By taking the ratio of Eqs. (26) and (27), one obtains the result

$$U(0, t_1')/U(0, t_1) = \{\exp(-\theta)\} \Big/ \left\{\exp(-\theta) - \frac{\alpha_1^2}{\alpha_2^2}\right\}, \quad (28)$$

where

$$\theta = D\alpha_2^2(t_1' - t_1)/r_0^2 = (\pi l k \alpha_2^2/C')(t_1' - t_1).$$

Solving Eq. (28) for $t_1' - t_1$ gives

$$(t_1' - t_1) = (C'/\pi l k \alpha_2^2) \ln[(\alpha_2^2/\alpha_1^2)(1 - U(0, t_1)/U(0, t_1'))]. \quad (29)$$

Equations (25) and (29) together result in

$$(1 - U(0, t_1)/U(0, t_1')) = (\alpha_1^2/\alpha_2^2)\{0.403(1 - \exp(-R't_1/C'))\}^{-1},$$

which along with Eq. (25) gives

$$\{(t_1'/t_1) - 1\}\{1 - U(0, t_1)/U(0, t_1')\}^{-1} = (0.403/\psi)(1 - \exp\psi) \ln\{0.403(1 - \exp\psi)\}, \quad (30)$$

where $\psi = -R't_1/C'$.

V. CONCLUSION

From the experimentally obtained heating and cooling curves, the quantities t_1, t_1', $U(0, t_1)$, $U(0, t_1')$, and Q are easily obtained. Equation (30) then allows for a calculation of the ratio R'/C'. Having obtained this quantity, one may then calculate R' from Eq. (26), and hence the heat capacity C'. Formulas (25) and (30) have been checked on heat capacity measurements taken in the temperature range 65°K–300°K, and yielded values of C' approximately 2 percent larger than the values obtained by applying the Keesom and Kok method to heating and cooling curves of the type shown in Fig. 2.

The author expresses his gratitude to Professor H. Grayson-Smith of the University of Alberta for his interest in the problem, and to the staff of the Theoretical Section of the Sylvania Physics Laboratories, where the problem was finally completed.

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A Statistical Approach to the Space-Charge Distribution in a Cut-Off Magnetron*

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When the anode potential of a nonoscillating magnetron is lower than the cut-off potential, the discrete electron-to-electron interaction has an effect on the space-charge distribution which would be negligible if only resulting in fluctuations about a known steady state. However the states proposed by Brillouin, Slater, and others are not steady states in this sense. The electron interaction must, therefore, be expected to produce a steady drift away from such initial states. This paper discusses qualitatively the final steady state to be expected and its dependence on the ratio of anode voltage to cut-off voltage. Because of the involved nature of the electron distribution function no attempt is made to calculate the space-charge distribution explicitly.

I. INTRODUCTION

As an initial state for the oscillating magnetron the nonconducting (or approximately nonconducting) condition of a dc magnetron has a considerable interest and has received appreciable attention, both of a theoretical and an experimental nature. Nonetheless, a satisfactory agreement has not been reached about the shape of the electron orbits and the detailed distribution of potential and electron density in such a space charge.

The solutions presented necessarily rest on postulates and assumptions that constitute idealizations of the real conditions. So does, of course, all physical theory, but in this case well-known factors have been neglected with the justification that their effect is judged to be so small that the approximate solution resulting will be close enough to be of value. Furthermore, the mathematical difficulties of a more rigorous approach appeared prohibitive.

It is the purpose of this paper to reconsider the conventional simplifications on which these solutions are based, to discuss whether or not they introduce appreciable errors, and to investigate the feasibility of a solution from a more realistic set of assumptions. Since the space charge is essentially a dilute gas formed by discrete electrons, classical statistical mechanics offers a logical approach to a fresh study of the problem.

We shall first place the problem into the framework of statistical mechanics. The next step is to consider the solutions proposed so far in this light and to show that they cannot possibly represent steady states. This discussion does not directly indicate whether the errors in the distribution of space charge and potential are large or small. Finally, the problem of finding a solution consistent with the laws of statistical mechanics will be investigated and the result compared with the previously suggested solutions.

