A phase cell approach to Yang–Mills theory. II. Analysis of a mode

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Properties of a mode announced in a previous paper are proved. This involves some complicated calculations in linear algebra, and observation of the structure of a function of several complex variables.

I. $A_1^i$ and $A_2^i$, LINEAR ALGEBRA

$A_1^i(p)$ is given by (3.12) of Ref. 1. We will consider a mode at level zero, specified by bond assignments at level zero, having only one bond assignment different from zero. We choose the bond to be the bond in the $+1$ direction from the origin, and the assignment to it to be unity. By invariance under certain interchanges of coordinate axes it is sufficient to find $A_1^i$ and $A_2^i$ to determine $A_1^i$.

With $i$ fixed, (3.12) of Ref. 1 is a matrix product, $ABC$, with $A$, $B$, $C$, respectively, a $(1 \times 6)$ matrix, $(6 \times 6)$ matrix, and $(6 \times 1)$ matrix. The coordinates of the six-dimensional space are labeled by oriented plaquette directions, for which we choose the ordering

$$(1,2), (2,3), (3,4), (1,3), (1,4), (2,4).$$

(1.1)

We find it convenient to introduce certain special notation and matrices. We let

$$f_i = (e^{i\rho_i} - 1)$$

(1.2)

and $D$ be the $6 \times 6$ diagonal matrix with

$$D_{(j), (j), (j)} = f_i f_j$$

(1.3)

(here labeling rows and columns of $D$ by the associated oriented plaquette directions). We let

$$\langle h(p) \rangle \equiv \sum \frac{1}{\pi} \frac{1}{(p + 2\pi n)^2} \prod_{i=1}^{4} \left| \frac{e^{i\rho_i} - 1}{(p_i + 2\pi n_i)} \right|^2 h(p + 2\pi n),$$

(1.4)

where $h$ is any function of $p = (p_1, p_2, p_3, p_4)$.

We now detail the matrices occurring in (3.12) of Ref. 1. We first view the $(6 \times 1)$ matrix

\[
\begin{array}{cccccc}
1/p_1^2 + 1/p_2^2 & -1/p_2^3 & 0 & 1/p_3^2 \\
-1/p_2^3 & 1/p_2^2 + 1/p_3^2 & -1/p_3^2 & 1/p_4^2 \\
0 & -1/p_3^2 & 1/p_3^2 + 1/p_4^2 & -1/p_4^2 \\
1/p_1^2 & 1/p_4^2 & -1/p_2^2 & 1/p_1^2 + 1/p_2^2 \\
1/p_1^2 & 0 & 1/p_4^2 & 1/p_1^2 \\
-1/p_3^2 & 1/p_2^2 & 1/p_4^2 & 0
\end{array}
\]

Here $(M_0)$ is a positive Hermitian matrix, and $w_i$ is in its range. We deal with the limit

$$\lim_{\epsilon \to 0} \epsilon^{-1} (M_0 + \epsilon)^{-1} w_i.$$  

(1.14)

\[
\beta \epsilon^{i, p} \equiv v_i = f_1^{-1} D w_i,
\]

(1.5)

$$w_i = (1,0,0,1,0)^T,$$

(1.6)

where the bars indicate complex conjugates. We next view the left $(1 \times 6)$ matrix, two different vectors depending on whether one is studying $A_1^i$ or $A_2^i$. With $r_L$ defined by

$$r_L = \frac{i}{2\pi} \prod_k \left( \frac{e^{i\rho_k} - 1}{\rho_k} \right),$$

(1.7)

we have

$$\bar{P}_1 = r_L (1/p_1) w_i^T D$$

(1.8)

and

$$\bar{P}_2 = r_L (1/p_2) w_i^T D$$

(1.9)

with

$$w_i = (-1,1,0,0,0,1)^T.$$  

(1.10)

At this point we may write (3.12) of Ref. 1 as

$$A_1^i(p) = (1/p_1) \bar{P}_1 M_0^{-1} v_i.$$  

(1.11)

Recall $M$ is singular; and $M^{-1}$ is defined on suitable vectors as $\lim_{\epsilon \to 0} (\epsilon + M)^{-1}$.

