Electrostatic screening*

Jeffrey Rauch and Michael Taylor

Department of Mathematics, The University of Michigan, Ann Arbor, Michigan 48104
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Using the methods of partial differential equations and functional analysis, we investigate the electric field in the presence of a screen composed of wires of radius r spaced at distance R spread over a surface S. In the limit as r and R converge to zero if $[R \ln r]^{-1} \to -\infty$, the field in the presence of the screen converges to the field with a conducting sheet spread over S. If $[R \ln r]^{-1} \to 0$, the field converges to the field with no conductors.

1. INTRODUCTION

It is well known that a region enclosed by a mesh of conducting wire is shielded from external static electric fields. In this sense the mesh acts like a solid sheet of conductor. On the other hand, it is clear that if the wires of the mesh are sufficiently narrow (for fixed mesh width), then they will have a negligible effect on the electric field. In this note we will study the problem of determining what range of physical parameters correspond to these two types of behavior. If the screen consists of wires of radius r whose axes are spaced at approximately distance R from each other, the critical parameter is $[R \ln r]^{-1} \equiv \delta$. We consider screens spread over a surface s in the limit as r and R approach zero and prove that for any charge distribution, if $\delta \to +\infty$, then the field in the presence of the screens converges to the field in the presence of a sheet of conductor spread over S (Theorem 2). In the opposite extreme case, if $\delta \to 0$, then the field converges to the field without any conductors present, that is, the screen becomes negligible.

2. VARIATIONAL FORMULATION OF THE BASIC BOUNDARY VALUE PROBLEM

We seek the electrostatic potential u in the exterior of a finite number of conductors $\kappa_1, \kappa_2, \cdots, \kappa_j$, arising from a charge distribution with density $\rho(x)$. For convenience we suppose that the whole system lies inside a very large but bounded region $\mathcal{R}$ whose boundary is kept at potential zero and is assumed to be smooth. With a little extra effort the problem in unbounded regions can also be handled by our methods. The boundary value problem for u is

$$\Delta u = -4\pi\rho \quad \text{in } \mathcal{R}\backslash \cup \kappa_i, \tag{1}$$

$$u = \text{const} \quad \text{on each } \kappa_i, \quad i, \cdots, j, \tag{2}$$

$$\int_{\partial\kappa_i} \frac{\partial u}{\partial \nu} = 0, \quad i = 1, \cdots, j, \tag{3}$$

$$u = 0 \quad \text{on } \partial\mathcal{R}. \tag{4}$$

From a mathematical standpoint the condition (3) which asserts that the conductors carry no charge is the most troublesome, and we will give a weak or variational formulation in which (3) becomes a natural boundary condition. Let $K = \cup \kappa_i$, $\Omega = \mathcal{R}\backslash K$, and $H_1(\Omega)$ the Sobolev space of functions on Ω which are square integrable together with their partial derivatives of order one.

Definition 1: $\mathcal{B}$ is the closed subspace of $H_1(\Omega)$ consisting of functions u which vanish on $\partial\mathcal{R}$ and in addition are constant on $\partial\kappa_i$, $i = 1, 2, \cdots, j$. For $u, v \in H_1(\Omega)$ let

$$a(u,v) = -\int_\Omega (\text{grad} u \cdot \text{grad} v).$$

It is not hard to show that u is a solution of (1)–(4) if and only if $u \in \mathcal{B}$ and

$$a(u,v) = 4\pi \int_\Omega \rho(x) v(x) dx \;\; \forall\, v \in \mathcal{B}. \tag{5}$$

Equation (5) is just the Euler–Lagrange equation associated with Thompson's principle: u minimizes $-a(u,u)/2 + \int_\Omega \rho u$ over all $u \in \mathcal{B}$. Note that (3) is a natural boundary condition. It is useful to notice that if $u \in \mathcal{B}$ satisfies (5), then $\Delta u = \rho$ in the sense of distributions and $u = \text{const}$ on $\partial\kappa_i$ so that the regularity theorems for the Dirichlet problem can be applied to show that u is smooth provided that the boundaries of the κ_i are smooth, which we will assume henceforth.

