Vector coherent state constructions of U(3) symmetric tensors and their SU(3)⊃SU(2)×U(1) Wigner coefficients

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Generalized vector coherent state constructions of totally symmetric U(3) tensors are used to gain new expressions for the SU(3)⊃SU(2)×U(1) Wigner coefficients for the coupling $(\lambda_1\mu_1)\times(\lambda_2 0)\rightarrow(\lambda_3\mu_3)$. These expressions show how the extremely simple formulas of Le Blanc and Biedenharn, involving a single 9-*j* coefficient, arise as special cases of a general result that involves 12-*j* coefficients. A simpler general result involving only 9-*j* coefficients and *K*-normalization factors is derived in a way that can, in principle, be generalized to the generic coupling with multiplicity.

I. INTRODUCTION

In the past few years, a vector coherent state theory[1–5] (VCS) and its associated *K*-matrix technique[1,6,7] have been used to great advantage to evaluate explicit expressions for the matrix representations of higher rank Lie algebras. The unitary groups in the canonical chain $U(n)\supset U(n-1)\supset\cdots\supset U(2)\supset U(1)$ form a particularly simple example,[8] and VCS techniques have been used to cast many results for the $U(n)$ Wigner–Racah calculus into new forms that reveal the structure of the Wigner and recoupling coefficients in a new light. It has been shown in particular[9] that the $U(n)$ elementary unit projective operators of Biedenharn and Louck[10] can be written down very simply in terms of $U(n-1)$ Racah coefficients and the simple *K*-normalization factors of VCS theory. Very recently Le Blanc and Biedenharn[11] have shown that some classes of $U(n)\supset U(n-1)\times U(1)$ reduced Wigner coefficients are simply products of $U(n-1)$ 9-*j* type recoupling coefficients and *K*-normalization factors. The question naturally arises: To what extent can the most general $U(n)\supset U(n-1)\times U(1)$ reduced Wigner coefficient be expressed in terms of $U(n-1)$ recoupling coefficients and the *K*-normalization factors of VCS theory? The earliest detailed applications of VCS theory have focused on the matrix elements of the generators of the algebra. Very recently[12,13] VCS theory has been generalized to include the Bargmann space realizations of more general operators lying outside the algebra. This generalization now makes it possible to examine the spectacularly simple class of $U(n)$ tensors of Le Blanc and Biedenharn[11] and show that they are a special case of a more general result. In this generalization, $U(n)\supset U(n-1)\times U(1)$ reduced Wigner coefficients are expressible in terms of summations involving $U(n-1)$ 12-*j* type recoupling coefficients. For the Le Blanc–Biedenharn case, these sums collapse to a single term in which the 12-*j* type coefficient collapses to a 9-*j* type coefficient. The ultimate aim of constructing a $U(n)$ tensor operator calculus in a unique (author-independent) way has not been fully implemented in the generic case with multiplicity.[14] It may therefore be useful to first reexamine the special multiplicity-free case of totally symmetric $U(n)$ tensors within the framework of the generalized VCS theory.[12,13] It is the purpose of this investigation to generalize the Le Blanc–Biedenharn result. To avoid the multiplicity problem, however, the investigation is restricted to the special case of totally symmetric tensors. To avoid some of the notational complexities of the Gel'fand notation needed for general $U(n)$, a further simplification to $n=3$ is made so that the $U(n-1)$ recoupling coefficients are expressible in terms of well-known angular momentum recoupling coefficients of 12-*j*, 9-*j*, or 6-*j* type. Three new expressions are given for the SU(3)⊃SU(2)×U(1) reduced Wigner coefficients for the most general coupling of type $(\lambda_1\mu_1)\times(\lambda_2 0)\rightarrow(\lambda_3\mu_3)$. In terms of their complexity and the number of required summations, these expressions are comparable to previously known[15] results. Since all results are expressed in terms of SU(2) recoupling coefficients and the *K*-normalization factors of VCS theory, these results reveal the structure of the SU(3)⊃SU(2)×U(1) reduced Wigner coefficients in a new light. They are derived by VCS techniques that can, in principle, be generalized to the generic case with multiplicity, the ultimate aim of this type of investigation.

The paper is organized in the following way. Section II gives the Bargmann space realization of totally symmetric U(3) tensor operators using the generalized VCS approach.[12] In this approach, a U(3) tensor operator is factored into two parts in an SU(2)-coupled basis: (1) an "intrinsic" operator acting only on the generalized VCS "vacuum" states, U(1) extremal states in the U(3)⊃SU(2)×U(1) scheme; and (2) a Bargmann space (**z**-space) operator that changes the U(1) weights. As soon as the "intrinsic" operator reduced matrix elements are known, the evaluation of SU(3)⊃SU(2)×U(1) reduced Wigner coefficients is reduced to an exercise in angular momentum coupling. The "intrinsic" operator reduced matrix elements are evaluated in Sec. III. The new expression for the SU(3)⊃SU(2)×U(1) reduced Wigner coefficient is then

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given in Sec. IV. For the coupling $(\lambda_1\mu_1)\times(\lambda_2 0)\rightarrow(\lambda_3\mu_2)$. The expression of Sec. IV is particularly simple in practice if n, the number of squares added to row 3 of the Young tableau for $(\lambda_1\mu_1)$ is very small compared with (λ_2-n) the number of squares added to rows 1 and 2. The case $n=0$ leads to the first Le Blanc–Biedenharn result. For the case when (λ_2-n) is small an alternate but similar expression may be more sueful. This is given in Sec. V. For the case $(\lambda_2-n)=0$ it collapses to the second Le Blanc–Biedenharn result. The general expressions of both Sec. IV and V involve 12-j coefficients. An even simpler expression, involving only 9-j coefficients, is derived in Sec. VI by a buildup process that compounds the two special Le Blanc–Biedenharn results. This final approach not only gives the simplest result from the point of view of actual computations but also shows the greatest promise for the needed generalization to the generic case with multiplicity.

II. VCS REALIZATION OF TOTALLY SYMMETRIC U(3) TENSORS

The U(3) generators E_{ij} can be realized in the usual way in terms of oscillator creation and annihilation operators, $\alpha^\dagger_{ip}$ and α_{ip}; with "spatial" index $i=1, 2, 3$, or x, y, z, and "particle index p, with $p=1, ..., n$:

$$E_{ij}=\frac{1}{2}\sum_{p=1}^{n}(\alpha^\dagger_{ip}\alpha_{jp}+\alpha_{jp}\alpha^\dagger_{ip}). \tag{1}$$

The complementary[16] $U(n)$ generators C_{pq} are

$$C_{pq}=\frac{1}{2}\sum_{i=1}^{3}(\alpha^\dagger_{ip}\alpha_{iq}+\alpha_{iq}\alpha^\dagger_{ip}). \tag{2}$$

For U(3) it is sufficient to choose $n=3$, and this choice will be made. However, the specific value of n plays very little role in the present investigation. [A restriction to SU(3) with $n=2$ has been shown to have some advantages by Le Blanc and Rowe[17] but would require some modification in the present construction.]

In the VCS theory, the U(3) generators are organized into (1) an Abelian nilpotent algebra of raising operators $E_{i3}\equiv A_i$, with $i=1, 2$; (2) an Abelian nilpotent algebra of lowering operators $E_{3i}=A^\dagger_i$, with $i=1, 2$; (3) The U(2) subgroup generators E_{ij} with $i,j=1, 2$; (4) the U(1) subgroup generator E_{33}.

The generators of U(2) ⊗ U(1) are called the core subgroup generators.

The U(3) state vectors can be specified by the Young frame integers $[m_{13}\ m_{23}\ m_{33}]$ with standard Gel'fand subgroup labels m_{12}, m_{22}, and m_{11}. Alternatively, they can be specified by the total number of oscillator quanta $N=m_{13}+m_{23}+m_{33}$, the Cartan SU(3) labels $\lambda=m_{13}-m_{23}$, $\mu=m_{23}-m_{33}$, and U(2)×U(1) subgroup labels given in the notation of Ref. 5 by angular momentum quantum numbers of I, M_I, and the U(1) label Y, the eigenvalue of $\frac{1}{3}(E_{11}+E_{22}-2E_{33})$, with $Y\frac{1}{3}=(\lambda+2\mu)-w$, $w=0, 1, ..., \lambda+\mu$. Note that $w=m_{13}+m_{23}-m_{12}-m_{22}$, $I=\frac{1}{2}(m_{12}-m_{22})$, $M_I=m_{11}-\frac{1}{2}(m_{12}+m_{22})$. Note also that w gives the eigenvalue of E_{33}. The set of generalized "vacuum" or "intrinsic" states of VCS theory will be chosen to be the states with $w=0$, $|N(\lambda\mu)w=0, I=\lambda/2, M_I=m\rangle$, for which

$$E_{i3}|N(\lambda\mu)w=0,\quad I=(\lambda/2)m\rangle=0,$$
$$\text{for } i=1,2;\ m=+\lambda/2,..., -\lambda/2. \tag{3}$$

Note that the operators E_{i3} are *raising* operators for Y. The state with $w=0$, $m=\frac{1}{2}\lambda$ is a *highest* weight state. In terms of the Elliott label[18] $\epsilon=-3Y$, however, the E_{i3} become *lowering* operators and the vacuum states become *lowest* ϵ-weight states. The words raising and lowering will therefore be avoided. The $E_{i3}\equiv A_i$ will be named annihilation operators instead since they annihilate the generalized vacuum states of Eq. (3), whereas the $E_{3i}\equiv A^\dagger_i$ can be named creation operators. In the U(3)×U(n) realization, with $n=3$, the generalized vacuum state with $m=+\frac{1}{2}\lambda$ has the form

$$|N(\lambda\mu)w=0,\quad I=m=\lambda/2\rangle$$
$$=\mathcal{N}_H(\alpha^\dagger_{11})^\lambda\begin{vmatrix}\alpha^\dagger_{11} & \alpha^\dagger_{12}\\ \alpha^\dagger_{21} & \alpha^\dagger_{21}\end{vmatrix}^\mu \times\begin{vmatrix}\alpha^\dagger_{11} & \alpha^\dagger_{12} & \alpha^\dagger_{13}\\ \alpha^\dagger_{21} & \alpha^\dagger_{22} & \alpha^\dagger_{23}\\ \alpha^\dagger_{31} & \alpha^\dagger_{32} & \alpha^\dagger_{33}\end{vmatrix}^{m_{33}}, \tag{4a}$$

with

$$\mathcal{N}_H=\sqrt{\frac{(\lambda+1)(\mu+1)(\lambda+\mu+2)}{(\lambda+\mu+m_{33}+2)!(\mu+m_{33}+1)!m_{33}!}}. \tag{4b}$$

The vector coherent state is built in terms of two complex variables z_i $(i=1,2)$, through the action of the creation operators E_{3i} on the generalized vacuum or intrinsic states:

$$|\mathbf{z};N(\lambda\mu)m\rangle=\exp(z_1^*E_{31}+z_2^*E_{32})|N(\lambda\mu)w=0,(\lambda/2)m\rangle. \tag{5}$$

Note that this coherent state carries the labels $N(\lambda\mu)$ and $m=+\lambda/2,..., -\lambda/2$. It forms a $(\lambda+1)$: dimensional array, i.e., it is a vector quantity.

State vectors can be expanded in terms of the U(3) basis vectors $|N(\lambda\mu)wIM_I\rangle$. In the VCS method these are mapped into their z-space functional realizations:

$$|N(\lambda\mu)wIM_I\rangle\rightarrow|N(\lambda\mu)wIM_I\rangle_{\rm VCS},$$
$$|N(\lambda\mu)wIM_I\rangle_{\rm VCS}$$
$$=\sum_m\langle N(\lambda\mu)0(\lambda/2)m|e^{\mathbf{z}\cdot\mathbf{A}}|N(\lambda\mu)wIM_I\rangle \times|N(\lambda\mu)0(\lambda/2)m\rangle\otimes|0\rangle, \tag{6}$$

with $\mathbf{z}\cdot\mathbf{A}=z_iA_i=z_1E_{13}+z_2E_{23}$. Note that this is a linear combination of intrinsic space standard kets, $|N(\lambda\mu)0(\lambda/2)m\rangle$, with coefficients that are functions of $\mathbf{z}$. For completeness the z-space vacuum ket $|0\rangle$ (for the action of the z bosons), has been included in Eq. (6). However, since its z-space realization is the simple number 1 it will usually be omitted throughout later sections.

Operators $\mathbf{O}$ are mapped into their z-space realizations $\Gamma(\mathbf{O})$ via

$$(\mathbf{O}|N(\lambda\mu)wIM_I\rangle)_{\rm VCS} + \sum_m \langle N(\lambda\mu)0\,\frac{\lambda}{2}m|e^{\mathbf{z}\cdot\mathbf{A}}\mathbf{O}e^{-\mathbf{z}\cdot\mathbf{A}}e^{\mathbf{z}\cdot\mathbf{A}}|N(\lambda\mu)wIM_I\rangle|N(\lambda\mu)0\,\frac{\lambda}{2}m\rangle\otimes|0\rangle$$

$$= \sum_m \sum_{\overline{w}\overline{I}\overline{M}_I} \langle N(\lambda\mu)0(\lambda/2)m|\Gamma(\mathbf{O})|N(\lambda\mu)\overline{w}\overline{I}\overline{M}_I\rangle\langle N(\lambda\mu)\overline{w}\overline{I}\overline{M}_I|e^{\mathbf{z}\cdot\mathbf{A}}|N(\lambda\mu)wIM_I\rangle|N(\lambda\mu)0\frac{\lambda}{2}\,m\rangle\otimes|0\rangle, \tag{7}$$

with

$$\Gamma(\mathbf{O}) = \mathbf{O} + [\mathbf{z}\cdot\mathbf{A},\mathbf{O}] + \tfrac{1}{2}[\mathbf{z}\cdot\mathbf{A},\,[\mathbf{z}\cdot\mathbf{A},\mathbf{O}]] + \cdots. \tag{8}$$

The z-space realizations for the generators E_{ij} are given in Refs. 5 and 8, but are repeated here for completeness.