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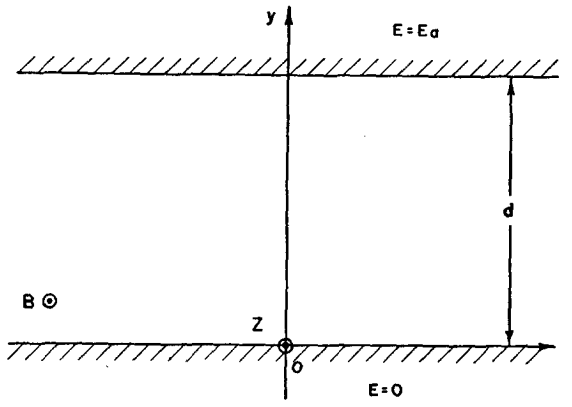

FIG. 1. Coordinate system for plane magnetron.

II. EQUATIONS FOR ELECTRON MOTION IN A MAGNETRON

Since we are going to apply the methods of statistical mechanics to the problem of electron motion in a magnetron, it is convenient to write the equations of motion in Hamiltonian form. The total energy of an electron is:

$$W=\left[\frac{1}{2m}\mathbf{p}-e\mathbf{A}\right]^2-eE, \tag{1}$$

where $\mathbf{p}$ is the momentum vector, $\mathbf{A}$ the vector potential of the constant magnetic field, and E the scalar electric potential. The momentum $\mathbf{p}$ is given by:

$$m=\mathbf{mv}+e\mathbf{A}, \tag{2}$$

where $\mathbf{v}$ is the velocity vector of the electron.

Equation (1) defines the "energy state" of the electron with reference to an average potential produced by the electrode potentials and a "smeared out" space charge. In describing the motion of an electron in an arbitrary but fixed energy state we shall disregard entirely the fluctuations in space-charge density and electric field so that we can define an ideal phase space or μ-space for the electron motion. Then we can proceed to study the flow of the electrons between different energy levels in μ-space, caused by the interchange of energy between the electrons or, which is the same, by the fluctuations in space-charge density and potential.

To simplify the mathematical processes, we choose a plane magnetron and a Cartesian coordinate system (Fig. 1). The cathode is represented by the plane $y=0$, the anode by the plane $y=d$. The uniform magnetic flux vector $\mathbf{B}$ is parallel to the positive z-axis. The potential of the cathode is taken to be zero and the potential of the anode positive is equal to E_a. No variations are taking place with x and z,

$$\partial/\partial x=\partial/\partial z=0. \tag{3}$$

The vector potential $\mathbf{A}$ then has only one component, A_x, such that

$$-\partial A_x/\partial y=B_z=B, \tag{4}$$

$$A_x=-By, \tag{5}$$

since we can choose A as well as E to be zero at the cathode.

The Hamiltonian (Eq. (1)) then is

$$W=\frac{1}{2m}\{(p_x+eBy)^2+p_y^2+p_z^2\}-eE, \tag{6}$$

and the canonical equations of motion

$$-\frac{dp_x}{dt}=\frac{\partial W}{\partial x}=0=m\frac{dv_x}{dt}+eBv_y, \tag{7}$$

$$-\frac{dp_y}{dt}=\frac{\partial W}{\partial y}=\frac{1}{2m}(p_x+eBy)\cdot eB-e\frac{\partial E}{\partial y}=eBv_x-e\frac{\partial E}{\partial y}, \tag{8}$$

$$-\frac{dp_z}{dt}=\frac{\partial W}{\partial z}=0, \tag{9}$$

$$\frac{dx}{dt}=\frac{\partial W}{\partial p_x}=\frac{1}{m}(p_x+eBy)=v_x, \tag{10}$$

$$\frac{dy}{dt}=\frac{\partial W}{\partial p_y}=v_y, \tag{11}$$

$$\frac{dz}{dt}=\frac{\partial W}{\partial p_z}=v_z. \tag{12}$$

The *constants of the motion*, or the parameters that determine the energy state of each electron, are W, p_x and p_z. Optionally, we can specify the state of an electron by p_x, p_z and p_{y0}, the last quantity being the value of p_y at the cathode ($y=0$). It should be noted that p_{y0} may be imaginary, since the orbit of an electron may not necessarily reach the cathode.