The quadratic form a on $L^2(\Omega)$ with domain $D(a) = \mathcal{B}$ is a closed symmetric and nonpositive. It is well known[1] that there is a self-adjoint operator Δ defined by the recipe

$$D(\Delta) = \{u \in \mathcal{B}: (\exists f \in L^2(\Omega)) \text{ such that } a(u,v) = (f,v)_{L^2(\Omega)}\},$$

$$\Delta u = f \quad \text{for } u \in D(\Delta).$$

With the aid of the regularity theorems mentioned above one can show that

$$D(\Delta) = \{u \in H^2(\Omega) : u \in \mathcal{B} \text{ and } \int_{\partial\kappa_i} \frac{\partial u}{\partial \nu} = 0, \; i = 1, \cdots, j\},$$

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i 2} \quad \text{for } u \in D(\Delta).$$

The solution to the electrostatics problem (1)–(4) is therefore $\Delta^{-1}(-4\pi\rho)$, the inverse of Δ applied to $-4\pi\rho$.

3. A THEOREM ON VANISHING SCREENS

We now pose the basic problem. For each integer n we consider the electrostatics problem in the presence of conductors $\kappa_1^n, \kappa_2^n, \cdots, \kappa_{j_n}^n$ and we ask whether the effect of the conductors has some limiting behavior as $n \to \infty$. In this section we prove a theorem which asserts that the effect of the conductors disappears as $n \to \infty$ provided they are sufficiently small. As an application we obtain the result on vanishing screens mentioned in the Introduction.

The appropriate measure of smallness turns out to be electrostatic capacity. Recall that for reasonable subsets Λ of $\mathbb{R}^3$, $\text{cap}(\Lambda)$ is defined as follows: Let v be the solution of the boundary value problems

$$\Delta v = 0 \text{ on } \mathbb{R}^3\backslash\Lambda,$$
$$v = O(1/|x|) \text{ as } |x| \to \infty,$$
$$v = 1 \text{ on } \partial\Lambda.$$

Then $-\int_{|x|=L} \partial v/\partial r$ is independent of L for L large and is the total charge on a conductor occupying the region Λ and raised to potential one. This quantity is $\text{cap}(\Lambda)$, the capacity of Λ.

Notations: Let Δ_n, a_n, $\mathcal{B}_n$ be the operator, form, and form domain on $\Omega_n \equiv \mathcal{R}\backslash \cup_i \kappa_i^n$ as defined in Sec. 2. In addition, for $v \in L^2(\mathcal{R})$ let $P_n v \in L^2(\Omega_n)$ be the restriction of v to Ω_n. Any element of $L^2(\Omega_n)$ is considered as an element of $L^2(\mathcal{R})$ by extending it to vanish on the union of the κ_i^n. Let $K(n) = \cup_i \kappa_i^n$ denote this union. We suppose all $K(n)$ are contained in some compact set $\Gamma \subset \mathcal{R}$.

The main tool we use to show that $K(n)$ vanishes is Theorem 3.1 of Ref. 2. This asserts that $f(\Delta_n)P_n u \to f(\Delta)u$ in $L^2(\mathcal{R})$ for all $u \in L^2(\mathcal{R})$ and any f bounded and continuous on $(-\infty, 0]$ provided the Ω_n satisfy mild regularity conditions, that the quadratic form $a(u,u)$ satisfies the coerciveness hypothesis $-a(u,u) \geq \int_{\Omega_n} |\text{grad} u|^2$ for all $u \in \mathcal{B}_n$, and the following two special assumptions:

(A) There exist extension operators E_n: $\mathcal{B}_n \to \mathcal{B}$ [the domain of the form $a(u,v)$ on $\mathcal{R}$ without conductors] with the properties

(i) $E_n u = u$ on Ω_n for all $u \in \mathcal{B}_n$,

(ii) there is a constant M such that for all n and $u \in \mathcal{B}_n$,

$$\|E_n u\|_{H_1(\mathcal{R})} \leq M\|u\|_{H_1(\Omega_n)};$$

and either

(B) $\text{Meas}(K(n)) \to 0$, and if $u \in \mathcal{B}$, there exist $u_j \to u$ in $\mathcal{B}$ such that $u_j|_{\Omega_j} \in \mathcal{B}_j$,

or

(B′) $\text{cap}(K(n)) \to 0$ as $n \to \infty$.

That (A), (B′) imply operator convergence is stated in Theorem 4.2 of Ref. 2; alternatively, condition (B′) implies condition (B).