With $i,j = 1,2$,

$$\Gamma(E_{i3}) = \Gamma(A_i) = \frac{\partial}{\partial z_i},$$

$$\Gamma(E_{ij}) = \mathbb{E}_{ij} - z_j\frac{\partial}{\partial z_i} = E^{\rm intr}_{ij} + E^{\rm coll.}_{ij} \tag{9}$$

$$\Gamma(E_{3i}) = \Gamma(A_i^\dagger) = \sum_{\alpha=1}^{2}\left(\mathbb{E}_{\alpha i}z_\alpha - \mathbb{E}_{33}z_i - z_i z_\alpha\frac{\partial}{\partial z_\alpha}\right).$$

Note that they are functions of the z-space operators z_i, $\partial/\partial z_j$, and intrinsic operators $\mathbb{E}_{ij}$ (denoted by double lines). These intrinsic operators are defined only through their action on the intrinsic states. They commute with the z-space operators z_i, $\partial/\partial z_j$, and in the matrix element of $\Gamma(\mathbf{O})$ of Eq. (7) they must be worked through to the left so that they can act on the intrinsic state. Since generators do not change the irreducible representation of the group, the $\mathbb{E}_{ij}$ connect intrinsic states only to intrinsic states (possibly with $m'\neq m$). The $\mathbb{E}_{ij}$ are thus defined through their pure intrinsic state matrix elements, e.g., with $\mathbb{E}_{12}\equiv\mathbb{I}_+$,

$$\langle N(\lambda\mu)0(\lambda/2)m|\mathbb{E}_{12}|N(\lambda\mu)0(\lambda/2)(m-1)\rangle$$
$$= \sqrt{(\lambda/2+m)(\lambda/2-m+1)}. \tag{10}$$

In the generalized VCS method[12,13] operators $\mathbf{O}$ outside the group algebra are to be included. Since the action of such operators can change the U(3) irreducible representations, Eq. (7) is to be generalized to

$$(\mathbf{O}|N(\lambda\mu)wIM_I\rangle)_{\rm VCS}$$
$$= \sum_{N'(\lambda'\mu')}\sum_{m'}\sum_{\overline{w}\overline{I}\overline{M}_I}\langle N'(\lambda'\mu')0\frac{\lambda'}{2}\,m'|$$
$$\times\Gamma(\mathbf{O})|N(\lambda\mu)\overline{w}\overline{I}\overline{M}_I\rangle$$
$$\times\langle N(\lambda\mu)\overline{w}\overline{I}\overline{M}_I|e^{\mathbf{z}\cdot\mathbf{A}}|N(\lambda\mu)wIM_I\rangle$$
$$\times|N'(\lambda'\mu')0(\lambda'/2)m'\rangle\otimes|0\rangle. \tag{11}$$

The oscillator creation and annihilation operators ($\alpha^\dagger_{ia}$, α_{ia}, for specific particle index a are of particular interest since more complicated operators can be built from these. We note first that the operators $\alpha^\dagger_{ia}$, $E_{i3}=A_i,z_i^*$, ($i=1,2$), transform as U(2) tensors of rank [10], whereas their conjugate partners α_{ia}, $E_{3i}=A_i^\dagger$, z_i transform as U(2) tensors [0–1]. In terms of standard spherical tensor T^I_M, therefore,

$$\{z_1^*,z_2^*\} = \{Z^{1/2}_{+1/2}(\mathbf{z}^*),\,Z^{1/2}_{-1/2}(\mathbf{z}^*)\}, \tag{12a}$$

whereas

$$\{z_1,z_2\} = \{+Z^{1/2}_{-1/2}(\mathbf{z}),\,-Z^{1/2}_{+1/2}(\mathbf{z})\}$$
$$= \{(-1)^{1/2-m}Z^{1/2}_{-m}(\mathbf{z})\}, \tag{12b}$$

and

$$\{\alpha^\dagger_{1a},\alpha^\dagger_{2a}\} = \{(\alpha^\dagger)^{1/2}_{+1/2},(\alpha^\dagger)^{1/2}_{-1/2}\};$$
$$\{\alpha_{1a},\alpha_{2a}\} = \{(\tilde{\alpha})^{1/2}_{-1/2},\,-(\tilde{\alpha}^{1/2}_{+1/2}\}. \tag{13}$$

Note also that $\alpha^\dagger_{3a}$ and α_{3a} are SU(2) scalars.

The z-space realizations of the operators $\alpha^\dagger_{ia}$, α_{ia} can now be constructed by the application of Eq. (8) to yield

$$\Gamma((\boldsymbol{\alpha}^\dagger_a)^{1/2}_m) = (\mathbb{a}^\dagger_a)^{1/2}_m, \tag{14a}$$

$$\Gamma(\alpha^\dagger_{3a}) = \mathbb{a}^\dagger_{3a} + \mathbb{a}^\dagger_{1a}z_1 + \mathbb{a}^\dagger_{2a}z_2$$
$$= (\mathbb{a}^\dagger_a)^0_0 - \sqrt{2}[\mathbb{a}^{\dagger 1/2}_a\times\mathbf{Z}^{1/2}(\mathbf{z})]^0_0, \tag{14b}$$

$$\Gamma(\alpha_{3a}) = \mathbb{a}_{3a} = (\mathbb{a}_a)^0_0, \tag{15a}$$

$$\Gamma((\boldsymbol{\alpha}_a)^{1/2}_m) = (\mathbb{a}_a)^{1/2}_m - Z^{1/2}_m(\mathbf{z})(\mathbb{a}_a)^0_0. \tag{15b}$$

After the application of Eq. (8) the operators $\alpha^\dagger_{ia}$, α_{ia} have been formally replaced by double-line intrinsic operators. These double-line operators again commute with the z_i and $\partial/\partial z_j$. They are again to be worked through to the left in the matrix element of $\Gamma(\mathbf{O})$ in Eq. (11) where they are then defined through their actions on the adjacent intrinsic state. Unlike the matrix elements of Eq. (10), however, they can convert an intrinsic state with $N'(\lambda'\mu')$ to a nonintrinsic state, (with $\overline{w}\neq 0$), in the representation with $N(\lambda\mu)$. The practical application of VCS theory thus depends on the evaluation of the matrix elements of double-line operators such as $\mathbb{a}^\dagger$, $\mathbb{a}$ between the purely intrinsic states on the left and the permitted states on the right (see Sec. III). Finally, note that the square bracket in Eq. (14) denotes angular momentum coupling using a *right to left* coupling order. This right to left coupling order convention simplifies phase factors in the VCS constructions and will be used throughout.

The VCS mappings of Eqs. (6) and (7) are nonunitary. The $\Gamma(\mathbf{O})$ are, in general, nonunitary realizations of the operator $\mathbf{O}$. Clearly, $\Gamma(A_i^\dagger)\neq(\Gamma(A_i))^\dagger$; and $\Gamma(\alpha^\dagger_{ia})\neq(\Gamma(\alpha_{ia}))^\dagger$. Similarly, the VCS state vector $|N(\lambda\mu)wIM_I\rangle_{\rm VCS}$ of Eq. (6) is not normalized. It will be instructive to give a specific evaluation of $|N(\lambda\mu)wIM_I\rangle_{\rm VCS}$, through

$$\langle N(\lambda\mu)0(\lambda/2)m|e^{\mathbf{z}\cdot\mathbf{A}}|N(\lambda\mu)wIM_I\rangle$$
$$= \langle N(\lambda\mu)wIM_I|e^{\mathbf{z}^*\cdot\mathbf{A}^\dagger}|N(\lambda\mu)0(\lambda/2)m\rangle^*$$
$$= \sum_{k,m_k}\langle N(\lambda\mu)wIM_I|Z^{k/2}_{m_k}(\mathbf{A}^\dagger)$$
$$\times|N(\lambda\mu)0(\lambda/2)m\rangle^* Z^{k/2}_{m_k}(\mathbf{z}), \tag{16}$$

where we have used

$$\mathbf{z}^*\cdot\mathbf{A}^\dagger = z_1^*E_{31} + z_2^*E_{32} = \sqrt{2}[\mathbf{A}^{\dagger 1/2}\times Z^{1/2}(\mathbf{z}^*)]^0_0, \tag{17}$$

and repeated use of the angular momentum recoupling transformation:

$$[[Z^{(k-1)/2}(\mathbf{A}^\dagger)\times Z^{(k-1)/2}(\mathbf{z}^*)]^0\times[\mathbf{A}^{\dagger 1/2}\times Z^{1/2}(\mathbf{z}^*)]^0]_0^0$$
$$=\begin{bmatrix}1/2 & 1/2 & 0\\ (k-1)/2 & (k-1)/2 & 0\\ k/2 & k/2 & 0\end{bmatrix}$$
$$\times[[Z^{(k-1)/2}(\mathbf{A}\dagger)\times Z^{1/2}(\mathbf{A}\dagger)]^{k/2}$$
$$\times[Z^{(k-1)/2}(\mathbf{z}^*)\times Z^{1/2}(\mathbf{z}^*)]^{k/2}]_0^0 \quad (18)$$
$$=\sqrt{\frac{k(k+1)}{2}}\,[Z^{k/2}(\mathbf{A}\dagger)\times Z^{k/2}(\mathbf{z}^*)]_0^0.$$

In Eq. (18) we have used the value of the unitary (square bracket) form of the 9-j recoupling coefficient and the buildup relation for the **z**-space boson polynomials:

$$[Z^{w_1/2}(\mathbf{z})\times Z^{w_2/2}(\mathbf{z})]_m^{w/2}$$
$$=\delta_{w,w_1+w_2}\sqrt{(w_1+w_2)!/w_1!w_2!}\,Z_m^{w/2}(\mathbf{z}). \quad (19)$$

In Eq. (16) we have also used the conjugation relation

$$(Z_{-m_k}^{k/2}(\mathbf{z}^*))^*=(-1)^{k/2+m_k}Z_{m_k}^{k/2}(\mathbf{z}). \quad (20)$$

The buildup relation together with Eq. (12) leads to the specific construction

$$Z_m^{w/2}(\mathbf{z})=\frac{(z_1)^{w/2-m}(-z_2)^{w/2+m}}{\sqrt{(w/2-m)!(w/2+m)!}}, \quad (21)$$

where this is an eigenfunction of $\langle I_0\rangle^{\text{coll}}=\frac{1}{2}(E_{11}^{\text{coll}}-E_{22}^{\text{coll}})$ and $(\mathbf{I}^{\text{coll}}\cdot\mathbf{I}^{\text{coll}})$ with eigenvalues m and $(w/2)(w/2+1)$; [see Eq. (9)]. The creation operator polynomial $Z(\mathbf{A}^\dagger)$ is obtained from $Z(\mathbf{z})$ by the replacement $z_i\to A_i^\dagger$. The *non*-normalized state $|\phi_{wIM}\rangle$, constructed through the action of w creation operators $\mathbf{A}^\dagger$, via

$$|\phi_{wIM_I}\rangle=[Z^{w/2}(\mathbf{A}^\dagger)\times|N(\lambda\mu)0(\lambda/2)\rangle]_{MI}^I$$
$$=\sum_m\langle(\lambda/2)m(w/2)(M_I-m)|IM_I\rangle$$
$$\times Z_{(M_I-m)}^{w/2}(\mathbf{A}^\dagger)|N(\lambda\mu)0(\lambda/2)m\rangle \quad (22a)$$

is orthogonal to states $|N(\lambda\mu)w'I'M'_I\rangle$ with $w'\neq w$, or $I'\neq I$ $M'_I\neq M_I$ but is not normalized. The normalization factor is given by the K-matrix element of VCS theory[5,8] [for the derivation, see Eq. (28) below]:

$$|N(\lambda\mu)wIM_I\rangle=[1/K(\lambda\mu)_{wI}]|\phi_{wIM_I}\rangle. \quad (22b)$$

Equations (6), (16), and (22) thus lead to

$$|N(\lambda\mu)wIM_I\rangle_{\text{VCS}}$$
$$=K(\lambda\mu)_{wI}\sum_m\langle\frac{\lambda}{2}m\frac{w}{2}(M_I-m)|IM_I\rangle$$
$$\times Z_{M_I-m}^{w/2}(\mathbf{z})|N(\lambda\mu)0(\lambda/2)m\rangle\otimes|0\rangle$$
$$=K(\lambda\mu)_{wI}|N(\lambda\mu)wIM_I). \quad (23)$$

The normalized **z**-space state vector will henceforth be denoted by

$$|N(\lambda\mu)wIM)=\left[Z^{w/2}(\mathbf{z})\times|(\lambda\mu)0\frac{\lambda}{2})\right]_M^I$$
$$\equiv|(\lambda\mu)\left[\frac{w}{2}\times\frac{\lambda}{2}\right]\mathrm{I,M}). \quad (24)$$

