On the nature of the Gardner transformation

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It is shown that every higher Korteveg-de Vries equation can be included in a one-parameter family of integrable equations.

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1. INTRODUCTION

The initial analysis of the Korteveg–de Vries (KdV) equation by Gardner, Green, Kruskal, Miura, and Zabusky, which later developed into a theory of the so-called "integrable systems", appears now to be a combination of some ingenious tricks (which have only been applied in the case of the KdV equation) and the general concept of "inverse scattering". While the general developments of the theory of inverse scattering are fairly well known (see, e.g., Refs. 1–4), the applicability of the ingenious tricks has not been examined in general; it is the goal of this note to discuss the degree of generality of the particular trick that has been called the "Gardner transformation" (see the historical remarks in Ref. 5, p. 422), which led to the discovery of an infinite number of conservation laws for the KdV equation.

2. THE GARDNER TRANSFORMATION FOR THE KdV EQUATION

We recall briefly the actual derivation, which is taken from Ref. 5. If v satisfies the modified KdV (MKdV) equation

$$v_t = 6v^2 v_x - v_{xxx}, \tag{1}$$

then

$$u = v^2 + v_x \tag{2}$$

satisfies the KdV equation

$$u_t = 6uu_x - u_{xxx}. \tag{3}$$

Equation (2) is called the "Miura transformation" (a general interpretation of it can be found in Ref. 6). Now (3) admits Galilean symmetry

$$t' = t, \qquad x' = x + 6ct, \qquad u' = u + c, \tag{4}$$

while (1) does not. Then a suitable combination of (2) and (4) shows that

$$u = w + \epsilon w_x + \epsilon^2 w^2, \tag{5}$$

the Gardner transformation, is a solution of (3) if w is a solution of

$$\begin{aligned} w_t &= 6ww_x - w_{xxx} + 6\epsilon^2 w^2 w_x \\ &= \partial(3w^2 - w_{xx} + 2\epsilon^2 w^3), \qquad \partial \equiv d/dx, \end{aligned} \tag{6}$$

which can be considered as a deformation of Eq. (3). Moreover, Eq. (6) also possesses an infinite number of conservation laws [because it becomes equivalent to (1) after a change of variables]. So we have an integrable deformation. Since such deformations are only very rarely integrable, the situation is quite intriguing. And even if all that were not enough, formula (5) tells us that Eqs. (3) and (6) are in fact equivalent as evolution fields [in other words, using (5) we can express w as a formal power series in ϵ with coefficients that are polynominals in $u, u_x, \cdots$ $w = u - \epsilon u_x + \epsilon^2(u_{xx} - u^2) + \cdots$. Since $w_t = \partial(3w^2 - w_{xx} + 2\epsilon^2 w^3)$ we obtain an infinite number of conservation laws for (3) by inverting (5); half of them are nontrivial and this completes the "classical" story of the Gardner transformation[7]].

3. INTERPRETATION

The Galilean invariance of the KdV equation which was used in an essential way in the construction of the deformation (5), is no longer available for the higher KdV equations which correspond to the Lax representation

$$L_t = [P, L], \tag{7}$$

with $L = -\partial^2 + u$. Therefore, if any deformation exists for the higher KdV equations it must be based on something other than Galilean invariance. We shall examine this in what follows.

First, we recall that one of the most important properties of the Miura map (2) is the fact that this map is a "canonical transformation" from the natural Hamiltonian structure $\partial(\delta/\delta v)$,

$$v_t = \partial(\delta H/\delta v), \tag{8}$$

of Eq. (1), namely,

$$v_t = 6v^2 v_x - v_{xxx} = \partial(\delta/\delta v)(\tfrac{1}{2}v^4 + \tfrac{1}{2}v_x^2), \tag{1'}$$

into the second Hamiltonian structure $B_2 = (-\partial^3 + 2\partial u + 2u\partial)(\delta/\delta u)$,

$$u_t = (-\partial^3 + 2u\partial + 2\partial u)(\delta h/\delta u) \tag{9}$$

of the KdV equation, namely

$$u_t = 6uu_x - u_{xxx} = (-\partial^3 + 2u\partial + 2\partial u)(\delta/\delta u)\tfrac{1}{2}u^2. \tag{3'}$$