Theorem 1: Suppose there is a compact set $\Gamma \subset R$ with $K(n) \subset \Gamma$ for all n and that $\text{cap}(K(n)) \to 0$ as $n \to \infty$. Then for any continuous function f on $(-\infty, 0)$ bounded at $-\infty$ and any $u \in L^2(\mathcal{R})$ we have

$$f(\Delta_n)P_n u \to f(\Delta)u \text{ in } L^2(\mathcal{R}),$$

where Δ is the operator on $\mathcal{R}$ without any conductors.

As a particular example, for $\rho \in L^2(\mathcal{R})$ with ρ supported in the exterior of all conductors, we can take $f(x) = 1/x$ to get $\Delta_n^{-1}(\rho) \to \Delta^{-1}(\rho)$ in $L^2(\mathcal{R})$. Thus the solutions of the electrostatics problems converge to the solution to the problem with no conductors at all.

Proof: Note that $\sigma(\Delta)$ and $\sigma(\Delta_n) \subset (-\infty, \delta)$ for some $\delta < 0$, so f can be altered to be bounded and continuous on $(-\infty, 0]$ without changing $f(\Delta_n)$ or $f(\Delta)$. To complete the proof, it is only necessary to verify hypothesis (A).

To describe E_n notice that if $u \in \mathcal{B}_n$, then u is constant on $\partial\kappa_i^n$, $i = 1, 2, \cdots, j_n$, say $u = c_i$ on $\partial\kappa_i^n$. Define $E_n u = c_i$ on κ_i^n. It is clear that $\int_R |\text{grad} E_n u|^2 = \int_{\Omega_n} |\text{grad} u|^2$. Furthermore, since $E_n u = 0$ on ∂R, we have

$$\int_R |E_n u|^2 \leq \int_R |\text{grad} E_n u|^2,$$

where $\lambda < 0$ is the largest eigenvalue of the Laplacian on $\mathcal{R}$ with Dirichlet boundary conditions on $\mathcal{R}$. Thus (ii) is satisfied with $M = 1 + \lambda^{-1}$ and the proof is complete. □

It is quite easy to apply this result to screens. The basic fact that is needed is that the capacity of a solid circular cylinder of length L and radius r is proportional to $-L/\ln r$. Similarly a not excessively curved piece of wire of length L and radius $\mathcal{R}$ has capacity $O(-L/\ln r)$. In addition, capacity is a subadditive set function, that is, $\text{cap}(\cup A_i) \leq \Sigma_i \text{cap}(A_i)$ for any countable union of sets. Thus the capacity of a curved screen of fixed area with wires of radius r and spacing R between axes of the wires is $O(-1/R\ln r)$. Thus if $K(n)$ is a screen as above with r and R approaching zero as $n \to \infty$ in such a way that $1/R\ln r \to 0$, then the effect of the screen is negligible for n large.

For the electrostatic problem, $\text{cap}K(n) \to 0$ is by no means a necessary condition for the $K(n)$ to have a negligible effect. Suppose, for example, that $K(n)$ consists of n balls, of radius r_n, and say their center ξ_{jn} are spaced at a distance at least $4r_n$. By defining extension operators E_n as in the proof of Theorem 1, it is easy to see that hypothesis (A) is satisfied. We show that hypothesis (B) is verified, assuming $\text{vol}K(n) = (4/3\pi)nr_n^3 \to 0$.

Define a continuous linear map Q: $H_1(B_2) \to H_1(B_2)$ ($B_2 = \{x: |x| \leq 2\}$) such that

(i) $Qu(x) = u(x)$ for $3/2 \leq |x| \leq 2$

(ii) $Qu(x)$ is constant for $|x| \leq 1$

(iii) $\int_{B_2} |Qu|^2 \leq C_0 \int_{B_2} |u|^2$

(iv) $\int_{B_2} |\text{grad} Qu|^2 \leq C_0 \int_{B_2} |\text{grad} u|^2$.

This is easy to arrange. Given this, you can scale B_2 to $B_{2r_n}(\xi_{jn}) = \{x: |x - \xi_{jn}| \leq 2r_n\}$ and get maps with the same properties as i–iv (same constant C_0). Thus you get maps Q_n: $\mathcal{B}_n \to \mathcal{B}$ such that

(i) $Q_n u(x) = u(x)$, $x \notin \bigcup_j B_{2r_n}(\xi_{jn})$,

(ii) $\|Q_n u\|^2_{H_1(B_{2r_n}(\xi_{jn}))} \leq C_0\|u\|^2_{H_1(B_{2r_n}(\xi_{jn}))}$,

(iii) $Q_n u|_{\Omega_n} \in \mathcal{B}_n$.