Substituting (1.5) and (1.8) or (1.9) into (1.11) we get

$$A_1^i = (-1/p^2) r_L f_1^{-1} (1/p_1) w_i^T (M_0)^{-1} v_i,$$

(1.12)

where

$$M = \bar{D} (M_0) D$$

(1.13)

and the inverse of $(M_0)$ taken in the same sense as the inverse of $M$. We now display $M_0$, itself, in its full glory:

\[
\begin{array}{cccccc}
1/p_1^2 & -1/p_2^3 & 0 & 1/p_3^2 \\
-1/p_2^3 & 1/p_2^2 + 1/p_3^2 & -1/p_3^2 & 1/p_4^2 \\
0 & -1/p_3^2 & 1/p_3^2 + 1/p_4^2 & -1/p_4^2 \\
1/p_1^2 & 1/p_4^2 & -1/p_2^2 & 1/p_1^2 + 1/p_2^2 \\
1/p_1^2 & 0 & 1/p_4^2 & 1/p_1^2 \\
-1/p_3^2 & 1/p_2^2 & 1/p_4^2 & 0
\end{array}
\]

This limit may be taken by the following procedure. Let $P$ be the (orthogonal) projection onto the range of $(M_0)$, $S$. Let

$$m = P (M_0) P,$$

a strictly positive Hermitian matrix. We invert $m$, in $S$, writing it as $m^{-1}$. One has for the result of (1.14)
The observations of this paragraph reduce the computations of (1.12) to calculations in a three-dimensional vector space. We note that $\mathcal{S}$ is spanned by $w_1$, $w_2$, and $w_3 = (0,0,1,0,1,1)^T$.

We first present the results of the computation in (1.15):

\[
(M_0)^{-1}w_1 = (1/D)(4bc + 4bd + 8cd, 4bd - 4cd, 4bc + 4bd + 4cd, 8bc + 4bd + 4cd, 4bd - 4cd)^T,
\]

where

\[
a = (1/p_1^2), \quad b = (1/p_2^2), \quad c = (1/p_3^2), \quad d = (1/p_4^2),
\]

and

\[
D = 16\sum_{k,j,l} \prod (1/p_j^2).
\]

Here $\mathcal{D}$ has arisen as the determinant of the $3 \times 3$ matrix involved in the computation of $m^{-1}$.

Collecting our results and substituting into (1.12) we easily find

\[
A_1' = \left( - r_L \tilde{f}^{-1} \frac{1}{p_1^2} \right) \frac{1}{D} \left( \frac{1}{p_1^2} \right) \left( \frac{1}{p_2^2} \right)
+ \left( \frac{1}{p_2^2} \right) \left( \frac{1}{p_3^2} \right) + \left( \frac{1}{p_3^2} \right) \left( \frac{1}{p_4^2} \right),
\]

\[
A_2' = \left( - r_L \tilde{f}^{-1} \frac{1}{p_1^2} \right) \frac{1}{D} \left( - \frac{1}{p_1^2} \right) \left( \frac{1}{p_2^2} \right)
+ \left( \frac{1}{p_3^2} \right) \left( \frac{1}{p_4^2} \right) + \left( \frac{1}{p_3^2} \right) \left( \frac{1}{p_4^2} \right).
\]