Note, in particular, that the state vector in Eq. (24) has been written with a round parenthesis. (Note also that the z-space vacuum vector $|0\rangle$ will be omitted henceforth for economy of notation; for the same reason the label N will henceforth be omitted but will be quietly understood.) The z-space state vectors of Eq. (24) form an orthonormal set with respect to the z-space integrations with the standard Bargmann measure.[5] In evaluating matrix elements, it will be very useful to indicate explicitly whether matrix elements are to be calculated through their z-space integrations (with the Bargmann measure) or in standard Hilbert space form. For this reason the state vector of Eq. (24) has been written with a round parenthesis, $|\cdots)$, whereas the state vectors of Eq. (22) are interpreted as standard Hilbert space vectors and are denoted by angular brackets, $|\cdots\rangle$. The appearance of round parentheses in a matrix element henceforth will automatically signal z-space integrations and pure intrinsic space operations. The appearance of angular brackets on the other hand, will signal standard Hilbert space operations. To transcribe the matrix element of an operator **O** between states of the orthonormal Hilbert space basis $|N(\lambda\mu)wIM_I\rangle$ to the corresponding z-space matrix element, we not only need to transcribe to the orthonormal z-space basis $|N(\lambda\mu)wIM_I)$ but also need to transform the nonunitary z-space realization $\Gamma(\mathbf{O})$ to a unitary realization of the operator to be denoted by $\gamma(\mathbf{O})$. In the VCS technique, this transformation is achieved via the K operator:[5]

$$\gamma(\mathbf{O})=K^{-1}\Gamma(\mathbf{O})K=(\gamma(\mathbf{O}^\dagger))^\dagger. \quad (25)$$

The Hilbert space matrix element of **O** can thus be transcribed to the z-space matrix element of $\gamma(\mathbf{O})$:

$$\langle(\lambda'\mu')w'I'M'|\mathbf{O}|(\lambda\mu)wIM\rangle$$
$$=((\lambda'\mu')w'I'M'|\gamma(\mathbf{O})|(\lambda\mu)wIM)$$
$$=K^{-1}(\lambda'\mu')_{w'I'}\Big((\lambda'\mu')\left[\frac{w'}{2}\times\frac{\lambda'}{2}\right]I'M'\Big|$$
$$\times\Gamma(\mathbf{O})\Big|(\lambda\mu)\left[\frac{w}{2}\times\frac{\lambda}{2}\right]IM\Big)K(\lambda\mu)_{wI}. \quad (26)$$

This is a basic relation that will be used repeatedly to calculate matrix elements of intrinsic operators. It can also be used to verify the normalization factor character of K in Eq. (22). Note that

$$\langle\phi_{wIM}|(\lambda'\mu')w'I'M'\rangle$$
$$=\sum_m\Big\langle\frac{\lambda}{2}(M-m)\frac{w}{2}m\Big|IM\Big\rangle$$
$$\times(-1)^{w/2-m}\langle(\lambda\mu)0(\lambda/2)(M-m)|$$
$$\times Z_{-m}^{w/2}(\mathbf{A})|(\lambda'\mu')w'I'M'\rangle. \quad (27a)$$

where we have used the conjugation properties of the Z of Eq. (20) to obtain $(Z_m^{w/2}(\mathbf{A}^\dagger))^\dagger=(-1)^{w/2-m}Z_{-m}^{w/2}(\mathbf{A})$. Equation (26) can then be used to give

$$\langle(\lambda\mu)0(\lambda/2)(M-m)|Z_{-m}^{w/2}(\mathbf{A})|(\lambda'\mu')w'I'M'\rangle$$
$$=((\lambda\mu)0(\lambda/2)(M-m)|$$
$$\times K^{-1}\Gamma(Z_{-m}^{w/2}(\mathbf{A}))K|(\lambda'\mu')w'I'M')$$
$$=1\times\Big((\lambda\mu)0\frac{\lambda}{2}(M-m)|\Big(Z_{-m}^{w/2}\Big(\frac{\partial}{\partial\mathbf{z}}\Big)\Big)$$
$$\times|(\lambda'\mu')w'I'M'\Big)K(\lambda'\mu')_{w'I'}$$

$$= (-1)^{w/2-m}((\lambda'\mu')w'I'M'|Z^{w/2}_m(\mathbf{z})|(\lambda\mu)w$$
$$=0(\lambda/2)(M-m))^*K(\lambda'\mu')_{w'I'}, \qquad (27b)$$

where we have used $\Gamma(A_i) = \partial/\partial z_i$, and the fact that $\partial/\partial z_i$ is the adjoint of z_i with respect to Bargmann integrations.[5] Finally, K or K^{-1} acts as a simple unit operator on the normalized intrinsic state. Equations (27a) and (27b), with the orthonormality of the states (24), then give

$$\langle\phi_{wIM}|(\lambda'\mu')w'I'M'\rangle$$
$$=((\lambda'\mu')w'I'M'|(\lambda\mu)wIM)^*K(\lambda'\mu')_{w'I'}$$
$$=\delta_{(\lambda'\mu')(\lambda\mu)}\delta_{w'w}\delta_{I'I}\delta_{M'M}K(\lambda\mu)_{wI}, \qquad (28a)$$

leading to

$$\langle\phi_{wIM}|\phi_{wIM}\rangle$$
$$=\sum_{(\lambda'\mu')}\sum_{w'I'M'}\langle\phi_{wIM}|(\lambda'\mu')w'I'M'\rangle$$
$$\times\langle(\lambda'\mu')w'I'M'|\phi_{wIM}\rangle$$
$$=|K(\lambda\mu)_{wI}|^2. \qquad (28b)$$

The K-factors have been evaluated by VCS techniques:[5,8]

$$K(\lambda\mu)_{wI}=\sqrt{\frac{(\lambda+\mu+1)!\mu!}{(\lambda/2+\mu+1-w/2+I)!(\mu+\lambda/2-w/2-I)!}}, \qquad (29a)$$

or, with $w=p+q$, $I=\frac{1}{2}\lambda-\frac{1}{2}p+\frac{1}{2}q$,

$$K(\lambda\mu)_{pq}=\sqrt{\frac{(\lambda+\mu+1)!\mu!}{(\lambda+\mu+1-p)!(\mu-q)!}}. \qquad (29b)$$

General U(3) tensors can be constructed from the oscillator creation and annihilation operators so that their VCS realizations follow from Eqs. (14) and (15). Note, however, that Eqs. (14) and (15) lead to the following form for the VCS realization of the group generators $E_{i3}=A_i$ of annihilation type: $\Gamma(A_i)=\Sigma_a \boldsymbol{\alpha}^\dagger_{ia}\boldsymbol{\alpha}_{3a}$; i.e., they are built entirely from intrinsic (double-line) operators. This is quite different from the "standard" realization of these group generators,[5,8] $\Gamma(A_i)=\partial/\partial z_i$, in which they are built from pure z-space operators. It is well known that coherent state realizations of operators are not unique due to the overcompleteness of coherent states. Both realizations must, however, give the same matrix elements. This was demonstrated explicitly in Ref. 13 for the analogous versions of the annihilation generators of the Sp(6)$\supset$U(3) algebra. It is to be noted, however, that the structure of the expressions for the matrix elements can be very simple for one type of coherent state realization and very cumbersome for another. In some cases a search for an optimal realization may therefore be needed.

Totally symmetric U(3) tensor operators can be constructed through the three-dimensional oscillator creation operators in a single particle variable a, say $a=1$, via the polynomials

$$T^{(\lambda_2 0)}_{w_2 I_2 M_2}(\boldsymbol{\alpha}^\dagger_a)=\left[(\alpha^\dagger_{3a})^{w_2}/\sqrt{w_2!}\right]P^{I_2=(1/2)(\lambda_2-w_2)}_{M_2}(\boldsymbol{\alpha}^\dagger_a), \qquad (30a)$$

with

$$P^{I_2}_{M_2}(\boldsymbol{\alpha}^\dagger_a)=\frac{(\alpha^\dagger_{1a})^{I_2+M_2}}{\sqrt{(I_2+M_2)!}}\frac{(\alpha^\dagger_{2a})^{I_2-M_2}}{\sqrt{(I_2-M_2)!}}. \qquad (30b)$$

(The subscript 2 on the quantum numbers λ, w, I, M is used for later applications.) The VCS realizations of these totally symmetric U(3) tensors could be obtained by repeated application of Eqs. (14) or preferably by the direct application of Eq. (8).

For the latter, the needed tools are the commutator

$$[\mathbf{z}\cdot\mathbf{A},\alpha^\dagger_{3a}]=\alpha^\dagger_{1a}z_1+\alpha^\dagger_{2a}z_2=-\sqrt{2}[\boldsymbol{\alpha}^{\dagger 1/2}\times Z^{1/2}(\mathbf{z})]^0_0, \qquad (31)$$

and the angular momentum recoupling transformation

$$[[P^{w/2}(\boldsymbol{\alpha}^\dagger)\times Z^{w/2}(\mathbf{z})]^0\times[\boldsymbol{\alpha}^{\dagger 1/2}\times Z^{1/2}]^0]^0_0$$
$$=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{w}{2} & \frac{w}{2} & 0\\ \frac{w+1}{2} & \frac{w+1}{2} & 0\end{bmatrix}$$
$$\times[[P^{w/2}(\boldsymbol{\alpha}^\dagger)\times(\boldsymbol{\alpha}^\dagger)^{1/2}]^{(w+1)/2}$$
$$\times[Z^{w/2}(\mathbf{z})\times Z^{1/2}]^{(w+1)/2}]^0_0$$
$$=\sqrt{(w+2)(w+1)/2}[P^{(w+1)/2}(\boldsymbol{\alpha}^\dagger)$$
$$\times Z^{(w+1)/2}(\mathbf{z})]^0_0, \qquad (32)$$

where we have again used the value of the unitary (square bracket) form of the 9-j recoupling coefficient and the build-up relation (19), together with the analogous relation

$$[P^{w_1/2}(\boldsymbol{\alpha}^\dagger)\times P^{w_2/2}(\boldsymbol{\alpha}^\dagger)]^{w/2}_m$$
$$=\delta_{w,w_1+w_2}\sqrt{(w_1+w_2)!/w_1!w_2!}P^{w/2}_m(\boldsymbol{\alpha}^\dagger). \qquad (33)$$

Repeated applications of Eqs. (31) and (32) in the νth commutator of Eq. (8) yields

$$\Gamma(T^{(\lambda_2 0)}_{w_2 I_2=(1/2)(\lambda_2-w_2)M_2}(\boldsymbol{\alpha}^\dagger_a))$$
$$=\Gamma\left(\frac{(\alpha^\dagger_{3a})^{w_2}}{\sqrt{w_2!}}P^{(1/2)(\lambda_2-w_2)}_{M_2}(\boldsymbol{\alpha}^\dagger_a)\right)$$
$$=\sum_{k=0}^{w_2}(-1)^k\sqrt{\frac{(\lambda_2+k-w_2+1)!}{(\lambda_2-w_2+1)!}\frac{w_2!}{k!(w_2-k)!}}$$

$$\times \frac{(\alpha_{3a}^\dagger)^{w_2-k}}{\sqrt{(w_2-k)!}}$$
$$\times [P^{(1/2)(\lambda_2-w_2+k)}(\alpha_a^\dagger)\times Z^{k/2}(\mathbf{z})]^{(1/2)(\lambda_2-w_2)}_{M_2}. \quad (34)$$

This is the needed VCS realization of the totally symmetric U(3) tensor.

III. MATRIX ELEMENTS OF INTRINSIC OPERATORS

In order to evaluate matrix elements of the totally symmetric U(3) tensors between states of arbitrary $(\lambda_1\mu_1)$ and $(\lambda_3\mu_3)$,

$$\langle(\lambda_3\mu_3)w_3I_3M_3|T^{(\lambda_2 0)}_{w_2I_2M_2}|(\lambda_1\mu_1)w_1I_1M_1\rangle$$
$$= \langle(\lambda_3\mu_3)w_3I_3\|T^{(\lambda_2 0)}_{w_2I_2}\|(\lambda_1\mu_1)w_1I_1\rangle$$
$$\times\langle I_1M_1I_2M_2|I_3M_3\rangle, \quad (35)$$

via VCS techniques through the use of the basic relation (26) and the $\Gamma(T^{(\lambda_2 0)})$ of Eq. (34) it is necessary to evaluate the matrix elements of the intrinsic operators $(\alpha_{3a}^\dagger)^\nu P^{(1/2)(\lambda_2-\nu)}(\alpha_a^\dagger)$ in the z-space basis. Note again that these intrinsic operators commute with z_i, $\partial/\partial z_i$. In a z-space matrix element they must be worked through to the left where they are then defined through their matrix elements between a pure intrinsic state on their left and the appropriate permitted states on their right.

