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# THE EFFECT OF THE ATMOSPHERE REFRACTIVE INDEXES ON THE ACCURACY OF DOVAP

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## ABSTRACT

This report discusses to what extent the refractive indexes in the atmosphere affect the DOVAP accuracy. The basic DOVAP equation is derived and applied to propagation under the conditions of varying refractive indexes due to air density and ionization. It is suggested that a small improvement in position determination can result if a variable average velocity of radio waves, as a function of altitude, is used. Calculations indicate that the ionosphere propagation will have only a very small effect on the accuracy at altitudes under 90 km. The effect of the curved propagation path, i.e., the difference between the actual propagation path and the straight-line distance, has been shown to be negligible, contrary to previous estimates of the effect.

# 1. INTRODUCTION

DOVAP (derived from Doppler Velocity and Position) is a method for accurately determining the trajectory of a missile. DOVAP provides for transmitting a frequency-stabilized continuous radio-frequency wave from a fixed ground station to the missile, doubling the frequency of this transmitted wave in the missile receiver, and transmitting the doubled frequency from the missile to four fixed ground receivers. The frequency of the missile signal as received at each ground station is heterodyned with twice the frequency of the original transmitted ground signal. This is accomplished by receiving this reference frequency on a separate receiver at each ground station and doubling the frequency by the same method used in the missile receiver. The beat, or doppler, frequency is produced by mixing the radio-frequency output from the two receivers. This frequency is recorded on magnetic tape or on film during the missile flight.

Each doppler-frequency cycle represents a change of one half the wavelength of the reference frequency in path length from the transmitter to the missile to the ground receiver. From the path lengths to three or more receivers, the spatial coordinates of the missile at any instant may be computed by solving a group of equations representing the intersection of three or more ellipsoids.

The reference frequency of the existing DOVAP system is approximately 37 mc.* Since the retransmitted carrier frequency is twice the received frequency, the missile-borne transmitter operates at about 74 mc.

Thus DOVAP is a missile-positioning system in which one-half of the wavelength of the reference frequency is the basic yardstick. This yardstick is affected by the refractive index on the path of propagation. The refractive index in the atmosphere depends on the air density and varies strongly with the degree of ionization. The problem discussed in this report is to what extent the refractive indexes in the atmosphere affect the DOVAP accuracy. Primary consideration is given to the use of DOVAP in the rocket-grenade experiment for upper-atmosphere temperature and winds.

*This frequency is assumed in all the computations, except that in Appendix A the actual frequency recorded in the pre-IGY flight SM1.01, 36,938,775 cps, is used. The frequency has been changed to about 38 mc for the IGY series of flights at Ft. Churchill.

## 2. NOTATION

| | |
|---|---|
| $v$ | velocity of the rocket |
| $c$ | velocity of light |
| $T(t)$ | number of the wave being emitted by the transmitter at the time t |
| $M(t)$ | number of waves propagating on the path transmitter-missile-receiver |
| $G(t)$ | number of waves propagating on the direct path transmitter-receiver |
| $R_m(t)$ | number of the wave entering the receiver from the missile path |
| $R_g(t)$ | number of the wave entering the receiver by the direct propagation from the transmitter |
| $S(t_1;t_o)$ | the DOVAP count in the time interval from $t_o$ to $t_1$ |
| $t$ | time |
| $t_o$ | initial time of trajectory computation |
| $t_1$ | time for which the rocket position is being computed |
| $dr$ | incremental distance taken along the path of propagation |
| $\lambda_1(r)$ | half a wavelength for frequency $f_1$ at point r |
| $c_1(r)$ | velocity of propagation for frequency $f_1$ at point r |
| $\lambda_2(r)$ | wavelength for frequency $2f_1$ at point r |
| $c_2(r)$ | velocity of propagation for frequency $2f_1$ at point r |
| $\lvert r_1 \rvert$ | transmitter-missile transmission distance |
| $\lvert r_1 - r_k \rvert$ | missile-receiver transmission distance |
| $t_p$ | time of propagation from the transmitter to the receiver |
| $r_1'$ | position of the rocket at $t_1'$ |
| $r_o'$ | position of the rocket at $t_o'$ |
| $c_2^*(r)$ | velocity of propagation of the path missile-receiver |

| | |
|---|---|
| $\nu_1(r)$ | twice the wave number at point r for frequency $f_1$ |
| $\nu_2(r)$ | the wave number at point r for frequency $2f_1$ |
| $f_1$ | frequency of the transmitter |
| $u_1$ | distance transmitter-missile-receiver at $t_1$ |
| $\bar{\nu}$ | average of $\bar{\nu}_1$ and $\bar{\nu}_2$ |
| $u_o$ | distance transmitter-missile-receiver at $t_o$ |
| $\bar{c}_1(r_1)$ | average velocity of propagation for frequency $f_1$ |
| $\bar{c}_2(r_1)$ | average velocity of propagation for frequency $2f_1$ |
| $\bar{\nu}_1(r_1)$ | twice the average wave-number for frequency $f_1$ |
| $\bar{\nu}_2(r_1)$ | average wave-number for frequency $2f_1$ |
| $c_o$ | velocity of light in vacuum, km/sec |
| $\rho$ | ambient density |
| $c(\rho)$ | velocity of radio waves as a function of the ambient density |
| f | frequency |
| $\lambda(\rho)$ | wavelength as a function of the ambient density |
| $\nu(\rho)$ | wave number as a function of ambient density |
| $\mu$ | refractive index of air |
| T | absolute temperature, °K |
| $H_s$ | scale height of the equivalent atmosphere |
| k | Boltzmann's constant |
| M | mean molecular mass |
| g | acceleration of gravity |
| $\bar{\mu}(H_1)$ | average refractive index for propagation to altitude $H_1$ |
| $H_1$ | altitude of the rocket |
| $H_o$ | altitude of the DOVAP array |

$p(H_o)$ pressure at the altitude of the DOVAP array

$p(H_1)$ pressure at altitude $H_1$

n electron density per cubic centimeter

e electron charge

m electron mass

$\epsilon$ increase in length of RF wavelength in the ionosphere as compared with its length in an un-ionized medium

$\Delta H/H$ error in thickness of the layer in DOVAP determination of altitude differences in a uniformly ionized layer

$\Delta T$ error in determination of the temperature

## 3. THE BASIC DOVAP EQUATION

In deriving the basic DOVAP equation, a Newtonian inertial frame of reference is assumed. Thus, any possible relativistic correction terms in $(v/c)^2$, where v is the velocity of the rocket and c is the velocity of light, are neglected. Even for a missile traveling with the orbiting velocity such terms are of the order of $10^{-9}$.

For the purpose of deriving the basic equation, each electromagnetic wave on the path transmitter-missile and on the direct path transmitter-receiver will be counted as two waves. This simplifies the presentation. It is assumed that the rocket does not spin.

