of nonrelativistic electrons in a neutral gas is investigated including the polarization and exchange effects. The intensity and spectrum of the bremsstrahlung and of the induced dipole radiation are obtained for a Maxwellian distribution of electron energy in terms of the elastic scattering cross section of the atom for electrons and its polarizability. The interference of the induced dipole radiation with the bremsstrahlung is also considered. It is found that the exchange effects and the induced dipole radiation are negligible as far as the total radiated power is concerned. The latter, however, may be important at the short-wave end of the spectrum. Finally, the absorption coefficient is obtained from the bremsstrahlung cross section. The results are evaluated explicitly for a Maxwellian distribution.

### I. INTRODUCTION

**B**REMSSTRAHLUNG of slow electrons in the field of neutral atoms has attracted interest in recent years in estimating the intensity of microwave radiation from slightly ionized gases and their free-free absorption coefficients. The radiation from a neutral gas containing free electrons is due to the deceleration of the free electrons in the field of a neutral atom and to the time-dependent dipole moment of neutral atoms induced by free electrons. In this paper, we refer to the former radiation mechanism as the "bremsstrahlung" and to the latter as the "induced dipole radiation." The spectra of the radiation due to these mechanisms are entirely different. The observed spectrum is a superposition of the bremsstrahlung and the induced dipole radiation. However, since these two emission mechanisms are not independent in so far as they are caused by the same collision event between a free electron and a neutral atom, the resultant radiation can not be obtained by simply adding the two intensities. The interference between these two radiation mechanisms should be taken into account.

Furthermore, the exchange effect due to the indistinguishability of the incident electron and the bound electrons in the neutral atom may also play a role in estimating the intensity of the observed radiation.

The aim of this paper is to derive an expression for the intensity and spectrum of the total radiation from a slightly ionized neutral gas taking into account the aforementioned effects.

Bremsstrahlung of slow electrons decelerated by neutral atoms was discussed previously by Firsov and Chibisov,<sup>1</sup> who argued classically that the in-

duced dipole radiation may account for the large portion of the radiation from a neutral gas. However, as is apparent in the text, their quantum-mechanical calculation includes neither the induced dipole radiation nor its interference with the bremsstrahlung. Their result gives the intensity of the bremsstrahlung only in terms of the elastic scattering cross section of an electron on a neutral atom. However, the exchange effects and the polarization of the atom by the field of the incident electron are included implicitly through the scattering cross section. In this paper, the induced dipole radiation and its interference with the bremsstrahlung are included in the quantum-mechanical calculations, and the relative magnitudes of bremsstrahlung, induced dipole radiation and the interference effects, are compared as a function of the gas temperature, assuming that the electrons and the neutral atoms are in thermal equilibrium. Moreover, the magnitude of the exchange effects on the radiation intensity is calculated explicitly, and its relative importance is discussed and shown to be negligible.

It is hardly necessary to mention that the calculations are approximate in view of the complexity of the problem. Most of the approximations used in the derivations are standard in the study of the elastic scattering of slow electrons by a neutral atom. Some of the approximations are made only to derive a simple practical formula that contains all the qualitative features of the phenomenon under consideration. They can easily be relaxed if numerical precision is required. These approximations enable one to relate the intensity of the observed radiation to some atomic parameters which are already known either experimentally or theoretically for many atoms, such as the elastic scattering cross section for slow electron scattering and coefficient of polarization.

<sup>1</sup> O. B. Firsov and M. I. Chibisov, *Zh. Eksperim. i Teor. Fiz.* **39**, 1770 (1960) [English transl.: *Soviet Phys.—JETP* **12**, 1235 (1961)].

The exchange effects are discussed in the case of hydrogen atom for simplicity. The calculation of the intensity of bremsstrahlung and induced dipole radiation is carried out in general for an arbitrary atom.

## II. GENERAL FORMULATION

The physical system under consideration consists of an atom situated at the origin of the system of reference and an incident electron. We consider a radiative transition of this system from an initial state  $|i\rangle$  to a final state  $|f\rangle$  with the emission of a photon. The energy intensity of the radiation emitted in all directions and in two polarization states per unit energy is

$$S_{i \rightarrow f}(\hbar\omega) = (4\omega^4 e^2 / 3c^3) |\langle f | \mathbf{R} | i \rangle|^2, \quad (1)$$

where

$$\mathbf{R} = \sum_{j=1}^{Z+1} \mathbf{r}_j \quad (2)$$

and where  $\mathbf{r}_j$  denotes the position of the  $j$ th electron with respect to the nucleus of the atom which is assumed to be at rest. There are  $Z$  electrons in the atom. The frequency  $\omega$  is given by  $\hbar\omega = \varepsilon_i - \varepsilon_f$ , where  $\varepsilon_i$  and  $\varepsilon_f$  are the energies of the initial and final states. The symbol  $|\langle f | \mathbf{R} | i \rangle|^2$  in (1) is to be interpreted as

$$|\langle f | \mathbf{R} | i \rangle|^2 = \sum_{\nu=1}^3 |\langle f | R_\nu | i \rangle|^2, \quad (3)$$

where  $R_\nu$  are the Cartesian coordinates of the vector  $\mathbf{R}$ .

The state vectors  $|i\rangle$  and  $|f\rangle$  are the solutions of the Schrödinger equation for the atom + electron,

$$(H^a + H^e + V - \varepsilon_i) |i\rangle = 0, \quad (4)$$

where  $H^a$  is the Hamiltonian of the atom,  $H^e$  is the kinetic energy of the incident electron, and  $V$  is the Coulomb interaction between the atom and the electron. The final state  $|f\rangle$  satisfies a similar equation.

We assume that the atom is in a ground state, and the energy of the incident electron is insufficient for the excitation of the atom. Thus, the atom is found in a ground state after the collision in which the incident electron is scattered from the initial momentum state  $|\mathbf{k}_i\rangle$  to the final momentum state  $|\mathbf{k}_f\rangle$ . Conservation of energy requires

$$\varepsilon_i = E_0 + (\hbar^2 k_i^2 / 2m), \quad (5)$$

$$\varepsilon_f = E_0 + (\hbar^2 k_f^2 / 2m), \quad (6)$$

where  $m$  is the electron mass, and  $E_0$  energy of the atom in the ground state.