In the study of the space charge in the magnetron we shall be interested in the distribution of the electrons in *phase space* or μ-space, i.e., a six-dimensional space with the coordinates x, y, z, p_x, p_y, p_z. Actually, a subspace y, p_x, p_y will contain all the boundary surfaces and orbit projections necessary for the study of the problem.

III. EQUILIBRIUM STATES

If the cathode of the magnetron is at constant temperature and no current flows to the anode, no energy is received or lost by the swarm of electrons in the tube. It should, under these hypothetic conditions, be possible to consider the magnetron as a closed system in thermal equilibrium. In this section we shall consider the conditions to be satisfied for thermal equilibrium in a magnetron.

On the other hand, when a minute current flows to the anode the problem becomes a transport problem, and

the solution is obtained from the thermal equilibrium by applying a small perturbation to the distribution function for the electrons. As the current is increased, the diffusion of electrons through the space charge from the cathode gradually changes to a steady flow of the whole space charge towards the anode. The intermediate conditions, with severely distorted distribution function but no coherent flow pattern, offer the greatest mathematical difficulties; unfortunately, there are reasons to believe that this is the actual state of affairs in a cut-off magnetron.

Since the space charge is assumed to be in equilibrium with the thermionic cathode emitter, the distribution density function at $y=0$ is necessarily of the Maxwell-Boltzmann type.

$$P_0=A\cdot\exp\left[-\frac{\alpha}{2m}(p_x^2+p_{y0}^2+p_z^2)\right]. \tag{13}$$

The distribution density function P at an arbitrary point approaches its equilibrium value under the influence of two simultaneous processes, convection by electron motion, and exchange of energy and momentum between the electrons because of Coulomb interaction.

We shall not include here the complete classical derivation of the distribution density function. It is only necessary to investigate the consequences of the magnetic field and the curvilinear orbits of the electrons.

Under equilibrium conditions the convection-current element formed by a certain group of electrons in phase space must be continuous throughout its path through x-y-z space. The expected value of a current element formed by electrons of y-directed momentum between p_{y0} and $p_{y0}+dp_{y0}$ at the cathode is, therefore, equal to the corresponding element at any other value of y,

$$eP_0p_{y0}dp_{y0}=eP p_y dp_y. \tag{14}$$

However, conservation of energy requires

$$W=\frac{1}{2m}\{p_x^2+p_{y0}^2+p_z^2\}$$
$$=\frac{1}{2m}\{(p_x+eBy)^2+p_y^2+p_z^2\}-eE, \tag{15}$$

and consequently,

$$p_{y0}dp_{y0}=p_y dp_y. \tag{16}$$

Continuity, therefore, requires that the distribution density at y is

$$Pdp_y=P_0dp_y. \tag{17}$$

It should be noted that the distribution density function P is given by Eq. (17) only for the regions in phase space accessible to electrons emitted from the cathode. We shall later map these regions.

The interaction between the electrons during thermal equilibrium must be such that the same number of electrons are removed from a certain cell in phase space as are entering into the cell during the same time interval.

Suppose that a certain encounter involves n electrons with the initial energy states $W_1\cdots W_n$, and after the encounter, the energy states $W_1'\cdots W_n'$. Conservation of energy requires that

$$W_1+W_2+\cdots W_n=W_1'+W_2'+\cdots W_n'. \tag{18}$$

The rates at which such encounters and the inverse encounters take place should be equal and are proportional to

$$P_1\cdot P_2\cdot P_3\cdots P_n=P_1'\cdot P_2'\cdots P_n'. \tag{19}$$

Since the exponential function transforms a sum into a product, it is obvious that the Maxwell-Boltzmann distribution function satisfies the requirements (18) and (19), as well as (13) and (17). Thus we can write:

$$P=A\cdot e^{-\alpha W}, \tag{20}$$

where A is a constant related to the total number of electrons in the space charge and $\alpha=1/kT$.