In accordance with the above, each half-wave broadcast by the transmitter is labeled with a consecutive real integer. Each half wave becomes a full wave in the frequency doubling, whether in the rocket or at the receiver, and still retains its identifying number. The labeling is done on an analog basis, i.e., the idea is extended to fractions of a wave and fractions of a number.

Let $T(t)$ be the number of the wave being emitted by the transmitter at the time t; let $M(t)$ be the number of waves propagating on the path transmitter-missile-receiver; and let $G(t)$ be the number of waves propagating on the direct path transmitter-receiver. The number of the wave entering the receiver from the missile path, $R_m(t)$, will be

$$R_m(t) = T(t) - M(t) \quad , \tag{3.01}$$

and the number of the wave entering the receiver by the direct propagation from the transmitter, $R_g(t)$, will be

$$R_g(t) = T(t) - G(t) \quad . \tag{3.02}$$

In the interval $t_1 - t_o$ a number of waves $R_m(t_1) - R_m(t_o)$ enter the receiver from the missile direction and a number of waves $R_g(t_1) - R_g(t_o)$ enter the receiver by the direct propagation path. The DOVAP count for this time interval, $S(t_1;t_o)$, represents the difference between those two numbers:

$$\begin{aligned} S(t_1;t_o) &= \{R_g(t_1) - R_g(t_o)\} - \{R_m(t_1) - R_m(t_o)\} \\ &= \{[T(t_1) - G(t_1)] - [T(t_o) - G(t_o)]\} \\ &- \{[T(t_1) - M(t_1)] - [T(t_o) - M(t_o)]\} \\ &= [M(t_1) - M(t_o)] - [G(t_1) - G(t_o)] \quad . \end{aligned} \tag{3.03}$$

We assume now that the frequency of the transmitter and the conditions of propagation on the direct path transmitter-receiver are constant. It follows that $G(t)$ = constant. Then we have

$$S(t_1;t_o) = M(t_1) - M(t_o) \quad . \tag{3.04}$$

Equation (3.01) was written on the assumption of no time delay or phase shift between missile receiver and missile transmitter. Equation (3.04) remains valid if a constant phase shift occurs between the missile receiver and the missile transmitter.

Assume first that the rocket was at rest with respect to the DOVAP array at the time $t_o$ and decelerated to rest before $t_1$. Let $r_1$ be the vector from the transmitter to the rocket (see Fig. 1) at $t_1$ and let $r_k$ be the vector from the transmitter to a receiver. Then we have, from the examination of the physical meaning of $M(t_1)$, which is the number of waves in propagation on the path transmitter-missile-receiver,

$$\begin{aligned} M(t_1) &= \int_o^{r_1} \frac{dr}{\lambda_1(r)} + \int_{r_1}^{r_k} \frac{dr}{\lambda_2(r)} \\ &= 2f_1 \left[ \int_o^{r_1} \frac{dr}{c_1(r)} + \int_{r_1}^{r_k} \frac{dr}{c_2(r)} \right] \quad , \end{aligned} \tag{3.05}$$

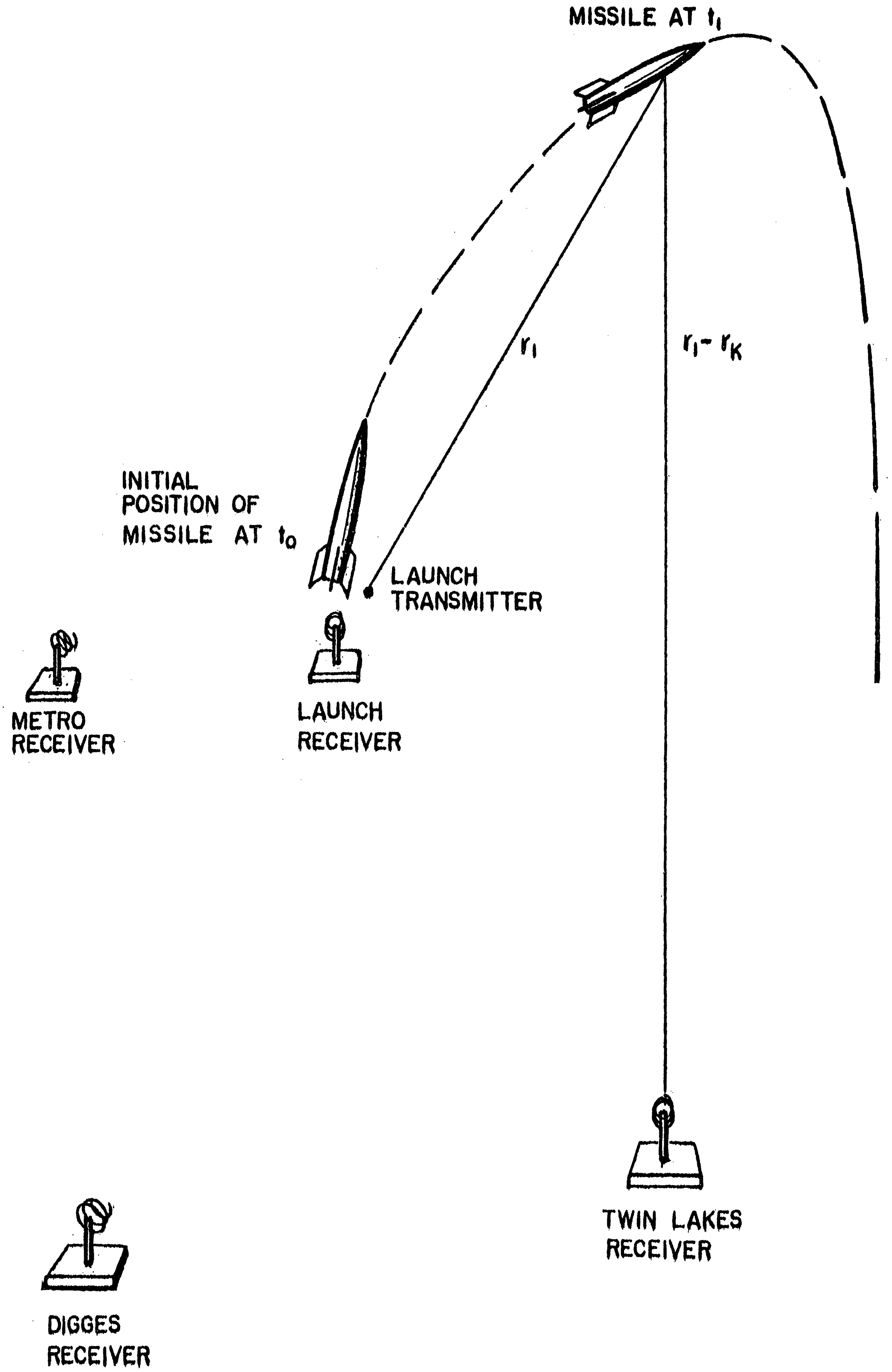


Fig. 1. Schematic presentation of the DOVAP array at Ft. Churchill.

where dr is incremental distance taken along the path of propagation; where $\lambda_1(r)$ and $c_1(r)$ are, respectively, half a wavelength and the velocity of propagation for frequency $f_1$ at point r; and where $\lambda_2(r)$ and $c_2(r)$ are, respectively, a wavelength and the velocity of propagation for frequency $2f_1$ at point r.