It is sufficient to describe the intrinsic states of $(\lambda_1\mu_1)$ by two-rowed tableaux. For the most general coupling $(\lambda_1\mu_1)\times(\lambda_2 0)\to(\lambda_3\mu_3)$ in which n squares are added to row 3 of this tableau [with $0\leqslant n\leqslant\min(\lambda_2,\mu_1)$], the intrinsic states for $(\lambda_3\mu_3)$ with $w_3=0$ will then have only n oscillator excitations of type 3. Since the $(\alpha_{3a}^\dagger)^\nu$ are defined through their left actions on the intrinsic states of $(\lambda_3\mu_3)$, and since the left action of $\alpha_{3a}^\dagger$ annihilates an oscillator excitation of type 3, only operators with $\nu\leqslant n$ will have nonzero matrix elements. The only terms of Eq. (34) that can contribute are those with $k=w_2, w_2-1,\ldots, w_2-n$. Note that, with $n=0$ and consequently $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2$, the basic relation of Eq. (26) leads to

$$\langle(\lambda_3\mu_3)w_3=0, I_3=(\lambda_3/2)\|T^{(\lambda_2 0)}_{w_2=0,I_2=\lambda_2/2}$$
$$\times\|(\lambda_1\mu_1)w_1=0, I_1=\lambda_1/2\rangle$$
$$=\langle(\lambda_3\mu_3)0(\lambda_3/2)\|P^{\lambda_2/2}(\alpha^\dagger)\|(\lambda_1\mu_1)0(\lambda_1/2)\rangle$$
$$=((\lambda_3\mu_3)0(\lambda_3/2)\|P^{\lambda_2/2}(\alpha^\dagger)\|(\lambda_1\mu_1)0(\lambda_1/2)). \quad (36)$$

In this case the only needed reduced matrix element of intrinsic operators is related immediately to a very simple reduced matrix element in ordinary space. This is the reason for the simplicity of the Le Blanc-Biedenharn result. Moreover, in this case the ordinary space angular momentum reduced matrix element [denoted by standard double lines in Eqs. (35) and (36)] can be reduced to an overall SU(3) reduced matrix element (to be denoted by both double lines and double brackets[9]) through an $SU(3)\supset SU(2)\times U(1)$ Wigner coefficient with value 1. With $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2$:

$$\langle(\lambda_3\mu_3)0(\lambda_3/2)\|P^{\lambda_2/2}(\alpha^\dagger)\|(\lambda_1\mu_1)0(\lambda_1/2)\rangle$$
$$=\langle(\lambda_1\mu_1)Y_1=\tfrac{1}{3}(\lambda_1+2\mu_1)I_1=\lambda_1/2;(\lambda_2 0)\tfrac{1}{3}\lambda_2,\tfrac{1}{2}\lambda_2\|(\lambda_3\mu_3)Y_3=\tfrac{1}{3}(\lambda_3+2\mu_3)I_3=\lambda_3/2\rangle$$
$$\times\langle\langle(\lambda_3\mu_3)\|T^{(\lambda_2 0)}(\alpha^\dagger)\|(\lambda_1\mu_1)\rangle\rangle=1\times\langle\langle(\lambda_3\mu_3)\|T^{(\lambda_2 0)}(\alpha^\dagger)\|(\lambda_1\mu_1)\rangle\rangle. \quad (37)$$

Although this $\langle\langle\|\ \|\rangle\rangle$ can be evaluated,[9] it will drop out of all final expressions andis therefore not needed. (Note also that the particle index a on $\alpha^\dagger$ has been dropped and will be omitted henceforth for economy of notation, although it is to be quietly understood.)

For the case of arbitrary n, the intrinsic operator matrix elements of operators $(\alpha_3^\dagger)^\nu P^{1/2(\lambda_2-\nu)}(\alpha^\dagger)$ with $\nu=0,1,\ldots,n$, will be related to the standard Hilbert space matrix element

$$\langle(\lambda_3\mu_3)0(\lambda_3/2)\|(\alpha_3^\dagger)^n P^{(1/2)(\lambda_2-n)}(\alpha^\dagger)\|(\lambda_1\mu_1)0(\lambda_1/2)\rangle$$

by an inductive process through repeated use of the basic relation (26). The matrix element of the intrinsic operator $(\alpha_3^\dagger)^\nu P^{(1/2)(\lambda_2-\nu)}(\alpha^\dagger)$ between a purely intrinsic state of $(\lambda_3\mu_3)$ on the left must have $n-\nu$ z excitations in the state of $(\lambda_1\mu_1)$ on the right. This type of matrix element is evaluated by a tranformation back to ordinary Hilbert space, via Eq. (26), where the states with $n-\nu$ excitations are constructed by ordinary Hilbert space creation operator excitations $A_i^\dagger$ through Eq. (22). The evaluation of the matrix elements in VCS space of the formal intrinsic operators is thus reduced to the evaluation of ordinary matrix elements in ordinary Hilbert space. The intrinsic operator matrix elements therefore become fully explicit and well defined.

The case $n=1$ will be illustrated in detail. In this case only intrinsic operators with $\nu=0$ and $\nu=1$ lead to nonzero matrix elements. With $w_2=0$, Eq. (34) leads to

$$\Gamma(P^{\lambda_2/2}_{M_2}(\alpha^\dagger))=P^{\lambda_2/2}_{M_2}(\alpha^\dagger), \quad (38)$$

and the basic relation (26) gives

$$\langle(\lambda_3\mu_3)w_3$$
$$=0(\lambda_3/2)\|\mathrm{P}^{\lambda_2/2}(\alpha^\dagger)\|(\lambda_1\mu_1)w_1=1\quad I_1=(\lambda'/2)\rangle$$
$$=\left((\lambda_3\mu_3)\ \frac{\lambda_3}{2}\|\gamma(P^{\lambda_2/2}(\alpha^\dagger))\|(\lambda_1\mu_1)\left[\frac{1}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda'}{2}\right)$$
$$=1\times\left((\lambda_3\mu_3)\ \frac{\lambda_3}{2}\|P^{\lambda_2/2}(\alpha^\dagger)\|(\lambda_1\mu_1)\left[\frac{1}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda'}{2}\right)$$
$$\times K(\lambda_1\mu_1)_{1(\lambda'/2)}, \quad (39)$$

where the z-space states are given by the angular momentum coupled notation, $|(\lambda\mu)[w/2\times\lambda/2]I,M)$, of Eq. (24). Note that the "collective" angular momentum of the intrinsic state, $w_3/2 = 0$, is to be omitted in this notation. Note also that the left action of the intrinsic operator $P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)$ on the intrinsic state $(\,(\lambda_3\mu_3)(\lambda_3/2)m|$ must create a state with one z-space excitation since $n = 1$ in the space of states with $\lambda_1 + 2\mu_1 = \lambda_3 + 2\mu_3 - \lambda_2 + 3$. By expressing the standard Hilbert space state with $w_1 = 1$ in terms of the $|\phi_{w_1 I_1 M_1}\rangle$ of Eq. (22), the needed intrinsic operator matrix element is given by

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\,\|P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big)$$
$$= \frac{1}{K^2(\lambda'\mu_1)_{1(\lambda'/2)}}\,\langle(\lambda_3\mu_3)w_3 = 0\,\frac{\lambda_3}{2}\,m_3\Big|$$
$$\times\Big[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\times\Big[\mathbf{A}^{\dagger 1/2}\times|(\lambda_1\mu_1)w_1 = 0\,\frac{\lambda_1}{2}\Big\rangle\Big]^{\lambda'/2}\Big]^{\lambda_3/2}_{m_3}, \tag{40}$$

where we have used $Z^{1/2}(\mathbf{A}^\dagger) = \mathbf{A}^{\dagger 1/2}$ and the reduced matrix element relation

$$\langle I_3\|T^{I_2}\|I_1\rangle = \langle I_3 M_3|[T^{I_2}|I_1\rangle]^{I_3}_{M_3}. \tag{41}$$

Angular momentum recoupling yields

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\,\|P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big)$$
$$= \frac{1}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}\,U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda'}{2}\,\frac{\lambda_2-1}{2}\Big)$$
$$\times\Big\langle(\lambda_3\mu_3)w_3 = 0\,\frac{\lambda_3}{2}\,m_3\Big|\Big[[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger),A^{\dagger 1/2}]^{(\lambda_2-1)/2}$$
$$\times\Big|(\lambda_1\mu_1)w_1 = 0\,\frac{\lambda_1}{2}\Big\rangle\Big]^{\lambda_3}_{m_3}, \tag{42}$$

where the U-coefficient is a Racah coefficient in unitary form, and where the angular momentum-coupled operator $[P^{\lambda_2/2}\times A^{\dagger 1/2}]^{(\lambda_2-1)/2}$ has been converted to an angular momentum-coupled commutator by using the fact that $\mathbf{A}^\dagger$ annihilates the intrinsic state in its left action on this state [cf. Eq. (3)]. The angular momentum-coupled commutator is defined by

$$[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger),A^{\dagger 1/2}]^{(\lambda_2-1)/2}_m$$
$$= \sum_{m_1}\Big\langle\frac{1}{2}\,m_1\,\frac{\lambda_2}{2}\,m-m_1\Big|\frac{\lambda_2-1}{2}\,m\Big\rangle$$
$$\times[P^{\lambda_2/2}_{m-m_1}(\boldsymbol{\alpha}^\dagger),A^{\dagger 1/2}_{m_1}], \tag{43}$$

with

$$\{A^{\dagger 1/2}_{-1/2},-A^{\dagger 1/2}_{+1/2}\},=\{A^\dagger_1,A^\dagger_2\}\equiv\{E_{31},E_{32}\}, \tag{44}$$

this gives

$$[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger),A^{\dagger 1/2}]^{(\lambda_2-1)/2}_m = \sqrt{(\lambda_2+1)}\,\alpha_3^\dagger P^{(\lambda_2-1)/2}_m(\boldsymbol{\alpha}^\dagger); \tag{45}$$

so that

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\,\|P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\lambda_1\Big]\frac{\lambda'}{2}\Big)$$
$$= \frac{\sqrt{(\lambda_2+1)}}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}\,U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda'}{2}\,\frac{\lambda_2-1}{2}\Big)$$
$$\times\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle, \tag{46}$$

which is the first intrinsic space matrix element needed for $n = 1$.

For the second needed intrinsic matrix element for the case $n = 1$, we use Eq. (34) to give

$$\Gamma(\alpha_3^\dagger P^{(\lambda_2-1)/2}_m(\boldsymbol{\alpha}^\dagger))$$
$$= \boldsymbol{\alpha}_3^\dagger P^{(\lambda_2-1)/2}_m(\boldsymbol{\alpha}^\dagger)$$
$$-\sqrt{(\lambda_2+1)}\,[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\times Z^{1/2}]^{(\lambda_2-1)/2}_m. \tag{47}$$

The basic relation (26) yields

$$\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle - \Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|\boldsymbol{\alpha}_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)$$
$$= -\sqrt{(\lambda_2+1)}\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\,Z^{1/2}]^{(\lambda_2-1)/2}\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)$$
$$= -\sqrt{(\lambda_2+1)}\sum_{\lambda'/2}U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda'}{2}\,\frac{\lambda_2-1}{2}\Big)\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big). \tag{48}$$

The last intrinsic matrix element in Eq. (48) is given by Eq. (46), so that

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|\boldsymbol{\alpha}_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)$$
$$= \Big\{1+(\lambda_2+1)\sum_{\lambda'/2}\frac{1}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}\,U^2\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda'}{2}\,\frac{\lambda_2-1}{2}\Big)\Big\}\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle$$
$$= \frac{(\lambda_3+\mu_3+3)(\mu_3+2)}{(\lambda_1+\mu_1+1)\mu_1}\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{\lambda_2/2-1/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle, \tag{49}$$

where we have used $\lambda_3 + 2\mu_3 = \lambda_1 + 2\mu_1 + \lambda_2 - 3$ to simplify the final result for the sum over possible $\lambda'/2$. Equations (46) and (49) give the two needed intrinsic operator matrix elements for the case $n = 1$.

The case $n = 2$ illustrates some additional features. Three intrinsic operator matrix elements are now needed. The analogs of Eqs. (39)–(46) yield

$$\left((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\left[1\times\frac{\lambda_1}{2}\right]\frac{\lambda''}{2}\right)$$
$$=\frac{\sqrt{\lambda_2(\lambda_2+1)}}{K^2(\lambda_1\mu_1)_{2(\lambda''/2)}}\,U\Big(\frac{\lambda_1}{2}\,1\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda''}{2}\,\frac{\lambda_2}{2}-1\Big)\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle, \tag{50}$$

where we have used $Z^1_m(\mathbf{A}^\dagger)=2^{-1/2}[A^{\dagger 1/2}\times A^{\dagger 1/2}]^1_m$, angular momentum recoupling, and a double application of the commutator Eq. (45).

For the next matrix element, Eqs. (26) and (47) yield

$$\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)w_1=1\,\frac{\lambda'}{2}\Big\rangle$$
$$=\Big\{\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|\boldsymbol{\alpha}_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big)$$
$$-\sqrt{(\lambda_2+1)}\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\times Z^{1/2}]^{(\lambda_2-1)/2}\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big)\Big\}\,K(\lambda_1\mu_1)_{1(\lambda'/2)}. \tag{51}$$

The left-hand side is evaluated by the technique used to evaluate the standard Hilbert space matrix element in Eq. (40) to give

$$\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\|\alpha_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)w_1=1\,\frac{\lambda'}{2}\Big\rangle$$
$$=\frac{1}{K(\lambda_1\mu_1)_{1(\lambda'/2)}}\sqrt{2\lambda_2}\,U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\,\frac{\lambda_2}{2}-1\Big)\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle. \tag{52}$$

Angular momentum recoupling together with Eq. (19) reduces the second term of the right-hand side of Eq. (51) to the intrinsic operator reduced matrix element of $P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)$ which is given by Eq. (50), so that

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|\boldsymbol{\alpha}_3^\dagger P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\|(\lambda_1\mu_1)\Big[\frac{1}{2}\times\frac{\lambda_1}{2}\Big]\frac{\lambda'}{2}\Big)$$
$$=\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle\Big\{\frac{\sqrt{2\lambda_2}}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\,\frac{\lambda_2}{2}-1\Big)$$
$$+\sum_{\lambda''/2}\frac{(\lambda_2+1)\sqrt{2\lambda_2}}{K^2(\lambda_1\mu_1)_{2(\lambda''/2)}}\,U\Big(\frac{\lambda'}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda''}{2}\,\frac{\lambda_2-1}{2}\Big)U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda''}{2}\,\frac{1}{2};\frac{\lambda'}{2}\,1\Big)U\Big(\frac{\lambda_1}{2}\,1\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda''}{2}\,\frac{\lambda_2}{2}-1\Big)\Big\}$$
$$=\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle$$
$$\times\frac{\sqrt{2\lambda_2}}{K^2(\lambda_1\mu_1-1)_{1(\lambda'/2)}}\,U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\,\frac{\lambda_2}{2}-1\Big)\frac{(\lambda_3+\mu_3+4)(\mu_3+2)}{(\lambda_1+\mu_1+1)\mu_1}, \tag{53}$$

where we have used $\lambda_1+2\mu_1+\lambda_2-6=\lambda_3+2\mu_3$ to simplify the final result for the sum over possible $\lambda''/2$.