The word canonical means that the corresponding Poisson brackets are compatible with the Miura map (2). Technically, this statement is equivalent to the equality

$$(2v + \partial)\partial(2v - \partial) = -\partial^3 + 2u\partial + 2\partial u, \tag{10}$$

where $2v + \partial = Du/Dv$ is the corresponding Fréchet derivative and $2v - \partial = (2v + \partial)^+$ is its adjoint operator (for details see Ref. 2 or 6).

Keeping this in mind it is natural to assume that the map (5) is also a canonical map from the natural Hamilton-

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ian structure $\partial(\delta/\delta w)$ of Eq. (6).

$$w_t = 6ww_x - w_{xxx} + 6\epsilon^2 w^2 w_x = \partial(\delta/\delta w)(w^3 + w_x^2/2 + \epsilon^2 w^4/2), \qquad (11)$$

into some other Hamiltonian structure, which must be nothing else but $[(Du/Dw)\partial(Du/Dw)^+]\delta/\delta u$. We have $(Du/Dw)\,\partial(Du/Dw)^+ = (1 + \epsilon\partial + 2\epsilon^2 w)\partial(1 - \epsilon\partial + 2\epsilon^2 w) = \partial - \epsilon^2\partial^3 + 2\epsilon^2[(w + \epsilon w_x + \epsilon^2 w^2)\partial + \partial(w + \epsilon w_x + \epsilon^2 w^2)] = \partial + \epsilon^2(-\partial^3 + 2u\partial + 2\partial u)$. Therefore the map (5) is indeed canonical into the Hamiltonian structure B:

$$B = [\partial + \epsilon^2(-\partial^3 + 2u\partial + 2\partial u)](\delta/\delta u), \qquad (12)$$

which is just the first Hamiltonian structure $B_1 = \partial(\delta/\delta u)$ of the KdV equation,

$$u_t = 6uu_x - u_{xxx} = \partial(\delta/\delta u)(u^3 + \tfrac{1}{2}u_x^2)$$

plus ϵ^2 times the second Hamiltonian structure B_2 of the KdV equation: $B = B_1 + \epsilon^2 B_2$.

Thus we have arrived at the combination (12) of two Hamiltonian structures for the higher KdV equations. This means that the Hamiltonians for these equations, if they exist, are formed in some way from the regular Hamiltonians H_n of the KdV equations. Recall that the sequence H_n of Hamiltonians is such that the equation

$$u_t = B_2(H_n) = B_1(H_{n+1}) \qquad (13)$$

is the higher KdV equation number n (see, e.g., Ref. 2), and all the Hamiltonians H_n commute in both the Hamiltonian structures B_1 and B_2.

Because the natural initial term $H_0 = u$ is such that $B_1(H_0) = 0$, we see that Eq. (13) can be written in terms of our mixed Hamiltonian structure (12) as

$$u_t(= B_2(H_n)) = B(\tilde{H}_n), \qquad (14)$$

where

$$\tilde{H}_n = \epsilon^{-2}\sum_{k=0}^{n}(-\epsilon^{-2})^k H_{n-k}. \qquad (15)$$

Notice that $\tilde{H}_n = \tilde{H}_n(\epsilon)$ is a *singular* function of ϵ. For example, the KdV equation could be written as

$$u_t = 6uu_x - u_{xxx} = [\partial + \epsilon^2(-\partial^3 + 2u\partial + 2\partial u)]\frac{\delta}{\delta u}\left(\frac{1}{\epsilon^2}\frac{u^2}{2} - \frac{1}{\epsilon^4}\frac{u}{2}\right),$$

and if we consider the linear combination of the KdV fields which corresponds to $H = \Sigma\, c_i H_i$,

$$u_t = B_2(H), \qquad (16)$$

then the same Eq. (16) in our mixed Hamiltonian structure B has the Hamiltonian

$$\tilde{H} = \sum_i c_i \epsilon^{-2}\sum_{k=0}^{i}(-\epsilon^{-2})^k H_{i-k}. \qquad (17)$$