Let $|r_1|$ and $|r_1 - r_k|$ be, respectively, the length of the transmitter-missile and the missile-receiver transmission path. (It will be shown later that straight-line propagation can be assumed as a very good approximation.) Then, if average velocities $\bar{c}_1(r_1)$ and $\bar{c}_2(r_1)$ are defined in the following way

$$\frac{|r_1|}{\bar{c}_1(r_1)} = \int_0^{r_1} \frac{dr}{c_1(r)} \quad , \tag{3.06}$$

and

$$\frac{|r_1 - r_k|}{\bar{c}_2(r_1)} = \int_{r_1}^{r_k} \frac{dr}{c_2(r)} \quad , \tag{3.061}$$

Eq. (3.05) can be rewritten:

$$M(t_1) = 2f_1 \left[ \frac{|r_1|}{\bar{c}_1(r_1)} + \frac{|r_k - r_1|}{\bar{c}_2(r_1)} \right] \quad . \tag{3.07}$$

Since the averaging is done over the inverse of velocity, it might be convenient to use the wave number instead. Let

$$\nu_1(r) = \frac{2f_1}{c_1(r)} \tag{3.08}$$

$$\nu_2(r) = \frac{2f_1}{c_2(r)} \quad . \tag{3.09}$$

In terms of wave numbers, Eq. (3.05) becomes:

$$M(t_1) = \int_0^{r_1} \nu_1(r)\, dr + \int_{r_1}^{r_k} \nu_2(r)\, dr \quad . \tag{3.10}$$

If we define:

$$\bar{\nu}_1(r_1) = \frac{\int_0^{r_1} \nu_1(r)\, dr}{|r_1|} \tag{3.11}$$

and

$$\bar{\nu}_2(r_1) = \frac{\int_{r_1}^{r_k} \nu_2(r)\, dr}{|r_k - r_1|} , \qquad (3.12)$$

then

$$M(t_1) = \bar{\nu}_1(r_1) \cdot |r_1| + \bar{\nu}_2(r_1) \cdot |r_k - r_1| . \qquad (3.13)$$

Now, let us no longer assume that the rocket is stationary at $t_1$. The argument that led to Eqs. (3.01), (3.02), and (3.03) is unchanged. However, the expression for $M(t_1)$ given before in Eq. (3.05) has to be reconsidered, since the path of propagation changes continuously.

It will be remembered that $M(t_1)$ is the difference between the number of the wave $T(t_1)$ being emitted by the transmitter at $t_1$ and the number of the wave $R_m(t_1)$ entering the receiver from the missile at $t_1$, Eq. (3.01). $M(t_1)$ is thus equal to the number of waves produced by the transmitter from the time wave number $R_m(t_1)$ was emitted by the transmitter until time $t_1$. This number is equal to twice the frequency of the transmitter, $2f_1'$, times the time of propagation, $t_p$, of the wave $R_m(t_1)$ from the transmitter to the receiver. This wave was received and sent by the missile at time $t_1' < t_1$, such that

$$\int_{r_1'}^{r_k} \frac{dr}{c_2^*(r)} = t_1 - t_1' , \qquad (3.14)$$

where $r_1'$ is the position of the rocket at $t_1'$ and $c_2^*(r)$ is the velocity of propagation on the path missile-receiver for frequency modified by the doppler effect. The time of propagation is therefore given by

$$t_p = \int_0^{r_1'} \frac{dr}{c_1(r)} + \int_{r_1'}^{r_k} \frac{dr}{c_2^*(r)} , \qquad (3.15)$$

and the equation for $M(t_1)$ is:

$$M(t_1) = 2f_1 \left[ \int_0^{r_1'} \frac{dr}{c_1(r)} + \int_{r_1'}^{r_k} \frac{dr}{c_2^*(r)} \right] . \qquad (3.16)$$

Comparing Eq. (3.16) with Eq. (3.05), we note that, first of all, the difference lies in the fact that the integration of the inverse of velocity is carried out on a different path: on path $0 - r_1' - r_k$, instead of $0 - r_1 - r_k$. A second difference is that the velocity of propagation in the $r_1' - r_k$ integral is $c_2^*(r)$ to denote that the propagation is at a frequency somewhat different from $2f_1$. The frequency is different from $2f_1$ because of a doppler shift in the rocket receiver due to the radial velocity of the rocket with respect to the transmitter, and because of the doppler shift for a stationary observer due to the radial velocity of the missile relative to the receiver.* However, the change in velocity caused by this frequency shift is quite insignificant.

Thus, we commit an insignificant error if we supplant $c_2(r)$ for $c_2^*(r)$ in Eq. (3.16). We obtain

$$M(t_1) = 2f_1 \int_0^{r_1'} \frac{dr}{c_1(r)} + \int_{r_1'}^{r_k} \frac{dr}{c_2(r)} dr \quad , \tag{3.17}$$

and for $S(t_1;t_0)$, if the rocket moves at $t_0$, we have

$$S(t_1;t_0) = M(t_1) - M(t_0)$$

$$= 2f_1 \left[ \int_0^{r_1'} \frac{dr}{c_1(r)} + \int_{r_1'}^{r_k} \frac{dr}{c_2(r)} - \int_0^{r_0'} \frac{dr}{c_1(r)} - \int_{r_0'}^{r_k} \frac{dr}{c_2(r)} \right] , \tag{3.18}$$

where $r_1'$ and $r_0'$, the positions of the missile at $t_1'$ and $t_0'$, respectively, are defined by

$$\int_{r_1'}^{r_k} \frac{dr}{c_2(r)} = t_1 - t_1' \quad , \tag{3.19}$$

and

$$\int_{r_0'}^{r_k} \frac{dr}{c_2(r)} = t_0 - t_0' \quad . \tag{3.20}$$

Let us compute the order of magnitude of the time interval $t_1 - t_1'$. For a rocket at 120-km altitude this will be

$$\frac{120 \text{ km}}{300{,}000 \text{ km/sec}} = 0.4 \cdot 10^{-3} \text{ sec}$$

for a station underneath the rocket. A rocket traveling at the orbiting velocity of about 8 km/sec will move during this time a distance of $|r_1 - r_1'| = 8000 \cdot 0.4 \cdot 10^{-3} = 3.2$ m. This is not a negligible amount, but smaller than the current accuracy of the DOVAP system.