Now with $u_n = Q_n u$ you get

$$\|u_n - u\|^2_{H_1(\mathcal{R})} = \sum_j \|u_n - u\|^2_{H_1(B_{2r_n}(\xi_{jn}))}$$
$$\leq 4C_0 \sum_j \|u\|^2_{H_1(B_{2r_n}(\xi_{jn}))} \to 0$$
$$\text{as } n \to \infty \text{ since meas } \bigcup_j B_{2r_n}(\xi_{jn}) \to 0.$$

This verifies hypothesis (B).

The conclusion is that if $K(n)$ consists of n "well spaced" balls of radius r_n, then $K(n)$ disappears as $n \to \infty$, *assuming only that* $\text{vol}K(n) \to 0$.

4. THE CASE OF ELECTROSTATIC SCREENING

In this section we will investigate the observed phenomenon of screens behaving like solid barriers. To be more precise, suppose that $K(n)$ is a conducting screen, with wires of radius r and spacing R, spread smoothly over the surface S and that r and R tend to zero as $n\to\infty$. If $(-R\ln r)^{-1}\to+\infty$ as $n\to\infty$, then for any charge distribution ρ on $\mathcal{R}$ the solutions, $\Delta_n^{-1}(\rho)$, of the electrostatics problems in $\mathcal{R}\backslash K(n)$ converge to the solution u of the problem where S is covered by a sheet of perfect conductor, that is,

$$\Delta u=-4\pi\rho \quad \text{in } \mathcal{R}\backslash s, \tag{6}$$

$$u=\text{const} \quad \text{on } S, \tag{7}$$

$$\int_S\left[\frac{\partial u}{\partial v}\right]=0 \quad \text{([ ] denotes jump on crossing } S) \tag{8}$$

$$u=0 \quad \text{on } \partial\mathcal{R}. \tag{9}$$

This result complements the result of Sec. 3 and confirms the idea that the parameter $(-R\ln r)^{-1}$ is a reasonable measure of the solidity of a screen. It is interesting to note the same parameter occurs in the clever special problem treated in §203 of Maxwell's treatise.[3] In addition, as Maxwell observed, a complete screen is not needed, just one family of parallel wires which are connected to each other in any way at all will suffice.

We must make precise the notion of a screen spread smoothly over S, where S is an open subset of a compact surface in the interior of $\mathcal{R}$. The intuitive idea is to take a piece of planar screen and give a mapping of the planar region to the surface. Precisely, if $s\in S$ and $\mathcal{O}$ is an open neighborhood of s in $\mathbb{R}^3$, then a mapping ψ: $\mathcal{U}\to\mathcal{O}$ is called a δ *bending* if

(i) $\mathcal{U}$ is a cube $|x_i|<\alpha$, $i=1, 2, 3$,

(ii) $\psi[\mathcal{U}\cap\{x_3=0\}]=S\cap\mathcal{O}$,

(iii) ψ is a diffeomorphism with $\|J_\psi\|$ and $\|J_{\psi^{-1}}\|$ less than δ where J is the Jacobian matrix.

Screens are laid on S by placing a screen in the $x_3=0$ plane of $\mathcal{U}$ and carrying it to S by the map ψ.

Definition 2: A *patch of* δ *bent screen* on S consisting of wires of radius r and spacing R is the set $\psi[\Sigma]$, where ψ: $\mathcal{U}\to\mathcal{O}$ is a δ bending and

$$\Sigma=\{x\in\mathcal{U}: (x_1-jR)^2+x_2^2\le r^2 \text{ for some } j\}.$$

In addition we require $R>3r$.

To form a picture, notice that the wires in Σ are parallel to the x_2 axis. The only interesting case of screening is when the screen has large gaps, that is, $R\gg r$.

Definition 3: A sequence of systems of conductors will be called *screens smoothly covering* S if there is a $\delta>0$, an $\alpha>0$, and an integer M such that (1) each system consists of at most M patches of δ bent screen on S, (2) the sets $\psi(\mathcal{U}_i)$, $i=1,\cdots,M$, cover S for each system, and (3) the lengths of the sides of the cubes are all greater than α.