II. $A_1'$ AND $X(p)$

From (1.19) and (1.20), and the invariance under interchange of coordinate directions—except for the special one-direction—we find our expression for $A_1'(p)$,

\[
A_1' = \left( - r_L \tilde{f}^{-1} \frac{1}{p_1^2} \right) \frac{1}{D} \left( \frac{1}{p_1^2} \right) \left( \frac{1}{p_2^2} \right) l_1,
\]

\[
l_1 = \left( \frac{1}{p_2^2} \right) \left( \frac{1}{p_3^2} \right) + \left( \frac{1}{p_2^2} \right) \left( \frac{1}{p_4^2} \right) + \left( \frac{1}{p_3^2} \right) \left( \frac{1}{p_4^2} \right),
\]

\[
l_i = - \prod_{j \neq i} \left( \frac{1}{p_j^2} \right), \quad i \neq 1.
\]

A little study of these expressions for the region near $p_1 \sim p_2 \sim p_3 \sim p_4 \sim 0$ shows that the $A_1'$ have a singularity in the complex four-dimensional region on the surface $p^2 = 0$, of course hitting the real axis. We define $A_1^\gamma(p)$, a gauge transformation of $A_1'(p)$, by

\[
A_1^\gamma(p) = A_1'(p) + p_i X(p),
\]

with

\[
X(p) = \left( - r_L \tilde{f}^{-1} \frac{1}{p^2} \right) \frac{1}{D} \frac{1}{p^2} \left( \frac{1}{p_i^2} \right),
\]

where $s$ is an arbitrary integer, sufficiently large. The analytic properties of $A_1^\gamma(p)$ will be studied in the next section.

III. ANALYTIC PROPERTIES OF $A_1^\gamma(p)$

The analysis we follow herein is analogous to the similar analysis due to Gawedzki and Kupiainen in the appendix of Ref. 2. We take the analytic extensions of the expressions in Sec. II from real $p$ to complex $p$. For example,

\[
f_i = e^{i\phi} - 1
\]

and

\[
\left| \frac{e^{-i\phi} - 1}{p_i + 2\pi n} \right|^2 \frac{e^{-i\phi} - 1}{p_i + 2\pi n} \frac{e^{-i\phi} - 1}{p_i + 2\pi n}
\]

We make the preliminary observation that $f_1, D, \mathcal{D},$ and $(\cdot)$ are periodic functions of $p$, invariant under

\[
p \rightarrow p + 2\pi n
\]

($n$ is a four-vector with integer components). We note that $r_L, p^2,$ and $p_i$ are not periodic functions of $p$.

Because of the periodicity mentioned in the last paragraph it is natural to divide our results into the following three theorems.

**Theorem 3.1 (Local analyticity):** There is an $\epsilon_0 > 0$ such that in the domain, $\mathcal{D}_L$, specified by

\[
-\pi < \text{Re} p_i < \pi, \quad |\text{Im} p_i| < \epsilon_0
\]

$A_1^\gamma(p)$ is analytic.

**Theorem 3.2 (Global analyticity):** $A_1^\gamma(p)$ is analytic in the domain, $\mathcal{D}_G$, specified by

\[
|\text{Im} p_i| < \epsilon_0
\]

**Theorem 3.3 (Boundedness):** Within the domain, $\mathcal{D}_B$, specified by

\[
|\text{Im} p_i| < \epsilon_0/2
\]

$A_1^\gamma(p)$ satisfies bounds of the form

\[
|A_1^\gamma(p)| < c \prod_{j} \frac{1}{|p_j| + 1} \frac{1}{|p_j|^2 + 1}.
\]

In Theorems 3.2 and 3.3 the $\epsilon_0$ is as defined in Theorem 3.1.

Theorems 3.2 and 3.3 are easy consequences of the form of $A_1^\gamma(p)$ and Theorem 3.1. We will devote our attention entirely to the proof of Theorem 3.1, which is carried out in Sec. V.

IV. CONCLUSIONS

Equations (3.13)–(3.15) of Ref. 1 follow directly from Theorems 3.2 and 3.3 of the last section by standard techniques.