*Actually, the spin will also affect the frequency on the path missile-receiver, but ordinarily by an even smaller amount.

Thus, for all present-day applications, Eqs. (3.05), (3.10), and (3.13) are sufficiently accurate. Strictly speaking, the use of these equations yields the position of the rocket at $t_1'$ rather than at $t_1$.

The question of the numerical values of the average velocity of light to be used in the equations will be discussed in later chapters. It should be pointed out here that Eqs. (3.06) and (3.061), or (3.11) and (3.12), for computing the average velocity of light, or the average wave number, are based on the knowledge of the propagation path transmitter-missile-receiver. Straight-line propagation will be assumed, since it is shown in Appendix B that the effect of bending can be neglected. Knowledge of the position of the rocket is necessary; however, since the velocity of light changes slowly, only an approximate knowledge of the position is required.

It should be pointed out further that Eq. (3.07) and Eq. (3.13) are equations in two variables: $|r_1|$ and $|r_2 - r_1|$. Equation (3.13) can be rewritten as follows:

$$M(t_1) = \frac{\bar{\nu}_1(r_1)+\bar{\nu}_2(r_1)}{2} \left(|r_1|+|r_k-r_1|\right) + \left[\frac{\bar{\nu}_1(r_1)-\bar{\nu}_1(r_1)}{2} \left(|r_1|-|r_k-r_1|\right)\right] \qquad (3.21)$$

Except for ionosphere propagation, $\bar{\nu}_1(r_1)$ is equal to $\bar{\nu}_2(r_1)$; the term in the square brackets will therefore vanish. For ionosphere propagation the term in square brackets can be considered as a small correction term. In either case, the equation can be considered as an equation in one variable: $|r_1| + |r_k-r_1| = u_1$. If the concept of

$$\bar{\nu} = \frac{\bar{\nu}_1 + \bar{\nu}_2}{2} \qquad (3.211)$$

is introduced, we have

$$M(t_1) \cong \frac{\bar{\nu}_1(r_1) + \bar{\nu}_2(r_1)}{2} u_1 = \bar{\nu}(r_1) \cdot u_1 \ . \qquad (3.22)$$

Equations (3.05) to (3.13) and (3.21) to (3.22) were written for $t_1$. A parallel set of equations can be written for $t_o$, and in particular

$$M(t_o) \cong \frac{\bar{\nu}_1(r_o) + \bar{\nu}_2(r_o)}{2} u_o = \bar{\nu}(r_o) \cdot u_o \ . \qquad (3.23)$$

If the velocities of propagation are known and the position of the rocket at $t = t_o$, then the distance transmitter-missile-receiver $u_1$ can be computed from Eq. (3.22), (3.23), and (3.04).

$$u_1 = \frac{S(t_1;t_o) + u_o \cdot \bar{\nu}(r_o)}{\bar{\nu}(r_1)} \qquad (3.24)$$

## 4. THE EFFECTS OF THE CHANGING AIR DENSITY

The velocity of light near sea level departs by about 3 parts in 10,000 from the velocity of light in a vacuum. At the present time, the determination of the position of a missile from the DOVAP cycle count is based on the use of a constant velocity of light, which is chosen as an average of the velocities in the vacuum and at the ground.

In this chapter, the problem of how the air density affects the numerical value of the average velocity of light to be used in Eq. (3.07), or the average wave number to be used in Eqs.(3.13) and (3.24), is considered.

From this discussion it follows that for greater accuracy a variable average velocity of light as a function of altitude of the rocket should be used. A formula for calculating this variable velocity is suggested. It is shown that, for rockets at high altitudes, the current practice can result in errors approximately equivalent to omitting two cycles for each ten thousand cycles counted. The suggested method of calculation assumes that an approximate altitude of the rocket is already known.

The velocity of light in vacuum, $c_0$, is thought to be known with an accuracy of about 1 part in 300,000. Its value is given by DuMond and Cohen[1] as 299,792.9 (± 0.8) km/sec, and this value will be used here.

For atmospheric propagation, the velocity of radio waves will be a function of the ambient density

$$c(\rho) = \frac{c_0}{\mu(\rho)} \quad , \qquad (4.01)$$

and so will be the wavelength

$$\lambda(\rho) = \frac{c(\rho)}{f} = \frac{c_0}{f\ \mu(\rho)} \quad , \qquad (4.02)$$

and the wave number

$$\nu(\rho) = \frac{f}{c(\rho)} = \frac{f\ \mu(\rho)}{c_0} \quad . \qquad (4.03)$$

The velocity is assumed independent of frequency (except for ionospheric propagation).

BRL Report No. 677[2] gives the refractive index of air as $\mu = 1.0002726$ computed for sea-level pressure and for the temperature of 59°F (15°C). This checks closely with a newer experimentally determined value of $(\mu-1)10^6 = 288.15$ for dry sea-level air at 0°C.[3] Since $\mu-1$ is approximately proportional to the density, we have for the sea-level propagation

$$\mu_{\text{sea level}} = 1 + 2.726 \cdot 10^{-4} \frac{288}{T} , \quad (4.04)$$

where T is the temperature in °K.

Thus, if the DOVAP array is substantially at sea level and if the initial position of the missile at $t_o$ is very close to the ground (this was the case in the IGY rocket firings in Fort Churchill, Canada), the number of waves $M(t_o)$ can be determined from [see Eq. (3.23)]

$$M(t_o) = \bar{\nu}(r_o) \cdot u_o = \frac{2f_1 u_o}{c_o} \cdot u_o = \frac{2f_1 u_o}{c_o}\left(1 + 2.726 \cdot 10^{-4} \frac{288}{T}\right) . \quad (4.05)$$

The calculations of the average velocity of light (i.e., the calculations of the average wave number) for different altitudes of the rocket are not complicated. The calculations are simplest for points at altitudes higher than about 40 km. For such trajectory points, more than 99% of the atmosphere lies under the altitude of the missile; and for the purpose of our calculation, it can be assumed that the missile is "outside the atmosphere." The calculations will be developed first for such points.

A well-known notion of an equivalent atmosphere, of uniform (sea-level) density, is thought to simplify the presentation. The thickness of such equivalent atmosphere is referred to as scale height (at sea level). The propagation up to one-scale-height altitude is assumed to be taking place with the sea-level velocity; and the propagation above the scale height and up to the missile is assumed to be taking place with the vacuum velocity of light. The height of the equivalent atmosphere (equal scale height) is given by[4]

$$H_s = \frac{kT}{Mg} = 2.926 \cdot 10^{-2}\, T \quad \text{km} , \quad (4.06)$$

where k is Boltzmann's constant, T is the absolute temperature in °K, M is the mean molecular mass, and g is the acceleration due to gravity.