It is important that the electrostatic potential be constant on the screen, not just on the individual wires from which it is constructed. There are two ways we could arrange this. In one approach, we could suppose that a few wires are added to the screen so that it becomes a connected set. In the second we just prescribe the constancy of the potential on the screen as a boundary condition. Both methods yield the same results and we will adopt the second so that the basic boundary value problem becomes (1)–(4) with $j=1$ and κ_1 the screen on S.

As in Sec. 2: the boundary value problem (6)–(9) can be given a variational formulation in which $u=\Delta_\infty^{-1}(-4\pi\rho)$, where Δ_∞ is the operator on $L^2(R)$ defined by the quadratic form

$$a(u,v)=\int_R \text{grad}u\cdot\text{grad}v,$$

$$D(a_\infty)=\{u\in H_1(R): u=0 \text{ on } \partial\mathcal{R} \text{ and } u \text{ is constant on } S\}.$$

Theorem 2: Suppose that $K(n)$, $n=1,2,\cdots$, are screens smoothly placed on S, where $K(n)$ consists of wires of radius r_n and spacing R_n. Let Δ_n be the operator on $L^2(\mathcal{R}\backslash K(n))$ as in Sec. 2 and P_n: $L^2(R)\to L^2(\mathcal{R}\backslash K(n))$ be the restriction mapping. If $(-R_n\ln r_n)^{-1}\to\infty$, then for any continuous function f on $(-\infty,0)$ bounded at $-\infty$, $f(\Delta_n)P_n\rho\to f(\Delta_\infty)\rho$ in $L^2(\mathcal{R})$ for any $\rho\in L^2(\mathcal{R})$.

Proof: We describe the modifications that are required to adopt the methods of our paper on wild perturbations[2] to this setting. For the remainder of the proof this paper is referred to as PSWPD. First we define uniformly bounded extension operators E_n: $\mathcal{B}_n\to\mathcal{B}_\infty\equiv D(a_\infty)$ by extending functions to be constant inside κ_i^n. As in our previous work (see the proof of Theorem 1.2 of PSWPD) it suffices to prove the result for $f=(1-x)^{-1}$. Imitating the proof of Theorem 4.4 of PSWPD, we notice that for $g\in L^2(\mathcal{R})$

$$\begin{aligned}\|(1-\Delta_n)^{-1}P_ng\|^2_{H_1(\mathcal{R}\backslash K(n))}&=((1-\Delta_n)(1-\Delta_n)^{-1}P_ng, (1-\Delta_n)^{-1}P_ng)_{\mathcal{R}\backslash K(n)}\\&=(P_ng, (1-\Delta_n)^{-1}P_ng)_{\mathcal{R}\backslash K(n)}\\&\le\|g\|^2_{L^2(\mathcal{R})}\end{aligned}$$

so that $w_n\equiv E_n(1-\Delta_n)^{-1}P_ng$ is a bounded sequence in $H_1(\mathcal{R})$. By using Eq. (5) on $\Omega=\mathcal{R}\backslash K(n)$ for the function w_n it is easy to show that if w is a limit point of the sequence $\{w_n\}$ in the weak topology for $H_1(\mathcal{R})$, then

$$\int_{\mathcal{R}}(wu-\text{grad}w\cdot\text{grad}u)=\int_{\mathcal{R}}gu \tag{10}$$

for all $u\in H_1(\mathcal{R})$ such that u is constant on a neighborhood of S. Since these u are dense in $\mathcal{B}_\infty=D(a)$, (10) holds for all $u\in\mathcal{B}_\infty$. To show that $w=(1-\Delta_\infty)^{-1}g$, it therefore suffices to prove that $w\in\mathcal{B}_\infty$, that is, $w=\text{const}$ on S and $w=0$ on $\partial\mathcal{R}$. The latter is true since $\{v\in H_1(\mathcal{R})|v=0 \text{ on } \partial\mathcal{R}\}$ is a closed linear subspace, hence weakly closed. That w is constant on S lies considerably deeper. The crucial inequality is the following:

Let $\mathcal{U}$, Σ, r, R be as in Definition 2 and let $\mathcal{U}_H=\mathcal{U}\cap\{|x_3|\le H\}$.