It is interesting to note that the presence of the magnetic field and the consequent curvature of the electron orbits do not affect the distribution function. The factors that determine the distribution arise from conservation of energy and isotropy of interaction.

In order to find a space-charge distribution compatible with thermal equilibrium, we integrate (20) with respect to p_x, p_y, and p_z from $-\infty$ or $+\infty$. The result is

$$\rho=\rho_0e^{\alpha eE}. \tag{21}$$

This relation is combined with Poisson's equation to

$$\frac{\partial^2E}{\partial y^2}=-\frac{\rho}{\epsilon_0}=-\frac{\rho_0e^{\alpha eE}}{\epsilon_0}, \tag{22}$$

$$\frac{1}{2}\left(\frac{\partial E}{\partial y}\right)^2=-\frac{\rho_0}{\alpha e\epsilon_0}e^{\alpha eE}+C_1. \tag{23}$$

In order to perform the second integration, we introduce

$$Q=e^{\alpha eE}, \tag{24}$$

$$dQ/Q(a+Q)^{\frac{1}{2}}=-2\rho_0\alpha e/\epsilon_0\,dy. \tag{25}$$

Since the first term in (23) is always positive, C_1 and a must be negative if a potential minimum exists.

Changing the sign of a and integrating, we get

$$2/(a)^{\frac{1}{2}}\tan^{-1}(Q-a/a)^{\frac{1}{2}}=by+C_2, \tag{26}$$

$$\rho=\rho_0\cdot a/\cos^2b(y-y_m)=\rho_0[\cos by_m/\cos b(y-y_m)]^2, \tag{27}$$

$$E=2/\alpha e\,\log\cos by_m/\cos b(y-y_m), \tag{28}$$

where

$$b=\left(\frac{-2\rho_0\alpha e}{\epsilon_0}\right)^{\frac{1}{2}}. \tag{29}$$

The boundary conditions are represented by the constants ρ_0, a, and y_m. For negative values of y_m the solution represents temperature-limited conditions.

Figure 2 indicates the space-charge distribution required for this thermal equilibrium. It obviously does not resemble the distribution in a cut-off magnetron, since it demands emission of electrons from the anode with the same temperature as those emitted from the cathode but with considerably higher density,

$$\rho_a = \rho_0 e^{\alpha e E_a}. \tag{30}$$

Nonetheless this equilibrium has a certain interest to us, since the difference between this distribution and the actual distribution in any particular volume element in real space may serve as a rough indication of the amount of diffusion that takes place in that element. A steady state is reached when the diffusion into every energy state is equal to the diffusion out of the same energy state.

Before we investigate more closely this diffusion process, we shall in the next section discuss the representation of electron energy states as points and orbits in phase space.

IV. INITIAL STATES AND PHASE-SPACE REPRESENTATION OF ELECTRON ENERGY STATES

When a magnetron is switched to a source with a voltage E smaller than the cut-off voltage of the magnetron, a charge Q will flow to the magnetron. The source supplies the energy EQ, but the stored energy in the magnetron is only $\frac{1}{2}EQ$. The difference is lost in the circuit resistance and, possibly, in a temporary increase in the electron temperature of the magnetron space charge above the cathode temperature. These transient effects we shall disregard and assume that the initial conditions are consistent with conservation of energy and momentum for each electron emitted from the cathode and with Poisson's law.

As stated earlier in this paper, the energy state of an electron is characterized by its energy W and two of the components of its momentum, p_x and p_z, these three

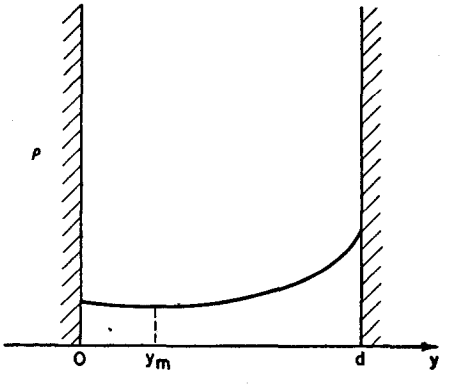

FIG. 2. Space-charge distribution at thermal equilibrium.

