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Three-Dimensional Linear Transport Theory*

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A recent technique for extending the singular eigenfunction method in linear transport theory to problems which are not strictly 1-dimensional is compared to a more naive approach based on the Fourier transform. The latter appears to have advantages with regard to simplicity and directness.

INTRODUCTION

The 1-dimensional form of the time-independent one-speed transport equation with isotropic scattering is conventionally solved by means of the "singular-eigenfunction" technique.[1] Recently a method for extending this technique to problems which are not strictly 1-dimensional has been proposed by Kaper.[2] This method, although evidently limited in its applicability to problems in which the boundary *surfaces* are no more complicated than parallel planes, does allow boundary *conditions* which vary over these planes. Thus the "reduced" 3-dimensional form of the transport equation is used throughout, and one may handle, for example, point sources, rather than only plane sources.

The extension of the 1-dimensional theory to this wider class of problems is far from trivial. Indeed, a fairly elaborate mathematical framework, involving the theory of generalized analytic functions,[3] is required. The solutions obtained from this framework are initially in the form of complex 2-dimensional integrals, the reduction of which to more readily useful form is, again, nontrivial.

Our purpose here is to compare the above method to a much more naive approach based on the Fourier transform. We hope to show that the latter has at least as wide a range of applicability as the singular eigenfunction method, while at the same time requiring much less sophisticated mathematics. Moreover, the Fourier-transform method seems in several ways more direct; in particular, it provides solutions directly in a form suitable for evaluation.

Our comparison of the two methods will be effected by examining their respective applications to two simple problems. Thus, in Sec. 1, after some brief remarks on the method of Kaper, we begin with the solutions obtained in Ref. 2 for the infinite-space Green's function and half-space albedo problems. We show how the 2-dimensional integrals which these solutions involve are to be reduced to much simpler contour integrals. The Fourier-transform method, which yields solutions in contour integral form directly, is presented in Sec. 2. In the conclusion, we attempt to summarize the essential differences between the two techniques.

1. THE SINGULAR EIGENFUNCTION SOLUTIONS

The reduced transport equation has the form

$$\left(1 - iB_y\Omega_y - iB_z\Omega_z + \mu\frac{\partial}{\partial x}\right)\hat{N} = \frac{c}{4\pi}\int d\Omega'\hat{N}. \tag{1.1}$$

Here, $\hat{N}(x, B_y, B_z, \mathbf{\Omega})$ is related to the neutron angular density $N(\mathbf{r}, \mathbf{\Omega})$ either by means of the ansatz

$$\hat{N} = e^{-iB_y y - iB_z z}N, \tag{1.2}$$

in which case B_y and B_z are the transverse buckling constants, or by means of a 2-dimensional Fourier transform. In either case, notice that Eq. (1.1) limits us to considering only problems in which the boundaries may be taken to be independent of y and z. The vector $\mathbf{\Omega}$ refers, of course, to the normalized velocity

$$\mathbf{\Omega} = \mathbf{v}/|\mathbf{v}| \tag{1.3}$$

$$= (\mu, (1-\mu^2)^{\frac{1}{2}}\cos\theta, (1-\mu^2)^{\frac{1}{2}}\sin\theta) \tag{1.4}$$

$$= (\mu, \mathbf{\Omega}_{(x)}), \tag{1.5}$$

where, in Eq. (1.5), we have introduced the convention of denoting the y and z components of a vector by a subscript (x).

Kaper[2] observes that, if we define a transformation of variables

$$(\mu, \theta) \to (\xi, \eta) \tag{1.6}$$

by means of the complex variable

$$\zeta = \xi + i\eta \tag{1.7}$$

$$\equiv \mu/(1 - i\mathbf{B}_{(x)}\cdot\mathbf{\Omega}_{(x)}) \tag{1.8}$$

and, furthermore, define a new angular density ψ by

$$\psi(x, \xi, \eta) \equiv (1 - i\mathbf{B}_{(x)} \cdot \mathbf{\Omega}_{(x)})\hat{N}(x, \mathbf{B}_{(x)}, \mathbf{\Omega}), \quad (1.9)$$

then Eq. (1.1) may be written in the form

$$\left(1 + \zeta \frac{\partial}{\partial x}\right) \psi(x, \zeta) = \iint_G g(\zeta')\psi(x, \zeta')\, d\xi'\, d\eta'. \quad (1.10)$$

Here,

$$g(\zeta) = \frac{c}{4\pi} \left| \frac{\partial(\mu, \theta)}{\partial(\xi, \eta)} \right| [1 - i\mathbf{B}_{(x)} \cdot \mathbf{\Omega}_{(x)}(\zeta)] \quad (1.11)$$

and $G = G^+ \cup G^-$ is a certain figure-eight-shaped region of the complex plane. (See Fig. 1, which is reproduced here from Ref. 2 for convenience.) Note that the convenient notation

$$f(\xi, \eta) = f(\zeta) \quad (1.12)$$

is by no means intended to imply that f is an analytic function of ζ.

The evident similarity between Eq. (1.10) and the standard form of the 1-dimensional transport equation[1] is exploited in Ref. 2 by applying to Eq. (1.10) a variation of the singular-eigenfunction technique. Thus, one seeks eigenfunctions of the form

$$\psi_\nu(x, \zeta) = e^{-x/\nu}\phi(\nu, \zeta), \quad (1.13)$$

where $\phi(\nu, \zeta)$ is a generalized function of the complex variable ζ, defined for test functions with support in G. Discrete and continuum "modes" are obtained from Eqs. (1.10) and (1.13), and these are then shown to possess the usual ("half-range" and "full-range") completeness and orthogonality properties. Without attempting to reproduce the work of Kaper here, we wish to remark on a few of the differences between the singular-eigenfunction theory of Eq. (1.10) and the conventional 1-dimensional analysis.

Primary among these differences is the mathematical complexity of the 3-dimensional dispersion function

$$\Lambda(\nu) = 1 + \iint_G \frac{\nu g(\zeta)}{\zeta - \nu}\, d\xi\, d\eta. \quad (1.14)$$

$\Lambda(\nu)$ is analytic "almost nowhere" inside the region G. Only for $\nu \notin G$ can it be written in closed form:

$$\Lambda(\nu) = \hat{\Lambda}(\nu), \quad \nu \notin G, \quad (1.15)$$

$$\hat{\Lambda}(\nu) \equiv 1 - \frac{c}{2(\nu^{-2} - B^2)^{\frac{1}{2}}} \ln \frac{1 + (\nu^{-2} - B^2)^{\frac{1}{2}}}{1 - (\nu^{-2} - B^2)^{\frac{1}{2}}}. \quad (1.16)$$

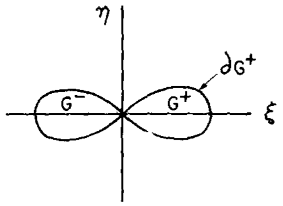

FIG. 1. The region $G = G^+ \cup G^-$ of the complex ζ plane.