There is a constant b independent of H such that for all $v\in H_1(\mathcal{U})$ with $v|_\Sigma=0$,

$$\int_{\mathcal{U}_H}|\text{grad}v|^2\Big/\int_H|v|^2\ge\frac{b}{H^2-HR\ln(r/R)} \tag{11}$$

provided $H>R>3r$.

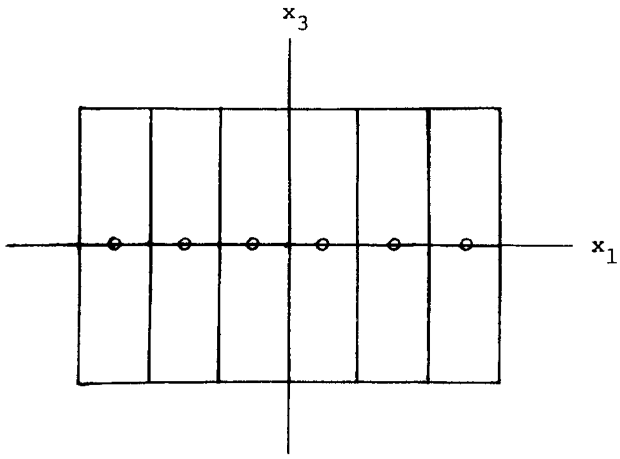

FIG. 1.

The verification of inequality (11) is postponed to the end of the proof. Let $\mathcal{U}_1^n, \cdots, \mathcal{U}_{j_n}^n$ be cubes with $\cup_i \psi_i^n(\mathcal{U}_i^n) \supset S$, the screen $K(n) = \cup_i \psi_i^n(\Sigma_i^n)$. Let $c_n = w_n|_{\text{screen}}$ and apply the inequality to $w_n \circ \psi_i^n - c_n$. In this case, $R_n \ln(r_n/R_n) \to 0$ as $n \to \infty$ so the right-hand side of (11) behaves like const H^{-2} for n large. Letting $S_H^n = \cup_i \psi_i^n(\mathcal{U}_H)$, we get

$$\int_{S_H^n} |w_n - c_n|^2 \le \text{const } H^2. \tag{12}$$

Let $S_H = \{x: \text{dist}(x, S) \le H\}$; then, since $|c_n - c_m| \le |w_n - c_n| + |w_n - c_m|$, it follows that $\int_{S_H} |c_n - c_m|^2 \le \text{const } H^2$ provided $\delta H > R_n$, R_m for some δ independent of n, m. Since vol (S_H) approaches zero like a multiple of H, we get $|c_n - c_m|^2 = O(H)$ for $H > R_n$, R_m. Letting n, m tend to infinity, we see that $\{c_n\}$ is a Cauchy sequence so that $c_n \to c$ for some c. Passing to the limit in (12) yields

$$\frac{1}{H}\int_{S_H} |w - c|^2 = O(H), \tag{13}$$

and it follows that $w = c$ on S, since

$$\int_S (w - c)^2 \le \text{const} \lim_{H \to 0} H^{-1} \int_{S_H} (w - c)^2.$$

We have now shown that w_n converges weakly in $H_1(\mathcal{R})$ to $w \equiv (1 - \Delta_\infty)^{-1} g$. Since $\|w_n\|_{H_1(\mathcal{R})}$ is bounded independent of n, it follows by the Rellich compactness theorem that $\{w_n\}$ is precompact in $L^2(\mathcal{R})$. Since w_n converges weakly to w in $L^2(\mathcal{R})$ it follows that $w_n \to w$ in $L^2(\mathcal{R})$ which is the desired result.

We now return to the proof of inequality (11). This is reduced to a two-dimensional problem by considering the $x_2 = \text{const}$ cross sections of $\mathcal{U}_H$. For these cross sections we prove that

$$\int_{\text{cross section}} \left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 dx_1 dx_3 \ge \frac{\text{const}}{H^2 - HR\log(r/R)} \int_{\text{cross section}} v^2 \tag{14}$$

for v which vanish on Σ. This in turn can be proven by chopping the cross section into punctured rectangles as in Fig. 1. It suffices to prove (14) where the integration is only over one of the punctured rectangles. The lower bound for the punctured rectangles is proved exactly as inequality (4.1) of PSWPD and the argument is not reproduced here. This completes the proof of Theorem 2. □

