Scale symmetries of spherical string fluids

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We consider homothetic maps in a family of spherical relativistic star models. A generalization of Vaidya's radiating metric provides a fluid atmosphere of radiation and strings. The similarity structure of the string fluid is investigated. © *1999 American Institute of Physics.* [S0022-2488(99)03308-3]

I. INTRODUCTION

Metric symmetries have always played a large role in the development of exact solutions to the Einstein field equations. Often a choice of metric symmetry is made based on an assumed symmetry of the matter distribution, i.e., spherical symmetry for astrophysical objects or cylindrical symmetry for a simple string.[1] A homothetic motion (homothety) describes the symmetry of scale transformations, and homothetic symmetry has been called ''similarity of the first kind'' by Cahill and Taub.[2] One must distinguish between *geometrical* and *physical* self-similarity. Geometrical similarity is a property of the space–time metric, whereas physical similarity is a property of the matter fields. These need not be equivalent and the relationship between them also depends on the nature of the matter. Yavuz and Yilmaz[3] recently investigated inheritance symmetries wherein the stress energy inherits metric symmetries of the type

$$\mathcal{L}_\xi g_{ab} = 2\Psi g_{ab},$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector ξ. Some of the possibilities are

$\Psi = \Psi(x^a)$, ξ is a conformal Killing vector,

$\Psi = 1$, ξ is a homothetic vector,

$\Psi = 0$, ξ is a Killing vector.

Carter and Henriksen[4] have introduced the idea of *kinematic self-similarity* in the context of relativistic fluid mechanics and an extended analysis has been given by Coley.[5] A kinematic self-similarity vector satisfies the conditions

$$\mathcal{L}_\xi u_a = \text{const}\, u_a,$$

$$\mathcal{L}_\xi h_{ab} = 2 h_{ab},$$

where $h_{ab} = g_{ab} - u_a u_b$ is the first fundamental form of the three-spaces orthogonal to u^a. The case const $\neq 1$ is called ''similarity of the second kind.''

In this work we apply the ideas of scaling and homothety to a string fluid atmosphere. Since our primary interest is in the extended Schwarzschild mass function $m(u,r)$ and the related string atmosphere, we apply scaling to the mass in two different ways. First, we assume diffusive mass transport and investigate the symmetries of the diffusion equation and second, we investigate the scaling properties of the metric and from those derive mass transport equations.

In Sec. II we briefly describe the Schwarzschild string fluid atmosphere. Section III studies the symmetry map which takes the diffusion equation to an ordinary differential equation. New

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diffusion solutions are found. Geometric symmetries, homothetic and conformal, are developed in Sec. IV. Mass transport is discussed in Sec. V. One of the results of the homothetic analysis are new self-similar solutions to the Einstein equations.

Our sign conventions are $2A_{c;[ab]} = A_e R^e{}_{cab}$, and $R_{ab} = R^e{}_{abe}$. Latin indices range over $(0,1,2,3) = (u,r,\vartheta,\varphi)$. Overdots abbreviate $\partial/\partial u$, and primes abbreviate $\partial/\partial r$. Overhead carets denote unit vectors. We use units where $G = c = 1$. Einstein's field equations are $G_{ab} = -8\pi T_{ab}$, and the metric signature is $(+,-,-,-)$.

II. STRING FLUID ATMOSPHERE

Recently, Glass and Krisch[6,7] showed that there can be a spherically symmetric string fluid atmosphere outside a Schwarzschild horizon. The space–time metric is

$$ds^2_{\text{GK}} = A\,du^2 + 2\,du\,dr - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2), \tag{1}$$

where $A = 1 - 2m(u,r)/r$. Initially $m(u,r) = m_0$ provides the vacuum Schwarzschild solution in the region $r > 2m_0$. The metric can be written in a natural basis as

$$g^{\text{GK}}_{ab} = \hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b - \hat{\vartheta}_a\hat{\vartheta}_b - \hat{\varphi}_a\hat{\varphi}_b, \tag{2}$$

where the unit vectors are defined by

$$\hat{v}_a dx^a = A^{1/2}\,du + A^{-1/2}\,dr, \quad \hat{v}^a\partial_a = A^{-1/2}\partial_u, \tag{3a}$$