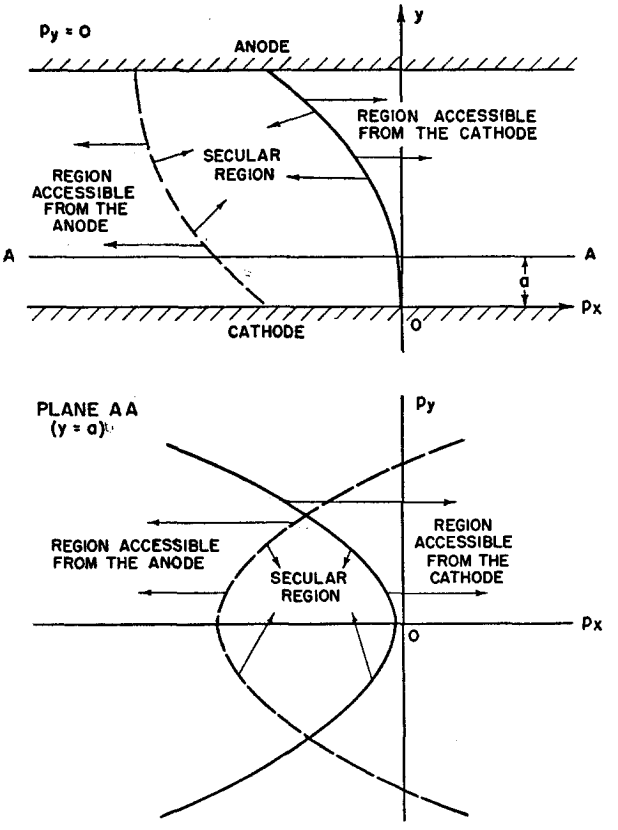

FIG. 3. Mapping of regions in phase space accessible to electrons whose orbits intersect the anode or cathode plane.

quantities being constants of the motion, as long as energy and momentum are conserved. On the other hand, the component p_y and the potential energy vary during the motion.

In order to map the regions in phase space accessible to the electrons emitted from the cathode it is sufficient to consider the space p_x, p_y, y.

The accessibility criterion is obtained from

$$2mW = p_x^2 + p_{y0}^2 + p_z^2 = (p_x + eBy)^2 + p_y^2 + p_z^2 - 2meE,$$

or

$$p_y^2 = p_{y0}^2 + 2meE - e^2B^2y^2 - 2p_x eBy, \tag{31}$$

where p_y and p_{y0} have to be real quantities.

Figure 3 shows a sketch of the boundary surface between the accessible and the inaccessible part of p_x-p_y-y-space. It is, of course, qualitative only, since $E(y)$ is not known but related to the space-charge distribution by Poisson's law. Integration of the distribution function (20) over the accessible space with respect to p_x and p_y would give the relation between p, E, and y that, combined with Poisson's equation, determines the initial conditions.

The inaccessible region has the shape of a distorted cone with its apex on the axis $p_y=0$, but in general not at $p_x=0$. The axis $p_y=0$ to the left of the apex is a generatrix of the surface. It should be pointed out that

the fact that the inaccessible region reaches the axis does not mean that electrons with $p_y=0$ and p_x in this range cannot escape from the cathode. They do, but only tangentially to the p_x-p_y plane, describe a broad elliptic orbit in the y-p_y plane and reach their turning point on the upper part of the boundary line of the accessible region or hit the anode.

In Fig. 3 the boundary for interception of the electrons by the anode is also indicated. In other words, any electron to the left of this surface will be removed by impact on the anode. The space-charge density can, therefore, be considered zero to the left of this boundary except where this region overlaps the region accessible from the cathode.