The interaction potential  $V$  in (4) is the Coulomb interaction between the electron and the atom,

$$V = -\frac{Ze^2}{r} + \sum_{j=1}^Z \frac{e^2}{|\mathbf{r} - \mathbf{r}_j|}, \quad (7)$$

where  $\mathbf{r}$  is the position vector of the incident electron.

The central problem is to compute the matrix elements  $\langle f | \mathbf{R} | i \rangle$  using the solution of the Schrödinger equation (4), for energies  $\varepsilon_i$  and  $\varepsilon_f$  given by (5). Let the wavefunction associated with initial and final states be denoted by  $\psi_i(\mathbf{r}_1, \sigma_1; \dots; \mathbf{r}_{Z+1}, \sigma_{Z+1})$  and  $\psi_f(\mathbf{r}_1, \sigma_1; \dots; \mathbf{r}_{Z+1}, \sigma_{Z+1})$ . These functions must be antisymmetric with respect to the interchange of any pair  $(\mathbf{r}_i, \sigma_i)$  and  $(\mathbf{r}_j, \sigma_j)$ , where  $\sigma_j$  denotes the spin of the  $j$ th electron. In the nonrelativistic theory, the total spin of the system is a constant of motion, and the spin state of the system is not altered by the collision. However, the symmetry with respect to the interchange of the position coordinates of the wavefunction depends upon the symmetry properties of the spin state. Therefore, the spin state of the system affects indirectly the intensity of the radiation (exchange effects). In order to discuss the exchange effects in a most simple way, we focus our attention to the hydrogen atom. However, the other effects are calculated for an arbitrary atom.

In the case of a two-electron system, there are two possible spin states: a triplet and a singlet state. The triplet state is symmetric, whereas the singlet state is antisymmetric with respect to the interchange of spins. Therefore, the coordinate wavefunction is antisymmetric in the triplet state while it is symmetric in the singlet state. Let us denote the symmetrized initial and final wavefunctions by

$$\psi_i^\pm(\mathbf{r}_1, \mathbf{r}_2), \quad \psi_f^\pm(\mathbf{r}_1, \mathbf{r}_2),$$

where the superscripts (+) and (−) indicate a symmetric and an antisymmetric wavefunction, respectively. Using symmetrized wavefunctions one can modify the intensity formula (1) as follows:

$$S_{i \rightarrow f}^\pm = (4\omega^4 e^2 / 3c^3) |\langle \psi_f^\pm | \mathbf{R} | \psi_i^\pm \rangle|^2. \quad (8)$$

In the case of an unpolarized incident electron, the probabilities of finding the system in a triplet and singlet state are  $\frac{3}{4}$  and  $\frac{1}{4}$ . Hence,  $S_{i \rightarrow f}^+$  and  $S_{i \rightarrow f}^-$  must be combined by the ratio 1 to 3 to yield the total intensity

$$S_{i \rightarrow f} = \frac{3}{4} S_{i \rightarrow f}^+ + \frac{1}{4} S_{i \rightarrow f}^-. \quad (9)$$

The symmetrized wavefunction can be constructed from an unsymmetrized wavefunction as

$$\psi^*(\mathbf{r}_1, \mathbf{r}_2) = [(1 \pm P_{12})/\sqrt{2}]\psi(\mathbf{r}_1, \mathbf{r}_2), \quad (10)$$

where  $P_{12}$  is the exchange operator. The matrix element of  $\mathbf{R}$  between two symmetrized wavefunctions can be expressed in terms of the unsymmetrized wavefunction as

$$\langle \psi_f^* | \mathbf{R} | \psi_i^* \rangle = \langle \psi_f | \mathbf{R} | \psi_i \rangle \pm \langle \psi_f | \mathbf{R} P_{12} | \psi_i \rangle, \quad (11)$$

where we have used the fact that the exchange operator  $P_{12}$  commutes with  $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$ , and that

$$(1 \pm P_{12})^2 = 2(1 - P_{12}). \quad (12)$$

Substituting (11) into (8) and combining the resulting equation with (9), one obtains

$$S_{i \rightarrow f}(\omega) = (4\omega^4 e^2 / 3c^3) [ |\langle \psi_f | \mathbf{R} | \psi_i \rangle|^2 + |\langle \psi_f | \mathbf{R} P_{12} | \psi_i \rangle|^2 - R_e \langle \langle \psi_f | \mathbf{R} | \psi_i \rangle^* \cdot \langle \psi_f | \mathbf{R} P_{12} | \psi_i \rangle \rangle ]. \quad (13)$$

The last two terms accounts for the exchange effects.

We now attempt to determine the wavefunction  $\psi_f$  and  $\psi_i$  in order to compute the matrix elements appearing in (13). For this purpose, one<sup>2,3</sup> expands  $\psi(\mathbf{r}_1, \mathbf{r}_2)$  into the atomic wavefunctions  $\Phi_n$ ,

$$\psi_{f,i} = \sum_n F_n^{f,i}(\mathbf{r}_2) \Phi_n(\mathbf{r}_1), \quad (14)$$

where  $F_n(\mathbf{r}_2)$  are the state functions of the scattered electron when the atom is in the state  $\Phi_n(\mathbf{r}_1)$ , and satisfies

$$[\nabla^2 + (k_n^{f,i})^2 - U_{nn}] F_n^{f,i} = \sum_{n' \neq n} U_{nn'} F_{n'}^{f,i}. \quad (15)$$

In this equation, one defines

$$U_{nn'} = (2m/\hbar^2) \langle \Phi_n | V | \Phi_{n'} \rangle, \quad (16)$$

$$(k_n^{f,i})^2 = k_{f,i}^2 - (2m/\hbar^2)(E_n - E_0), \quad (17)$$

where  $E_n$  is the energy of atomic state  $|\Phi_n\rangle$ .

In (14), the summation over  $n$  includes the integration over continuous spectrum also.