$\hat{\Lambda}(\nu)$, like the 1-dimensional dispersion function which it resembles, has the two roots

$$\hat{\Lambda}(\pm\nu_0) = 0, \quad (1.17)$$

where, denoting the known[1] 1-dimensional roots by $\pm L$, we have

$$\nu_0^{-2} = B^2 + L^{-2}. \quad (1.18)$$

However, because of the qualification $\nu \notin G$ on Eq. (1.15), one cannot conclude in general that the $\pm\nu_0$ will be roots of Λ. This technical difficulty (which, as we shall see below, has only a temporary significance) is dealt with in Ref. 2 by defining the function

$$\chi_c(B^2) = 1, \quad 0 < B^2 < 1 - L^{-2},$$
$$= 0, \quad B^2 \geq 1 - L^{-2}. \quad (1.19)$$

Now when $\chi_c(B^2) = 1$, it is clear from Eq. (1.18) that $\pm\nu_0 \notin G$, so that Eq. (1.15) holds and

$$\Lambda(\pm\nu_0) = O, \quad \chi_c(B^2) = 1. \quad (1.20)$$

Equation (1.20) yields, in the usual way, the discrete modes

$$\psi_\pm(x, \zeta) = e^{\mp x/\nu_0}\phi_\pm(\zeta), \quad (1.21)$$

$$\phi_\pm(\zeta) = \pm\nu_0 g(\pm\nu_0)/(\pm\nu_0 - \zeta). \quad (1.22)$$

But it must be observed that these modes do not occur in the case $\chi_c(B^2) = 0$.

Equation (1.10) always possesses a continuum of eigenvalues. Here, the continuum is 2-dimensional: all $\nu \in G$ and the corresponding eigenfunctions

$$\psi_\nu(x, \zeta) = e^{-x/\zeta}\phi(\nu, \zeta), \quad (1.23)$$

$$\phi(\nu, \zeta) = \nu g(\nu)/(\nu - \zeta) + \Lambda(\nu)\delta(\nu - \zeta) \quad (1.24)$$

are generalized analytic functions, with the definitions

$$((\nu - \zeta)^{-1}, \psi(\zeta)) \equiv \iint_G \frac{\psi(\zeta)}{\nu - \zeta}\, d\xi\, d\eta, \quad (1.25)$$

$$(\delta(\nu - \zeta), \psi(\zeta)) \equiv \psi(\nu), \quad \nu \in G,$$
$$\equiv 0, \quad \nu \notin G. \quad (1.26)$$

The formal similarity of Eq. (1.24) to the familiar 1-dimensional continuum modes is perhaps deceptive. Note, for example, that the integral of Eq. (1.25) exists without any principal-value interpretation.

The proof of full-range orthogonality for the eigenfunctions of Eqs. (1.21)–(1.24) is as trivial here as it is in the 1-dimensional theory. However, rather complicated arguments are required to prove their full-range completeness, half-range orthogonality, and half-range completeness. Essential to these arguments is a "theorem" which, ignoring details of rigor, may be stated in the form[4]

$$\frac{\partial}{\partial \bar{\zeta}} \frac{1}{\zeta} = \pi \delta(\zeta). \tag{1.27}$$

In particular, it follows from Eq. (1.27) that

$$\begin{aligned} \frac{\partial \Lambda}{\partial \bar{\zeta}} &= 0, \qquad \zeta \notin G, \\ &= -\pi \zeta g(\zeta), \quad \zeta \in G. \end{aligned} \tag{1.28}$$

This theorem plays a role here somewhat analogous to that of the Plemelj formulas in the conventional theory. It is used in the half-range orthogonality proof, for example, to find a function $X(\zeta)$ which is analytic for $\zeta \notin G^+$, continuous on the boundary ∂G^+ of G^+, and which satisfies

$$\Lambda(\zeta) \frac{\partial X}{\partial \bar{\zeta}} - X(\zeta) \frac{\partial \Lambda}{\partial \bar{\zeta}} = 0, \quad \zeta \in G^+. \tag{1.29}$$

Of course, the degree of complexity of the completeness and orthogonality proofs has little bearing on the task of solving specific problems. Indeed, Kaper shows that this task proceeds in a quite straightforward manner. Given a problem of the general form of Eq. (1.1), with the addition perhaps of inhomogeneous terms, and conditions specified on boundary planes perpendicular to the x axis, we first transform the angular density and angle variables according to Eqs. (1.7)–(1.9). Then, as in the 1-dimensional theory, we expand the unknown transformed density ψ in terms of the eigenfunctions of Eqs. (1.21)–(1.24), and expect the given boundary conditions, together with the appropriate (half-range or full-range) orthogonality relations, to provide the expansion coefficients. For purposes of comparison with a quite different approach to be described below, we wish now to examine the solutions Kaper obtains in this way to two very simple problems.

A. The Infinite-Space Green's Function

The infinite-space Green's function satisfies, for all $\mathbf{r}$,

$$(\mathbf{\Omega} \cdot \nabla + 1) N_g(\mathbf{r}, \mathbf{\Omega}) = \frac{c}{4\pi} \int d\mathbf{\Omega}' N_g(\mathbf{r}, \mathbf{\Omega}') + \delta(\mathbf{r}) \delta(\mathbf{\Omega} - \mathbf{\Omega}_0) \tag{1.30}$$

and is found by Kaper to have the representation

$$N_g(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{(2\pi)^2} \int d\mathbf{B}_{(x)} e^{-i\mathbf{B}_{(x)} \cdot \mathbf{r}_{(x)}} \frac{\psi_g(x, \zeta)}{1 - i\mathbf{B}_{(x)} \cdot \mathbf{\Omega}_{(x)}}, \tag{1.31}$$

$$\psi_g(x, \zeta) = \pm \chi_c(B^2) a_\pm \psi_\pm(x, \zeta) \pm \iint_{G^+} A(\zeta') \psi_{\zeta'}(x, \zeta)\, d\xi' d\eta', \quad x \gtrless 0, \tag{1.32}$$

where[5]

$$a_\pm = g(\zeta_0) \left| \frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)} \right| N_\pm^{-1} \phi_\pm(\zeta_0), \tag{1.33}$$

$$A(\zeta) = g(\zeta_0) \left| \frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)} \right| [N(\zeta)]^{-1} \phi(\zeta, \zeta_0), \tag{1.34}$$

and the N's are certain quantities defined in Ref. 2.

What is especially interesting for our purposes here is the transformation of the solution afforded by Eqs. (1.32)–(1.34) into a form directly suitable for evaluation. In particular, we wish to express the integral over the region G^+ or G^- [in Eq. (1.32)] in the much more tractable form of a line integral. Assuming for definiteness that $x > 0$, we have

$$\psi_g(x, \zeta) = \chi_c(B^2) \frac{c}{4\pi} \frac{\zeta_0}{\mu_0} \frac{1}{N_+} \phi_+(\zeta_0) \psi_+(x, \zeta) + \frac{c}{4\pi} \frac{\zeta_0}{\mu_0} I_1, \tag{1.35}$$

where

$$I_1 = \iint_{G^+} \frac{\phi(\zeta', \zeta_0)}{N(\zeta')} \psi_{\zeta'}(x, \zeta)\, d\xi'\, d\eta' \tag{1.36}$$

and we have noted that

$$g(\zeta_0) \left| \frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)} \right| = \frac{c}{4\pi} \frac{\zeta_0}{\mu_0}. \tag{1.37}$$