For the rocket altitude of $H_1$, we average the refractive index in the following way: propagation in vacuum (i.e., refractive index equal to unity) over the altitude $H_1 - H_s$, and propagation at the ground density for an altitude $H_s$.

$$\bar{\mu}(H_1) = \frac{1(H_1 - H_s) + \mu_{\text{sea level}} \cdot H_s}{H_1} = 1 + (\mu_{\text{sea level}} - 1) \frac{H_s}{H_1} . \quad (4.07)$$

From this equation and from Eq. (4.04), it follows that

$$\bar{\mu}(H_1) = 1 + 2.726 \cdot 10^{-4} \left(\frac{288}{T}\right) \frac{H_s}{H_1} , \quad (4.08)$$

and when the value of the scale height $H_s$ is introduced from Eq. (4.06), we have

$$\bar{\mu}(H_1) = 1 + 2.726 \cdot 10^{-4} \frac{288}{T} \frac{2.926 \cdot 10^{-2} T}{H_1} = 1 + 2.726 \cdot 10^{-4} \frac{8.427}{H_1} , \tag{4.09}$$

where $H_1$ is the altitude in km. It can readily be seen that the value of the average coefficient of refraction is independent of the temperature. The formula is written in this way to bring out the fact that 8.427 km is the scale height for the temperature of 288°K, for which the value of $(\mu_{sea\ level} - 1)$ is $2.726 \cdot 10^{-4}$.

Equation (4.09) applies for the trajectory points outside the atmosphere, i.e., at altitudes greater than several scale heights, with the transmitter and receiver located at sea level. For points lower than 40 km and when the DOVAP array is located at altitude $H_0$ different from sea level, the average refractive index will be given by the formula:

$$\bar{\mu}(H_1) = \frac{\int_{H_0}^{H_1} \mu(\rho) dH}{H_1 - H_0} = \frac{\int_{H_0}^{H_1} \left(1 + \frac{\mu_{sea\ level} - 1}{\rho_{sea\ level}} \rho\right) dH}{H_1 - H_0}$$

$$= \frac{H_1 - H_0}{H_1 - H_0} + \frac{(\mu_{sea\ level} - 1) \int_{H_0}^{H_1} \rho dH}{(H_1 - H_0) \rho_{sea\ level}} = 1 - \frac{(\mu_{sea\ level} - 1) \int_{H_0}^{H_1} dp}{(H_1 - H_0) \rho_{sea\ level} \cdot g}$$

$$= 1 + \frac{(\mu_{sea\ level} - 1) [p(H_0) - p(H_1)]}{(H_1 - H_0) \rho_{sea\ level} \cdot g} = 1 + \frac{p(H_0) - p(H_1)}{(H_1 - H_0) \rho_{sea\ level} \cdot g} \cdot 2.726 \cdot 10^{-4} \frac{288}{T} , \tag{4.10}$$

and since

$$\rho_{sea\ level} = \frac{p_{sea\ level} M}{kT} , \tag{4.11}$$

we have

$$\bar{\mu}(H_1) = 1 + \frac{p(H_0) - p(H_1)}{(H_1 - H_0) p_{sea\ level}} \cdot \frac{kT}{gM} \cdot 2.726 \cdot 10^{-4} \frac{288}{T}$$

$$= 1 + \frac{p(H_0) - p(H_1)}{(H_1 - H_0) p_{sea\ level}} H_s \cdot 2.726 \cdot 10^{-4} \frac{288}{T} = 1 + 2.726 \cdot 10^{-4} \frac{8.427}{H_1 - H_0} \cdot \frac{p(H_0) - p(H_1)}{p_{sea\ level}} . \tag{4.12}$$

In the simpler case of Eq. (4.09), $H_0$ is zero, $p(H_0)$ is taken to be the sea-level pressure, and $p(H_1)$ is taken to be vanishingly small.

The average refractive indexes [or the average wave numbers to be used in Eq. (3.24)] are functions of altitude. The error which results from using the current approach is calculated for one trajectory point of Aerobee SM1.01 in Appendix A.

## 5. THE EFFECTS OF THE IONOSPHERE

The problem of errors caused by the ionosphere in the DOVAP determination of a rocket trajectory is discussed by deBey and Hoffleit in BRL Report No. 677, already mentioned.[5] They discuss the effect of change of wavelength, i.e., effective change of the yardstick, and the effect of bending, i.e., the difference in length between the curved-path propagation and straight-line transmitter-receiver distance.

Calculations were carried out which indicated that the BRL figures regarding the magnitude of the bending effect were apparently too large by several orders of magnitude. The actual differences between the curved path and the straight-line transmitter-receiver distance appear to be negligible compared with the change of wavelength effect (see Appendix B). Accordingly, this second effect is not considered further in the body of this report.

The index of refraction, $\mu$, neglecting the effects of the earth's magnetic field and electron collisions, is given by the formula

$$\mu = (1 - ne^2/m\pi f^2)^{1/2} = (1 - 80.5 \cdot 10^6\ n/f^2)^{1/2} , \qquad (5.01)$$

where n is the electron density per cubic centimeter, e the electron charge, m the electron mass, and f the radio frequency.

Thus, the RF wavelength becomes longer in the ionosphere by a fraction $\epsilon$ as compared with its length in an un-ionized medium:

$$\epsilon = \frac{1}{\mu} - 1 = \frac{1}{(1 - 80.5 \cdot 10^6\ n/f^2)^{1/2}} - 1 , \qquad (5.02)$$

and for $\epsilon \ll 1$

$$\epsilon \cong 40.25 \cdot 10^6\ n/f^2 . \qquad (5.03)$$

For the transmitter-missile path we have

$$\epsilon \cong 40.25 \cdot 10^6 \, n/37^2 \cdot 10^{12} = 2.94 \, n \cdot 10^{-8} \quad , \qquad (5.04)$$

and for the missile-receiver path

$$\epsilon \cong 40.25 \cdot 10^6 \, n/74^2 \cdot 10^{12} = 0.74 \, n \cdot 10^{-8} \quad . \qquad (5.05)$$

The actual error in missile position obviously depends on the geometry of the missile DOVAP system. The magnitude of the error can be readily computed for a nearly vertical trajectory if it is borne in mind that at a high altitude the angle between the transmitter-missile path and the missile-receiver path is small. A horizontally stratified ionosphere is assumed.

Under these conditions, the DOVAP determination of altitude differences in an uniformly ionized layer will be approximately in error by a fraction:

$$\frac{\Delta H}{H} = \frac{2.94 + 0.74}{2} 10^{-8} \, n = 1.84 \cdot 10^{-8} \, n \quad . \qquad (5.06)$$

The determination of the layer thickness will be too low, and therefore the determination of the velocity of sound will be too high by the same fraction $1.84 \cdot 10^{-8}$ n, where n is the average electron density per cubic centimeter in the layer between two successive grenade explosions. The temperature will be too high by the fraction $2 \cdot 1.84 \cdot 10^{-8} \, n = 3.68 \cdot 10^{-8} \, n$ or (in the 80-90-km region) by about $205 \cdot 3.68 \cdot 10^{-8} \, n = 7.54 \cdot 10^{-6} \, n \, {}^\circ K$.