One observes in (17) that  $(k_n^{f,i})^2 < 0$  for all  $n \neq 0$ , and only  $(k_0^{f,i})^2 > 0$ , since the incident electron energy is assumed to be insufficient for excitation of the atom. In other words, the interaction of the electron with the atom is an elastic scattering collision. The solution of (15) with the asymptotic condition

$$F_n = \exp(i\mathbf{k} \cdot \mathbf{r}) \delta_{n0} + f_{n0}(\boldsymbol{\Omega}) \exp(ikr)/r \quad (18)$$

is standard in the study of elastic scattering of

electrons by neutral atoms<sup>2,3</sup> and is not repeated here. The relevant results are<sup>2</sup>

$$F_n^f \approx -[V_{nf}/(E_n - E_0)]F_0^f, \quad (19)$$

$$(\nabla^2 + k_f^2 - U_{ff} - U_p^f)F_0^f = 0, \quad (20)$$

where

$$U_p^f(\mathbf{r}_2) = - \sum_{n \neq f} \frac{|V_{nf}|^2}{E_n - E_0}, \quad (21a)$$

$$U_{ff}(\mathbf{r}_2) = (2m/\hbar^2) \langle \Phi_f | V | \Phi_f \rangle. \quad (21b)$$

Similar equations are obtained for  $F_n^i(\mathbf{r}_2)$  by replacing the index  $f$  by  $i$ . The symbols  $|\Phi_i\rangle$  refer to the initial and final ground states of the atom belonging to the energy  $E_0$ . They may differ from each other in their magnetic quantum numbers, viz.,  $|\Phi_f\rangle = |\tau_i J_i M_f\rangle$  and  $|\Phi_i\rangle = |\varepsilon_i J_i M_i\rangle$ , where  $J_i$  and  $M_i$  refer to the total orbital angular momentum and its projection, respectively. The  $\tau_i$  denote the remaining quantum number describing the ground state of the atom. When the latter is an  $s$  state, i.e., when  $J_i = 0$  and thus  $M_i = M_f = 0$ , there is no distinction between the initial and final ground states. Such atoms are referred to as spherically symmetric. Although the hydrogen atom which is being used for the discussion of the exchange effect is spherically symmetric, we retain the distinction between the initial and final ground states, because most of the results here are used for an arbitrary atom. It is to be noted that the additional potential energy  $U_p^f(\mathbf{r}_2)$  in (20) represents<sup>2</sup> the effect of the polarization of the atom by the field of the incident electron when the atom is in the state  $|\Phi_f\rangle$ . The potential  $U_{ff}(\mathbf{r}_2)$  represents the mean potential, or the "rigid" potential of the atom in the state  $|\Phi_f\rangle$ . Thus, (20) yields the wavefunction of the electron in the potential field  $U_{ff}(\mathbf{r}_2) + U_p^f(\mathbf{r}_2)$ .

We now return to the calculation of the matrix element  $\langle \psi_f | \mathbf{R} | \psi_i \rangle$  appearing in (13). Substitution of the expansion of  $\psi_f^f$  and  $\psi_i^i$  in (14) and the use of the orthogonality of  $\Phi_n$ 's yield

$$\langle \psi_f | \mathbf{R} | \psi_i \rangle = \sum_{n,n'} \langle F_n^f | F_{n'}^i \rangle \langle \Phi_n | \mathbf{r}_1 | \Phi_{n'} \rangle + \sum_n \langle F_n^f | \mathbf{r}_2 | F_n^i \rangle. \quad (22)$$

Note that the functions  $F_n^f$  and  $F_n^i$  are essentially the expansion coefficients in (14) and are not orthogonal.

The double sum in (22) contains the matrix elements of the dipole operator  $e\mathbf{r}_1$  associated with the bound electron and represents the induced dipole radiation. Similarly, the second term in (22) contains

<sup>2</sup> G. G. Drukarev, *The Theory of Electron-Atom Collisions* (Academic Press Inc., New York, 1965).

<sup>3</sup> N. F. Mott and H. W. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, England, 1950), 2nd ed.

the matrix elements of the dipole moment of the incident electron between various states of the scattered electron and represents the bremsstrahlung in the field of the neutral atom. The cross term which appears in the expression of  $|\langle \psi_f | \mathbf{R} | \psi_i \rangle|^2$  accounts for the interference of the induced dipole radiation and the bremsstrahlung.

The second and third terms in (13) represent the exchange effect as indicated by the presence of the exchange operator  $P_{12}$ .

It is in order to mention at this point the simplification introduced by considering the hydrogen atom for the discussion of the exchange effect. The crucial problem in the discussion of the exchange effect is the construction of the  $(Z + 1)$ -electron functions from the  $Z$  electron and the incident electron wavefunctions. This problem has been discussed in detail for an arbitrary atom in Ref. 2. The coordinate wavefunction for the system of the electron plus the atom for a given total spin is rather complicated even in the case of the helium atom. It has a simple form only in the case of a 2-electron system already indicated by (10). Since the exchange effect is expected to be small as far as the radiation intensity at low electron energies is concerned, its inclusion for an arbitrary atom is considered as an unwarranted complication in the present analysis. However, the magnitude of the error due to the neglect of the exchange effect is estimated quantitatively in the case of hydrogen atom as a guide by considering the last two terms in (13).

### III. BREMSSTRAHLUNG

The radiation due to the deceleration of the incident electron by the neutral atom is represented in (22) by

$$\sum_n \langle F'_n | \mathbf{r} | F'_n \rangle,$$

where we replaced  $\mathbf{r}_2$  by  $\mathbf{r}$ , which we recall refers to the position of the incident electron. Since the energy of the latter is insufficient to excite the atom, the dominant contribution comes from the first term

$$\mathbf{I}_b \equiv \langle F'_0(\mathbf{r}) | \mathbf{r} | F'_0(\mathbf{r}) \rangle, \quad (23)$$

where  $F'_0(\mathbf{r})$  and  $F_0(\mathbf{r})$  are the wavefunctions of the scattered electron when the atom is in the initial and final states and satisfy (20). They can be expanded into spherical harmonics as

$$F'_0(\mathbf{r}) = 4\pi \sum_{l,m} (i)^l F_l(k_f r) Y_{lm}(\hat{\mathbf{k}}_f) Y_{lm}^*(\hat{\mathbf{r}}), \quad (24)$$

where  $F_l(k_f r)$  are the solution of the radial Schrö-

ding equation. Substituting (24) and the similar expansion for  $F_0(\mathbf{r})$  into (23), performing the angular integration and retaining only the terms in the resulting equation containing the product  $F_0 F_1$ , one obtains