4. DEFORMATION

The next step is to make sure that every deformed equation

$$w_t = \partial(\delta/\delta w)\tilde{H}^*, \qquad (18)$$

where $\tilde{H}^*(w) = \tilde{H}(u)$ at $u = w + \epsilon^2 w^2 + \epsilon w_x$ and H is taken from (17), is indeed a deformation of the "unperturbed" Eq. (16) in exactly the same manner as (6) is a deformation of (3). Of course, $B|_{\epsilon=0} = B_1$ but there is a potential source of difficulties in the singular dependence of $\tilde{H}$ upon ϵ in (17).

Theorem. The r.h.s. of the modified Eq. (18) is a polynomial in ϵ for $\tilde{H}$ taken from (17).

Proof: It suffices to check the claim for $\tilde{H} = \tilde{H}_n$ from (15). In this case the r.h.s. of (18) is clearly a finite polynomial in ϵ and ϵ^{-1}; we wish to show that this polynomial contains no terms which involve negative powers of ϵ. This follows from the *regular* invertibility of the deformation (5): We can write $w = u + \Sigma_{k=1}^{\infty}\epsilon^k P_k$, P_k being finite polynomial in $u, u_x, \ldots$, and therefore $w_t = \partial_t(u + \Sigma\,\epsilon^k P_k) = F + \Sigma\,\epsilon^k\,\Sigma(\partial P_k/\partial u^{(s)})\partial^s F$, where F is the r.h.s. of (16) and $u^{(s)} = d^s u/dx^s$. Substituting $u = w + \epsilon^2 w^2 + \epsilon w_x$ in the last expression we find w_t as a formal series in non-negative powers of ϵ only. Q.E.D.

Example: The next KdV equation after (3) is

$$\begin{aligned} u_t &= (-\partial^3 + 2u\partial + 2\partial u)(\delta/\delta u)(u^3 + u_x^2/2) \\ &= \partial[u^{(4)} - 10uu^{(2)} \quad - 5u^{(1)2} + 10u^3] \\ &= \partial(\delta/\delta u)(5/2u^4 + 5uu^{(1)2} + \tfrac{1}{2}u^{(2)2}). \end{aligned} \qquad (19)$$

Corresponding $\tilde{H} = \epsilon^{-2}(u^3 + u_x^2/2) - \epsilon^{-4}(u^2/2) + \epsilon^{-6}(u/2)$; this gives a deformation of Eq. (19) in the form

$$w_t = \partial(\delta/\delta w)\{\tfrac{5}{2}w^4 + 5ww^{(1)2} + \tfrac{1}{2}w^{(2)2} + \epsilon^2[5w^2w^{(1)2} + 3w^5] + \epsilon^4 w^6\}. \qquad (20)$$

Note that the (usual) modified equation associated with (19) via the Miura map (2) is

$$v_t = \partial(\delta/\delta v)\{v^6 + 5v^2v^{(1)2} + \tfrac{1}{2}v^{(2)2}\}. \qquad (21)$$

5. RELATIONS BETWEEN THE GARDNER AND MIURA TRANSFORMATIONS

In Secs. 3 and 4 we showed that the deformation (5) is indeed valid for the whole KdV hierarchy. We now wish to understand its relation with the Miura map (2).

The map (2) is canonical between the Hamiltonian structures $\partial(\delta/\delta v)$ and B_2 and therefore it is canonical for the Hamiltonian structures $\epsilon^2\partial(\delta/\delta v)$ and $\epsilon^2 B_2$. If we now make the translation $\tilde{u} = u + c\epsilon^{-2}$, then the map