Making the appropriate substitutions from Eqs. (1.21)–(1.24) and (1.34), and integrating the δ-function terms, we can write Eq. (1.36) as

$$I_1 = \chi_+(\zeta) \frac{e^{-x/\zeta}}{(\zeta - \zeta_0)\Lambda(\zeta)} + \chi_+(\zeta_0) \frac{e^{-x/\zeta_0}}{(\zeta_0 - \zeta)\Lambda(\zeta_0)} + \chi_+(\zeta_0) \delta(\zeta - \zeta_0) \frac{e^{-x/\zeta_0}}{\zeta_0 g(\zeta_0)} + I_2, \tag{1.38}$$

where

$$\begin{aligned} \chi_+(\zeta) &= 1, \quad \zeta \in G^+, \\ &= 0, \quad \zeta \notin G^+, \end{aligned} \tag{1.39}$$

and

$$I_2 = \iint_{G^+} e^{-x/\zeta'} \frac{\zeta' g(\zeta')}{\Lambda^2(\zeta')(\zeta' - \zeta)(\zeta' - \zeta_0)}\, d\xi'\, d\eta'. \tag{1.40}$$

I_2 can be made to appear much less formidable if we avail ourselves of the "theorem" (1.27) and its corollary (1.28). These are easily seen to imply that the integrand in Eq. (1.40) may be written in the form

$$\frac{1}{\pi}\frac{\partial}{\partial\zeta'}\left(\frac{e^{-x/\zeta'}}{\Lambda(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)}\right) - \frac{e^{-x/\zeta'}}{(\zeta-\zeta_0)\Lambda(\zeta')}[\delta(\zeta'-\zeta_0)-\delta(\zeta-\zeta_0)]. \quad (1.41)$$

We observe that the δ-function terms in the expression (1.41) are precisely such as to cancel (upon integration over G^+) the first two terms in Eq. (1.38). Furthermore, the integral over G^+ of the first term in (1.41) is readily converted, by means of what is essentially Stokes' theorem, into an integral over the boundary ∂G^+. In this way we find that Eqs. (1.38) and (1.40) reduce to

$$I_1 = \chi_+(\zeta_0)\delta(\zeta-\zeta_0)\frac{e^{-x/\zeta_0}}{\zeta_0 g(\zeta_0)} + \frac{1}{2\pi i}\int_{\partial G^+}\frac{e^{-x/\zeta'}\,d\zeta'}{\Lambda(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)}. \quad (1.42)$$

With regard to the line integral in Eq. (1.42), one further significant manipulation is allowed: Since $\zeta' \in \partial G^+$, we may substitute for $\Lambda(\zeta')$ the much more analytically tractable function $\hat{\Lambda}(\zeta')$ [cf. Eq. (1.16)]. It is clear that the latter function has its only singularities, branch points, at

$$\zeta' = \pm\alpha, \quad (1.43)$$

$$\alpha = (1+B^2)^{-\frac{1}{2}}. \quad (1.44)$$

We choose the cut $l = l^+ + l^-$ as joining these two points along the real axis as in Fig. 2. Now we define a region of the complex plane $\hat{G}_+$ to be the region G^+ with the exclusion of a small neighborhood of the cut l, and observe that the integrand in Eq. (1.42) is analytic throughout the interior of $\hat{G}_+$ except for the following:

(i) There will be a pole at $\zeta' = \zeta(\zeta' = \zeta_0)$ unless $\zeta(\zeta_0) \in G^-$ or $\zeta(\zeta_0) \in l$;

(ii) there will be a pole at $\zeta' = \nu_0$ unless $\chi_c(B^2) = 1$. [It is important to note, from Eq. (1.33) and the known fact[1] that $|L| > 1$, that in any case $\nu_0 \notin l$.]

With these remarks, it is evident that we may write

$$\frac{1}{2\pi i}\int_{\partial G^+}\frac{e^{-x/\zeta'}\,d\zeta'}{\hat{\Lambda}(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)} = \frac{e^{-x/\zeta}}{\hat{\Lambda}(\zeta)(\zeta-\zeta_0)}\hat{\chi}_+(\zeta) + \frac{e^{-x/\zeta_0}}{\hat{\Lambda}(\zeta_0)(\zeta_0-\zeta)}\hat{\chi}_+(\zeta_0) + \frac{e^{-x/\nu_0}[1-\chi_c(B^2)]}{\hat{\Lambda}'(\nu_0)(\nu_0-\zeta)(\nu_0-\zeta_0)} + \frac{1}{2\pi i}\int_{l^{+\prime}}\frac{e^{-x/\zeta'}\,d\zeta'}{\hat{\Lambda}(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)}. \quad (1.45)$$

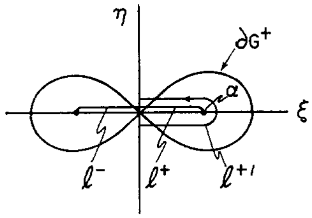

FIG. 2. Cuts and integration contours.

Here, of course,

$$\hat{\chi}_+(\zeta) \equiv 1, \quad \zeta \in \hat{G}_+,$$
$$\equiv 0, \quad \zeta \in G^- \cup l^+, \quad (1.46)$$

and $l^{+\prime}$ is a path surrounding the cut l^+ (cf. Fig. 2). Note that we have recovered in Eq. (1.45) both the previously cancelled terms of Eq. (1.38), in somewhat altered form, and the discrete term in the case $\chi_c(B^2) = 0$.

The right-hand side of Eq. (1.45) represents an improvement over the left-hand side for two reasons:

(i) The integration path for the line integral is much simpler. Indeed, when neither ζ nor ζ_0 is on l^+, we may write

$$\frac{1}{2\pi i}\int_{l^+}\frac{e^{-x/\zeta'}\,d\zeta'}{\hat{\Lambda}(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)} = \frac{1}{2\pi i}\int_0^\alpha d\zeta'\frac{e^{-x/\zeta'}}{(\zeta'-\zeta)(\zeta'-\zeta_0)}\left(\frac{1}{\hat{\Lambda}^-(\zeta')}-\frac{1}{\hat{\Lambda}^+(\zeta')}\right), \quad (1.47)$$

where the $\hat{\Lambda}^\pm$ are the boundary values of $\hat{\Lambda}$ on its cut. In case either ζ or ζ_0 is on l^+, we avoid the pole by a small semicircle and may write the integral in terms of a principal value plus pole contributions in the conventional way.

(ii) More significantly, we have isolated in Eq. (1.45) the asymptotically dominant contribution to the integral over ∂G^+. Indeed, it is evident from Eqs. (1.18) and (1.44), that the discrete term [which must, regardless of the magnitude of B^2, occur in $\psi(x,\zeta)$] always dominates the integral of Eq. (1.45) for large x.

Substituting Eqs. (1.42) and (1.45) into Eq. (1.35) and reverting to the more physical quantity N_g, we conclude

$$\hat{N}_g(x,\mathbf{\Omega}) = \chi_+(\zeta_0)\left|\frac{\partial(\xi_0,\eta_0)}{\partial(\mu_0,\theta_0)}\right|\mu_0^{-1}e^{-x/\zeta_0}\delta(\zeta-\zeta_0) + \frac{c}{4\pi}\frac{\zeta\zeta_0}{\mu\mu_0}\left(\frac{e^{-x/\nu_0}}{\hat{\Lambda}'(\nu_0)(\nu_0-\zeta)(\nu_0-\zeta_0)} + \hat{\chi}_+(\zeta)\frac{e^{-x/\zeta}}{\hat{\Lambda}(\zeta)(\zeta-\zeta_0)} + \hat{\chi}_+(\zeta_0)\frac{e^{-x/\zeta_0}}{\hat{\Lambda}(\zeta_0)(\zeta_0-\zeta)} + \frac{1}{2\pi i}\int_{l^{+\prime}}\frac{e^{-x/\zeta'}\,d\zeta'}{\hat{\Lambda}(\zeta')(\zeta'-\zeta)(\zeta'-\zeta_0)}\right), \quad x > 0, \quad (1.48)$$

where the correct functional dependence is ultimately to be obtained by means of Eq. (1.8). We omit this final step and merely draw attention to the fact that Eq. (1.48) involves only $\hat{\Lambda}$,[6] which, unlike Λ, can be written in closed form; there are no 2-dimensional integrals, explicit or implicit, in our final expression for $\hat{N}$.