$$\mathbf{I}_b = 4\pi i \left( \hat{\mathbf{k}}_i \int_0^\infty dr r^3 F_0^*(k_f r) F_1(k_i r) - \hat{\mathbf{k}}_f \int_0^\infty dr r^3 F_1^*(k_f r) F_0(k_i r) \right), \quad (25)$$

where  $\mathbf{k}_i = \mathbf{k}_i/k_i$  and  $\mathbf{k}_f = \mathbf{k}_f/k_f$ . As pointed out by Firsov and Chibisov,<sup>1</sup> the terms involving  $F_l$  for  $l \geq 2$  correspond to electrons at a large distance from the atom at low energies and do not interact appreciably with it. Therefore only the terms in (25) are of significance. Following this reference, we use

$$F_0(kr) = \sin(kr + \delta_0)/kr, \quad (26a)$$

$$F_1(kr) = j_1(kr), \quad (26b)$$

where  $\delta_0$  is the phase shift of the  $s$  wave and  $j_1(kr)$  is the spherical Bessel function. We ignore the phase shift  $\delta$ , associated with the  $p$  wave which is justified at small incident-electron energies. The phase angle  $\delta_0$  is a function of  $k$ , and represents the interaction of the electron with the atom. It is related to the elastic-scattering cross section by

$$\sigma(k) = (4\pi/k^2) \sin^2 \delta_0(k). \quad (27)$$

Substituting (26) into (25) and performing the indicated integrals, one obtains

$$\mathbf{I}_b = 4\pi^{\frac{1}{2}} i (\hbar/2m\omega)^2 \{ [\sigma(k_f)]^{\frac{1}{2}} \hat{\mathbf{k}}_i - [\sigma(k_i)]^{\frac{1}{2}} \hat{\mathbf{k}}_f \}. \quad (28)$$

This can be further simplified if the cross section does not change appreciably in the region  $(0 - k_i)$  then  $\sigma(k_f) \approx \sigma(k_i) \approx \sigma(0)$ , and

$$\mathbf{I}_b = 4\pi^{\frac{1}{2}} i (\hbar/2m\omega)^2 [\sigma(0)]^{\frac{1}{2}} \mathbf{q}, \quad (29)$$

where

$$\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f. \quad (30)$$

The intensity of the bremsstrahlung alone can be calculated substituting (29) into (13), multiplying the resulting equation by the density of final electron states per unit electron energy, viz.,  $\sqrt{2}(m/\hbar^2)^{\frac{1}{2}}(E_f)^{\frac{1}{2}}/(2\pi)^3$ , and integrating over the direction of  $\mathbf{k}_f$ , one obtains

$$S_b(\hbar\omega) = N_a \frac{4\sqrt{2}}{3\pi} \alpha \sigma(0) c \left( \frac{E_i}{mc^2} \right)^{\frac{1}{2}} \left( 2 - \frac{\hbar\omega}{E_i} \right) \left( 1 - \frac{\hbar\omega}{E_i} \right)^{\frac{1}{2}}, \quad (31)$$

where  $\alpha$  is the fine structure constant ( $e^2/\hbar c$ ),  $E_i$  is the incident electron energy, and finally  $N_a$  is the number of neutral atoms per unit volume.

The spectral density for a Maxwellian electron distribution is obtained from (31) by averaging it with  $2\pi(\pi\Theta)^{-1}N_e E_i^{\frac{1}{2}} \exp(-E_i/\Theta)$  as

$$\bar{S}_b(\hbar\omega) = N_a N_e \frac{4\alpha}{3\pi} \left(\frac{2}{\pi m\Theta}\right)^{\frac{1}{2}} \cdot \frac{\sigma(0)}{m} \left(\frac{\hbar\omega}{c}\right)^2 K_2\left(\frac{\hbar\omega}{2\Theta}\right) e^{-\hbar\omega/2\Theta}, \quad (32)$$

where  $N_e$  is the number of electrons per unit volume,  $\Theta$  is the temperature of the gas, and  $K_2(x)$  is the modified Hankel function. Equation (32) gives the intensity of bremsstrahlung per unit energy from a unit volume of gas containing  $N_a$  atoms and  $N_e$  electrons. The ratio of  $N_a/N_e$  can be obtained from the Saha equation at the specified temperature.

The total radiated power in all energies is obtained by integrating (32) over  $\hbar\omega$ . The result is

$$\bar{S}_b = \frac{256\alpha}{15\pi} \frac{\sigma(0)}{mc^2} \left(\frac{2\Theta}{\pi m}\right)^{\frac{1}{2}} \Theta^2 N_a N_e. \quad (33)$$

$$I_d = - \sum_{n \neq 0} \frac{\langle F'_0 | V_{fn} | F'_0 \rangle \langle \phi_n | \mathbf{D} | \phi_i \rangle + \langle F'_0 | V_{ni} | F'_0 \rangle \langle \phi_f | \mathbf{D} | \phi_n \rangle}{E_n - E_0}, \quad (36)$$

where

$$V = -\frac{Ze^2}{r} + \sum_{j=1}^Z \frac{e^2}{|\mathbf{r} - \mathbf{r}_j|}.$$

In order to calculate the integration with respect to  $\mathbf{r}$  appearing in  $\langle F'_0 | V_{fn} | F'_0 \rangle$  and  $\langle F'_0 | V_{ni} | F'_0 \rangle$ , we approximate  $F'_0$  and  $F'_i$  by plane waves. The result is

$$I_d = -i \frac{4\pi e^2}{q^2} \sum_{n \neq 0} \frac{\langle \phi_f | \mathbf{q} \cdot \mathbf{D} | \phi_n \rangle \langle \phi_n | \mathbf{D} | \phi_i \rangle + \langle \phi_n | \mathbf{q} \cdot \mathbf{D} | \phi_i \rangle \langle \phi_f | \mathbf{D} | \phi_n \rangle}{E_n - E_0}. \quad (38)$$

Using Wigner-Echart theorem, one evaluates (38) as follows:

$$I_d = -i \frac{4\pi}{q} \left\{ \begin{array}{l} \alpha_0(M_i) \hat{e}_z, \quad \text{for } M_f = M_i \\ \beta_+(-\hat{e}_x + i\hat{e}_y), \quad \text{for } M_f = M_i + 1 \\ \beta_-(-\hat{e}_x + i\hat{e}_y), \quad \text{for } M_f = M_i - 1 \end{array} \right\}, \quad (39)$$

where  $\hat{e}_z \parallel \hat{q}$ , and where

$$\alpha_0(M_i) \equiv a_0 - b_0 M_i^2 \equiv 2e^2 \sum_{n \neq 0} \frac{|\langle \phi_i | D_z | \phi_n \rangle|^2}{E_n - E_0}, \quad (40)$$