For the final intrinsic operator matrix element for $n=2$, the basic relation (26) gives

$$\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle$$
$$=\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\Big\|\frac{(\boldsymbol{\alpha}_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\boldsymbol{\alpha}^\dagger)\Big\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)-\sqrt{2\lambda_2}\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|\boldsymbol{\alpha}_3^\dagger[P^{(\lambda_2-1)/2}(\boldsymbol{\alpha}^\dagger)\times Z^{1/2}]^{\lambda_2/2-1}\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)$$
$$+\sqrt{\lambda_2(\lambda_2+1)}\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\|[P^{\lambda_2/2}(\boldsymbol{\alpha}^\dagger)\times Z^1]^{\lambda_2/2-1}\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big). \tag{54}$$

Angular momentum recoupling reduces the last two terms to the form that follows from Eqs. (50) and (53), so that

$$\Big((\lambda_3\mu_3)\,\frac{\lambda_3}{2}\Big\|\frac{(\boldsymbol{\alpha}_3^\dagger)^2}{\sqrt{2}}P(\boldsymbol{\alpha}^\dagger)^{\lambda_2/2-1}\Big\|(\lambda_1\mu_1)\,\frac{\lambda_1}{2}\Big)$$
$$=\Big\langle(\lambda_3\mu_3)0\,\frac{\lambda_3}{2}\Big\|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P(\boldsymbol{\alpha}^\dagger)^{\lambda_2/2-1}\Big\|(\lambda_1\mu_1)0\,\frac{\lambda_1}{2}\Big\rangle\Big\{1+2\lambda_2\sum_{\lambda'/2}\frac{1}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}U^2\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\,\frac{\lambda_2}{2}-1\Big)$$
$$+2\lambda_2(\lambda_2+1)\sum_{\lambda''/2}\Big[\sum_{\lambda'/2}U\Big(\frac{\lambda_1}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\,\frac{\lambda_2}{2}-1\Big)U\Big(\frac{\lambda'}{2}\,\frac{1}{2}\,\frac{\lambda_3}{2}\,\frac{\lambda_2}{2};\frac{\lambda''}{2}\,\frac{\lambda_2-1}{2}\Big)$$

$$U\left(\frac{\lambda_1}{2}\frac{1}{2}\frac{\lambda''}{2}\frac{1}{2};\frac{\lambda'}{2}1\right)\Big]\frac{1}{K^2(\lambda_1\mu_1)_{2(\lambda''/2)}}U\left(\frac{\lambda_1}{2}1\frac{\lambda_3}{2}\frac{\lambda_2}{2};\frac{\lambda''}{2}\frac{\lambda_2}{2}-1\right)$$

$$\times -\lambda_2(\lambda_2+1)\sum_{\lambda''/2}\frac{1}{K^2(\lambda_1\mu_1)_{2(\lambda''/2)}}U^2\left(\frac{\lambda_1}{2}1\frac{\lambda_3}{2}\frac{\lambda_2}{2};\frac{\lambda''}{2}\frac{\lambda_2}{2}-1\right)\Big\}. \quad (55)$$

It is now advantageous to carry out the sum over $\lambda'/2$ first in the third term, where

$$\sum_{\lambda'/2}U\left(\frac{\lambda_1}{2}\frac{1}{2}\frac{\lambda_3}{2}\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\frac{\lambda_2}{2}-1\right)U\left(\frac{\lambda'}{2}\frac{1}{2}\frac{\lambda_3}{2}\frac{\lambda_2}{2};\frac{\lambda''}{2}\frac{\lambda_2-1}{2}\right)\times U\left(\frac{\lambda_1}{2}\frac{1}{2}\frac{\lambda''}{2}\frac{1}{2};\frac{\lambda'}{2}1\right)=U\left(\frac{\lambda_1}{2}1\frac{\lambda_3}{2}\frac{\lambda_2}{2};\frac{\lambda''}{2}\frac{\lambda_2}{2}-1\right), \quad (56)$$

to obtain

$$\left((\lambda_3\mu_3)\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)\frac{\lambda_1}{2}\right)$$
$$=\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle$$
$$\Big\{1+2\lambda_2\sum_{\lambda'/2}\frac{1}{K^2(\lambda_1\mu_1)_{1(\lambda'/2)}}U^2\left(\frac{\lambda_1}{2}\frac{1}{2}\frac{\lambda_3}{2}\frac{\lambda_2-1}{2};\frac{\lambda'}{2}\frac{\lambda_2}{2}-1\right)$$
$$\lambda_2(\lambda_2+1)\sum_{\lambda''/2}\frac{1}{K^2(\lambda_1\mu_1)_{2(\lambda''/2)}}U^2\left(\frac{\lambda_1}{2}1\frac{\lambda_3}{2}\frac{\lambda_2}{2};\frac{\lambda''}{2}\frac{\lambda_2}{2}-1\right)\Big\}$$
$$=\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^2}{\sqrt{2}}P^{\lambda_2/2-1}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle\frac{(\lambda_3+\mu_3+4)(\lambda_3+\mu_3+3)(\mu_3+3)(\mu_3+2)}{(\lambda_1+\mu_1+1)(\lambda_1+\mu_1)\mu_1(\mu_1-1)}. \quad (57)$$

For the case of general n (with $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2-3n$), the corresponding result is

$$\left((\lambda_3\mu_3)\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^n}{\sqrt{n!}}P^{(\lambda_2-n)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)\frac{\lambda_1}{2}\right)$$
$$=\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^n}{\sqrt{n!}}P^{(\lambda_2-n)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle\sum_{\nu=0}^{n}\sum_{\lambda''/2}\frac{(\lambda_2+1-\nu)!}{(\lambda_2+1-n)!}\frac{n!}{(n-\nu)!\nu!}\frac{1}{K^2(\lambda_1\mu_1)_{(n-\nu)(\lambda''/2)}}$$
$$\times U^2\left(\frac{\lambda_1}{2}\frac{n-\nu}{2}\frac{\lambda_3}{2}\frac{\lambda_2-\nu}{2};\frac{\lambda''}{2}\frac{\lambda_2-n}{2}\right)$$
$$=\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^n}{\sqrt{n!}}P^{(\lambda_2-n)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle$$
$$\times\frac{(\lambda_3+\mu_3+n+2)!(\mu_3+n+1)!(\lambda_1+\mu_1+1-n)!(\mu_1-n)!}{(\lambda_3+\mu_3+2)!(\mu_3+1)!(\lambda_1+\mu_1+1)!\mu_1!}. \quad (58)$$

The techniques illustrated in detail for $n=1$ and $n=2$ lead to the general result for the needed intrinsic operator matrix elements

$$\left((\lambda_3\mu_3)\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}P^{(\lambda_2-\nu)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)\left[\frac{n-\nu}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda''}{2}\right)$$
$$=\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^n}{\sqrt{n!}}P^{(\lambda_2-n)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle\sqrt{\frac{(\lambda_2+1-\nu)!}{(\lambda_2+1-n)!}\frac{n!}{(n-\nu)!\nu!}\frac{1}{K^2(\lambda_1,\mu_1-\nu)_{(n-\nu)\lambda''/2}}}$$
$$\times U\left(\frac{\lambda_1}{2}\frac{n-\nu}{2}\frac{\lambda_3}{2}\frac{\lambda_2-\nu}{2};\frac{\lambda''}{2}\frac{\lambda_2-n}{2}\right)\frac{(\lambda_3+\mu_3+n+2)!(\mu_3+n+1)!(\lambda_1+\mu_1+1-\nu)!(\mu_1-\nu)!}{(\lambda_3+\mu_3+n-\nu+2)!(\mu_3+n-\nu+1)!(\lambda_1+\mu_1+1)!\mu_1!}. \quad (59)$$

Note that Eq. (59) reduces properly to Eq. (58) for $n=\nu$.

Finally, there remains the evaluation of the standard Hilbert space matrix element, with $\nu=n$. This is given by

$$\left\langle(\lambda_3\mu_3)0\frac{\lambda_3}{2}\left|\left|\frac{(\alpha_3^\dagger)^n}{\sqrt{n!}}P^{(\lambda_2-n)/2}(\alpha^\dagger)\right|\right|(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\rangle$$
$$=\left\langle(\lambda_1\mu_1)Y_1=\frac{1}{3}(\lambda_1+2\mu_1)I_1=\frac{\lambda_1}{2};(\lambda_20)\frac{1}{3}\lambda_2-n,\frac{1}{2}\lambda_2-\frac{n}{2}\|(\lambda_3\mu_3)Y_3=\frac{1}{3}(\lambda_3+2\mu_3)I_3=\frac{\lambda_3}{2}\right\rangle$$
$$\times\langle\langle(\lambda_3\mu_3)\|T^{(\lambda_2 0)}(\alpha^\dagger)\|(\lambda_1\mu_1)\rangle\rangle. \quad (60)$$

Since the double-bar, double-bracket factor will not be needed, it suffices to evaluate the SU(3) ⊃ SU(2) × U(1) reduced

Wigner coefficient with both $w_3=0$, $w_1=0$. In Sec. IV it will be shown that the square of this Wigner coefficient is given by the inverse of the double sum of Eq. (58). With a generalized Condon–Shortley phase convention, we therefore get, with $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2-3n$:

$$\left\langle(\lambda_1\mu_1)Y_1=\frac{1}{3}(\lambda_1+2\mu_1)I_1=\frac{\lambda_1}{2};\quad(\lambda_2 0)\,\frac{1}{3}\lambda_2-n,\ \frac{\lambda_2-n}{2}\right|\left|(\lambda_3\mu_3)Y_3=\frac{1}{3}(\lambda_3+2\mu_3)I_3=\frac{\lambda_3}{2}\right\rangle$$
$$=\sqrt{\frac{(\lambda_3+\mu_3+2)!(\mu_3+1)!(\lambda_1+\mu_1+1)!\mu_1!}{(\lambda_3+\mu_3+n+2)!(\mu_3+n+1)!(\lambda_1+\mu_1+1-n)!(\mu_1-n)!}}. \tag{61}$$

IV. THE SU(3)⊃SU(2)×U(1) WIGNER COEFFICIENT. FORM I

The general SU(3)⊃SU(2)×U(1) reduced Wigner coefficient for the product $(\lambda_1\mu_1)\times(\lambda_2 0)\to(\lambda_3\mu_3)$ follows from the general matrix element

$$\left\langle(\lambda_3\mu_3)w_3I_3\right|\left|\frac{\alpha_3^{\dagger w_2}}{\sqrt{w_2!}}P^{(\lambda_2-w_2)/2}(\boldsymbol{\alpha}^\dagger)\right|\left|(\lambda_1\mu_1)w_1I_1\right\rangle$$
$$=\left\langle(\lambda_1\mu_1)\mathrm{Y}_1=\frac{1}{3}(\lambda_1+2\mu_1)-w_1I_1;(\lambda_2 0)\,\frac{1}{3}\lambda_2-w_2,\frac{\lambda_2-\omega_2}{2}\right|\left|(\lambda_3\mu_3)Y_3\right.$$
$$\left.=\frac{1}{3}(\lambda_3+2\mu_3)-w_3I_3\right\rangle\langle\langle(\lambda_3\mu_3)\|T^{(\lambda_2 0)}\|(\lambda_1\mu_1)\rangle\rangle, \tag{62}$$

with $w_3=w_1+w_2-n$ for the coupling with n squares added to row 3 so that $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2-3n$. The basic relation (26) together with the VCS realization of the totally symmetric tensor, Eq. (34), gives

$$\left\langle(\lambda_3\mu_3)w_3I_3\right|\left|\frac{(\alpha_3^\dagger)^{w_2}}{\sqrt{w_2!}}P^{(\lambda_2-w_2)/2}(\boldsymbol{\alpha}^\dagger)\right|\left|(\lambda_1\mu_1)w_1I_1\right\rangle$$
$$=\frac{K(\lambda_1\mu_1)_{w_1I_1}}{K(\lambda_3\mu_3)_{w_3I_3}}\sum_{\nu=0}^{\min(n,w_2)}(-1)^{w_2-\nu}\sqrt{\frac{(\lambda_2-\nu+1)!}{(\lambda_2-w_2+1)!}\frac{w_2!}{(w_2-\nu)!\nu!}}$$
$$\times\left((\lambda_3\mu_3)\left[\frac{w_3}{2}\times\frac{\lambda_3}{2}\right]I_3\right|\left|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}\left[P^{(\lambda_2-\nu)/2}(\boldsymbol{\alpha}^\dagger)Z^{(w_2-\nu)/2}(\mathbf{z})\right]^{I_2=(\lambda_2-w_2)/2}\right|\left|(\lambda_1\mu_1)\left[\frac{w_1}{2}\times\frac{\lambda_1}{2}\right]I_1\right). \tag{63}$$