The electron-density data at altitudes under 100 km seemed somewhat contraversial. Kuiper[6] quotes Piggott (1950) to the effect that the noon values of n at 90 km are about $1.5 \cdot 10^4/cm^3$. On the other hand, Berning[7] reported electron densities as high as $7 \cdot 10^4/cm^3$ at 60 km.

From conversation with Berning (October, 1956), it transpired that he considered his electron-density data not reliable in the D-region. In view of this fact, the density of $2 \cdot 10^4/cm^3$ will be used as the basis for calculating the temperature error. This figure was given by J. C. Seddon (private correspondence) as the outside value for noon-time electron density at 90 km at White Sands Proving Ground, New Mexico.

Assuming that this density prevails throughout the layer between two successive grenade explosions, the determination of the thickness of the layer will be in error by

$$\frac{\Delta H}{H} = 1.84 \cdot 10^{-8} \cdot 2 \cdot 10^4 = 3.68 \cdot 10^{-4} = 0.037\% \quad , \qquad (5.07)$$

and the temperature will be in error by:

$$\Delta T = 7.54 \cdot 10^{-6} \cdot 2 \cdot 10^4 = 0.158^\circ \quad . \qquad (5.08)$$

This is only about 1/35 of the average probable error for the temperature data (± 5°) of the 1950-1953 rocket-grenade experiments carried out at White Sands[8] and 1/3 of the probable error (± 0.4°) for the most accurate experiment of the series (24 April 1956). The ionosphere error, as calculated above, is expected to occur only close to 90 km, where the accuracy of the experiment might be lower than at lower altitudes. Thus, the conclusion might be drawn that the ionosphere does not greatly influence the accuracy of the experiment.

It should be pointed out that, at high latitudes, transient ionospheric abnormalities are very pronounced.[9] Electron-density profiles are not available, however, at the present but might become available during the International Geophysical Year.

At altitudes higher than those of interest in the rocket-grenade experiment, the ionospheric effect can be much more pronounced. Assuming an electron density of $3 \cdot 10^6$ electrons/cc to exist in a sporadic E-layer,[10] the DOVAP determination of the thickness of a layer will be in error by

$$\frac{\Delta H}{H} = 1.84 \cdot 10^{-8} \cdot 3 \cdot 10^{6} = 5.5 \cdot 10^{-2} = 5.5\% \quad . \qquad (5.09)$$

## 6. CONCLUSIONS

The effect of the atmospheric refractive indexes on the accuracy of the DOVAP system has been discussed. It has been shown that a small improvement in position determination can result if a variable average velocity of the radio waves, as a function of altitude, is used. This is due to effect of air density.

The calculations indicate that the ionosphere propagation will have only an extremely small effect at the altitudes of the rocket-grenade experiment. This conclusion, however, should be further checked when more information about the ionosphere at high latitudes is available.

The effect of the curved propagation path, i.e., the difference between the actual propagation path and the straight-line distance, has been shown to be negligible, contrary to previous estimates of the effect.

## 7. ACKNOWLEDGMENT

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APPENDIX A

# THE DETERMINATION OF THE DISTANCES TRANSMITTER-MISSILE-RECEIVER FOR ONE SM1.01 TRAJECTORY POINT USING VARIABLE AVERAGE VELOCITY OF THE RADIO WAVES

NOTATION

| | |
|---|---|
| $\bar{\lambda}(H_o)$ | ground wavelength, m |
| $\bar{\mu}(61.2)$ | average refractive index for 61.2-km altitude |
| $\bar{\lambda}(61.2)$ | average wavelength at 61.2-km altitude, m |
| $\lambda_{BRL}$ | average wavelength recommended by BRL, m |
| $S(t_1;t_o)$ | spin-corrected cycle count from wire break up to the time $t_1$ of the last explosion. |
| $u(t_1)$ | distance transmitter-missile-receiver at the time of the last explosion |
| $u(t_o)$ | distance transmitter-missile-receiver at the wire-break time |
| $\Delta u(t_1)$ | differences between the variable-velocity method and the current method |

In this Appendix the numerical differences between the computation by the current method and the computation by the suggested method of variable average velocity of the radio waves (as a function of altitude) will be demonstrated for a single trajectory point of the SM1.01 rocket flight, namely, the explosion of the last grenade. The comparison shows that the differences in the distance transmitter-missile-receiver for various receiving stations are equivalent to between three and five cycles.

At the time of Aerobee SM1.01 firing, on November 12, 1957, the temperature at the ground level at Fort Churchill, Manitoba, Canada, was about -25°C. The recorded frequency of the DOVAP transmitter was 36,938,775 cps. The explosion of the last grenade occurred at about 61.2 km above msl.

The refractive index for propagation near the ground, in accordance with Eq. (4.04), will be

$$\mu = \mu_{sea\ level} = 1 + 2.726 \cdot 10^{-4}\ 288/T$$
$$= 1 + 2.726 \cdot 10^{-4}\ 288/248 = 1.0003165 \qquad \text{(A.01)}$$

and the ground wavelength

$$\bar{\lambda}(H_o) = \frac{299,792.9}{1.0003165 \cdot 2 \cdot 36938775} = 4.056686 \text{ m} \quad . \qquad (A.02)$$

The average refractive index for the altitude of 61.2 km will be, in accordance with (4.09),

$$\bar{\mu}(61.2) = 1 + 2.726 \cdot 10^{-4} \frac{8.427}{61.2} = 1.00003754 \quad , \qquad (A.03)$$

and the average wavelength

$$\bar{\lambda}(61.2) = \frac{299,792.9}{1.00003754 \cdot 2 \cdot 36938775} = 4.057818 \text{ m} \quad . \qquad (A.04)$$

The average wavelength recommended by BRL for SM1.01 data reduction is

$$\lambda_{BRL} = 4.05736248 \text{ m} \quad . \qquad (A.05)$$

The distances $\mu(t_o)$ for wire-break time and the spin-corrected cycle counts from wire-break time up to time $t_1$ of the last explosion are tabulated in meters here for the four receiving stations.

| | Launch | Digges | Twin Lakes | Metro |
|---|---|---|---|---|
| $u(t_o)$ | 320.51 | 28,300.34 | 13,149.42 | 14,769.52 |
| $S(t_1;t_o)$ | 31,391.34 | 24,139.68 | 27,722.12 | 27,886.57 |

Computation of the distance $u(t_1)$ by the current method is carried out according to the formula

$$u(t_1) = \lambda_{BRL}\, S(t_1;t_o) + u(t_o) \quad , \qquad (A.06)$$

and the results are tabulated in meters below.