$$\hat{r}_a\,dx^a = A^{-1/2}\,dr, \quad \hat{r}^a\partial_a = A^{-1/2}\partial_u - A^{1/2}\partial_r, \tag{3b}$$

$$\hat{\vartheta}_a\,dx^a = r\,d\vartheta, \quad \hat{\vartheta}^a\partial_a = -r^{-1}\partial_\vartheta, \tag{3c}$$

$$\hat{\varphi}_a\,dx^a = r\sin\vartheta\,d\varphi, \quad \hat{\varphi}^a\partial_a = -(r\sin\vartheta)^{-1}\partial_\varphi. \tag{3d}$$

$\hat{v}^a$ is hypersurface-orthogonal with h_{ab} the first fundamental form of the hypersurface,

$$h_{ab}dx^a dx^b = (g^{\text{GK}}_{ab} - \hat{v}_a\hat{v}_b)dx^a dx^b = -A^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \tag{4}$$

The kinematics of the $\hat{v}^a$ flow are described by

$$\hat{v}^a_{;b} = a^a\hat{v}_b + \sigma^a{}_b - (\Theta/3)(\hat{r}^a\hat{r}_b + \hat{\vartheta}^a\hat{\vartheta}_b + \hat{\varphi}^a\hat{\varphi}_b), \tag{5}$$

where

$$a^a = [\dot{m}/r + A\partial_r(m/r)]A^{-3/2}\hat{r}^a, \tag{6a}$$

$$\sigma^a{}_b = (\Theta/3)(-2\hat{r}^a\hat{r}_b + \hat{\vartheta}^a\hat{\vartheta}_b + \hat{\varphi}^a\hat{\varphi}_b), \tag{6b}$$

$$\Theta = (\dot{m}/r)A^{-3/2}. \tag{6c}$$

The string distribution is described by a string bivector Σ_{ab}. Spherical symmetry demands that the averaged string bivector will describe a world-sheet in either the (u,r) or the (ϑ,φ) plane. The string bivector is timelike and given by

$$\Sigma^{ac} = \hat{r}^a\hat{v}^c - \hat{r}^c\hat{v}^a, \tag{7}$$

where $\Sigma^{ac}\Sigma_c{}^b = \hat{v}^a\hat{v}^b - \hat{r}^a\hat{r}^b$. The two-surfaces spanned by Σ_{ab} are orthogonally transitive to the two-surfaces spanned by the dual bivector

$$\Sigma^*_{ab} = \hat{\vartheta}_a\hat{\varphi}_b - \hat{\vartheta}_b\hat{\varphi}_a, \tag{8}$$

which follows from the Frobenius surface-forming condition satisfied by Σ_{ab}. It is also true that $\Sigma_a^{*c}\Sigma_{cb}^* = \hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b$.

The Einstein tensor computed from (1) can be written as a two-fluid system $G_{ab}^{\mathrm{null}} + G_{ab}^{\mathrm{matter}}$.

$$G_{ab} = (2\dot{m}/r^2)l_a l_b - (2m'/r^2)(\hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b) + (m''/r)(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b), \tag{9}$$

where $l_a dx^a = du$. The Einstein field equations $G^a{}_{b;a} = 0$ are satisfied for arbitrary $m(u,r)$.

In Glass and Krisch[6,7] mass transport was modeled by diffusion, and the diffusion equation used is given by

$$\dot{\rho} = \mathcal{D}r^{-2}\partial_r(r^2\partial_r\rho), \tag{10}$$

where $\mathcal{D}$ is the positive coefficient of self-diffusion (taken to be constant).

III. SIMILARITY MAP OF THE DIFFUSION EQUATION

There is a similarity technique explained by Bluman and Cole[8] that maps the diffusion equation into an ordinary differential equation. Our primary interest is in the Schwarzschild mass function $m(u,r)$. New functional solutions for $m(u,r)$ are new solutions to the field equations for the parameter extended radiating atmosphere. The behavior of $m(u,r)$ describes the string fluid atmosphere beyond the Schwarzschild horizon through the relations $\dot{m} = 4\pi\mathcal{D}r^2\rho'$ and $4\pi\rho = m'/r^2$. The mass function obeys a diffusion equation

$$\dot{m} = \mathcal{D}r^2\partial_r(r^{-2}\partial_r m) \tag{11}$$

with homogeneous solution $m_{\mathrm{hom}}(r) = m_0 + \frac{4}{3}\pi r^3\rho_0$ which can be added to each time-dependent solution.