The volume between the two boundaries contains all the possible energy states whose electron orbits reach neither the anode nor the cathode. We shall refer to this volume as the *secular* region since the life of an electron energy state in this region is very long compared with the period of the cyclic motion of the electron. It should be noted that:

1. *Only an infinitesimal change in momentum is required for an electron to cross the boundary into this volume.*

2. *The electron population of this volume will continue to increase until the current flowing to the anode through the opposite boundary equals the current entering the volume from the cathode-accessible region.*

3. *Whether the discrete electron-electron interaction is weak or strong determines primarily the time required to reach a steady state but not necessarily the final space-charge distribution.*

Before discussing the possibility of determining at least roughly the final space-charge distribution, we shall give some brief comments on previously proposed solutions to the space-charge distribution in the cut-off magnetron. Common to them all is that the solution is identified with some form of what we have called initial conditions; sometimes the justification is given that the interaction between discrete electrons is small enough to be neglected. The three underlined conditions above show that this reasoning is not correct because of the effect of the interaction, however weak, is cumulative. In other words, these distributions are unstable, since the perturbations due to electron interaction do not produce fluctuations about the initial state, but a drift away from this state.

The Brillouin or single-stream space-charge distribution[1–3] in phase space (Fig. 4) is limited to the line AB and independent of y within this range. At first sight this distribution may appear to be a possible equilibrium at 0°K temperature. However, it would be strange

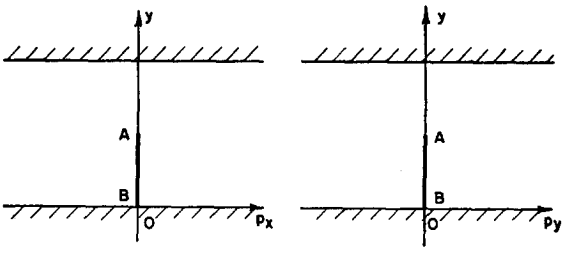

FIG. 4. Brillouin distribution in phase space.

indeed if all electrons occupied the same point in momentum space, although lower energy states certainly are possible. But we must consider the possibility that the state be metastable because of the lack of a process whereby the unoccupied energy states could be populated. Obviously such a process does exist. Since the space charge is formed by discrete electrons whose velocity varies with y, even though their orbits do not intersect, the electric field at the edge of the space-charge cloud necessarily fluctuates in magnitude and direction both in time and space. A diffusion will therefore take place, both out into the unoccupied space and towards the cathode. The energy required for this random motion is, of course, supplied by the dc electric field.

The double-stream distribution described by Slater,[4] Page and Adams,[5,6] and others is represented by an approximately elliptic line charge in phase space (Fig. 5). The space-charge density in real space at the cathode and at the edge of the swarm is infinite; a finite minimum is located at an intermediate plane. This is also a 0°K distribution, since all electrons have the same energy although occupying a line rather than a point in momentum space. There can be no question about a metastable state in this case, since the electron orbits intersect with considerable relative velocity so that exchange of energy and momentum is inevitable.

Twiss[7] has considered the modification in the initial space-charge distribution produced by the initial velocities of the electrons. The result is essentially the initial state mentioned earlier in this section as obtain-

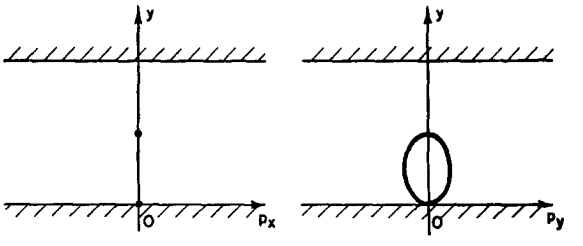

FIG. 5. Double-stream distribution in phase space.

[1] L. Brillouin, "Electronic theory of the plane magnetron," *Advances in Electronics* (Academic Press, Inc., New York, 1951), Vol. 3, p. 85.

[2] L. Brillouin and F. Bloch, "Electronic theory of the cylindrical magnetron," *Advances in Electronics* (Academic Press, Inc., New York, 1951), Vol. 3, p. 145.

[3] A. W. Hull, Phys. Rev. **18**, 31 (1921).

[4] J. C. Slater, *Microwave Electronics* (D. Van Nostrand Company, Inc., New York, 1950).

[5] L. Page and N. J. Adams, Phys. Rev. **69**, 492 (1946).

[6] L. Page and N. J. Adams, Phys. Rev. **69**, 494 (1946).