B. The Albedo Problem

If $\hat{N}_a(x, \mathbf{B}_{(x)}, \mathbf{\Omega})$ satisfies the homogeneous equation (1.1) for $x > 0$ and the boundary condition

$$\hat{N}_a(0, \mathbf{B}_{(x)}, \mathbf{\Omega}) = \delta(\mu - \mu_0)\delta(\theta - \theta_0), \quad \text{for } \mu > 0, \quad \mu_0 < 0, \tag{1.49}$$

then the corresponding ψ_a satisfies Eq. (1.10) for $x > 0$ with[7]

$$\psi_a(0, \zeta) = \frac{\mu_0}{\zeta_0}\frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)}\delta(\zeta - \zeta_0), \quad \zeta, \zeta_0 \in G^+. \tag{1.50}$$

Using the half-range completeness and orthogonality relations, Kaper obtains from Eqs. (1.10) and (1.50) the solution

$$\psi_a(x, \zeta) = b_+\psi_+(x, \zeta) + \iint_{G^+} B(\zeta')\psi_{\zeta'}(x, \zeta)\, d\xi'\, d\eta', \tag{1.51}$$

where

$$B(\zeta) = \frac{\mu_0}{\zeta_0}\left|\frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)}\right|\frac{1}{\Lambda(\zeta_0)}\bigg(\delta(\zeta - \zeta_0) - \frac{\zeta_0 g(\zeta_0)(\zeta_0 - \nu_0)(\zeta - 1)e^{\omega(\zeta_0) - \omega(\zeta)}}{\Lambda(\zeta)(\zeta - \nu_0)(\zeta_0 - 1)(\zeta_0 - \zeta)}\bigg), \tag{1.52}$$

$$b_+ = \frac{\mu_0}{\Lambda(\zeta_0)}\left|\frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)}\right|\frac{g(\zeta_0)e^{\omega(\zeta_0)}(\nu_0 - 1)}{g(\nu_0)e^{\omega(\nu_0)}(\zeta_0 - 1)}. \tag{1.53}$$

Here we have used the function

$$\omega(\zeta) = -\frac{1}{\pi}\iint_{G^+}\frac{\partial\Lambda(\zeta')}{\partial\bar{\zeta}'}\frac{1}{\Lambda(\zeta')}\frac{1}{\zeta' - \zeta}\, d\xi'\, d\eta', \tag{1.54}$$

which arises in the solution of Eq. (1.29). By methods similar to those used in regard to Eqs. (1.38)–(1.45), we may write ω in the form

$$\omega(\zeta) = \hat{\chi}_+(\zeta)\ln\Lambda(\zeta) - \hat{\chi}_+(\zeta)\ln\hat{\Lambda}(\zeta) + \gamma(\zeta) - \ln(\zeta - \alpha)/(\zeta - 1), \tag{1.55}$$

where

$$\gamma(\zeta) \equiv -\frac{1}{2\pi i}\int_{l^+}\frac{\ln\hat{\Lambda}(\zeta')}{\zeta' - \zeta}\, d\zeta'. \tag{1.56}$$

The last term in Eq. (1.56) arises from an integration along the cut of $\ln\hat{\Lambda}$, which cut is taken to be along the real axis inside G and is to be distinguished from the cut of $\hat{\Lambda}$ itself. [Throughout this discussion of the albedo problem we assume for convenience that $\chi_c(B^2) = 1$; the actual value of $\chi_c(B^2)$ is, of course, as irrelevant to the final answer here as it was in the infinite-space case discussed above.]

Our task now is to substitute Eqs. (1.52) and (1.53) into Eq. (1.51), and to reduce the integral over G^+ to a simple branch-cut integral. These manipulations are somewhat lengthier than, but otherwise very similar to, the procedure we performed above for the infinite-space Green's function. Omitting both the detailed calculation and the general result, we state here only the most physically interesting result, namely, that the emergent angular density, which according to Eqs. (1.51)–(1.53) is given by

$$\hat{N}_a(0, \mathbf{B}_{(x)}, \mathbf{\Omega}) = \frac{\psi_a(\zeta)}{1 - i\mathbf{B}_{(x)}\cdot\mathbf{\Omega}_{(x)}}, \quad \mu < 0, \tag{1.57}$$

$$\psi_a(0, \zeta) = b_+\phi_+(0, \zeta) + \frac{\mu_0}{\zeta_0}\left|\frac{\partial(\xi_0, \eta_0)}{\partial(\mu_0, \theta_0)}\right|\frac{1}{\Lambda(\zeta_0)} \times \bigg(\psi_{\zeta_0}(0, \zeta) - \frac{\zeta_0 g(\zeta_0)e^{\omega(\zeta_0)}(\zeta_0 - \nu_0)}{(\zeta_0 - 1)} \times \iint_{G^+}\frac{(\zeta' - 1)e^{-\omega(\zeta')}\psi_{\zeta'}(0, \zeta)}{(\zeta' - \nu_0)(\zeta_0 - \zeta')\Lambda(\zeta')}\, d\xi'\, d\eta'\bigg), \quad \zeta \in G^-, \tag{1.58}$$

ultimately reduces to

$$\hat{N}_a(0, \mathbf{B}_{(x)}, \mathbf{\Omega}) = -\frac{c}{4\pi}\frac{\zeta\zeta_0(\zeta_0 - \nu_0)(\zeta - \alpha)e^{\gamma(\zeta_0) - \gamma(\zeta)}}{[\hat{\Lambda}(\zeta_0)](\zeta - \nu_0)(\zeta_0 - \alpha)(\zeta - \zeta_0)}, \quad \mu < 0. \tag{1.59}$$

We have placed the factor $\hat{\Lambda}(\zeta_0)$ in brackets to indicate that it is to be replaced by 1 in the case $\zeta_0 \in l^+$. The fact that Eq. (1.59), like Eq. (1.48), involves $\hat{\Lambda}$ rather than Λ follows essentially from Eq. (1.55).

It is perhaps worth mentioning that the crucial step in obtaining Eq. (1.59) from Eq. (1.58) depends upon the observation that Eq. (1.54) implies

$$\frac{\partial\omega(\zeta)}{\partial\bar{\zeta}} = 0, \qquad \zeta \in G^+,$$
$$= \frac{-\pi\zeta g(\zeta)}{\Lambda(\zeta)}, \quad \zeta \in G^+. \tag{1.60}$$

Using Eq. (1.60), the surface integral in Eq. (1.58) may be written in terms of some δ-function contributions plus an integral over ∂G^+; since $x = 0$, the contour for the latter may be deformed into a contour at infinity, with some residues. The final result is Eq. (1.59).

Let us consider an alternative formulation.