The similarity technique (for a fully general analysis see Bluman and Kumei[9]) requires one to introduce an independent dimensionless variable. A standard choice in diffusion problems is the Boltzmann transformation:[10]

$$\eta = r(4\mathcal{D}u)^{-1/2}. \tag{12}$$

[Note that as a mapping from (u,r) to $(u^{-1/2},\eta)$ the Jacobian is singular implying a breakdown of the $1-1$ mapping along r.] The argument of the equation, $m(u,r)$, is replaced by a dimensionless function $F(\eta)$: We look for a general solution of the form

$$m(u,r) := c_0 r^\alpha u^\beta F(\eta). \tag{13}$$

The constant c_0 is intended to map the dimensions of $r^\alpha u^\beta$ to mass for arbitrary constants α and β. Upon substituting Eq. (13) into the diffusion equation (11) we obtain the ordinary differential equation

$$F_{\eta\eta} + 2[(\alpha-1)\eta^{-1} + \eta]F_\eta + [\alpha(\alpha-3)\eta^{-2} - 4\beta]F = 0, \tag{14}$$

where F_η abbreviates $dF/d\eta$.

There are many analytic solutions of Eq. (14) which depend on the values of α and β. The choice $\alpha = \beta = 0$ has the differential equation $F_{\eta\eta} + 2(\eta - 1/\eta)F_\eta = 0$ with solution

$$F(\eta) = k_0 + k_1[-\eta e^{-\eta^2} + (\sqrt{\pi}/2)\mathrm{erf}(\eta)], \tag{15}$$

where $\mathrm{erf}(\eta) := (2/\sqrt{\pi})\int_0^\eta \exp(-s^2)ds$, $\lim_{\eta\to 0}\mathrm{erf}(\eta) = 2\eta/\sqrt{\pi}$. This is the mass solution given in Eq. (40) of Glass and Krisch[7] (with $k_0 = 0$ and with the homogeneous solution m_{hom} added). At fixed time u, it describes a mass with value $m_{\mathrm{hom}} + c_0 k_1$ as $\eta\to\infty$. At late times $c_0 k_1$ is radiated away. There is no length scale in this description so the m_{hom} atmosphere is unbounded.

Other choices can be made, for example $\alpha = n$, $\beta = -n/2$. This choice has const$\times r^n u^{-n/2}$ $= \eta^n$ and one can solve Eq. (14) or see directly from Eq. (13) that

$$F(\eta) = \eta^{-n}.$$

If we write $F(\eta) = \eta^{-n} H(\eta)$ then $H(\eta)$ satisfies the case $\alpha = \beta = 0$ and we have a new family of solutions parametrized by n:

$$F(\eta) = \eta^{-n}[k_0 + k_1[-\eta e^{-\eta^2} + (\sqrt{\pi}/2)\mathrm{erf}(\eta)]]. \tag{16}$$

Solution (15) is included here when $n = 0$.

IV. SYMMETRIES

Because the string fluid naturally lives on a two-dimensional world sheet, the question of the symmetries of these two-dimensional subspaces is interesting. We examine how the mass distribution and stress energy content reflect the separate two-surface symmetries

$$\mathcal{L}_\xi(\hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b) = 2\mu(\hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b),$$
$$\tag{17}$$
$$\mathcal{L}_\xi(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b) = 2\nu(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b).$$

For similarity of the second kind, the map action must be

$$\mathcal{L}_\xi\hat{v}_a = \gamma\hat{v}_a, \quad \gamma \neq 1,$$
$$\mathcal{L}_\xi\hat{r}_a = \hat{r}_a, \tag{18}$$
$$\mathcal{L}_\xi(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b) = 2(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b).$$