The reduced matrix element in Eq. (63), rhs, can be evaluated directly by the expansion

$$\left((\lambda_3\mu_3)\left[\frac{w_3}{2}\times\frac{\lambda_3}{2}\right]I_3\right|\left|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}\left[P^{(\lambda_2-\nu)/2}(\boldsymbol{\alpha}^\dagger)\times Z^{(w_2-\nu)/2}(\mathbf{z})\right]^{I_2=(\lambda_2-\omega_2)/2}\right|\left|(\lambda_1\mu_1)\left[\frac{w_1}{2}\times\frac{\lambda_1}{2}\right]I_1\right)$$
$$=\frac{1}{(2I_3+1)}\sum_{\text{all }M_im_im_{w_i}}\langle I_1M_1I_2M_2|I_3M_3\rangle\left\langle\frac{\lambda_1}{2}m_1\frac{w_1}{2}m_{w_1}\middle|I_1M_1\right\rangle\left\langle\frac{\lambda_2-\nu}{2}m_2\frac{w_2-n}{2}m_{w_2}\middle|I_2M_2\right\rangle$$
$$\times\left\langle\frac{\lambda_3}{2}m_3\frac{w_3}{2}m_{w_3}\middle|I_3M_3\right\rangle\left(\frac{\lambda_3}{2}m_3\middle|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}P^{(\lambda_2-\nu)/2}_{m_2}(\boldsymbol{\alpha}^\dagger)(Z^{w_3/2}_{w_{m_3}}(\mathbf{z}))^\dagger Z^{(\omega_2-\nu)/2}_{m_{w_2}}(\mathbf{z})Z^{w_1/2}_{m_{w_1}}(\mathbf{z})\middle|\frac{\lambda_1}{2}m_1\right), \tag{64}$$

and by using

$$\left(\frac{\lambda_3}{2}m_3\right|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}P^{(\lambda_2-\nu)/2}_{m_2}(\boldsymbol{\alpha}^\dagger)$$
$$=\sum_{\lambda'/2,m'}\left(\frac{\lambda_3}{2}m_3\middle|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}P^{(\lambda_2-\nu)/2}_{m_2}(\boldsymbol{\alpha}^\dagger)\middle|(\lambda_1\mu_1)\left[\frac{n-\nu}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda'}{2}m'\right)\left((\lambda_1\mu_1)\left[\frac{n-\nu}{2}\times\middle|\frac{\lambda_1}{2}\middle|\frac{\lambda'}{2}m'\right.\right.$$
$$=\sum_{\lambda'/2}\sum_{m'm_1'm_{n-\nu}}\left((\lambda_3\mu_3)\frac{\lambda_3}{2}\right|\left|\frac{(\alpha_3^\dagger)^\nu}{\sqrt{\nu!}}P^{(\lambda_2-\nu)/2}(\boldsymbol{\alpha}^\dagger)\right|\left|(\lambda_1\mu_1)\left[\frac{n-\nu}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda'}{2}\right)$$
$$\times\left\langle\frac{\lambda'}{2}m'\frac{\lambda_2-\nu}{2}m_2\middle|\frac{\lambda_3}{2}m_3\right\rangle\left\langle\frac{\lambda_1}{2}m_1'\frac{n-\nu}{2}m_{n-\nu}\middle|\frac{\lambda'}{2}m'\right\rangle\left(\frac{\lambda_1}{2}m_1'\middle|(Z^{(n-\nu)/2}_{m_{n-\nu}}(\mathbf{z}))^\dagger.\right. \tag{65}$$

Finally, using Eq. (19) and the orthonormality of the $Z^{w/2}_m(\mathbf{z})$ in z space and the $|\lambda_1/2m_1)$ in intrinsic space

$$\left(\frac{\lambda_1}{2}m_1'\middle|(Z^{(n-\nu)/2}_{m_{n-\nu}}(\mathbf{z}))^\dagger(Z^{w_3/2}_{m_{w_3}}(\mathbf{z}))^\dagger Z^{(w_2-\nu)/2}_{m_{w_2}}(\mathbf{z})Z^{w_1/2}_{m_{w_1}}(\mathbf{z})\middle|\frac{\lambda_1}{2}m_1\right)$$
$$=\sqrt{\frac{(w_3+n-\nu)!}{w_3!(n-\nu)!}\frac{(w_1+w_2-\nu)!}{w_1!(w_2-\nu)!}}\left\langle\frac{w_1}{2}m_{w_1}\frac{w_2-\nu}{2}m_{w_2}\middle|\frac{w_1+w_2-\nu}{2}(m_{w_1}+m_{w_2})\right\rangle$$

$$\times\left\langle \frac{w_3}{2} m_{w_3} \frac{n-\nu}{2} m_{n-\nu} \middle| \frac{w_3+n-\nu}{2} (m_{w_3}+m_{n-\nu})\right\rangle \delta_{m_1' m_1} \delta_{(m_{w_3}+m_{n-\nu})(m_{w_1}+m_{w_2})}, \tag{66}$$

with $w_3 = w_1 + w_2 - n$, Eqs. (64)–(66) give

$$\left((\lambda_3\mu_3)\left[\frac{w_3}{2}\times\frac{\lambda_3}{2}\right]I_3 \middle\| \frac{(\mathscr{A}_3^\dagger)^\nu}{\sqrt{\nu!}} [P^{(\lambda_2-\nu)/2}(\mathscr{A}^\dagger)\times Z^{(w_2-\nu)/2}(\mathbf{z})]I_2 = \frac{\lambda_2-w_2}{2} \middle\| (\lambda_1\mu_1)\left[\frac{w_1}{2}\times\frac{\lambda_1}{2}\right]I_1\right)$$

$$= \sum_{\lambda'/2} \left((\lambda_3\mu_3)\,\frac{\lambda_3}{2} \middle\| \frac{(\mathscr{A}_3^\dagger)^\nu}{\sqrt{\nu!}} P^{(\lambda_2-\nu)/2}(\mathscr{A}^\dagger) \middle\| (\lambda_1\mu_1)\left[\frac{n-\nu}{2}\times\frac{\lambda_1}{2}\right]\frac{\lambda'}{2}\right) \frac{(w_1+w_2-\nu)!}{\sqrt{(w_1+w_2-n)!(n-\nu)!(w_2-\nu)!w_1!}} \left\{\Sigma\right\}, \tag{67}$$

where $\{\Sigma\}$ is shorthand for the sum over the product of eight angular momentum Wigner coefficients that can be expressed in terms of a 12-j recoupling coefficient in unitary form:

$$\left\{\Sigma\right\} = \frac{1}{(2I_3+1)} \sum_{\text{all } m\text{'s}} \langle I_1 M_1 I_2 M_2 | I_3 M_3\rangle$$

$$\times\left\langle \frac{\lambda_1}{2} m_1 \frac{w_1}{2} m_{w_1} \middle| I_1 M_1\right\rangle \left\langle \frac{\lambda_2-\nu}{2} m_2 \frac{w_2-\nu}{2} m_{w_2} \middle| I_2 M_2\right\rangle \left\langle \frac{\lambda_3}{2} m_3 \frac{w_3}{2} m_{w_3} \middle| I_3 M_3\right\rangle \left\langle \frac{\lambda'}{2} m' \frac{\lambda_2-\nu}{2} m_2 \middle| \frac{\lambda_3}{2} m_3\right\rangle$$

$$\times\left\langle \frac{\lambda_1}{2} m_1 \frac{n-\nu}{2} m_{n-\nu} \middle| \frac{\lambda'}{2} m'\right\rangle \left\langle \frac{w_1}{2} m_{\omega_1} \frac{w_2-\nu}{2} m_{w_2} \middle| \frac{w_1+w_2-\nu}{2} (m_{w_1}+m_{w_2})\right\rangle$$

$$\times\left\langle \frac{w_3}{2} m_{w_3} \frac{n-\nu}{2} m_{n-\nu} \middle| \frac{w_3+n-\nu}{2} (m_{w_1}+m_{w_2})\right\rangle$$

$$= \sqrt{\frac{(2I_1+1)(2I_2+1)(w_1+w_2-\nu+1)(\lambda_3+1)}{(2I_3+1)(\lambda_1+1)(\lambda_2-\nu+1)(w_3+1)}} (-1)^{\lambda_1/2-\lambda'/2-\nu/2-n/2+w_2}$$

$$\times\begin{bmatrix} I_2 & I_1 & I_3 & \frac{w_3}{2} \\ \frac{w_2-\nu}{2} & \frac{w_1}{2} & \frac{w_1+w_2-\nu}{2} & \frac{\lambda'}{2} \\ \frac{\lambda_2-\nu}{2} & \frac{\lambda_1}{2} & \frac{\lambda_3}{2} & \frac{n-\nu}{2} \end{bmatrix}$$

$$= \sum_{I''} \sqrt{\frac{(2I''+1)(\lambda_3+1)}{(2I_3+1)(\lambda'+1)}}\, U\left(I_1 \frac{w_2-\nu}{2} I_3 \frac{\lambda_2-\nu}{2}; I'' \frac{\lambda_2-w_2}{2}\right) U\left(\frac{\lambda_2-\nu}{2} \frac{\lambda_3}{2} I'' \frac{w_3}{2}; \frac{\lambda'}{2} I_3\right)$$

$$\times U\left(\frac{\lambda_1}{2} \frac{w_1}{2} I'' \frac{w_2-\nu}{2}; I_1 \frac{w_1+w_2-\nu}{2}\right) U\left(\frac{\lambda_1}{2} \frac{n-\nu}{2} I'' \frac{w_3}{2}; \frac{\lambda'}{2} \frac{w_1+w_2-\nu}{2}\right), \tag{68}$$

where the 12-j coefficient in unitary (square bracket) form is the standard transformation coefficient[19,20]

$$\begin{bmatrix} j_a & j_b & J_{ab} & J' \\ j_c & j_d & J_{cd} & J'' \\ J_{ac} & J_{bd} & J_r & J \end{bmatrix}$$

$$= \langle [[J_r\times[j_a\times j_b]J_{ab}]J'\times[j_c\times j_d]J_{cd}]JM | [[J_r\times[j_a\times j_c]J_{ac}]J''\times[j_b\times j_d]J_{bd}]JM\rangle, \tag{69}$$

where the square brackets denote angular momentum coupling, for now in the conventional left to right coupling order.

Since the intrinsic operator reduced matrix elements needed for Eq. (67) are known from Eq. (59); Eqs. (63) and (67) can be combined to give the desired SU(3)$\supset$SU(2)$\times$U(1) Wigner coefficient in terms of the matrix elements of $(\alpha_3^\dagger)^n P^{(\lambda_2-n)/2}(\boldsymbol{\alpha}^\dagger)$ between extremal states with $w_1 = w_3 = 0$, where the double-line, double-bracket reduced matrix element is eliminated via Eqs. (60) and (62):

$$\frac{\langle(\lambda_1\mu_1) Y_1(w_1) I_1; (\lambda_2 0) Y_2 = \tfrac{1}{3}\lambda_2 - w_2, I_2 = (\lambda_2-w_2)/2 \| (\lambda_3\mu_3) Y_3(w_3) I_3\rangle}{\langle(\lambda_1\mu_1)\tfrac{1}{3}(\lambda_1+2\mu_1)(\lambda_1/2); (\lambda_2 0)\tfrac{1}{3}\lambda_2 - n, (\lambda_2-n)/2 \| (\lambda_3\mu_3)\tfrac{1}{3}(\lambda_3+2\mu_3)(\lambda_3/2)\rangle}$$

$$= \sum_{\nu=0}^{\min(n,w_2)} (-1)^\nu \frac{K(\lambda_1\mu_1)_{w_1 I_1}}{K(\lambda_3\mu_3)_{w_3 I_3}} \sqrt{\frac{(\lambda_2-\nu+1)!w_2!(\lambda_2+1-\nu)!n!}{(\lambda_2-w_2+1)!\nu!(w_2-\nu)!(\lambda_2+1-n)!(n-\nu)!\nu!}}$$

$$\times \frac{(w_1+\omega_2-\nu)!}{\sqrt{(n-\nu)!w_1!(w_2-\nu)!(w_1+w_2-n)!}}$$

$$\times \sum_{\lambda'/2} \frac{1}{K^2(\lambda_1,\mu_1-\nu)_{(n-\nu)(\lambda'/2)}} U\left(\frac{\lambda_1}{2} \frac{n-\nu}{2} \frac{\lambda_3}{2} \frac{\lambda_2-\nu}{2}; \frac{\lambda'}{2} \frac{\lambda_2-n}{2}\right)$$

$$\times\begin{bmatrix} I_2 & I_1 & I_3 & \frac{w_3}{2} \\ \frac{w_2-\nu}{2} & \frac{w_1}{2} & \frac{w_1+w_2-\nu}{2} & \frac{\lambda'}{2} \\ \frac{\lambda_2-\nu}{2} & \frac{\lambda_1}{2} & \frac{\lambda_3}{2} & \frac{n-\nu}{2} \end{bmatrix}$$
$$\times\sqrt{\frac{(2I_1+1)(2I_2+1)(\lambda_3+1)(w_1+w_2-\nu+1)}{(2I_3+1)(\lambda_1+1)(\lambda_2-\nu+1)(w_3+1)}}(-1)^{\lambda_1/2-\lambda'/2-n/2-\nu/2}$$
$$\times\frac{(\mu_3+n+1)!(\lambda_3+\mu_3+n+2)!(\mu_1-\nu)!(\lambda_1+\mu_1+1-\nu)!}{(\mu_3+n-\nu+1)!(\lambda_3+\mu_3+n-\nu+2)!\mu_1!(\lambda_1+\mu_1+1)!}. \tag{70}$$

Setting $w_3=0$, we obtain a relation between the Wigner coefficient with $w_3=0$ but arbitrary $w_1I_1(w_2=n-w_1)$ and the starting coefficient [Eq. (61)], with both $w_1=0$, $w_3=0$, $(w_2=n)$. Using relations such as those illustrated by Eqs. (56) and the sum over $\lambda''/2$ in Eq. (53), the orthonormality sum over w_1 and I_1 can be put into the form

$$\sum_{w_1,w_2I_1}\left\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_2\mu_2)\frac{\lambda_2}{3}-w_2\frac{\lambda_2-w_2}{2}\,||\,(\lambda_3\mu_3)Y_3(0)\frac{\lambda_3}{2}\right\rangle^2=1$$
$$=\left\langle(\lambda_1\mu_1)\frac{1}{3}(\lambda_1+2\mu_1)\frac{\lambda_1}{2};(\lambda_2\mu_2)\frac{\lambda_2}{3}-n\frac{\lambda_2-n}{2}\,\middle\|\,(\lambda_3\mu_3)\frac{1}{3}(\lambda_3+2\mu_3)\frac{\lambda_3}{2}\right\rangle^2$$
$$\times\sum_{\nu=0}^{n}\left\{\frac{(\lambda_2+1-\nu)!n!}{(\lambda_2+1-n)!(n-\nu)!\nu!}\sum_{I_1}\frac{1}{K^2(\lambda_1\mu_1)_{(n-\nu)I_1}}U^2\left(\frac{\lambda_1}{2}\frac{n-\nu}{2}\frac{\lambda_3}{2}\frac{\lambda_2}{2};I_1\frac{\lambda_2-n}{2}\right)\right\}. \tag{71}$$

The sum is the same as that which has been evaluated in Eq. (58) and leads at once to the needed starting Wigner coefficient with both $w_1=0$, $w_3=0$, as quoted in Eq. (61). It is interesting to note that the square of this starting Wigner coefficient is given by the ratio of the reduced matrix element of $(\alpha_3^\dagger)^nP^{(\lambda_2-n)/2}(\alpha^\dagger)$ between ordinary Hilbert space states with both $w_1=0$, $w_3=0$ to that of the intrinsic operator $(\mathscr{a}_3^\dagger)^n\mathrm{P}^{(\lambda_1-n)/2}(\mathscr{a}^\dagger)$ between pure intrinsic states.