| | Launch | Digges | Twin Lakes | Metro |
|---|---|---|---|---|
| $u(t_o)$ | 320.51 | 28,300.34 | 13,149.42 | 14,769.52 |
| $\lambda_{BRL} \cdot S(t_1;t_o)$ | 127,366.05 | 97,943.43 | 112,478.69 | 113,145.92 |
| $u(t_1)$ | 127,686.56 | 126,243.77 | 125,628.11 | 127,915.44 |

Computation of the distances $u(t_1)$ by the method of variable light velocity is carried out according to the formula [see Eq.(3.24)]

$$u(t_1) = \bar{\lambda}(H_1)\left[S(t_1;t_o) + \frac{u(t_o)}{\bar{\lambda}(H_o)}\right] = \bar{\lambda}(H_1)\, S(t_1;t_o) + \frac{\bar{\lambda}(H_1)}{\bar{\lambda}(H_o)} \cdot u(t_o) \quad , \tag{A.07}$$

and the results are tabulated in meters below.

| | Launch | Digges | Twin Lakes | Metro |
|---|---|---|---|---|
| $u_o \frac{\bar{\lambda}(61.2)}{\bar{\lambda}(H_o)}$ | 320.60 | 28,308.24 | 13,153.09 | 14,773.64 |
| $\bar{\lambda}(61.2) \cdot S(t_1;t_o)$ | 127,380.34 | 97,954.43 | 112,491.32 | 113,158.63 |
| $u(t_1)$ | 127,700.94 | 126,262.67 | 125,644.41 | 127,932.27 |

The differences $\Delta u(t_1)$ between the variable-velocity method and the current method are (in meters):

| Launch | Digges | Twin Lakes | Metro |
|---|---|---|---|
| + 14.38 | + 18.90 | + 16.30 | + 16.83 |

In general, comparison of Eqs. (A.06) and (A.07) will give for $\Delta u(t_1)$

$$\Delta u(t_1) = [\bar{\lambda}(H_1) - \lambda_{BRL}]\, S(t_1;t_o) + \frac{\bar{\lambda}(H_1) - \bar{\lambda}(H_o)}{\bar{\lambda}(H_o)}\, u(t_o) \; . \tag{A.08}$$

## APPENDIX B

## COMPUTATION OF THE DIFFERENCE BETWEEN THE CURVED RADIATION PATH AND THE ACTUAL DISTANCE FROM THE MISSILE TO THE RECEIVER

NOTATION

| | |
|---|---|
| M | missile |
| R | receiver |
| MR | actual radiation path missile-receiver |
| MO | line through M, parallel to the portion of the path with the largest zenith angle $\beta_1$ |
| $\beta_1$ | largest zenith angle |
| $\beta_2$ | smallest zenith angle |
| $\beta$ | zenith angle of the missile as seen from the receiver |
| RO | line tangent to the actual radiation path at R |
| $\ell$ | curved path $\overline{RO} + \overline{OM}$ |
| H | vertical elevation of the missile over the receiver |
| h | vertical elevation of point T over the receiver |
| $\Delta\beta$ | angle $\beta_2 - \beta_1$ |
| d | distance OT |
| $\alpha_1$ | angle $\beta_1 - \beta$ |
| $\alpha_2$ | angle $\beta - \beta_2$ |
| $\mu = 1 - \Delta\mu$ | smallest index of refraction throughout the path |

This computation presents a simple formula for estimating the effect of the curved radiation path. The assumptions are such that the formula provides an outside limit rather than the correct numerical value.

The calculations are carried out for the missile-receiver propagation path. The axes are vertical and horizontal lines at the receiver.

Let M represent the missile (see Fig. 2), R the receiver; and let the curved path MR be the actual radiation path. Let MO be a line through M parallel to the portion of the path with the largest zenith angle $\beta_1$ (in the region of smallest refractive index, i.e., the greatest electron density), and let RO (zenith angle $\beta_2$) be tangent to the actual radiation path at R. The angle $\beta$ is the zenith angle of the missile as seen from the receiver.

It can be seen (the sum of the lengths of two sides of a triangle is greater than the length of the third side; see Fig. 3) that $\overline{RO} + \overline{OM} = \ell$ is greater than the actual curved path RM.

Let $\overline{OT}$ be perpendicular to $\overline{RM}$ and let H and h be the vertical elevations of the missile and point T, respectively, over the receiver.

From the geometry in Fig. 2, and because $\Delta\beta = \beta_1 - \beta_2$ is a small angle, we have (we designate distance TO as d)

$$\ell = \overline{RO} + \overline{OM} = \sqrt{\overline{RT}^2 + \overline{TO}^2} + \sqrt{\overline{TM}^2 + \overline{TO}^2}$$

$$= \sqrt{(h/\cos\beta)^2 + d^2} + \sqrt{(H-h/\cos\beta)^2 + d^2} \qquad \text{(B.01)}$$

$$\ell \cong \frac{h}{\cos\beta}\left[1 + 1/2\left(\frac{d}{h}\cos\beta\right)^2\right] + \frac{H-h}{\cos\beta}\left\{1 + 1/2\left(\frac{d}{H-h}\cos\beta\right)^2\right\}$$

$$= \frac{H}{\cos\beta} + \frac{Hd^2\cos\beta}{2(H-h)h}$$

$$= \frac{H}{\cos\beta}\left\{1 + \frac{d^2\cos^2\beta}{2(H-h)h}\right\} \qquad \text{(B.02)}$$

$$d = \overline{TO} \cong \frac{\alpha_2 h}{\cos\beta} \cong \frac{\alpha_1(H-h)}{\cos\beta} \qquad \text{(B.03)}$$

$$\alpha_1 + \alpha_2 = \Delta\beta \qquad \text{(B.04)}$$

$$\alpha_1(H-h) \cong (\Delta\beta - \alpha_2)(H-h) \cong \alpha_2 h \qquad \text{(B.05)}$$