The phenomenon just considered has a great deal in common with the behavior of the Dirichlet problem, although the proof in the case of the electrostatic boundary problem is a little more involved. It is interesting to note that the electrostatic problem can exhibit behavior markedly different from that of the Dirichlet problem. For example, suppose the wire screen described above consists of wires which are not connected, that is, not at a common potential. If the wires are parallel to a vector field X on the surface S, and if $-(R_n \log r_n)^{-1} \to \infty$, then the u_n converge to a solution to the problem

$$\Delta u = -4\pi\rho \quad \text{on } \mathcal{R}\backslash S, \tag{14}$$

$$[u] = 0 \quad \text{on } S, \tag{15}$$

$$Xu = 0 \quad \text{on } S, \tag{16}$$

$$\int_S \left[\frac{\partial u}{\partial v}\right] v = 0 \quad \text{for all } v \in C^\infty(S) \text{ with } Xv = 0, \tag{17}$$

$$u = 0 \quad \text{on } \partial\mathcal{R}. \tag{18}$$

Since this is not a straightforward application of previously stated results, we indicate a proof. Let $u_n = \Delta_n^{-1}(-4\pi\rho)$, where Δ_n is defined on $\mathcal{R}$ with electrostatic boundary conditions on the wires $K(n)$, and u_n is extended as a constant on each wire. As usual, $\{u_n\}$ is bounded in $\mathring{H}_1(\mathcal{R})$, and so has a weak limit point $u \in \mathring{H}_1(\mathcal{R})$. Clearly u satisfies (14), (15), and (18) above so that we need to prove (16) and (17). Furthermore, we need only consider those ρ which vanish in a neighborhood of S since these are dense in $L^2(\mathcal{R})$. If we prove

$$-a(u, v) = (-4\pi\rho, v) \quad \text{for all } v \in B$$
$$\text{where } B = \{v \in H_1(\mathcal{R}): Xv = 0 \text{ on } S\}, \tag{19}$$

then (17) will arise as a natural boundary condition.

To prove (19), we need only observe that for each $v \in B$ there exist $v_n \in B$ such that v_n is constant on each wire of $K(n)$ and $v_n \to v$ in B as $n \to \infty$. Then (19) holds for v_n and we may pass to the limit. The existence of such v_n is proven by constructing operators analogous to the Q's at the end of Sec. 3.

It only remains to prove that $Xu = 0$ on S, i.e., that $u \in B$. Indeed, by previous calculations

$$\frac{1}{H}\int_{S_H} |u_n - c_n|^2 \le b\left(H - R_n \ln\frac{r_n}{R_n}\right).$$

This time, c_n is not a constant, but it is constant on the wires of $K(n)$, and in the direction normal to S. It merely varies from wire to wire. Thus $c_n \in L^2(S)$, $Xc_n = 0$. A trivial estimate is

$$\frac{1}{H}\int_{S_H} |u_n - \tilde{u}_n|^2 \le \beta(H) \to 0 \quad \text{as } H \to 0,$$

where $\tilde{u}_n = u_n|_S$ extended to S_H as a function independent of the normal variable. Putting these together and letletting $H \to 0$ yields

$$\int_S |\tilde{u}_n - c_n|^2 \to 0 \quad \text{as } n \to \infty.$$

Since $\tilde{u}_n \in H^{1/2}(S)$ is bounded, passing to a subsequence you get $\tilde{u}_n \to u|_S$ in $L^2(S)$. Hence $c_n \to u|_S$ in $L^2(S)$, so that $Xu = 0$ on S, as desired.

An even greater disparity is observed if $K(n)$ consists of n balls of radius r_n, with center ξ_{jn} lying on S and spaced apart a distance at least $4r_n$ [or $K(n)$ could consist of discs, the intersection of S with these balls]. If these obstacles are connected, say by arbitrarily thin wires, arguments as in the proof of Theorem 2 show that $K(n)$ behaves in the limit as a solid screen S, provided $nr_n \to \infty$. For this proof Lemma 4.5 of Ref. 2 is needed in place of (11). On the other hand, surely $\mathrm{vol}K(n) \to 0$, so if $K(n)$ is not connected, as we have seen at the end of Sec. 3, the obstacles disappear as $n \to \infty$.

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