[7] R. Q. Twiss, *On the Steady-State and Noise Properties of Linear and Cylindrical Magnetrons*, doctoral thesis (Massachusetts Institute of Technology, 1949).

able by integration of the distribution function (20) over the accessible region of momentum space. It avoids the discontinuities and singularities of the two distributions mentioned above, but is otherwise subject to the same criticism; it is an initial state but not a steady state.

V. DISCUSSION OF THE MAGNETRON SPACE-CHARGE DISTRIBUTION AS A DIFFUSION PROBLEM

A calculation of the steady state in the cut-off magnetron from the initial state previously indicated is complicated by the following circumstances:

1. The relationships governing the transfer of momentum between the electrons, including close encounters as well as distant encounters, are rather involved even when the state of the space charge differs very little from thermal equilibrium.

2. The actual distribution density is very irregular because of the initially abrupt variation at the accessibility boundaries.

3. The distribution density function is furthermore distorted by the automatic "sorting" of electrons that have gained or lost energy, respectively, in the process of interaction. Most of the electrons that have gained energy will return to the cathode, while those that have lost energy will not. On the average, therefore, the returning electrons will have a slightly higher temperature than those leaving the cathode. (We use the term "temperature" for convenience although the distribution is not regular Maxwell-Boltzmann.)

We shall here try to arrive at a qualitative understanding of the way in which a steady state is reached without resolving quantitatively the complications enumerated above.

The drift of electrons through phase space is such that if the system were temporarily closed its state would gradually approach thermal equilibrium.

During thermal equilibrium the number of electrons that move from one particular volume element in phase space to another such element is equal to the number that moves in the opposite direction. We should expect it to be possible to calculate the instantaneous value of the interaction current from one volume element in the phase space to another by comparing the instantaneous distribution with the equilibrium distribution. Here, however, we should not consider thermal equilibrium, which is determined by the boundary conditions of the whole space charge, but the standard normal distribution of (p_x+eBy), p_y and p_z that has the same total population, the same energy, and the same center of gravity as the initial distribution. The total interaction current across a certain boundary surface in momentum space would be obtained by a double integration over these two distribution density functions. In addition to a function of the densities of the two volume elements in both distributions the integrand must necessarily contain a weighting function expressing the probability of the required change of momentum between the two volume elements. At least two simultaneous asymptotic processes should be considered that give quite different weighting functions. The first one operates by close binary encounters and is a discrete random process of well-known type. The influx to a certain volume element in phase space because of its density deficiency is largely determined by the distance of the element from the center of gravity of the distribution density functions. In this case the weighting function accounts for the impact parameter and the angle between the relative velocities of the two electrons with respect to their common center of gravity.

The second process is the result of interaction between a large number of electrons at considerably larger distances. The weighting function in this case permits only very small changes of momentum. The interaction current density across a certain surface in momentum space is consequently determined chiefly by a relation between the actual density gradient at this surface and the equilibrium density gradient there.

It has been shown[8] that when the interaction is due to Coulomb forces, neither one of these two processes can in general be neglected in comparison with the other.

In enumerating the factors that determine the steady state we should begin with the boundary conditions at the cathode. It is natural to assume that the mass and thermal capacity of the cathode are so large that the temperature and distribution of the emitted electrons are independent of the temperature and distribution of the returning electrons, that is to say, the energy and momentum of the electron cloud are not conserved at the cathode. If momentum were at least partly conserved, the center of gravity of the distribution could not fall on the line $p_x=-eBy$ but somewhere between this line and $p_x=0$. The assumption made here appears satisfactory as long as the emission is strictly thermionic; if secondary emission is appreciable, the boundary conditions become much more difficult to state.

Between the cathode and the potential minimum the inaccessible part of the phase space is likely to be small and not to include dense regions of the distribution. The potential and space-charge density then drops approximately according to (21) until roughly the Brillouin density is reached.