2. THE FOURIER-TRANSFORM METHOD

A rather general problem in linear transport theory may be stated as follows:

Let V be some (bounded or unbounded) region of 3-dimensional space with boundary S. We are to find that function $\phi(\mathbf{r}, \boldsymbol{\Omega})$, for $\mathbf{r} \in V$ and $|\boldsymbol{\Omega}| = 1$, which satisfies

$$(\boldsymbol{\Omega} \cdot \nabla + 1)\phi(\mathbf{r}, \boldsymbol{\Omega}) = \frac{c}{4\pi} \rho(\mathbf{r}) + q(\mathbf{r}, \boldsymbol{\Omega}), \quad \mathbf{r} \in V, \tag{2.1}$$

given the boundary data $\phi_s(\mathbf{r}_s, \boldsymbol{\Omega})$ and $\boldsymbol{\Omega}$ inward to V; that is,

$$\phi(\mathbf{r}_s, \boldsymbol{\Omega}) = \phi_s(\mathbf{r}_s, \boldsymbol{\Omega}), \quad \boldsymbol{\Omega} \text{ inward}, \quad \mathbf{r}_s \in S. \tag{2.2}$$

In Eq. (2.1), $\phi(\mathbf{r}, \boldsymbol{\Omega})$ and $\rho(\mathbf{r})$, where

$$\rho(\mathbf{r}) \equiv \int d\boldsymbol{\Omega} \phi(\mathbf{r}, \boldsymbol{\Omega}), \tag{2.3}$$

are the angular density and density, respectively. q represents any external sources which may be present, and, like ϕ_s, is presumed to be given.

In order to express Eq. (2.1) as an integral equation, let $G(\mathbf{r} - \mathbf{r}', \boldsymbol{\Omega})$, defined for all $\mathbf{r}$ and $\mathbf{r}'$, satisfy

$$(-\boldsymbol{\Omega} \cdot \nabla + 1)G(\mathbf{r} - \mathbf{r}', \boldsymbol{\Omega}) = \delta(\mathbf{r} - \mathbf{r}'). \tag{2.4}$$

Then, by a conventional argument, Eqs. (2.1) and (2.4) imply

$$\phi(\mathbf{r}, \boldsymbol{\Omega}) = \int_V d\mathbf{r}' G(\mathbf{r}' - \mathbf{r}, \boldsymbol{\Omega})\left(\frac{c}{4\pi} \rho(\mathbf{r}') + q(\mathbf{r}', \boldsymbol{\Omega})\right) + \boldsymbol{\Omega} \cdot \int_S \hat{\mathbf{n}}_i \, d\mathbf{r}_s G(\mathbf{r}_s - \mathbf{r}, \boldsymbol{\Omega})\phi_s(\mathbf{r}_s, \boldsymbol{\Omega}), \tag{2.5}$$

where $\hat{n}_i$ is the inward normal to V. Note that we have put the known function $\phi_s(\mathbf{r}_s, \boldsymbol{\Omega})$ in the integrand, instead of $\phi(\mathbf{r}_s, \boldsymbol{\Omega})$; this is justified by the easily verified [cf. Eq. (2.10) below] fact that

$$G(\mathbf{r}_s - \mathbf{r}, \boldsymbol{\Omega}) = 0, \quad \text{for} \quad \mathbf{r} \in V, \quad \boldsymbol{\Omega} \text{ outward}. \tag{2.6}$$

Equation (2.5) holds, of course, only for $\mathbf{r} \in V$, the domain of definition of $\phi(\mathbf{r}, \boldsymbol{\Omega})$. We now extend this domain by assuming Eq. (2.5) to hold for all $\mathbf{r}$, so that we may take its 3-dimensional Fourier transform. With the convention

$$\tilde{f}(k) = \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{2.7}$$

and the definition

$$\tilde{\rho}_V(\mathbf{k}) \equiv \int_V d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} \rho(\mathbf{r}), \tag{2.8}$$

the transform of Eq. (2.5) takes the form

$$\tilde{\phi}(\mathbf{k}, \boldsymbol{\Omega}) = \frac{c}{4\pi} \frac{\tilde{\rho}_V(\mathbf{k})}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} + \frac{\tilde{q}(\mathbf{k}, \boldsymbol{\Omega})}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} + \frac{\boldsymbol{\Omega}}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} \cdot \int_S \hat{\mathbf{n}}_i \, d\mathbf{r}_s e^{i\mathbf{k}\cdot\mathbf{r}_s} \phi_s(\mathbf{r}_s, \boldsymbol{\Omega}). \tag{2.9}$$

In obtaining Eq. (2.9), we used the representation

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int d\mathbf{k} \frac{e^{-i\mathbf{k}\cdot(\mathbf{k}-\mathbf{r}')}}{1 + i\mathbf{k} \cdot \boldsymbol{\Omega}} \tag{2.10}$$

which follows immediately from Eq. (2.4), and assumed that

$$q(\mathbf{r}, \boldsymbol{\Omega}) = 0, \quad \mathbf{r} \notin V. \tag{2.11}$$

Upon integrating Eq. (2.9) over all directions $\boldsymbol{\Omega}$, we obtain the equation

$$\tilde{\rho}(\mathbf{k}) = [1 - \tilde{\Lambda}(\mathbf{k})]\tilde{\rho}_V(\mathbf{k}) + \tilde{B}(\mathbf{k}) + \tilde{Q}(\mathbf{k}), \tag{2.12}$$

where

$$\tilde{\Lambda}(\mathbf{k}) \equiv 1 - \frac{c}{4\pi} \int \frac{d\boldsymbol{\Omega}}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} \tag{2.13}$$

is the 3-dimensional dispersion function and

$$\tilde{B}(\mathbf{k}) \equiv \int_S d\mathbf{r}_s e^{i\mathbf{k}\cdot\mathbf{r}_s} \int d\mathbf{r} \frac{\hat{\mathbf{n}}_i \cdot \boldsymbol{\Omega}}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} \phi_s(\mathbf{r}_s, \boldsymbol{\Omega}), \tag{2.14}$$

$$\tilde{Q}(\mathbf{k}) \equiv \int d\boldsymbol{\Omega} \frac{\tilde{q}(\mathbf{k}, \boldsymbol{\Omega})}{1 - i\mathbf{k} \cdot \boldsymbol{\Omega}} \tag{2.15}$$

result of course from the *given* boundary and external source contributions, respectively.

A "general" prescription for solving any transport problem of the form (2.1) can now be given:

(i) In some way (perhaps only approximately), we are to solve Eq. (2.12) for $\tilde{\rho}_V(\mathbf{k})$.

(ii) Equation (2.9) then immediately provides $\tilde{\phi}(\mathbf{k}, \boldsymbol{\Omega})$.

(iii) Finally, we take the inverse Fourier transform of $\tilde{\phi}$.

Of course only step (ii) is trivial. In fact, the practicality of this prescription is in general very questionable. All we hope to show is that for at least that class of problems considered in the previous section, namely, problems in which the boundaries depend upon only one space variable, the method outlined above is indeed workable.

Our hopes for solving Eq. (2.12) rest essentially in the observation that

$$\tilde{\rho}_V(\mathbf{k}) = \int d\mathbf{k}' \tilde{\rho}(\mathbf{k}') \Delta_V(\mathbf{k} - \mathbf{k}'), \tag{2.16}$$

where

$$\Delta_V(\mathbf{k} - \mathbf{k}') \equiv \frac{1}{(2\pi)^3} \int_V d\mathbf{r}' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}'}, \tag{2.17}$$

so that Eq. (2.12) can be written, in general, as an integral equation for $\tilde{\rho}(\mathbf{k})$.[8] The geometry of a particular problem enters solely through the kernel $\Delta_V(\mathbf{k})$; this will be, for the problems considered here, highly singular (i.e., a generalized function), with the result that our solutions of Eq. (2.12) will depend more on analyticity arguments than on the Fredholm theory. Therefore, before proceeding further, we briefly examine the properties of $\tilde{\Lambda}(\mathbf{k})$ as an analytic function of, say, k_x.