With the evaluation of this starting Wigner coefficient, the general Wigner coefficient is given by

with $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2-3n$, $\quad w_3=w_1+w_2-n$,

Form I:

$$\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_20)Y_2(w_2)I_2=(\lambda_2-w_2)/2\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle$$
$$=\frac{K(\lambda_1\mu_1)_{w_1I_1}}{K(\lambda_3\mu_3)_{w_3I_3}}\sqrt{\frac{w_3!(\lambda_2+1-n)!(\lambda_3+\mu_3+2)!(\mu_3+1)!(\mu_3+n+1)!(\lambda_3+\mu_3+n+2)!}{w_1!w_2!n!(\lambda_2+1-w_2)!(\lambda_1+\mu_1+1)!\mu_1!(\mu_1-n)!(\lambda_1+\mu_1+1-n)!}}$$
$$\times\sum_{\nu=0}^{\min(n,w_2)}(-1)^\nu\frac{w_2!(\lambda_2+1-\nu)!(w_1+w_2-\nu)!n!(\mu_1-\nu)!(\lambda_1+\mu_1+1-\nu)!}{(w_2-\nu)!(\lambda_2+1-n)!w_3!(n-\nu)!\nu!(\lambda_3+\mu_3+2+n-\nu)!(\mu_3+n+1-\nu)!}$$
$$\times\sum_{\lambda'/2}\frac{1}{K^2(\lambda_1,\mu_1-\nu)_{(n-\nu)\lambda'/2}}U\left(\frac{\lambda_1}{2}\frac{n-\nu}{2}\frac{\lambda_3}{2}\frac{\lambda_2-\nu}{2};\frac{\lambda'}{2}\frac{\lambda_2-n}{2}\right)$$
$$\times\begin{bmatrix} I_2 & I_1 & I_3 & \frac{w_3}{2} \\ \frac{w_2-\nu}{2} & \frac{w_1}{2} & \frac{w_1+w_2-\nu}{2} & \frac{\lambda'}{2} \\ \frac{\lambda_2-\nu}{2} & \frac{\lambda_1}{2} & \frac{\lambda_3}{2} & \frac{n-\nu}{2} \end{bmatrix}(-1)^{\lambda_1/2-\lambda'/2-\nu/2-n/2}$$
$$\times\sqrt{\frac{(2I_1+1)(2I_2+1)(w_1+w_2-\nu+1)(\lambda_3+1)}{(2I_3+1)(\lambda_1+1)(\lambda_2-\nu+1)(w_3+1)}}. \tag{72}$$

This form is particularly economical, if n or w_2 are small so that the number of terms in the ν sum is small. In particular, for $n=0$, the sums collapse to a single term, the 12-j coefficient collapses to a 9-j coefficient, and Eq. (72) reduces to the first special Le Blanc–Biedenharn result.

With $n=0$, $\lambda_3+2\mu_3=\lambda_1+2\mu_1+\lambda_2$, $w_3=w_1+w_2$,

$$\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_20)Y_2(w_2)I_2$$
$$=\frac{\lambda_2-w_2}{2}\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle$$

$$= \frac{K(\lambda_1\mu_1)_{w_1I_1}K(\lambda_2 0)_{w_2I_2}}{K(\lambda_3\mu_3)_{w_3I_3}}\sqrt{\frac{(w_1+w_2)!}{w_1!w_2!}}\begin{bmatrix} \frac{\lambda_1}{2} & \frac{w_1}{2} & I_1 \\ \frac{\lambda_2}{2} & \frac{w_2}{2} & I_2 \\ \frac{\lambda_3}{2} & \frac{w_1+w_2}{2} & I_3 \end{bmatrix}. \quad (73)$$

V. FORM II FOR THE SU(3)⊃SU(2)×U(1) WIGNER COEFFICIENT

The expression of Sec. IV is particularly simple if n, the number of squares added to row 3 of the Young tableau for $(\lambda_1\mu_1)$, is very small compared with $\lambda_2 - n$. In the case when $\lambda_2 - n \ll n$, it is advantageous to evaluate the Wigner coefficient

$$\langle(\lambda_3\mu_3)Y_3(w_3)I_3;(0\lambda_2) - \tfrac{1}{3}\lambda_2 + w_2, \quad I_2 = (\lambda_2 - w_2)/2\|(\lambda_1\mu_1)Y_1(w_1)I_1\rangle \quad (74)$$

via the reduced matrix element of the operator

$$T^{(0\lambda_2)}_{Y_2 = -(1/3)\lambda_2 + w_2, I_2 M_2} = \frac{(\alpha_{3a})^{w_2}}{\sqrt{w_2!}} P^{I_2 = \lambda_2/2 - w_2/2}_{M_2}(\boldsymbol{\alpha}_a). \quad (75)$$

The VCS realization of this operator is (again omitting the specific particle index, a)

$$\Gamma\left(\frac{(\alpha_3)}{\sqrt{w_2!}} P^{(\lambda_2 - w_2)/2}_{M_2}(\boldsymbol{\alpha})\right)$$
$$= \sum_{k=0}^{\lambda_2 - w_2} (-1)^k \sqrt{\frac{(\lambda_2 - w_2)!(w_2+k)!}{(\lambda_2 - w_2 - k)!k!w_2!}} \frac{(\mathscr{A}_3)^{w_2+k}}{\sqrt{(w_2+k)!}} [P^{(\lambda_2 - w_2 - k)/2}(\boldsymbol{\mathscr{A}}) \times Z^{k/2}(\mathbf{z})]^{(\lambda_2 - w_2)/2}_{M_2}. \quad (76)$$

The needed intrinsic operator reduced matrix elements can be evaluated by the techniques illustrated in Sec. II. Now with $\lambda_1 + 2\mu_1 = \lambda_3 + 2\mu_3 + 2\lambda_2 - 3n'$,

$$\left((\lambda_1\mu_1)\frac{\lambda_1}{2}\right\|\left|\frac{(\mathscr{A}_3)^{\lambda_2 - \nu}}{\sqrt{(\lambda_2-\nu)!}} P^{\nu/2}(\boldsymbol{\mathscr{A}})\right|\left\|(\lambda_3\mu_3)\left[\frac{n'-\nu}{2}\times\frac{\lambda_3}{2}\right]\frac{\lambda''}{2}\right)$$
$$= \left\langle(\lambda_1\mu_1)0\frac{\lambda_1}{2}\right\|\left|\frac{\alpha_3^{\lambda_2 - n'}}{\sqrt{(\lambda_2 - n')!}} P^{n'/2}(\boldsymbol{\alpha})\right|\left\|(\lambda_3\mu_3)0\frac{\lambda_3}{2}\right\rangle$$
$$\times \sqrt{\frac{(\lambda_2-\nu)!n'!}{(\lambda_2-n')!(n'-\nu)!\nu!} \frac{K^2(\lambda_3,\mu_3+\lambda_2+1-\nu)_{n'(\lambda_1/2)}}{K^2(\lambda_3,\mu_3+\lambda_2+1-\nu)_{(n'-\nu)(\lambda''/2)}K^2(\lambda_3\mu_3)_{n'(\lambda_1/2)}}}. \quad (77)$$

With these intrinsic operator matrix elements the recoupling techniques of Sec. IV can give the general Wigner coefficient (74). The symmetry property

$$\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_2 0)(\lambda_2/3) - w_2, I_2 = (\lambda_2 - w_2)/2\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle$$
$$= \sqrt{[\dim(\lambda_3\mu_3)/\dim(\lambda_1\mu_1)][(2I_1+1)/(2I_1+1)]}\,(-1)^{\lambda_1+\mu_1+\lambda_2+\lambda_3+\mu_3+I_2+I_3-I_1}$$
$$\times\langle(\lambda_3\mu_3)Y_3(w_3)I_3;(0\lambda_2) - \lambda_2/3 + w_2, I_2\|(\lambda_1\mu_1)Y_1(w_1)I_1\rangle \quad (78)$$

gives the needed coefficient. Renaming $n' = \lambda_2 - n$ to be in agreement with the notation of Sec. IV we obtain the new form of the Wigner coefficient: with $\lambda_3 + 2\mu_3 = \lambda_1 + 2\mu_1 + \lambda_2 - 3n$, $w_3 = w_1 + w_2 - n$.

Form II:

$$\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_2 0)(\lambda_2/3) - w_2, I_2 = (\lambda_2 - w_2)/2\|(\lambda_3\mu_3)Y_3(w_3)I_3)\rangle$$
$$= (-1)^{\mu_1+\mu_3+n+I_2+I_3-I_1}\sqrt{\frac{\dim(\lambda_3\mu_3)}{\dim(\lambda_1\mu_1)}\frac{2I_1+1}{2I_3+1}\frac{K(\lambda_3\mu_3)_{w_3I_3}}{K(\lambda_1\mu_1)_{w_1I_1}}}$$
$$\times[K(\lambda_3\mu_3)_{(\lambda_2-n)I_3=\lambda_1/2}K(\lambda_3,\mu_3+1+n)_{\lambda_2-n,I=\lambda_1/2}]^{-1}$$
$$\times\sum_{\nu=0}^{\min(\lambda_2-n,\lambda_2-w_2)}\sqrt{\frac{(\lambda_2-w_2)!(\lambda_2-n)!}{w_1!w_2!w_3!n!}}(-1)^\nu\frac{(\lambda_2-\nu)!(w_1+\lambda_2-n-\nu)!}{\nu!(\lambda_2-w_2-\nu)!(\lambda_2-n-\nu)!}$$

$$\times \sum_{\lambda''/2} \frac{K^2(\lambda_3,\mu_3+\lambda_2+1-\nu)_{(\lambda_2-n)(\lambda_1/2)}}{K^2(\lambda_3,\mu_3+\lambda_2+1-\nu)_{(\lambda_2-n-\nu)(\lambda''/2)}} U\left(\frac{\lambda_3}{2}\,\frac{\lambda_2-n-\nu}{2}\,\frac{\lambda_1}{2}\,\frac{\nu}{2};\frac{\lambda''}{2}\,\frac{\lambda_2-n}{2}\right)$$

$$\times(-1)^{\lambda_1/2-\lambda''/2-\lambda_2/2+n/2+\nu/2}\sqrt{\frac{(\lambda_1+1)(2I_2+1)(w_1+\lambda_2-n-\nu+1)}{(\lambda_3+1)(2I_2-\nu+1)(w_1+1)}}$$

$$\times\begin{bmatrix} \frac{\lambda_1}{2} & \frac{\nu}{2} & \frac{\lambda''}{2} & \frac{\lambda_3}{2} \\ I_1 & I_2 & I_3 & \frac{w_1+\lambda_2-n-\nu}{2} \\ \frac{w_1}{2} & I_2-\frac{\nu}{2} & \frac{\lambda_2-n-\nu}{2} & \frac{w_3}{2} \end{bmatrix}. \tag{79}$$

In the special case with $n=\lambda_2$ the sum in this expression collapses to a single term. The 12-*j* coefficient collapses to a 6-*j* coefficient, and with $(\lambda_1\mu_1)=(\lambda_3,\mu_3+n)$:

$$\langle(\lambda_3,\mu_3+n)Y_1(w_1)I_1;(n0)Y_2=n/3-w_2,I_2=(n-w_2)/2\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle$$

$$=\frac{K(\lambda_3\mu_3)_{w_3I_3}}{K(\lambda_3,\mu_3+n)_{w_1I_1}}\sqrt{\frac{(\lambda_3+\mu_3+2)(\mu_3+1)n!(w_1+1)!}{(\lambda_3+\mu_3+n+2)(\mu_3+n+1)(n-w_2)!w_2!(w_1+w_2-n+1)!}}$$

$$\times U\left(I_3\frac{n-w_2}{2}\,\frac{\lambda_3}{2}\,\frac{w_1}{2};I_1\,\frac{w_3}{2}\right) \tag{80}$$

This is the second special case of Le Blanc and Biedenharn.[11]

VI. FORM III. AN EXPRESSION WITH 9-*j* COEFFICIENTS ONLY

In the general case both forms I and II have the same complexity. Both involve two summations, the ν sum and a sum over an angular momentum quantum number. Both involve 12-*j* coefficients in the general case. Since 12-*j* coefficients may not be readily available a simpler expression would be useful. Such an expression can be derived by a buildup process in which the representation $(\lambda_2 0)$ is obtained from a stretched coupling of the representations $(n0)$ and $(\lambda_2-n,0)$, the first adding n squares to row 3 of the Young tableau for $(\lambda_1\mu_1)$ to make the representation (λ_1,μ_1-n), the second λ_2-n squares to rows 1 and 2 to make the final $(\lambda_3\mu_3)$:

$$\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(\lambda_2 0)\tfrac{1}{3}\lambda_2-w_2,I_2=(\lambda_2-w_2)/2\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle U_{\mathrm{SU}_3}$$

$$=\sum_{w',I'}\langle(\lambda_1\mu_1)Y_1(w_1)I_1;(n0)\,\frac{n}{3}-w'',\frac{n-w''}{2}\|(\lambda_1,\mu_1-n)Y'(w')I'\rangle$$

$$\times\langle(\lambda_1,\mu_1-n)Y'(w')I';(\lambda_2-n,0)\,\frac{\lambda_2-n}{3}-w,\frac{\lambda_2-n-w}{2}\|(\lambda_3\mu_3)Y_3(w_3)I_3\rangle$$

$$\times\left\langle(n0)\,\frac{n}{3}-w''\,\frac{n-w''}{2};(\lambda_2-n,0)\,\frac{\lambda_2-n}{2}-w\,\frac{\lambda_2-n-w}{2}\right\|\left|(\lambda_2 0)\,\frac{\lambda_2}{3}-w_2\,\frac{\lambda_2-w_2}{2}\right\rangle$$

$$\times U\left(I_1\,\frac{n-w''}{2}\,I_3\,\frac{\lambda_2-n-w}{2};I'\,\frac{\lambda_2-w_2}{2}\right), \tag{81}$$

with $w_3=w_1+w_2-n$, $w''=w'+n-w_1$, $w=w_3-w'$.