The most interesting region in real space is between the potential minimum and the edge of the cloud. The boundary of the secular volume in phase space on the cathode side is likely to be almost parallel to the y-axis here, since the potential distribution is probably not radically different from that of the Brillouin solution (see Fig. 6). The space-charge density in the secular volume obviously must be such that the net number of electrons entering from the right equals the net number leaving at the left boundary. This density is obviously smaller than the one required for ideal thermal equilibrium; the center of gravity of the distribution must,

[8] Cohen, Spitzer, and Routly, Phys. Rev. **80**, 230 (1950).

therefore, be expected to be to the right of the line $p_x=-eBy$, that is, closer to the right-hand boundary than to the left-hand one. The space-charge density required for a steady state is therefore closer to the density at the right boundary than to the one at the left, which is close to zero. To calculate this space-charge distribution is evidently very difficult, but it should be clear that its density is by no means negligible as far as calculation of space-charge distribution in real space is concerned.

Twiss maintains that a double-stream motion with considerably more than thermal energy exists here in the cathode-accessible volume of phase space. The square of the y-directed momentum of an electron can be written (31)

$$p_y^2=p_{y0}^2+2mE-e^2B^2y^2-2p_xeBy\cong p_{y0}^2-2p_xeBy. \quad (32)$$

The orbits of the electrons in the p_y-y-plane are parabolas, and since $eBy\gg p_x$, p_y may be of a different order of magnitude than p_{y0} for negative values of p_x. The orbits do not extend to the anode, because outside the edge of the space-charge swarm $e^2B^2y^2$ is considerably larger than $2mE$, so that the electrons turn back into the cloud.

At first sight these conclusions seem to be inconsistent with the view presented in Sec. IV, that the initial energy distribution at any point is an incomplete Maxwell-Boltzmann distribution of cathode temperature. The answer is that as the electrons move in the positive y-direction and gain kinetic energy from the electric field, they also move towards the high energy fringes of a distribution that, if it were complete, would have a much higher space-charge density than at the cathode.

The high energy electrons will be much closer to the center of gravity of the actual, considerably distorted distribution than to the center of gravity of the Maxwell-Boltzmann distribution of which they form a small part. Consequently, the average kinetic energy or "temperature" of the actual distribution will be much higher than at the cathode.

This increase of the average electron energy with y no doubt contains the clue to a number of magnetron problems. Considering the cut-off magnetron as a resistor, we should expect the increased electron temperature to increase the noise output. Twiss has attributed the noise level to a "noise amplification" caused by the double-stream interaction in the space-charge cloud. The points of view may not be equivalent, but they are certainly related.

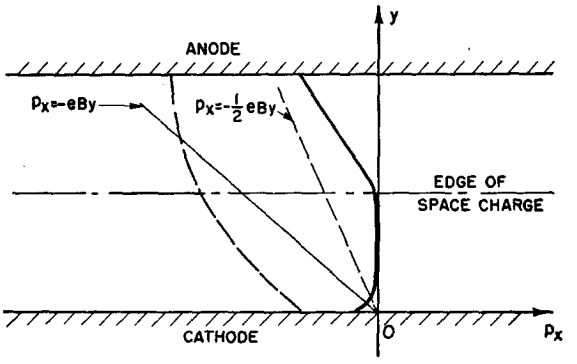

FIG. 6. Space-charge-limited conditions.

The difference between the space-charge distribution discussed here and the Brillouin or Twiss solution is probably not very large when the volume of the secular region in phase space is small, i.e., when the tube is operated close to the cut-off voltage. The anode current will be appreciably larger than the direct convection current from cathode to anode, however, because of the diffusion through the secular region.

When the anode voltage is far below the cut-off voltage, on the other hand, the secular volume is large, and the total space charge there may be large enough to affect the space-charge density in real space and the potential distribution appreciably.

The mathematical difficulties discourage any attempt to predict numerically the space-charge distribution in a cut-off magnetron; we must, therefore, look to experimental investigations for quantitative information. Reverdin[9] has described an interesting electron-optical method to explore the magnetron space charge. His results show that the steady-state formation is sensitive to cathode geometry and temperature, but his data are too scanty to yield any definite quantitative information as to the effect of the various parameters of the problem.

[9] D. L. Reverdin, J. Appl. Phys. **22**, 257 (1951).