#### IV. INDUCED DIPOLE RADIATION

The dominant contribution to the double sum

$$\sum_{nn'} \langle F'_n | F'_n \rangle \langle \phi_n | \mathbf{r}_1 | \phi_{n'} \rangle$$

in (22), representing the induced dipole radiation comes from terms for which either  $n = 0$  or  $n' = 0$ ,

$$I_d = \sum_n [\langle F'_0 | F'_n \rangle \langle \phi_f | \mathbf{D} | \phi_n \rangle + \langle F'_n | F'_0 \rangle \langle \phi_n | \mathbf{D} | \phi_i \rangle], \quad (34)$$

where

$$\mathbf{D} = \sum_{i=1}^Z \mathbf{r}_i. \quad (35)$$

In (34) we replaced  $\mathbf{r}_1$  by  $\mathbf{D}$  such that the subsequent analysis is valid for an arbitrary atom. The neglect of the terms for which both  $n \neq 0$  and  $n' \neq 0$  can be justified by observing that they contain the product of two interaction potentials, viz.,

$$V_{nf} V_{in'} / (E_n - E_0)(E_{n'} - E_0),$$

whereas the terms in (34) are proportional to  $V_{ni} / (E_n - E_0)$ . Substituting  $F'_n$  and  $F'_i$  from (19) into (34) one obtains

$$\begin{aligned} \langle F'_0 | V_{fn} | F'_0 \rangle &\cong \frac{4\pi e^2}{q^2} \langle \phi_f | -Z + \sum_i \exp(i\mathbf{q} \cdot \mathbf{r}_i) | \phi_n \rangle \\ &\approx i \frac{4\pi e^2}{q^2} \mathbf{q} \cdot \langle \phi_f | \mathbf{D} | \phi_n \rangle, \end{aligned} \quad (37)$$

where the last step is obtained by approximating  $\exp(i\mathbf{q} \cdot \mathbf{r}_i)$  by  $1 + i\mathbf{q} \cdot \mathbf{r}_i$ . This is justified when  $\mathbf{q} \cdot \mathbf{r}_i \ll 1$  which is the case for low-incident electron energies. Substituting (37) into (36) yields

$$\begin{aligned} \beta_+(M_i) &\equiv -\beta_-(-M_i) \\ &\equiv \frac{1}{4} b_0 (2M_i + 1) [(J_i - M_i)(J_i + M_i + 1)]^{\frac{1}{2}}. \end{aligned} \quad (41)$$

The expression  $a_0$  and  $b_0$  in terms of reduced matrix elements of  $\mathbf{D}$  can be obtained from (40). In the case of a spherical atom for which  $J_i = M_i = 0$ ,

$$I_d = -i(4\pi/q^2) a_0 q. \quad (42)$$

The quantity  $a_0$  is the polarizability of the atom, which is  $(9/2)r_0^3$  for hydrogen.<sup>2</sup>

The intensity of the induced dipole radiation alone is obtained by substituting (39) into (13),

summing over  $M_f$ , averaging over  $M_i$ , and integrating over  $\mathbf{k}_f$ ,

$$S_d(\hbar\omega) = N_a \frac{8\alpha}{3} \left(\frac{E_i}{\hbar c}\right)^4 \left(\frac{2mc^4}{E_i}\right)^{\frac{1}{2}} \eta^2 x^4 \cdot \ln \left( \frac{1 + (1-x)^{\frac{1}{2}}}{1 - (1-x)^{\frac{1}{2}}} \right), \quad (43)$$

where

$$\eta^2 \equiv \frac{1}{2J_i + 1} \sum_{M_i} (\alpha_0^2 + 2\beta_+^2 + 2\beta_-^2) = a_0^2 + b_0 \cdot \left( -2a_0 + \frac{b_0}{2} (2J_i^2 + 2J_i - 1) \right) \frac{J_i(J_i + 1)}{3}, \quad (44)$$

and where  $x = (\hbar\omega/E_i)$ .

One observes in (43) that the spectrum vanishes when  $\hbar\omega \rightarrow 0$  in contrast to bremsstrahlung of the incident electron given by (31). Certainly both spectra vanish at  $\hbar\omega = E_i$ . The spectrum of the dipole radiation has a peak at approximately  $\hbar\omega = 0.97E_i$ . The major portion of the total induced dipole radiation for a given electron energy  $E_i$  is emitted in the frequency range under this peak.

The spectrum for a Maxwellian distribution of incident electron energy is found as

$$\bar{S}_d(\hbar\omega) = N_a N_e \frac{1}{3} \alpha \left(\frac{\omega}{c}\right)^4 \left(\frac{2mc^4}{\pi\Theta}\right)^{\frac{1}{2}} \eta^2 e^{-\pi\omega/2\Theta} K_0 \left(\frac{\hbar\omega}{2\Theta}\right) \quad (45)$$

and the total radiated power

$$\bar{S}_d = N_a N_e \frac{2^{15}\alpha}{315} \left(\frac{2mc^4\Theta}{\pi}\right)^{\frac{1}{2}} \left(\frac{\Theta}{\hbar c}\right)^4 \eta^2. \quad (46)$$

It is interesting to compare the total radiated power in the case of bremsstrahlung and induced dipole radiation. From (33) and (46) one obtains their ratio as

$$\frac{\bar{S}_d}{\bar{S}_b} = \pi \frac{2^7}{21} \frac{\eta^2}{\sigma(0)} \left(\frac{mc^2}{\hbar c}\right)^2 \Theta^2. \quad (47)$$

The magnitude of  $\eta$  and  $\sigma(0)$  are of the order of  $10^{-24}\text{cm}^3$  and  $10^{-16}\text{cm}^2$ , respectively, for most atoms. For example  $\eta = 2.2 \times 10^{-24}\text{cm}^3$  and  $\sigma(0) = 48 \times 10^{-16}\text{cm}^2$  for hydrogen.<sup>2</sup> For  $\Theta = 0.5\text{eV}$  which corresponds to a temperature 5000 °K this ratio is less than 8% (for hydrogen 0.7%) indicating that the induced dipole radiation will be insignificant in most cases. This statement is particularly true in the microwave range of the spectrum because of the difference in the shape of the spectrum in the two cases as discussed above. This conclusion is at variance with that given in Ref. 1 by classical arguments.