A. The Dispersion Function

For $\mathbf{k}$ real, the integral of Eq. (2.13), like that of Eq. (1.14), may be written in closed form. We thus find the analog of Eq. (1.15):

$$\tilde{\Lambda}(\mathbf{k}) = \Lambda_3(k_x, B), \quad \mathrm{Im}\,(k_x) = 0, \tag{2.18}$$

where

$$B = |\mathbf{k}_{(x)}| \tag{2.19}$$

and

$$\Lambda_3(k, B) \equiv 1 + \frac{ic}{2(k^2 + B^2)^{\frac{1}{2}}} \ln \frac{1 + i(k^2 + B^2)^{\frac{1}{2}}}{1 - i(k^2 + B^2)^{\frac{1}{2}}}. \tag{2.20}$$

The correspondence between these functions and the dispersion function used in Sec. 1 is clear:

$$\tilde{\Lambda}(k_x = -i/\zeta) = \Lambda(\zeta), \tag{2.21}$$

$$\Lambda_3(k = -i/\zeta) = \hat{\Lambda}(\zeta). \tag{2.22}$$

It is again to be remarked that, just as Eq. (1.15) is false for $\zeta \in G$, Eq. (2.18) is invalid outside a certain neighborhood of the real k_x axis. However, since in the present formulation our basic equations are true for real k_x, we may ignore the pathological behavior of $\tilde{\Lambda}(\mathbf{k})$ for general complex k_x and work exclusively with the much more analytically tractable function $\Lambda_3(k_x, B)$, which function is to be taken as defined by Eq. (2.20) for complex k_x also.

We use the notation

$$\Lambda_3(k_x, B) = \Lambda_3(k), \tag{2.23}$$

where, of course, $k = k_x$.

$\Lambda_3(k)$ has its only singularities, branch points, at

$$k = \pm i\beta, \tag{2.24}$$

where

$$\beta \equiv (B^2 + 1)^{\frac{1}{2}} \tag{2.25}$$

and we take the branch cuts $l_\pm$ as extending to $\pm i\infty$ along the imaginary axis (cf. Fig. 3). Because of the resemblance of Eq. (2.20) to the form of the 1-dimensional dispersion function, we may infer that $\Lambda_3(k)$ has only two simple zeros, namely,

$$\Lambda_3(\pm i\kappa_0) = 0, \tag{2.26}$$

where

$$\kappa_0 = (B^2 + L^{-2})^{\frac{1}{2}}. \tag{2.27}$$

Here the $\pm L$ are as in Sec. 1. We may conclude further from the 1-dimensional theory that

(i) $c < 1 \Rightarrow |\kappa_0| > B$ and $\mathrm{Im}\,(\kappa_0) = 0$,
(ii) $c > 1$ and $B > |L^{-1}| \Rightarrow |\kappa_0| < B$ and $\mathrm{Im}\,(\kappa_0) = 0$,
(iii) $c > 1$ and $B < |L^{-1}| \Rightarrow |\kappa_0| < B$ and $\mathrm{Re}\,(\kappa_0) = 0$,

and, perhaps most importantly, that

$$\mathrm{Im}\,(\kappa_0) = 0 \Rightarrow |\kappa_0| < \beta, \tag{2.28}$$

i.e., the roots of $\Lambda_3(k)$ never lie on its cuts. [All these remarks are of course merely the translation, according to Eq. (2.22), of similar facts concerning $\hat{\Lambda}(\zeta)^2$.]

We are now prepared to apply the method outlined to specific transport problems.

B. The Infinite Space Green's Function

It is clear (and in fact well known) that this problem is almost trivially solved by the Fourier transform. Since the region V is all space and we have a point source with direction $\mathbf{\Omega}_0$ at $\mathbf{r}_0$, we find

$$\Delta_V(\mathbf{k}) = \delta(\mathbf{k}), \tag{2.29}$$

$$B(\mathbf{k}) = 0, \tag{2.30}$$

$$\tilde{Q}(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}_0}/(1 - i\mathbf{k}\cdot\mathbf{\Omega}_0). \tag{2.31}$$

Our basic equation (2.12) thus takes the form

$$\tilde{\rho}_G(\mathbf{k}) = [1 - \tilde{\Lambda}(\mathbf{k})]\tilde{\rho}_G(\mathbf{k}) + e^{i\mathbf{k}\cdot\mathbf{r}_0}/(1 - i\mathbf{k}\cdot\mathbf{\Omega}_0) \tag{2.32}$$

or

$$\tilde{\rho}_G(\mathbf{k}) = \frac{1}{\tilde{\Lambda}(\mathbf{k})} \frac{e^{i\mathbf{k}\cdot\mathbf{r}_0}}{1 - i\mathbf{k}\cdot\mathbf{\Omega}_0}, \tag{2.33}$$

i.e., in this translation-invariant case, Eq. (2.12) may be solved algebraically. Equation (2.19) yields

$$\tilde{\phi}_G(\mathbf{k}, \mathbf{\Omega}) = \frac{c}{4\pi} \frac{e^{i\mathbf{k}\cdot\mathbf{r}_0}}{(1 - i\mathbf{k}\cdot\mathbf{\Omega})(1 - i\mathbf{k}\cdot\mathbf{\Omega}_0)\tilde{\Lambda}(\mathbf{k})} + \frac{e^{i\mathbf{k}\cdot\mathbf{r}_0}}{1 - i\mathbf{k}\cdot\mathbf{\Omega}_0} \delta(\mathbf{\Omega} - \mathbf{\Omega}_0). \tag{2.34}$$

The 2-dimensional Fourier transform of the angular density

$$\hat{\phi}_G(x, \mathbf{k}_{(x)}, \mathbf{\Omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x e^{-ik_x x} \tilde{\phi}_G(\mathbf{k}, \mathbf{\Omega}) \tag{2.35}$$

corresponds to the function $\hat{N}_g$ discussed in Sec. 1. With the convenient abbreviations

$$\omega(\mathbf{\Omega}) \equiv -(i/\mu)(1 - i\mathbf{k}_{(x)}\cdot\mathbf{\Omega}_{(x)}), \tag{2.36}$$

$$\omega_0 = \omega(\mathbf{\Omega}_0), \tag{2.37}$$

Eq. (2.35) takes the form

$$\hat{\phi}_G(x, \mathbf{k}_{(x)}, \mathbf{\Omega}) = -\frac{e^{-i\mathbf{k}_{(x)}\cdot\mathbf{r}_{(x)0}}}{2\pi}\int_{-\infty}^{\infty} dk \frac{e^{-ik(x-x_0)}}{\mu\mu_0} \times \left(\frac{c}{4\pi}\frac{1}{(k-\omega)(k-\omega_0)\Lambda_3(k)} + \frac{\delta(\mathbf{\Omega}-\mathbf{\Omega}_0)}{k-\omega_0}\right). \tag{2.38}$$