$$\tilde{u} = c\epsilon^{-2} + v_x + v^2 \qquad (2')$$

is canonical between $\epsilon^2\partial(\delta/\delta v)$ and $\epsilon^2[-\partial^3 + 2(\tilde{u} - c\epsilon^{-2})\partial + 2\partial(u - c\epsilon^{-2})]\delta/\delta\tilde{u} = [\epsilon^2(-\partial^3 + 2\tilde{u}\partial + 2\partial\tilde{u}) - 4c\partial](\delta/\delta\tilde{u})$, which is exactly $B(\tilde{u})$ when $c = -\frac{1}{4}$. To eliminate ϵ^2 in $\epsilon^2\partial(\delta/\delta v)$ we set $v = \epsilon\tilde{v}$; then (2') becomes

$$\tilde{u} = -\epsilon^{-2}/4 + \epsilon^2\tilde{v}^2 + \epsilon\tilde{v}_{x'}, \qquad (2'')$$

which is a canonical map between $\partial(\delta/\delta\tilde{v})$ and $B(\tilde{u})$. To convert (2″) into a regular map we observe that the Hamiltonian structure $\partial(\delta/\delta\tilde{v})$ has *constant* coefficients and hence is invariant under translations of $\tilde{v}$. So if we let $\tilde{v} = w + b$, (2″) will become

$$\tilde{u} = -\epsilon^{-2}/4 + \epsilon^2 w^2 + 2\epsilon^2 wb + \epsilon^2 b^2 + \epsilon w_x. \qquad (2''')$$

Now the regularity condition for (2‴),

$$\epsilon^2 b^2 = 1/4\epsilon^2, \tag{22}$$

yields

$$b = 1/2\epsilon^2, \tag{23}$$

and (2‴) becomes (5).

6. DISCUSSION

For each higher KdV Eq. (13) we constructed its deformation (18) which has the following properties:

(i) There exists the *reduction map* (5) of the deformed equation into the undeformed one. Therefore the *deformed system is also integrable* (meaning: has an infinity of integrals), because all conservation laws (c.l.'s) of the undeformed equation become c.l.'s of the deformed equation after pull back;

(ii) The deformed equation (18) is Hamiltonian; it has now *only one* Hamiltonian structure, $\partial(\delta/\delta w)$. In this structure, all integrals of the deformed equations commute, since they are preimages of the c.l.'s which were in involution already, and the reduction map is *canonical*. Note that there is no such thing as "Lenard relations" (13) for the deformed equations.

It will be important to understand whether there exists any general "integrable deformations" pattern in the theory of integrable systems. The answer is undoubtedly yes and will be dealt with elsewhere. Here I shall make brief remarks.[8]

A) If one begins with the arbitrary scalar Lax equation (7) with

$$L = \partial^{n+2} + \sum_{i=0}^{n} u_i \partial^i \tag{24}$$

then one can construct the deformation theory, and generalizations of both properties (i) and (ii) from the above discussion remain true.

B) When an integrable equation is not bi-Hamiltonian, Hamiltonian formalism is of little help either to find a deformation or to interpret it.

Examples:

1) If

$$p_t = 6p_x C(1+\epsilon^2 C) - p_{xxx} + 2\epsilon^2 \nu^2 p_x^3, \tag{25}$$

where

$$C = (\text{sh} 2\epsilon\nu p)/(2\epsilon\nu) + (\text{ch} 2\epsilon\nu p - 1)/(2\epsilon^2), \tag{26}$$

then

$$W = C + \nu p_x \tag{27}$$

satisfies (6). Thus (25) represents the *second* deformation of the KdV equation (3).

(2) If

$$q_t = \partial[2q^3 - q_{xx} + 6\epsilon^2 q q_x^2/(1+4\epsilon^2 q^2)], \tag{28}$$

then

$$w = [(1+4\epsilon^2 q^2)^{1/2} - 1]/2\epsilon^2 + q_x(1+4\epsilon^2 q^2)^{-1/2} \tag{29}$$

satisfies (6),

$$v = q + \epsilon q_x (1+4\epsilon^2 q^2)^{-1/2} \tag{30}$$

satisfies (1), and we have the commutative diagram (2)·(30) = (5)·(27). Thus (28) is the deformation of the MKdV equation (1), (30) is the reduction map, and (29) is the *deformation of the Miura map*. This suggests that not only integrable systems but also their relationships are objects of deformations.