## V. INTERFERENCE EFFECTS

The interference of the dipole radiation and the bremsstrahlung is determined by the cross term in the expression of  $|\mathbf{I}_d + \mathbf{I}_b|^2 = |\mathbf{I}_b|^2 + |\mathbf{I}_d|^2 + 2R_e(\mathbf{I}_d \cdot \mathbf{I}_b^*)$ . We have already discussed the first two terms. Using (29) and (39) we find that

$$I_{\text{int}} = 2R_e(\mathbf{I}_d \cdot \mathbf{I}_b^*) = -32\pi^{\frac{3}{2}} (\hbar/2m\omega)^2 \alpha_0(M_i) [\sigma(0)]^{\frac{1}{2}}. \quad (48)$$

Note that there is no contribution for  $M_f \neq M_i$ . The interference correction to the radiation intensity is obtained by averaging  $I_{\text{int}}$  over  $M_i$ , multiplying it by the density of final electron states, integrating over  $\mathbf{k}_f$ , and finally substituting the resulting expression into (13). The spectrum for a given electron  $E_i$ , the averaged spectrum and the total radiated power are calculated as

$$S_{\text{int}}(\hbar\omega) = -\frac{16\sqrt{2}}{3\pi^{\frac{1}{2}}} \alpha \left(\frac{\omega}{c}\right)^2 \bar{\alpha}_0 \left(\frac{\sigma(0)}{m}\right)^{\frac{1}{2}} (E_i - \hbar\omega)^{\frac{1}{2}}; \quad (49)$$

$$\bar{S}_{\text{int}}(\hbar\omega) = -\frac{16\sqrt{2}}{3\pi} \alpha \frac{\bar{\alpha}_0 [\sigma(0)]^{\frac{1}{2}}}{\hbar^2 c^2 (m\Theta)^{\frac{1}{2}}} (\hbar\omega)^3 e^{-\pi\omega/2\Theta} K_1 \left(\frac{\hbar\omega}{2\Theta}\right), \quad (50)$$

$$\bar{S}_{\text{int}} = -\frac{2^{12}\sqrt{2}}{105\pi} \alpha \frac{\Theta^{7/2}}{\hbar^2 c^2 m^{\frac{1}{2}}} \bar{\alpha}_0 [\sigma(0)]^{\frac{1}{2}}, \quad (51)$$

where

$$\bar{\alpha}_0 = \frac{1}{2J_i + 1} \sum_{M_i} \alpha_0(M_i) = a_0 - \frac{1}{3} b_0 J_i (J_i + 1). \quad (52)$$

In performing the average over  $M_i$  we have approximated the average of  $\alpha_0(M_i) [\sigma(0)]^{\frac{1}{2}}$  by the product of the averages. When the atom is spherically symmetric in the ground state this approximation becomes unnecessary because there is no  $M_i$  dependence in  $\alpha_0(M_i) = a_0$  and  $[\sigma(0)]^{\frac{1}{2}}$ .

We now compare the combined effect of the induced dipole radiation and the interference term to the bremsstrahlung intensity. Using (33), (46), and (51) we obtain

$$\frac{\bar{S}_d + \bar{S}_{\text{int}}}{\bar{S}_b} = \frac{2^7\pi}{21} \frac{\bar{\alpha}_0^2}{\sigma(0)} \left(\frac{mc^2}{\hbar c}\right)^2 \Theta \cdot \left[ \Theta - \frac{3}{8\pi^{\frac{1}{2}}} \left(\frac{\hbar^2 c^2}{mc^2}\right) \frac{[\sigma(0)]^{\frac{1}{2}}}{\bar{\alpha}_0} \right]. \quad (53)$$

It is interesting to observe that this ratio depends on the type of the neutral atom only through the ratio  $[\bar{\alpha}_0^2/\sigma(0)]$ . To estimate the relative error we again use  $\bar{\alpha}_0 \approx 10^{-24}\text{cm}^3$  and  $\sigma(0) \approx 10^{-16}\text{cm}^2$  as

typical values in (53) and obtain  $0.3 \Theta(\Theta - 1.7)$ , where  $\Theta$  is in electron volts. In the range of validity of the foregoing derivations, which require the incident electron energy to be small, i.e.,  $\Theta < 1$  eV, one finds that the effect of the interference term on the radiation intensity is more important than the induced dipole radiation alone. It tends to decrease the total intensity as indicated by its negative sign. For  $\Theta = 0.5$  eV, the above ratio becomes 18% (for hydrogen it is less than 3%) which is probably an upper estimate for many atoms. It can be concluded that the dipole radiation and its interference with the bremsstrahlung, which are associated with the polarization of the atom by the field of the incident electron, are insignificant as far as the total radiation intensity is concerned. The relative error at a given frequency can be easily discussed with the foregoing formulas for the various spectra. At low photon energies the spectrum of the bremsstrahlung is flat, whereas that of the induced dipole radiation and the interference term decreases as  $\hbar^2 \omega^2$ . Hence, the emission due to the polarization of the atom can be ignored at low photon energies.