Here we have used Eq. (2.18). The equivalence of Eqs. (2.38) and (1.48) is easily seen. Indeed, for $x > x_0$, we may close the integration contour of Eq. (2.38) in the lower half-plane by means of the path γ_- which excludes the cut l_- of $\Lambda_3(k)$ (cf. Fig. 3) and find

$$\hat{\phi}_G(x, \mathbf{k}_{(x)}, \mathbf{\Omega}) = \hat{\Theta}(\mu_0)\frac{\delta(\mathbf{\Omega}-\mathbf{\Omega}_0)}{\mu_0} e^{i\mathbf{k}_{(x)}\cdot\mathbf{r}_{(x)0}} e^{-i\omega_0(x-x_0)} + \frac{ice^{i\mathbf{k}_{(x)}\cdot\mathbf{r}_{(x)0}}}{4\pi\mu_0\mu}\left(\frac{e^{-\kappa_0(x-x_0)}}{(i\kappa_0+\omega)(i\kappa_0+\omega_0)\Lambda_3'(-i\kappa_0)} + \hat{\Theta}(\mu_0)\frac{e^{-i\omega_0(x-x_0)}}{(\omega_0-\omega)\Lambda_3(\omega_0)} + \frac{\hat{\Theta}(\mu)e^{-i\omega(x-x_0)}}{(\omega-\omega_0)\Lambda_3(\omega)} - \frac{1}{2\pi i}\int_{\gamma_-} dk \frac{e^{-ik(x-x_0)}}{(k-\omega)(k-\omega_0)\Lambda_3(k)}\right), \tag{2.39}$$

where

$$\hat{\Theta}(\mu) \equiv 1, \quad \mu > 0, \quad \omega \notin l_-,$$
$$\equiv 0, \quad \mu < 0 \quad \text{or} \quad \omega \in l_-. \tag{2.40}$$

Equation (2.39), with $\mathbf{r}_0 = 0$, differs from Eq. (1.48) only in notation; for example,

$$\omega = -i/\zeta,$$
$$\kappa_0 = 1/\nu_0, \tag{2.41}$$

and the integration variables of the branch-cut integrals in Eqs. (2.39) and (1.48) are related by

$$k = -i/\zeta'. \tag{2.42}$$

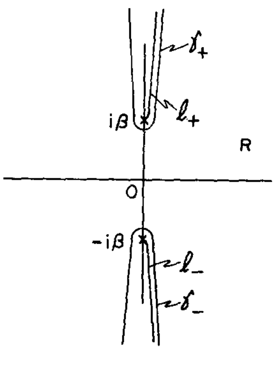

FIG. 3. The region R of the complex k-plane.

C. Half-Space Problems

Somewhat less trivial, but still quite straightforward, is the application of the method based on Eq. (2.12) to problems involving half-spaces.

When V is the region $\{x > 0, -\infty < y < \infty, -\infty < z < \infty\}$, we have

$$\tilde{\rho}_V(\mathbf{k}) = \frac{1}{(2\pi)^3}\int_0^{\infty} dx e^{ik_x x}\int_{-\infty}^{\infty} d\mathbf{r}_{(x)} e^{i\mathbf{k}_{(x)}\cdot\mathbf{r}_{(x)}}\rho(x, \mathbf{r}_{(x)}) \tag{2.43}$$

$$= \tilde{\rho}_+(k), \tag{2.44}$$

where $k \equiv k_x$ and we omit reference to the transverse variables $\mathbf{k}_{(x)}$. In Eq. (2.44) we use a common notation: the subscript indicates analyticity in the upper half k-plane, which property is clear from Eq. (2.43). If we similarly use the function

$$\tilde{\rho}_-(k) = \tilde{\rho}(k) - \tilde{\rho}_+(k) \tag{2.45}$$

which is analytic for Im $(k) < 0$, then Eq. (2.12) may be written in the form

$$\Lambda_3(k)\rho_+(k) = -\rho_-(k) + \tilde{B}(k) + \tilde{Q}(k), \quad k \text{ real}. \tag{2.46}$$

Equation (2.46) is of a standard form and may be solved by well-known methods.[9] It is convenient to begin by finding a "Wiener–Hopf factorization" of the function $\Lambda_3(k)$:

$$\Lambda_3(k) = \Lambda_+(k)/\Lambda_-(k), \tag{2.47}$$

where $\Lambda_\pm(k)$ is analytic for Im $(k) \gtrless 0$ and both factors have at most polynomial growth at $k \sim \infty$. We find the $\Lambda_\pm(k)$ by a conventional[10] procedure. First define the region R of the complex plane as being the entire plane with the exclusion of small neighborhoods of the cuts l_+ and l_-. Thus the boundary of R consists of two contours, γ_+ and γ_-, enclosing the lines l_+ and l_-, as in Fig. 3. From our discussion of $\Lambda_3(k)$ above, it is clear that the function

$$L(k) \equiv \ln[\Lambda_3(k)(k^2+\beta^2)/(k^2+\kappa_0^2)] \tag{2.48}$$

is analytic in R, and that (with the proper branch choice)

$$L(k) \xrightarrow[k\to\infty]{} 0, \quad k \in R. \tag{2.49}$$

Hence we may write

$$L(k) = L_+(k) + L_-(k), \quad k \in R, \tag{2.50}$$

where

$$L_\pm(\ ) = \frac{1}{2\pi i}\int_{\gamma_\mp}\frac{L(k')}{k'-k}dk'. \tag{2.51}$$

Equation (2.50) implies that

$$\Lambda_3(k) = [(k^2 + \kappa_0^2)/(k^2 + \beta^2)]e^{L_+(k)+L_-(k)}, \quad k \in R, \tag{2.52}$$

so that the choice

$$\Lambda_+(k) = [(k + i\kappa_0)/(k + i\beta)]e^{L_+(k)}, \tag{2.53}$$

$$\Lambda_-(k) = [(k - i\beta)/(k - i\kappa_0)]e^{-L_-(k)} \tag{2.54}$$

satisfies both the analyticity requirements and Eq. (2.47). Furthermore, it is evident that

$$\Lambda_\pm(k) \xrightarrow[k\to\infty]{} 1. \tag{2.55}$$

We will take the $\Lambda_\pm(k)$ as being defined by Eqs. (2.53) and (2.54) even for $k \notin R$ (i.e., $k \in l_+$ or $k \in l_-$). With this convention, it should be noted that Eq. (2.47) holds only for $k \in R$. It is not hard to show that, for $k \in R$,

$$\frac{1}{2\pi i}\int_{\gamma_-} \frac{\ln[(k'^2 + \beta^2)/(k'^2 + \kappa_0^2)]}{k' - k}\,dk' = 0, \tag{2.56}$$

whence

$$L_+(k) = \Gamma_+(k), \quad \text{for} \quad k \in R, \tag{2.57}$$

where

$$\Gamma_+(k) = \frac{1}{2\pi i}\int_{\gamma_-} \frac{\ln \Lambda_3(k')}{k' - k}\,dk'. \tag{2.58}$$

$\Gamma_+(k)$ is clearly related to the function $\gamma(\zeta)$ of Eq. (1.57), and for this reason will be found useful below.