A. Homothetic map

The similarity vector which preserves the distinct two-surfaces of the matter distribution in Eq. (17) is

$$\xi^a\partial_a = [\nu u_0 + (2\mu - \nu)u)]\partial_u + \nu r\partial_r, \tag{19}$$

with kinematic transformations

$$\mathcal{L}_\xi\hat{v}_a = \mu\hat{v}_a, \quad \mathcal{L}_\xi\hat{r}_a = \mu\hat{r}_a, \tag{20a}$$

$$\mathcal{L}_\xi\hat{\vartheta}_a = \nu\hat{\vartheta}_a, \quad \mathcal{L}_\xi\hat{\varphi}_a = \nu\hat{\varphi}_a, \tag{20b}$$

when the metric function A satisfies ($\kappa := 2\mu/\nu - 1$)

$$\psi\dot{A}/A + rA'/A + \kappa - 1 = 0, \tag{21}$$

with $\psi(u) := u_0 + \kappa u$. The constraint (21) requires the mass function to have the form

$$r - 2m(u,r) = \psi^{(2-\kappa)/\kappa} f(\psi/r^\kappa), \tag{22}$$

where f is an arbitrary function.

If $\mu = \nu = \kappa = 1$ then the map is homothetic with $\mathcal{L}_\xi g_{ab} = 2g_{ab}$.

B. Another homothetic map

The case $\kappa=0$ requires a separate solution. The metric function A satisfies

$$u_0\dot{A}/A + rA'/A = 1. \tag{23}$$

Constraint (23) has the integral

$$r - 2m(u,r) = r_1 e^{2u/u_0}\tilde{f}(e^{u/u_0}r_0/r) \tag{24}$$

with $\tilde{f}$ an arbitrary function. When $\nu=1$ and $\mu=1/2$ the u dependence is eliminated from ξ^a and the transformation acts on the (ϑ,φ) two-surfaces homothetically

$$\mathcal{L}_\xi(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b) = 2(\hat{\vartheta}_a\hat{\vartheta}_b + \hat{\varphi}_a\hat{\varphi}_b)$$

but preserves the scale of the string two-surfaces,

$$\mathcal{L}_\xi(\hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b) = \hat{v}_a\hat{v}_b - \hat{r}_a\hat{r}_b.$$

C. Interpreting the scale parameter

Under the action of the homothety $\xi^a\partial_a = (u_0+u)\partial_u + r\partial_r$ the acceleration of $\hat{v}^a$, given in Eq. (6a), has the following Lie derivative: with $a^b = a\hat{r}^b$, $a := [\dot{m}/r + A\partial_r(m/r)]A^{-3/2}$,

$$\mathcal{L}_\xi a^b = \left(\frac{a_\xi}{a} - 1\right)a^b, \tag{25}$$

where $a_\xi := [(u_0+u)\partial_u + r\partial_r]a$. Similarly the rate-of-shear given in Eqs. (6b) and (6c) obeys

$$\mathcal{L}_\xi\sigma^a{}_b = \left(\frac{\Theta_\xi}{\Theta} - 1\right)\sigma^a{}_b, \tag{26}$$

where $\Theta_\xi = [(u_0+u)\partial_u + r\partial_r]\Theta$.

There is no information to be gained by analyzing the scaling properties of the Raychaudhuri equation

$$a^b_{;b} - \sigma_{ab}\sigma^{ab} - \Theta^2/3 - \Theta_{,a}\hat{v}^a = -R_{ab}\hat{v}^a\hat{v}^b,$$

since it is identically satisfied by g^{GK}_{ab}.

D. Conformal map

The case $\mu = \nu = \kappa = 1$, with $\psi = u_0+u$, has an interesting conformal symmetry. We see from Eq. (22) that $A = (\psi/r)f(\psi/r) = F(\psi/r)$. Metric (1) is written as

$$ds^2_{GK} = F(\psi/r)du^2 + 2\,du\,dr - r^2\,d\Omega^2. \tag{27}$$

We define a new coordinate $y := r/\psi$ and rewrite (27) as

$$ds^2_{GK} = [F(1/y) + 2y]du^2 + 2\psi\,du\,dy - y^2\psi^2\,d\Omega^2. \tag{28}$$