It must be pointed out at this point that the polarization of the atom affects the intensity of the bremsstrahlung considerably. However this effect is taken into account through the elastic scattering cross section  $\sigma(0)$ , which is to be calculated from the asymptotic form of  $F_0(\mathbf{r})$ . The latter is the solution of (20) which includes the effect of polarization of the atom through the additional interaction energy  $U_p(\mathbf{r})$ . The elastic scattering cross section decreases<sup>2</sup> when the effect of polarization of the atom is added to the rigid potential scattering. Hence, the intensity of the bremsstrahlung is also smaller when the atom is polarizable, than when it is rigid, by a ratio which may be as high as  $\frac{1}{2}$  as is the case for hydrogen.<sup>2</sup>

VI. EXCHANGE EFFECTS

This section is devoted to the investigation of the exchange effects in the calculations of the bremsstrahlung intensity for the hydrogen atom. The matrix element associated with the exchange effect in (13) is  $\langle \psi_f(\mathbf{r}_1, \mathbf{r}_2) | \mathbf{r}_1 + \mathbf{r}_2 | \psi_i(\mathbf{r}_2, \mathbf{r}_1) \rangle$ . To evaluate this matrix element we substitute the expansion (14) for  $\psi_f$  and  $\psi_i$  and retain the terms for which  $n = 0$  and  $n' = 0$ ,

$$I_e = \langle F'_0(\mathbf{r}_2) | \Phi(\mathbf{r}_2) \rangle \langle \Phi_0(\mathbf{r}_1) | \mathbf{r}_1 | F'_0(\mathbf{r}_1) \rangle + \langle F'_0(\mathbf{r}_2) | \mathbf{r}_2 | \Phi_0(\mathbf{r}_2) \rangle \langle \Phi_0(\mathbf{r}_1) | F'_0(\mathbf{r}_1) \rangle. \quad (54)$$

In order to calculate  $\gamma \equiv \langle F'_0 | \Phi_0 \rangle$  we use the ex-

pansion of  $F'_0$  into spherical harmonics given by (24). Using the fact that  $\Phi_0(\mathbf{r}_2)$  is a function of  $|\mathbf{r}_2|$ , one obtains

$$\gamma = \langle F'_0 | \Phi_0 \rangle = 4\pi \int_0^\infty F_0^*(k,r) \Phi_0(r) r^2 dr.$$

Substituting  $F_0(kr) = \sin(kr + \delta_0)/kr$  as before one gets

$$\gamma = \frac{4\pi}{k_f} \left[ \cos \delta_0 \int_0^\infty r dr \Phi_0(r) \sin k_f r + \sin \delta_0 \int_0^\infty r dr \Phi_0(r) \cos k_f r \right].$$

Since we are dealing with slow electrons whose wavelength is larger than the size of the atom,  $k_f r \ll 1$ . Then, using  $\cos \delta_0 \approx 1$ ,  $\sin k_f r \approx k_f r$ , and  $\cos k_f r \approx 1$ , we find

$$\gamma = 4\pi \int_0^\infty r^2 dr \Phi_0(r) + [4\pi\sigma(0)]^{\frac{1}{2}} \int_0^\infty r dr \Phi_0(r). \quad (55)$$

For the hydrogen atom, the wavefunction of the ground state is  $\Phi_0(r) = (\pi r_0^3)^{\frac{1}{2}} \exp(-r/r_0)$ , where  $r_0 = \hbar^2/(me^2) = 5.3 \times 10^{-9}$  cm, the Bohr radius. Hence for the hydrogen atom

$$\gamma = 2r_0^{\frac{1}{2}} \{4r_0\pi^{\frac{1}{2}} + [\sigma(0)]^{\frac{1}{2}}\}. \quad (56)$$

In Ref. 2, the elastic scattering cross section is given as  $\sigma(0) \approx 60\pi r_0^2$ . With this value of  $\sigma(0)$ , we obtain  $\gamma \cong 23.48r_0^{\frac{1}{2}}\pi^{\frac{1}{2}}$ .

Next we consider  $\langle \Phi_0 | \mathbf{r} | F'_0 \rangle$ . Using again the expansion of  $F'_0$  into spherical harmonics, and the fact that  $\Phi_0$  is a function of  $|\mathbf{r}|$  only, we find

$$\langle \Phi_0 | \mathbf{r} | F'_0 \rangle = i4\pi \mathbf{k}_i \int_0^\infty F_1(k,r) \Phi_0(r) r^3 dr.$$

Putting  $F_1(k,r) = j_1(k,r)$  and using the asymptotic form of  $j_1(k,r)$  for small arguments we get

$$\langle \Phi_0 | \mathbf{r} | F'_0 \rangle = i\frac{4}{3}\pi \delta \mathbf{k}_i, \quad (57)$$

where

$$\delta = \int_0^\infty r^4 \Phi_0(r) dr. \quad (58)$$

For hydrogen atom,  $\delta = 4!r_0^3(r_0/\pi)^{\frac{1}{2}}$ .

Repeating similar calculations for the second term in (54), we obtain the desired matrix element as

$$I_e = i\mu \mathbf{q}, \quad (59)$$

where

$$\mu = \frac{4}{3}\pi\gamma \delta = 64\pi r_0^4 \{4r_0 + [\sigma(0)/\pi]^{\frac{1}{2}}\}. \quad (60)$$

The corresponding radiation intensity is obtained as

$$S_c(\hbar\omega) = N_a \frac{2\sqrt{2}}{3\pi^3} \left(\frac{m}{\hbar^2}\right)^{5/2} \frac{\omega^4 e^2}{c^3} \mu^2 \cdot (2E_i - \hbar\omega)(E_i - \hbar\omega)^{\frac{1}{2}}. \quad (61)$$

One observes again as in the case of dipole radiation that the spectrum vanishes when  $\hbar\omega \rightarrow 0$ . The total power for a maxwellian distribution of incident electron energy follows from (61) as

$$\bar{S}_c = 4.54 \times 10^{-2} \frac{4}{3\pi^2} \alpha \frac{(2m\Theta)^{\frac{1}{2}}}{\pi} \mu^2 \frac{m^2 \Theta^5}{c^2 \hbar^8} N_a N_e. \quad (62)$$

Using the value of the elastic-scattering cross section given in Ref. 2, i.e.,  $\sigma(0) = 60\pi r_0^2$ , one finds that the ratio  $(\bar{S}_c/\bar{S}_b) \approx 10^{-6}\Theta^4$ , where  $\Theta$  is in electron volts. Hence the contribution of this term is negligible for  $\Theta < 1$  eV.