(3) Deformations phenomenon is not the privilege of only the Lax equations. Consider, e.g., the Benney equations for long waves on a two-dimensional surface[9]

$$a_{n,t} = a_{n+1,x} + n a_{n-1} a_{0,x}, \qquad n = 0,1,2,\cdots \tag{31}$$

for the sequence of functions $a_n(x,t)$. This system has an infinity of integrals $h_n \in a_n + \mathbb{Z}[a_0,\ldots,a_{n-2}]$ (see Ref. 2).

Proposition 32: Let

$$A_{n,t} = A_{n+1,x} + nA_{n-1}A_{0,x} + \epsilon[A_0 A_{n,x} + (n+1)A_n A_{0,x} + nA_{n-1}(A_{1,x} - \epsilon A_0 A_{0,x}/2)]/2, \qquad n = 0,1,2,\cdots. \tag{33}$$

Denote by $H_n \in A_n + \mathbb{Q}[A_0,\ldots,A_{n-1}]$ the integral $\#n$ for (33). Then the map

$$a_n = A_n + O(\epsilon), \tag{34}$$

such that

$$h_n = H_n + \epsilon H_{n+1}, \tag{35}$$

maps solutions of (33) into solutions of (31).

C) Evidently conservation laws survive deformations, i.e., remain nontrivial under deformations. Therefore it is important to know which integrals of the undeformed equations were nontrivial in the first place.

Let us consider, as an example, the well-known case of the KdV hierarchy (13). Then the r.h.s. of (18) shows that for the deformed equation w is the c.l. Therefore inverting the Gardner transformation (5)

$$w = \sum_{n=0}^{\infty} h_n \epsilon^n \tag{36}$$

one gets an infinity of c.l.'s $h_n \in \mathscr{A}$ where $\mathscr{A}$ is the ring of polynomials in $u, u_x, \cdots,$. If $f, g, \in \mathscr{A}$, let us write $f \approx g$ is $f(u,0,0,\cdots) = g(u,0,\cdots)$, and $f \sim 0$ if $f = \partial g$.

Proposition 37: $h_{2n+1} \sim 0, h_{2n} \not\sim 0$.

Proof: 1) Write $w = w^+ + w^-$ where $w^+ = \Sigma\, h_{2n}\epsilon^{2n}$, $w^- = \Sigma\, h_{2n+1}\epsilon^{2n+1}$ and substitute this into (5). Then the part which is odd in ϵ yields $w^- - \epsilon w_x^+ - 2\epsilon^2 w^+ w^- = 0$, or $w^- = -(2\epsilon)^{-1}\partial \ln(1 - 2\epsilon^2 w^+)$. 2) From (5) one gets $u \approx w + \epsilon^2 w^2$, so $w \approx (2\epsilon^2)^{-1}[1 - (1 - 4\epsilon^2 u)^{1/2}] = \Sigma_{\infty}^{n=0} c_n \epsilon^{2n} \mu^{n+1}$, all c_n's are different from zero. Thus $h_{2n} \approx c_n u^{n+1}$. Note now that if $f \sim 0$ then $f \approx 0$. Q.E.D.

[1] I. M. Krichever, Russ. Math. Surv. **32**, 185–213 (1977).
[2] Yu. I. Manin, J. Sov. Math. **11**, 1–122 (1979).
[3] G. Wilson, Math. Proc. Cambridge Philos. Soc. **86**, 131–143 (1979).
[4] I. M. Gel'fand and L. A. Dikii, Russ. Math. Surv. **30**, 67 (1975).
[5] R. M. Miura, SIAM Review **18**, 412 (1976).
[6] B. A. Kupershmidt and G. Wilson, "Modifying Lax Equations and the second Hamiltonian structure," Invent. Math. (to appear).
[7] R. M. Miura, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. **9**, 1204 (1968).
[8] B. A. Kupershmidt, "Deformations of Integrable Systems" (to appear).
[9] D. J. Benney, Stud. Appl. Math. **52**, 45 (1973).
[10] B. A. Kupershmidt, "Integrable Two-Dimensional Systems" (to appear).