Now we factor out ψ^2 and introduce a new time coordinate $dw := du/\psi$ to obtain

$$ds^2_{GK} = \psi^2[(F+2y)dw^2 + 2\,dw\,dy - y^2\,d\Omega^2].$$

Upon choosing $F(1/y) = 1 - 2M(y)/y - 2y$, $M(y)$ arbitrary, we have

$$ds^2_{\mathrm{GK}}=e^{2w}[(1-2M/y)dw^2+2\,dw\,dy-y^2d\Omega^2]. \tag{29}$$

The argument above shows that the similarity transformation generated by vector $\xi^a\partial_a=(u_0+u)\partial_u+r\partial_r$ conformally relates the radiating string atmosphere of metric (1) to a previously identified family of static string atmospheres,[7] i.e.,

$$\mathcal{L}_\xi g^{\mathrm{GK}}_{ab}=2e^{2w}g^{\mathrm{static}}_{ab}. \tag{30}$$

E. Similarity of the second kind

For similarity vector $\xi^a\partial_a=(u+u_0)\partial_u+r\partial_r$ the metric function A must satisfy

$$(u_0+u)\dot{A}/A+rA'/A=\gamma-1. \tag{31}$$

Equation (31) has solution

$$r-2m(u,r)=r_2(u_0+u)^\gamma h[(u_0+u)/r], \tag{32}$$

where h is an arbitrary function and $\gamma\neq 1$.

V. MASS TRANSPORT

The mass functions found by similarity analysis obey certain transport equations. Most of the transport equations have the form of the ''telegrapher'' equation. This can describe dispersive and lossy electromagnetic wave motion.[11] Some forms have been interpreted by Kac[12] as a random Poisson process. Mass transport through the atmosphere is affected by the homothetic symmetries. The transport equations can be constructed from the similarity solutions of Sec. IV.

A. $\kappa=0$ homothety

Differentiation of Eq. (23), a constraint on the mass function, yields an inhomogeneous wave equation

$$\ddot{A}-3\dot{A}/u_0-(r/u_0)^2\nabla^2A=-2A/u_0^2. \tag{33}$$

B. $\kappa=1$ homothety

Recall metric function $A=1-2m(u,r)/r$. One can see directly from Eq. (22) that $A=(\psi/r)f(\psi/r)=F(\psi/r)$ with $\psi=u_0+u$. A, and thus m/r, satisfies a wave equation on the flat tangents to the $\hat{v}_a\hat{v}_b-\hat{r}_a\hat{r}_b$ two-spaces. Defining $\tau=\ln(\psi)$ and $z=\ln(r)$ we have

$$A=F(\tau-z). \tag{34}$$

It is clear that A, generated by homothety $\xi^a\partial_a=(u_0+u)\partial_u+r\partial_r$, satisfies the wave equation

$$\frac{\partial^2A}{\partial\tau^2}-\frac{\partial^2A}{\partial z^2}=0. \tag{35}$$

Alternatively, we can find a wave equation on the curved manifold by writing

$$\partial F/\partial u=(1/r)\hat{F},\quad \partial F/\partial r=-(\psi/r^2)\hat{F}$$

where $\hat{F}$ is the derivative of F with respect to its argument. It follows that

$$\ddot{A}=(1/r^2)\hat{F},\quad (r^2A')'=(\psi^2/r^2)\hat{F}.$$

Thus

$$\ddot{A} - v_s^2 \nabla^2 A = 0 \tag{36}$$

where $\nabla^2 = r^{-2}(\partial/\partial r)r^2(\partial/\partial r)$ and $v_s = r/\psi$. The wave speed varies with u and r.

If v_s were constant v, then Eq. (36) would have the general solution

$$A(u,r) = \frac{f(r - vu)}{r} + \frac{g(r + vu)}{r} \tag{37}$$

in terms of two arbitrary functions f and g. Substituting $A = f/r$ into (36) one finds

$$(v^2 - v_s^2)\hat{\ddot{f}} = 0.$$

This reflects "damped, yet relatively undistorted, progressing wave solutions,"[13] a special case of the telegrapher's equation.