Further identities and simplifications concerning the $\Lambda_\pm(k)$ are easily deduced[11]; but we wish to return our attention to Eq. (2.48), which may now be written in the form

$$\tilde{\rho}_+(k)\Lambda_+(k) + \tilde{\rho}_-(k)\Lambda_-(k) = [\tilde{B}(k) + \tilde{Q}(k)]\Lambda_-(k), \quad k \text{ real}. \tag{2.59}$$

Thus if we define the function

$$F(k) \equiv \tilde{\rho}_+(k)\Lambda_+(k), \quad \operatorname{Im}(k) > 0,$$
$$\equiv -\tilde{\rho}_-(k)\Lambda_-(k), \quad \operatorname{Im}(k) < 0, \tag{2.60}$$

then $F(k)$ is analytic in the plane cut along the real axis, and has a discontinuity along the cut given by

$$F^+(k) - F^-(k) = [\tilde{B}(k) + \tilde{Q}(k)]\Lambda_-(k). \tag{2.61}$$

It follows in a well-known[9] way that (with properly behaved $\tilde{B}$ and $\tilde{Q}$)

$$F(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{[\tilde{B}(k') + \tilde{Q}(k')]\Lambda_-(k')}{k' - k} + P(k). \tag{2.62}$$

Here $P(k)$ is an entire function which can generally be taken to be zero.[12] Thus, from Eq. (2.60),

$$\tilde{\rho}_+(k) = \frac{1}{2\pi i\Lambda_+(k)}\int_{-\infty}^{\infty} dk' \frac{[\tilde{B}(k') + \tilde{Q}(k')]\Lambda_-(k')}{k' - k - i0} \tag{2.63}$$

and the general half-space problem is solved.

As a particular example we consider again the albedo problem, in which there is no external source

$$\tilde{Q}(k) = 0 \tag{2.64}$$

but a boundary condition of the form

$$\phi_s(0, y, z, \mathbf{\Omega}) = \delta(y)\delta(z)\delta(\mathbf{\Omega} - \mathbf{\Omega}_0), \quad \mu, \mu_0 > 0, \tag{2.65}$$

$$\Rightarrow \tilde{B}(k) = i/(k - \omega_0), \quad \operatorname{Im}\omega_0 < 0. \tag{2.66}$$

The integral of Eq. (2.63) is entirely trivial and we find[13]

$$\tilde{\rho}_{a+}(k) = i\Lambda_-(\omega_0)/(k - \omega_0)\Lambda_+(k), \tag{2.67}$$

whence [from Eq. (2.9)]

$$\tilde{\phi}_a(\mathbf{k}, \mathbf{\Omega}) = \frac{i\delta(\mathbf{\Omega} - \mathbf{\Omega}_0)}{k - \omega_0} - \frac{c}{4\pi}\frac{\Lambda_-(\omega_0)}{\mu\Lambda_+(k)(k - \omega_0)(k - \omega)}. \tag{2.68}$$

To check that Eq. (2.68) agrees with Eq. (1.59), it is only necessary to observe that, for $\mu < 0$, $\tilde{\phi}_a$ is analytic in the upper-half $k = k_x$ plane except for a pole at $k = \omega$. Hence, for $x = 0$, the inverse k_x transform involves only a simple residue calculation. Using the explicit forms of the $\Lambda_\pm$ functions, we find the emergent angular density

$$\hat{\phi}_a(0, \mathbf{k}_{(x)}, \mathbf{\Omega}) = -\frac{ic}{4\pi\mu}\frac{(\omega_0 + i\kappa_0)(\omega + i\beta)e^{\Gamma_+(\omega_0)-\Gamma_+(\omega)}}{(\omega_0 + i\beta)(\omega + i\kappa_0)(\omega - \omega_0)[\Lambda_3(\omega_0)]}, \quad \mu < 0. \tag{2.69}$$

It is easy to verify that

$$\Gamma_+(-i/\zeta) - \Gamma_+(0) = \gamma(\zeta), \tag{2.70}$$

whence, in view of the relations (2.49) and the fact that $\beta = \alpha^{-1}$, the equivalence of Eqs. (2.69) and (1.60) becomes evident.

We could now go on to consider problems in which the region V is finite in the x direction, i.e., slab problems. It is in fact possible in this case also to derive from Eq. (2.12) a general prescription ("general" in the sense that the particular sources and boundary data need not be specified beforehand) which provides, at least in the usual wide slab approximation, the desired transformed densities.[11] This prescription involves a combination of function-theoretic techniques with a Fredholm-like iterative procedure. But we will not delve into these, or other still more complicated problems here; it is hoped that the above examples suffice to demonstrate the workability of the technique we have outlined with regard to that class of problems to which the singular eigenfunction technique based on Eq. (1.10) is applicable.

CONCLUSION

Our comparison of the two methods for solving problems in what might be called "quasi-3-dimensional" linear transport theory may be summarized in the following remarks—remarks which can be expected to apply regardless of the particular problem considered.

By expanding in terms of the set of functions $\{e^{ikx}; k \text{ real}\}$ we are led naturally to the dispersion function $\Lambda_3(k)$, which can be expressed in closed form and which has fairly simple analytic properties. On the other hand, use of the set of functions provided by Eqs. (1.21)-(1.24) (the orthogonality and completeness of which is neither well known nor trivial to prove) leads one to the much more mathematically formidable function $\Lambda(\zeta)$; one must then use the theory of generalized analytic functions to eliminate the pathologies of the latter, so that final answers ultimately may be expressed in terms of $\hat{\Lambda}(\zeta) = \Lambda_3(-i/\zeta)$.

A second and, perhaps, more significant difference between the two methods is that, according to the prescription given in Sec. 2, one finds the transformed density $\tilde{\rho}(k)$ [from which the transformed angular density $\tilde{\phi}(\mathbf{k}, \mathbf{\Omega})$ is trivially obtained] first, rather than, as in the singular-eigenfunction technique, determining the angular density first. The result is that the former method is almost entirely free of the mathematical complexities associated with 2-dimensional angular integrals.

Of course both these distinctions are without force in the 1-dimensional $\mathbf{B}_{(x)} = 0$ theory.[14] But when $\mathbf{B}_{(x)} \neq 0$, they have the effect of making the Fourier-transform analysis substantially more elementary and direct.

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[1] See, for example, K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, Reading, Mass., 1967).

[2] H. G. Kaper, J. Math. Phys. **10**, 286 (1969).

[3] I. N. Vekua, *Generalized Analytic Functions* (Addison-Wesley, Reading, Mass., 1962).

[4] Note that Eq. (1.27) is simply the statement that $(1/\pi) \ln \zeta$ is the Green's function for the 2-dimensional Laplacian.

[5] The Jacobians in Eqs. (1.33) and (1.34) were omitted in Ref. 2. These equations also differ from those in the Reference by a trivial factor of 4π, due to different source normalizations.

[6] That final "answers" can *in general* be expressed in terms of Λ only will become evident below; this circumstance allows one to ignore such apparently difficult questions (see Ref. 2) as the possible vanishing of Λ in G.

[7] The factor multiplying the δ function in Eq. (1.50), which is called for by Eqs. (1.9) and (1.49), was omitted in Ref. 2. For this reason the coefficients we give in Eqs. (1.52) and (1.53) differ slightly from those of Kaper.

[8] This fact is exploited, in particular, when the region V is finite in the x direction. (See R. D. Hazeltine, thesis, University of Michigan, 1968.)

[9] See, for example, N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Groningen, The Netherlands, 1953).

[10] See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part 1.

[11] R. D. Hazeltine, thesis, University of Michigan, 1968.

[12] Nonzero $P(k)$ clearly implies δ-function singularities in the density $\rho(x)$.

[13] The result, Eq. (2.67), could in this case have been obtained by a simple function-theoretic argument directly from Eq. (2.59) without using the general integral prescription.

[14] That the two methods are equivalent in this case is well known; cf. K. M. Case, Ann. Phys. (N.Y.) 7, 349 (1959).