V. LOCAL ANALYTICITY

We must study $A_1^\gamma(p)$ on $\mathcal{D}_L$ [of (3.2)]. We write $A_1^\gamma(p)$ from (2.1)–(2.5),

\[
A_1^\gamma(p) = - r_L \tilde{f}^{-1} \frac{1}{p^2} \frac{1}{D} l_1 + \frac{1}{p^2} \left( \frac{1}{p_i^2} \right) \left( 1 + p^2 \right)
\]

In studying analyticity we may remove the factor $- r_L$ and replace $\tilde{f}_i^{-1}$ by $\rho_i^{-1}$. We thus study
We introduce some notation
\[ p_i^2 (1/p_i^2) \equiv r_0(p) e_i(p) (1/p_i^2), \tag{5.3} \]
where \( e_i(0) = 1 \) and
\[ r_0(p) = \frac{4}{\pi} \sum_{i=1}^{4} \left| e_i - \frac{1}{p_i} \right|^2. \tag{5.4} \]
It will be sufficient to study \( a_i(p) \) and \( a_2(p) \). With the notation (5.3) and (5.4) we have
\[
\begin{align*}
  r_0(p) a_i(p) &= \frac{1}{\sum_{k \neq k} \frac{1}{p_i^2 \Pi_{j \neq k} e_j}} \left( p_i^2 e_i e_j + p_j^2 e_i e_j + p_k^2 e_i e_j \right) \\
  &\quad + \frac{1}{p_i^2 e_i (1 + p_i^2)^4}, \\
  r_0(p) a_2(p) &= \frac{-p_2 p_1}{\sum_{k \neq k} \frac{1}{p_i^2 \Pi_{j \neq k} e_j}} e_i e_k + \frac{p_2 p_1}{p_i^2 e_i (1 + p_i^2)^4}. \tag{5.5}
\end{align*}
\]
We now write \( e_i(p) \) as
\[ e_i(p) = 1 + \delta_i(p), \tag{5.6} \]
near \( p = 0 \). \( \delta_i \) is small. In (5.5) we substitute (5.7) and set \( \delta_i = 0 \) to get
\[
\begin{align*}
  r_0(p) a_i^2(p) &= 1 + \frac{p_i^2}{p_i^2 (1 + p_i^2)} - 1, \\
  r_0(p) a_2^2(p) &= \frac{p_2 p_1}{p_i^2 (1 + p_i^2)^4} - 1. \tag{5.8}
\end{align*}
\]
Writing
\[ a_i(p) = a_i^0(p) + R_i(p), \tag{5.10} \]
we see from (5.8) and (5.9) that \( a_i(p) \) are analytic in \( \mathcal{D}_L \). This is the main "algebraic miracle" involved in showing local analyticity of \( A_i(p) \). This algebraic miracle motivated, of course, our choice of \( X_i(p) \), which, however, was far from unique. We turn our attention to \( R_i(p) \). By showing the analyticity in \( \mathcal{D}_L \), of \( R_i(p) \), we will have completed the proof. (We have already used the analyticity of \( A_i(p) \) of \( r_L, p, f_L, \) and \( r_0 \).)

All hinges on certain properties of \( e_i \) and \( \delta_i \) in \( \mathcal{D}_L \), which we now pursue.

**VI. PROPERTIES OF THE \( e_i(p) \) IN \( \mathcal{D}_L \)**

1. \( e_i(p) \) is analytic, \tag{6.1}
2. \( \delta_i(p) = p_i^2 p_i h_i(p) \) with \( h_i(p) \) analytic, \tag{6.2}
3. \( e_i^{-1}(p) \) is analytic, \tag{6.3}
4. \[ \left[ \sum_{k \neq k} \frac{1}{p_i^2 \Pi_{j \neq k} e_j} - 1 \right] \text{ is analytic.} \tag{6.4} \]

It is quite immediate that (6.1)–(6.4) imply the analyticity of \( R_i(p) \). We are faced with our final task, the proof of these properties. (1) and (2) follow from the form of \( e_i(p) \) upon inspection. It is only (3) and (4) that must be studied.