The coefficient for the coupling $(n0)\times(\lambda_2-n,0)\to(\lambda_2 0)$ is a special case of the 1st class of Le Blanc–Biedenharn. The needed 9-*j* coefficient is related by symmetry to a trivial 9-*j* coefficient with all stretched angular momentum couplings leading to the simple result (with $w_2=w+w''$)

$$\left\langle(n0)\,\frac{n}{3}-w'',\frac{n-w''}{2};(\lambda_2-n,0)\,\frac{\lambda_2-n}{3}-w,\frac{\lambda_2-n-w}{2}\right\|\left|(\lambda_2 0)\,\frac{\lambda_2}{3}-w_2,\frac{\lambda_2-w_2}{2}\right\rangle$$

$$=\sqrt{\frac{(\lambda_2-n)!(\lambda_2-w_2)!n!w_2!}{(\lambda_2-n-w)!\lambda_2!(n-w'')!w!w''!}}. \tag{82}$$

Using Eqs. (73) and (80) for the remaining SU(3) Wigner coefficients, we obtain the right-hand side of Eq. (81).

The coefficient U_{SU_3} for the left-hand side of Eq. (81) can be treated as a normalization factor. It is the SU(3) U coefficient for the SU_3 recoupling implied by relation (81). It it were not known it could now be obtained from the right-hand side of Eq. (81) for the special values $w_1 = 0$, $w_3 = 0$, $w_2 = n$ for which the 9-j coefficient and the U coefficients for the right-hand side are all unity. (In this case $w' = 0$ only and $I' = \lambda_1/2$ only.) From the known Wigner coefficient with $w_1 = w_3 = 0$ of Eq. (61), we can thus determine U_{SU_3} via this special case:

$$U_{\mathrm{SU}_3}((\lambda_1\mu_1)(n0)(\lambda_3\mu_3)(\lambda_2 - n,0);(\lambda_1,\mu_1 - n)(\lambda_2 0)) = \sqrt{\frac{n!(\lambda_2 - n)!(\mu_1 - n + 1)!(\lambda_1 + \mu_1 + 2 - n)!}{\lambda_2!(\mu_1 + 1)!)(\lambda_1 + \mu_1 + 2)!} \frac{(\mu_3 + n + 1)!(\lambda_3 + \mu_3 + n + 2)!}{(\mu_3 + 1)!(\lambda_3 + \mu_3 + 2)!}}. \tag{83}$$

With this value we get the general result for the $\mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ Wigner coefficient.

Form III (with $\lambda_3 + 2\mu_3 = \lambda_1 + 2\mu_1 + \lambda_2 - 3n$, $w_3 = w_1 + w_2 - n$):

$$\begin{aligned}
&\left\langle (\lambda_1\mu_1) Y_1(w_1) I_1;(\lambda_2 0)\, \frac{1}{3}\lambda_2 - w_2, I_2 = \frac{\lambda_2 - w_2}{2} \right\| (\lambda_3\mu_3) I_3 \Big\rangle \\
&= \sqrt{\frac{(\lambda_1 + \mu_1 + 1)!\mu_1!(\mu_3 + 1)!(\lambda_3 + \mu_3 + 2)!(\lambda_2 - n + 1)!(\lambda_2 - w_2)!}{(\lambda_1 + \mu_1 + 1 - n)!(\mu_1 - n)!(\mu_3 + n + 1)!(\lambda_3 + \mu_3 + n + 2)!}} \\
&\times \sum_{w'} \sum_{I'} \sqrt{\frac{n!(w_1 + 1)!w_3!w_2!}{(\lambda_2 - n - w_3 + w' + 1)(w' + 1)}} \frac{1}{w'!(\lambda_2 - n - w_3 + w')!(w_1 - w')!(n + w' - w_1)!(w_3 - w')!} \\
&\times \frac{K^2(\lambda_1,\mu_1 - n)_{w'I'}}{K(\lambda_1\mu_1)_{w_1 I_1} K(\lambda_3\mu_3)_{w_3 I_3}} \begin{bmatrix} \frac{\lambda_1}{2} & \frac{w'}{2} & I' \\ \frac{\lambda_2 - n}{2} & \frac{w_3 - w'}{2} & \frac{\lambda_2 - n - w_3 + w'}{2} \\ \frac{\lambda_3}{2} & \frac{w_3}{2} & I_3 \end{bmatrix} \\
&\times U\left(I_1 \frac{w_1 - w'}{2} I_3 \frac{\lambda_2 - n - w_3 + w'}{2};I' \frac{\lambda_2 - w_2}{2}\right) U\left(I' \frac{w_1 - w'}{2} \frac{\lambda_1}{2} \frac{w_1}{2};I_1 \frac{w'}{2}\right).
\end{aligned} \tag{84}$$

For the special case with $w_3 = 0$ this collapses to the simple result:

$$\begin{aligned}
&\left\langle (\lambda_1\mu_1) Y_1(w_1) I_1;(\lambda_2 0)\, \frac{1}{3}\lambda_2 - w_2, I_2 = \frac{\lambda_2 - w_2}{2} \right\| (\lambda_3\mu_3)\, \frac{1}{3}(\lambda_3 + 2\mu_3) I_3 = \frac{\lambda_3}{2} \Big\rangle \\
&= (-1)^{\lambda_1/2 + w_1/2 - I_1} \sqrt{\frac{(2I_1 + 1)n! - (\lambda_2 - w_2)!}{(\lambda_1 + 1)(\lambda_2 - n)!w_1!(n - w_1)!}} \\
&\times \sqrt{\frac{(\lambda_1 + \mu_1 + 1)!\mu_1!(\lambda_3 + \mu_3 + 2)!(\mu_3 + 1)!}{(\lambda_1 + \mu_1 + 1 - n)!(\mu_1 - n)!(\lambda_3 + \mu_3 + n + 2)!(\mu_3 + n + 1)!}} \\
&\times \frac{1}{K(\lambda_1\mu_1)_{w_1 I_1}} U\left(I_1 \frac{w_1}{2} \frac{\lambda_3}{2} \frac{\lambda_2 - n}{2};\frac{\lambda_1}{2} \frac{\lambda_2 - w_2}{2}\right).
\end{aligned} \tag{85}$$

For the special case with $w_1 = 0$, on the other hand, Eq. (84) leads to

$$\begin{aligned}
&\left\langle (\lambda_1\mu_1)\, \frac{1}{3}(\lambda_1 + 2\mu_1), I_1 = \frac{\lambda_1}{2};(\lambda_2 0)\, \frac{\lambda_2}{3} - w_2, I_2 = \frac{\lambda_2 - w_2}{2} \right\| (\lambda_3\mu_3) Y_3(w_3) I_3 \Big\rangle \\
&= \sqrt{\frac{(\lambda_2 - n + 1)!w_2!(\lambda_1 + \mu_1 + 1)!\mu_1!(\lambda_3 + \mu_3 + 2)!(\mu_3 + 1)!}{(\lambda_2 - w_2 + 1)!n!(\lambda_1 + \mu_1 + 1 - n)!(\mu_1 - n)!(\mu_3 + n + 1)!(\lambda_3 + \mu_3 + n + 2)!w_3!}} \\
&\times \frac{1}{K(\lambda_3\mu_3)_{w_3 I_3}} U\left(\frac{\lambda_1}{2} \frac{\lambda_2 - n}{2} I_3 \frac{w_3}{2};\frac{\lambda_3}{2} \frac{\lambda_2 - w_2}{2}\right).
\end{aligned} \tag{86}$$

Finally, by interchanging the order of the coupling to $(\lambda_2 - n,0) \times (n0)$ in the analog of Eq. (81) still another form can be obtained for the totally symmetric $\mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ Wigner coefficient.

Form III′ (with $\lambda_3 + 2\mu_3 = \lambda_1 + 2\mu_1 + \lambda_2 - 3n$, $w_3 = w_1 + w_2 - n$):

$$
\left\langle (\lambda_1\mu_1) Y_1(w_1) I_1; (\lambda_2 0)\, \frac{1}{3}\lambda_2 - w_2, I_2 = \frac{\lambda_2 - w_2}{2} \right\| \left| (\lambda_3\mu_3) Y_3(w_3) I_3 \right\rangle
$$

$$
= \sqrt{\frac{(\lambda_3+\mu_3+2)(\mu_3+1)(\lambda_3+\mu_3+n+1)!(\mu_3+n)!(\mu_1-n)!(\lambda_1+\mu_1+1-n)!}{(\lambda_3+\mu_3+n+2)(\mu_3+n+1)(\lambda_3+\mu_3+1)!\mu_3!\mu_1!(\lambda_1+\mu_1+1)!}}
$$

$$
\times \sqrt{\frac{(\lambda_2-n+1)!(\lambda_2-w_2)!n!w_2!}{w_1!(w_3+1)!}} \sum_{w,I} \frac{(w_1+w)!}{w!(w_2-w)!(n-w_2+w)!(\lambda_2-n-w)!}
$$

$$
\times \sqrt{\frac{(w_1+w+1)}{(\lambda_2-n+1-w)}} \times \frac{K(\lambda_1\mu_1)_{w_1 I_1} K(\lambda_3\mu_3)_{w_3 I_3}}{K^2(\lambda_3,\mu_3+n)_{w+w_1,I}} \begin{bmatrix} \frac{\lambda_1}{2} & \frac{w_1}{2} & I_1 \\ \frac{\lambda_2-n}{2} & \frac{w}{2} & \frac{\lambda_2-n-w}{2} \\ \frac{\lambda_3}{2} & \frac{w_1+w}{2} & I \end{bmatrix}
$$

$$
\times U\left(I_1 \frac{\lambda_2-n-w}{2} I_3 \frac{n-w_2+w}{2}; I \frac{\lambda_2-w_2}{2}\right) U\left(I_3 \frac{n-w_2+w}{2} \frac{\lambda_3}{2} \frac{w_1+w}{2}; I \frac{w_3}{2}\right). \quad (87)
$$

VII. SUMMARY

Three types of expressions have been derived within the framework of generalized VCS theory for the SU(3) $\supset$SU(2)$\times$U(1) Wigner coefficients for the multiplicity-free coupling $(\lambda_1\mu_1)\times(\lambda_2 0)\rightarrow(\lambda_3\mu_3)$ involving totally symmetric U(3) tensors. All three involve two summations and are therefore comparable to previously known results[15] as far as their complexity is concerned. All results are expressed in terms of SU(2) recoupling coefficients and the simple K-normalization factors of VCS theory and therefore throw new light on the structure of such coefficients. Two of the expressions, given by Eqs. (72) and (79), involve 12-j coefficients. Their main value lies in the fact that they illustrate how the spectacularly simple special cases of Le Blanc and Biedenharn[11] arise as special cases of very general results. Simpler expressions involving only 9-j coefficients and Racah coefficients of 6-j type are given by Eqs. (84) and (87). These are derived by a coupling process which compounds the two special Le Blanc–Biedenharn results. In this process a U(3) tensor that can add squares only to row 3 of the starting tableau is combined with a U(3) tensor that can add squares only to rows 1 and 2. Since tensors for the generic case with multiplicity[14] can also be built in this fashion the VCS techniques used in this investigation may be useful for the general coupling with multiplicity. Special values for the SU(3) Wigner coefficients for the coupling $(\lambda_1\mu_1)\times(\lambda_2 0)\rightarrow(\lambda_3\mu_3)$ in which either the $(\lambda_1\mu_1)$ or $(\lambda_3\mu_3)$ states are restricted to highest weight have also been given in a new form involving a simple Racah coefficient, see Eqs. (85) and (86).

Note added in proof: The Wigner coefficients of this investigation use the phase convention of Draayer and Akiyama (see Ref. 5). To convert to the phase convention of Biedenharn and Louck (Refs. 8–10), the state vector of Eq. (24) must be multiplied by the phase $(-1)^{\lambda/2-\omega/2-I}$ leading to an additional overall phase of $(-1)^{\lambda_1/2-\omega_1/2-I_1-\lambda_3/2+\omega_3/2+I_3}$ to convert the Wigner coefficients of this investigation to the Biedenharn–Louck convention.

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