For new time coordinate $t \mapsto u + u_0 = e^{t/t_0}$ and with $A_t := \partial A/\partial t$, Eq. (36) transforms to

$$A_{tt} - A_t/t_0 - (r/t_0)^2 \nabla^2 A = 0. \tag{38}$$

C. $\kappa \geq 2$ two-surface symmetry

With $A = 1 - 2m(u,r)/r = \psi^{(2-\kappa)/\kappa} r^{-1} f(\psi/r^\kappa)$ we can write

$$A = r^{1-\kappa} H(\psi/r^\kappa), \quad H := (\psi/r^\kappa)^{(2-\kappa)/\kappa} f.$$

Differentiation yields

$$\ddot{A} = \kappa^2 r^{1-3\kappa} \hat{H}$$

and

$$(r^2 A')' = (1-\kappa)(2-\kappa)r^{1-\kappa} H - 3\kappa(1-\kappa)\psi r^{1-2\kappa}\hat{H} + \kappa^2 \psi^2 \hat{H}.$$

It follows that

$$(r^2 A')' = (1-\kappa)(2-\kappa)A - 3(1-\kappa)\psi \dot{A} + \psi^2 \ddot{A}.$$

Transforming to a new time coordinate $e^{t/t_0} = \psi = u_0 + \kappa u$ yields the inhomogeneous wave equation

$$A_{tt} + (2 - 3/\kappa)A_t/t_0 - (r/t_0)^2 \nabla^2 A = (1 - 1/\kappa)(2/\kappa - 1)A/t_0^2, \tag{39}$$

where $A_t := \partial A/\partial t$.

D. Similarity of the second kind

Differentiation of the constraint on metric function A, Eq. (31), yields the homogeneous wave equation

$$\ddot{A} + v_s\left(\frac{\gamma - 1}{r}\right)\dot{A} - v_s^2 \nabla^2 A + v_s^2\left(\frac{\gamma - 1}{r^2}\right)(rA') = 0, \tag{40}$$

where $v_s = r/(u_0 + u)$ and $\gamma \neq 1$. As above, the wave speed varies with u and r.

VI. DISCUSSION

Similarity is physically important since scaling behavior may offer clues about possible relationships between macroscopic and microscopic physics (i.e., Ehrenfest's classical adiabatic in-

variants and quantization rules). Using scaling, one can model long term behaviors with single solutions to the field equations in which only the scaling variable changes as a function of time. Self-similar behavior is an important aspect of many evolutionary processes both linear and nonlinear.[14] The simplifications of the nonlinear field equations of general relativity are a good example of the value of similarity methods. In addition, we have seen that the special homothety of fluid two-surfaces can be associated with self-similar behavior in the fluid parameters.

In this paper our primary interest is in the extended Schwarzschild mass function $m(u,r)$ and the related string atmosphere. We applied scaling to the mass in two different ways. First, we assumed a mass transport and investigated the scaling properties and second, we investigated the scaling properties of the metric and from those derived mass transport equations. In the first case, assuming diffusive mass transport with a Boltzmann scaling variable, we developed a new family of diffusive mass functions and the associated family of string atmospheres. In the second case, we examined the scaling symmetries of the orthogonal two-surfaces (u,r) and (ϑ,φ). A two-parameter similarity generator acted separately on the (u,r) two-surface containing the string fluid and the orthogonal (ϑ,φ) two-surface subject to the mass parameter obeying a constraining first order differential equation. The similarity map affects all metric components equally when the parameters are both equal to 1. For this case, where the transformation is a homothety for the entire space–time, the mass constraint conformally relates a radiating string atmosphere and a static atmosphere. Other parameter choices could be made, for example, the choices which remove time dependence from the generator. This time independent mapping acts on the (ϑ,φ) two-surface homothetically while preserving the scale of the (u,r) string two-surface. For all the parameter choices associated with the scaling action of the generator, a mass transport equation is implied. This equation is, in general, the telegrapher's equation. The telegrapher's equation and the diffusion equation have both macroscopic and microscopic interpretations.[12,15,16] The appearance of both of these mass transport equations in conjunction with the description of a macroscopic string fluid atmosphere is suggestive of the quantum nature of the fundamental string fluid bits. The classical continuum fluid describes only the averaged fluid behavior, with the mass transport equations suggesting the underlying quantum nature of the fluid.

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