We write \( e_i(p) \) in elaborate fashion \([e_i(p) \text{ and } e_i(p), i \neq 1, \text{ have the same properties}].\)

\[
\begin{align*}
  e_i(p) &= 1 + p_i^2 \sum \chi_{\nu}(1) \prod_{\nu} p_{\nu} K_{\nu}(p) \\
  &\quad + \chi_{\nu(1)} p_i^2 \prod_{\nu} K_{\nu}^{-1}(p), \tag{6.5}
\end{align*}
\]

\[
\begin{align*}
  K_e(p) &= \sum_{n \neq 0} \frac{1}{(p + 2\pi n)^2} \prod_{\nu} \frac{1}{(p_i + 2\pi n_i)^2}, \tag{6.6}
  K_f(p) &= \sum_{n \neq 0} \frac{1}{(p + 2\pi n)^2} \prod_{\nu} \frac{1}{(p_i + 2\pi n_i)^2}
  \times \frac{1}{(p_i + 2\pi n_i)^2}, \tag{6.7}
\end{align*}
\]

where
\[
\begin{align*}
  \chi_{\nu}(1) &= \begin{cases} 1 & 1 \in \nu, \\
                     0 & 1 \notin \nu, \end{cases} \\
  \chi_{\nu(1)}(1) &= \begin{cases} 0 & 1 \in \nu, \\
                        1 & 1 \notin \nu, \end{cases}
\end{align*}
\]

(a) \( \rho \) is a proper subset of \((1,2,3,4)\);
(b) \( \chi_{\nu}(1) = \begin{cases} 1 & 1 \in \nu, \\
                             0 & 1 \notin \nu, \end{cases} \)
(c) \( n \sim \rho \) means that \( \rho = \{i, n_i = 0\} \).

We easily see, for \( p \in \mathcal{D}_L \), that
\[
|K_e(p)| < m, \quad |K_f(p)| < m, \tag{6.8}
\]
for some fixed \( m \). And that
\[
|\text{Arg } K_e(p)| < \epsilon', \quad |\text{Arg } K_f(p)| < \epsilon', \tag{6.9}
\]
where \( \epsilon' \) can be made arbitrarily small by choosing \( \epsilon_0 \) of (3.2), suitably small. Again we have
\[
\begin{align*}
  |\text{Arg } p_i^2| < \epsilon^* & \quad \text{if } |p_i|^2 > \bar{\epsilon}, \\
  |\text{Arg } p_i^2| < \epsilon^* & \quad \text{if } |p_i|^2 > \bar{\epsilon}, \tag{6.10}
\end{align*}
\]
where \( \epsilon^* \) and \( \bar{\epsilon} \) can be fixed arbitrarily small, choosing \( \epsilon_0 \) small enough. Picking \( \epsilon^*, \epsilon^* \), and \( \bar{\epsilon} \) small enough, and using (6.8), we see \( e_i(p) \) is invertible in \( \mathcal{D}_L \) and so analytic implying (3), (6.3). (The terms on the right side of (6.5), individually, will either be small, or have small argument.)

We turn to the study of (4), (6.4). We write the brackets in (6.4) as
\[
\frac{1}{p_i^2 (1 + g)} = \frac{1}{p_i^2} - \frac{g}{p_i^2 (1 + g)} \tag{6.11}
\]
this relation defining \( g \). Property (4) will be proved by showing
\[
\begin{align*}
  (5) & \quad \frac{g}{p_i^2} \text{ is analytic}, \\
  (6) & \quad (1 + g) \text{ is analytic.} \tag{6.12}
\end{align*}
\]
For \( g \) we find the formula
\[
g = \sum \frac{p_i^2}{p_i^2} \left[ \prod_{i \neq k} (1 + \delta_i) - 1 \right]. \tag{6.14}
\]
From (6.2) we see that (5) [ (6.12) ] holds. The proof that \( (1 + g) \) is invertible, and so analytic, follows quite immediately from the proof above that \( e_i(p) \) is invertible.
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