Stochastic Approach to the Theory of Fluctuations in Plasmas

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A stochastic (in contrast to kinetic theoretic) approach to the calculation of correlation functions in fully ionized plasmas is investigated. Formally different results are obtained. The question as to whether or not the differences are qualitatively and/or quantitatively significant is raised, but not answered.

I. INTRODUCTION

The purpose of this study is to attempt to place the theory of fluctuations in plasmas in a slightly different perspective. Specifically, less reliance is placed on the conventional expansion in inverse powers of the number of particles in a Debye sphere. Instead, at a certain point in the analysis, a stochastic assumption is introduced which enables the dynamical problem to be completely solved in terms of static correlation functions. The question is thus raised—though not answered here—as to whether the formal solution so obtained is valid regardless of the number of particles in a Debye sphere. But whatever the answer to this question, the present approach provides a framework in which new approximation schemes for the solution of problems in kinetic theory can perhaps be developed.

A sort of derivative benefit to be gained from the present analysis is mathematical compactness and simplicity. Sometimes this can be a matter of some consequence, for occasionally it exposes the physics in new and clarifying perspectives.

In Sec. II we give a statement of the problem to be solved, and a brief discussion of the relevance of the solutions to the interpretation of measurements. Here we also present much of the notation and formalism to be employed in the succeeding sections.

In Sec. III we obtain the formal solution to a fairly general phase-space correlation problem.

In Sec. IV we discuss some of the implications and applications of the solution obtained in Sec. III.

II. STATEMENT OF THE PROBLEM

In the present work we only employ classical mechanics. The generalizations required for the inclusion of quantum effects are largely accomplished in a straightforward manner. These generalizations are not ignored here because they add significant complexity to the analysis, but rather because they are apparently unnecessary for present purposes.

The quantity to be computed in this and subsequent sections is a phase-space correlation function. Define a singlet density operator for particles of kind A in phase space as[1]

$$g_1^A(\mathbf{x}, \mathbf{v}, t) \equiv \sum_{\sigma}^{N_A} \delta[\mathbf{x} - \mathbf{x}^\sigma(t)]\, \delta[\mathbf{v} - \mathbf{v}^\sigma(t)],$$

or (1)

$$g_1^A(\mathbf{Q}, t) \equiv \sum_{\sigma}^{N_A} \delta[\mathbf{Q} - \mathbf{Q}^\sigma(t)],$$

where evidently $\mathbf{Q} = (\mathbf{x}, \mathbf{v})$. For purposes of illustration, we will assume a plasma containing electrons and one kind of ion only. The generalization required to account for an arbitrary number of kinds of ions is formally trivial. Now define the column matrix

$$\Psi(\mathbf{x}, \mathbf{v}, t) \equiv \begin{bmatrix} g^E(\mathbf{x}, \mathbf{v}, t) \\ g^I(\mathbf{x}, \mathbf{v}, t) \end{bmatrix}, \tag{2}$$

and take the Fourier transform in configuration space and time to obtain

$$\Psi(\mathbf{k}, \mathbf{v}, \omega) \equiv \int dt\, d^3x\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\Psi(\mathbf{x}, \mathbf{v}, t) = \begin{bmatrix} g^E(\mathbf{k}, \mathbf{v}, \omega) \\ g^I(\mathbf{k}, \mathbf{v}, \omega) \end{bmatrix}. \tag{3}$$

A correlation matrix of considerable utility is then defined to be

$$\Lambda(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega) \equiv \langle \Psi^*(\mathbf{k}, \mathbf{v}', \omega)\tilde{\Psi}(\mathbf{k}, \mathbf{v}, \omega)\rangle = \int d\tau\, \rho\Psi^*(\mathbf{k}, \mathbf{v}', \omega)\tilde{\Psi}(\mathbf{k}, \mathbf{v}, \omega), \tag{4}$$

where ρ is the basic probability distribution governed by the Liouville equation and $d\tau$ is an element of

[1] See, for example, T. H. Dupree, Phys. Fluids **6**, 1714 (1963).

volume in the $6(N^E + N^I)$-dimensional phase space of particle coordinates and momenta. In (4), Ψ^* is the complex conjugate of Ψ and $\tilde{\Psi}$ is the transpose of Ψ.

One of the more interesting and familiar applications of the correlation matrix defined in Eq. (4) is to the interpretation of light scattering.[2] The photon scattering cross section may be displayed as

$$\sigma(\omega, \mathbf{\Omega}; \omega'; \mathbf{\Omega}') = \frac{\omega'}{\omega} \sigma_T(\mathbf{\Omega}\cdot\mathbf{\Omega}') \frac{e^{\hbar\Delta\omega/2\theta}}{\pi N^E} \operatorname{sech} \frac{\hbar\Delta\omega}{2\theta} \cdot \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-i\Delta\omega t} \int d^3v'\, d^3v \,\langle[g^{E\dagger}(\boldsymbol{\kappa}, \mathbf{v}', 0), g^E(\boldsymbol{\kappa}, \mathbf{v}, t)]_+\rangle. \tag{5}$$