As a final step, we consider the last term  $-2 R_s[(\psi_f | \mathbf{R} | \psi_i)^* \cdot (\psi_f | \mathbf{R} P_{12} | \psi_i)]$  in (13), which is equal to  $-2 R_s[(\mathbf{I}_d + \mathbf{I}_b) \cdot \mathbf{I}^*]$ . Ignoring the induced dipole term  $\mathbf{I}_d$  one finds the spectrum associated with this term as

$$S_{c, \text{int}}(\hbar\omega) = -\frac{2\sqrt{2}}{3\pi^2} \frac{\alpha}{\pi^{\frac{1}{2}}} \mu[\sigma(0)]^{\frac{1}{2}} \hbar^2 \omega^2 (2E_i - \hbar\omega) \cdot (E_i - \hbar\omega)^{\frac{1}{2}}. \quad (63)$$

The total average power is

$$\bar{S}_{c, \text{int}} = -\frac{2^{11}\sqrt{2}}{63\pi^3} \alpha \mu[\sigma(0)]^{\frac{1}{2}} \left(\frac{m\Theta}{c^4 \hbar^8}\right)^{\frac{1}{2}} \Theta^4 N_a N_e. \quad (64)$$

The ratio of this term to the intensity of the bremsstrahlung is approximately equal to  $\frac{1}{10}\Theta^2$  where  $\Theta$  is in electron volts. Hence, for  $\Theta < 1$ , this ratio is less than 10%, and decreases rapidly with the gas temperature.

The final conclusion is that the exchange effects can be ignored completely in estimating the intensity as well as the absorption of radiation in a neutral gas containing slow electrons.

## VII. THE EFFECTIVE ABSORPTION COEFFICIENT DUE TO FREE-FREE TRANSITIONS

We have seen that, for the low electron energy range, the only processes responsible for transitions between states of the whole system (neutral atom + electron + radiation field) are bremsstrahlung and inverse bremsstrahlung. It is possible to use the energy intensity of emission due to bremsstrahlung of Eq. (32) to calculate the effective absorption coefficient for radiation. This calculation is now performed.

If it is assumed that the medium is isotropic, then

the effective absorption coefficient for unpolarized photons is<sup>4</sup>

$$\alpha_{\text{eff}} = \frac{1}{8\pi c} \sum_{\lambda} \int d\Omega_{\mathbf{k}} [\epsilon_{\lambda}(\mathbf{k}) - \alpha_{\lambda}(\mathbf{k})], \quad (65)$$

where  $\lambda$  is the index of the polarization state,  $\epsilon_{\lambda}$  is the transition probability per unit time for emission of a photon of polarization  $\lambda$  into direction  $\Omega_{\mathbf{k}}$ , and  $\alpha_{\lambda}$  is the transition probability per unit time for absorption of a photon of polarization  $\lambda$  traveling in direction  $\Omega_{\mathbf{k}}$ .

It is possible to show that for a Maxwellian electron distribution the absorption and emission transition probabilities are related by the equation

$$\alpha_{\lambda}(\mathbf{k}) = e^{-\hbar\omega/\Theta} \epsilon_{\lambda}(\mathbf{k}); \quad (66)$$

also, the total radiation intensity  $S(\hbar\omega)$  per neutral atom per unit energy interval is related to  $\epsilon_{\lambda}(\mathbf{k})$  by

$$S(\hbar\omega) = \frac{\omega^3}{c^3(2\pi)^3} \int d\Omega_{\mathbf{k}} \sum_{\lambda} \epsilon_{\lambda}(\mathbf{k}). \quad (67)$$

The use of Eqs. (66) and (67) in Eq. (65) yields for the effective absorption coefficient

$$\alpha_{\text{eff}} = (c^3/\omega^3)\pi^2 S(\hbar\omega)(1 - e^{-\hbar\omega/\Theta}).$$

Substitution of  $S(\hbar\omega)$  which was obtained previously in Eq. (32) by ignoring polarizability and exchange effects and approximation of  $K_2(\hbar\omega/2\Theta)$  and exp  $(\hbar\omega/\Theta)$  in the result yields for  $\alpha_{\text{eff}}$

$$\alpha_{\text{eff}} = \frac{32}{3} \frac{\sqrt{2}}{\pi} \frac{e^2}{m^{\frac{1}{2}} c} \sigma(0)\Theta^{\frac{1}{2}} N_a N_e \frac{1}{\omega^2}. \quad (68)$$

It is interesting to note that, if one defines an "effective collision frequency" in the standard way [cf. Ref. 5 where  $\nu_{\text{eff}} = 8/3(\sqrt{2}/\pi)\sigma(0)N_a(\Theta^{\frac{1}{2}}/m)$ ], the absorption coefficient may be seen to agree with the power absorption coefficient  $\alpha_{\text{eff}}$  of the Maxwell-Lorentz theory<sup>5</sup> for nondispersive media and small collision frequencies ( $\nu_{\text{eff}}^2 \ll \omega^2$ )

$$\alpha_{\text{eff}} = (\nu_{\text{eff}}/c)(\omega_p^2/\omega^2),$$

where the plasma frequency

$$\omega_p^2 = 4\pi e^2 N_e/m.$$

## VIII. CONCLUSION

The present analysis indicates that the dominant contribution to the intensity of radiation from a neutral gas containing slow electrons comes from the deceleration of the free electrons by the field

<sup>4</sup> E. H. Klevans, The University of Michigan, Radiation Laboratory Report No. 2764-12-T (1962).

<sup>5</sup> J. M. Anderson and L. Goldstein, Phys. Rev. **100**, 1037 (1955).



of the neutral atoms. The contribution of the induced dipole radiation is always negligible. However, the interference of the induced dipole radiation with the bremsstrahlung may decrease the total intensity as much as 18%, depending on the ratio of the polarizability of the atom and its elastic scattering cross section for slow electrons with energies less than 1 eV. The induced dipole radiation may become appreciable in the short-wave limit of the emitted spectrum.

It is also found that the exchange effect in the calculation of radiation intensity for the hydrogen atom is much less than 10%, and can be ignored entirely. The same conclusion is expected to be also true for a multi-electron atom.

The intensity of the bremsstrahlung of the free electrons in the field of the neutral atoms which is the dominant emission mechanism is shown to be proportional to the elastic-scattering cross section of the atom for slow electrons in the limit of zero incident energy. This cross section includes the effect

of the polarization of the atom by the field of the incident free electron. Thus, the polarizability of the atom affects, and decreases, the intensity of the bremsstrahlung, although the induced dipole radiation which is due to the polarization of the atom is negligible.

In view of these results, it is concluded that the free-free absorption of the microwaves in a slightly ionized neutral gas is predominantly due to the inverse bremsstrahlung. The absorption coefficient has been found to be in good agreement, above the plasma frequency, with that given by the classical Lorentz formula of electromagnetic theory for non-dispersive media.

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