$$\alpha_2 \cong \frac{H-h}{H}\Delta\beta \qquad \text{(B.06)}$$

$$d \cong \frac{h}{\cos\beta}\frac{(H-h)}{H}\Delta\beta \qquad \text{(B.07)}$$

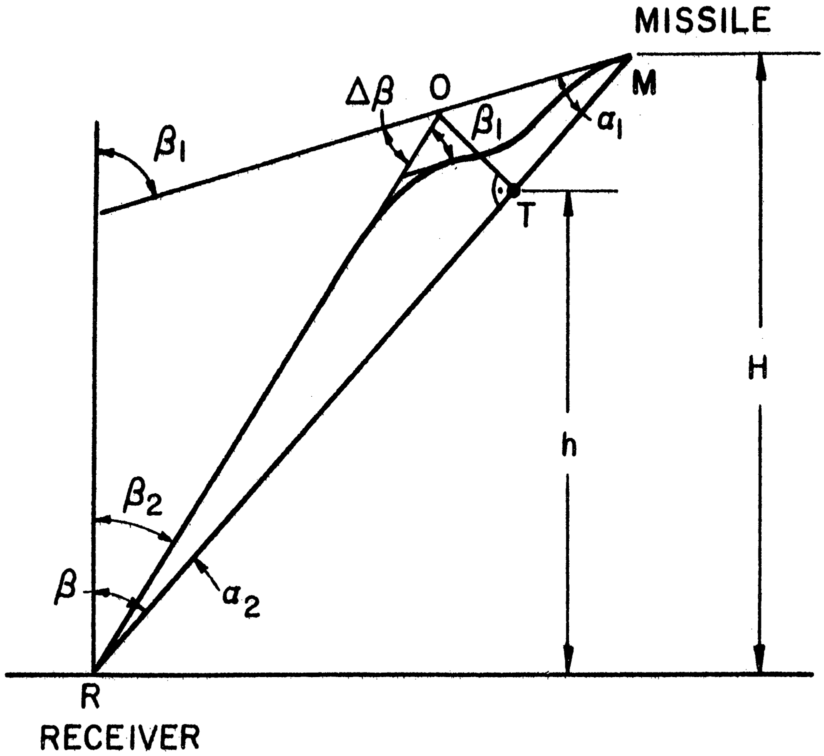


Fig. 2. Geometry of ionospheric propagation.

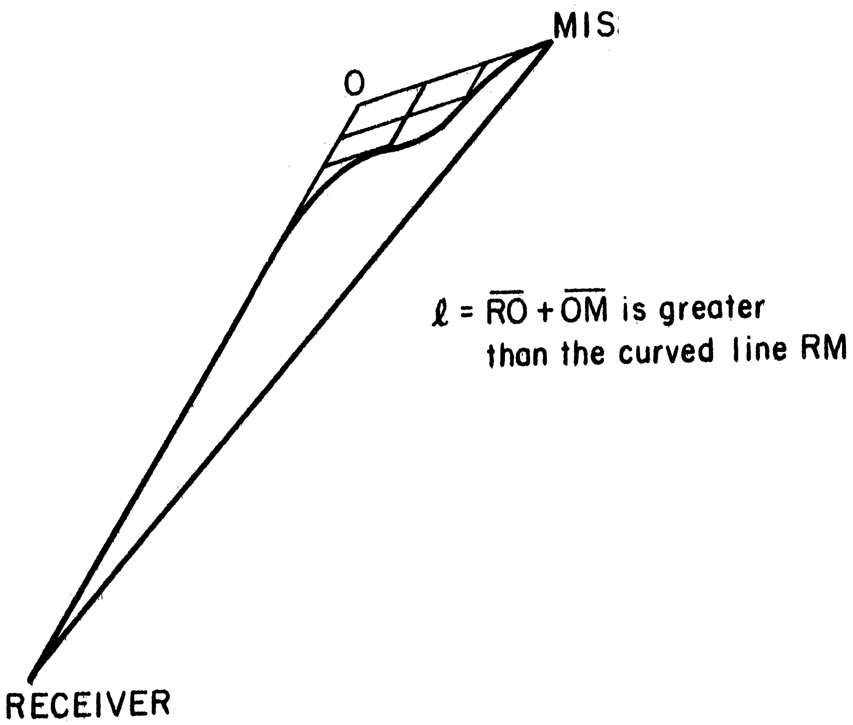


Fig. 3. Comparison of distance $\ell$ and the actual path.

$$\ell \cong \frac{H}{\cos\beta}\left\{1 + \frac{d^2 \cos^2\beta}{2(H-h)h}\right\} = \frac{H}{\cos\beta}\left\{1 + \frac{(H-h)h\ (\Delta\beta)^2}{2H^2}\right\}. \qquad (B.08)$$

In the following, $\mu = 1 - \Delta\mu$ designates the smallest index of refraction throughout the path.

$$\frac{\sin\beta_2}{\sin\beta_1} = \mu = 1 - \Delta\mu = \frac{\sin\beta_1 \cos\Delta\beta - \cos\beta_1 \sin\Delta\beta}{\sin\beta_1} \cong \frac{\sin\beta_1 - \cos\beta_1 \cdot \Delta\beta}{\sin\beta_1}$$

(B.09)

$$\Delta\beta \cong \Delta\mu \tan\beta_1 \cong \Delta\mu \tan\beta \qquad (B.10)$$

$$\ell \cong \frac{H}{\cos\beta}\left[1 + \frac{(H-h)h\ (\Delta\mu)^2 \tan^2\beta}{2H^2}\right] . \qquad (B.11)$$

The expression $(H-h)h/H^2$ takes a maximum value of 1/4 for $h = H/2$. Assuming that this maximum exists, we have

$$\ell = \frac{H}{\cos\beta}\left[1 + \frac{(\Delta\mu)^2}{8} \tan^2\beta\right] . \qquad (B.12)$$

The difference in path length $\Delta\ell$ between the curved path $\ell$ and the straight-line distance is

$$\Delta\ell = \frac{H}{8 \cos\beta} (\Delta\mu)^2 \tan^2\beta . \qquad (B.13)$$

Expressed in percentage, $\Delta\ell/\ell$ is

$$\frac{\Delta\ell}{\ell} \leqq 12.5\ (\Delta\mu)^2 \tan^2\beta\ \% . \qquad (B.14)$$

The numerical example given on pages 116 and 117 of BRL Report No. 677 by deBey and Hoffleit discusses the ionospheric refraction in the following case:

$$H \cong 100 \text{ km}$$

$$\Delta\mu = .00348$$

$$\tan\beta = 0.20, \quad \cos\beta = 0.98 .$$

The above reference estimates the difference between the curved radiation path and the distance from the missile to the receiver to be approximately 30 m, whereas actually

$$\Delta \ell \leq \frac{10^5 \cdot 3.48^2 \cdot 10^{-6} \cdot 0.2^2}{8 \cdot 0.98}$$

$$= 0.6 \cdot 10^{-2} \text{ m}$$

$$= 0.6 \text{ cm} \quad .$$

The formulas (B.13) and (B.14) are applicable to the effect of curved path due to changing atmospheric density, with the difference that $\Delta\mu = \mu-1$, where $\mu$ represents the largest index of refraction throughout the path. The numerical example given on pages 114 and 115 of BRL Report No. 677 discusses the refraction due to the atmospheric density in the following case:

$$H \cong 140 \text{ km}$$

$$\Delta\mu = 0.00027$$

$$\tan \beta = 1, \cos \beta = 0.707 \quad .$$

The reference estimates the effect of the curved path to be 4 m, whereas actually

$$\Delta \ell \leq \frac{1.4 \cdot 10^5 \cdot 2.7^2 \cdot 10^{-8}}{8 \cdot 0.707}$$

$$= 1.8 \cdot 10^{-3}$$

$$= 0.18 \text{ cm} \quad .$$

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