The quantity σ_T is just the Thompson cross section, and $[\ ]_+$ implies an anti-commutator. A classical calculation of the scattering cross section can now be accomplished by displaying

$$\frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-i\Delta\omega t} \int d^3v'\, d^3v \,\langle[g^{E\dagger}(\boldsymbol{\kappa}, \mathbf{v}', 0), g^E(\boldsymbol{\kappa}, \mathbf{v}, t)]_+\rangle = \int d^3v'\, d^3v \int dt\, e^{-i\Delta\omega t}\,\langle g^{E*}(\boldsymbol{\kappa}, \mathbf{v}', 0) g^E(\boldsymbol{\kappa}, \mathbf{v}, t)\rangle, \tag{6}$$

where now the operator g^E is defined as in Eq. (1). Note that, according to (4),

$$\Lambda_{11}(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega) = \langle g^{E*}(\mathbf{k}, \mathbf{v}', \omega) g^E(\mathbf{k}, \mathbf{v}, \omega)\rangle = \int_{-T/2}^{T/2} dt_1\, e^{i\omega t_1} \int_{-T/2}^{T/2} dt_2\, e^{-i\omega t_2} \cdot\langle g^{E*}(\mathbf{k}, \mathbf{v}', t_1) g^E(\mathbf{k}, \mathbf{v}, t_2)\rangle. \tag{7}$$

For stationary systems (and, of course, for large T) (7) becomes

$$\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\, e^{-i\omega(t_2 - t_1)} \cdot\langle g^{E*}(\mathbf{k}, \mathbf{v}', 0) g^E(\mathbf{k}, \mathbf{v}, t_2 - t_1)\rangle = T\int_{-T/2}^{T/2} dt\, e^{-i\omega t}\langle g^{E*}(\mathbf{k}, \mathbf{v}', 0) g^E(\mathbf{k}, \mathbf{v}, t)\rangle. \tag{8}$$

Evidently, therefore,

$$\sigma(\omega, \mathbf{\Omega}; \omega', \mathbf{\Omega}') = \frac{\omega'}{\omega}\sigma_T(\mathbf{\Omega}\cdot\mathbf{\Omega}') \cdot \frac{e^{\hbar\Delta\omega/2\theta}}{\pi N^E} \operatorname{sech}\frac{\hbar\Delta\omega}{2\theta} \int d^3v'\, d^3v\, \Lambda_{11}(\boldsymbol{\kappa}, \mathbf{v}', \mathbf{v}, \Delta\omega)/T, \tag{9}$$

in the limit as $T \to \infty$.

Note also that current–current correlation tensors can be obtained from

$$\int d^3v'\, d^3v\, v_r' v_l \Lambda(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega). \tag{10}$$

Such tensors are useful for the study of plasma conductivities and absorptivities.

III. CALCULATION OF THE CORRELATION MATRIX

The equations describing the density operators in fully ionized plasmas in the absence of external fields are well known, e.g.,

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}\right) g_1(\mathbf{x}, \mathbf{v}, t) - \frac{1}{M_E}\frac{\partial}{\partial v_i}\int d^3x'\, d^3v' \cdot\frac{\partial V^{EE}(|\mathbf{x} - \mathbf{x}'|)}{\partial x_i} g_2^{EE}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) - \frac{1}{M_E}\frac{\partial}{\partial v_i} \cdot\int d^3x'\, d^3v' \frac{\partial V^{EI}(|\mathbf{x} - \mathbf{x}'|)}{\partial x_i} \cdot g_2^{EI}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) = 0, \tag{11}$$

where the doublet density operator is defined analogously to Eq. (1), i.e.,

$$g_2^{AB}(\mathbf{x}, \mathbf{v}, \mathbf{x}', \mathbf{v}', t) \equiv \sum_{\sigma}^{N_A} \sum_{\alpha}^{N_B\,\prime} \delta[\mathbf{x} - \mathbf{x}^\sigma(t)] \cdot\delta[\mathbf{v} - \mathbf{v}^\sigma(t)]\, \delta[\mathbf{x}' - \mathbf{x}^\alpha(t)]\, \delta[\mathbf{v}' - \mathbf{v}^\alpha(t)]. \tag{12}$$

The prime on the double sum means delete terms $\sigma = \alpha$ if $A = B$. We rewrite Eq. (11) in a compressed notation as

$$\left(\frac{\partial}{\partial t} + D(1)\right) g_1^E(1, t) + \Omega^{EE}(1, 2) g_2^{EE}(1, 2, t) + \Omega^{EI}(1, 2) g_2^{EI}(1, 2, t) = 0. \tag{13}$$

Fluctuation operators are now defined by

$$\delta g_1^A(1, t) \equiv g_1^A(1, t) - \langle g_1^A(1, t)\rangle, \tag{14a}$$

and

$$\delta g_2^{AB}(1, 2, t) \equiv g_2^{AB}(1, 2, t) - \langle g_2^{AB}(1, 2, t)\rangle = \langle g_1^A(1, t)\rangle\, \delta g_1^B(2, t) + \delta g_1^A(1, t)\langle g_1^B(2, t)\rangle + \delta G_2^{AB}(1, 2, t). \tag{14b}$$

[2] For theory see for example: E. E. Salpeter, Phys. Rev. **120**, 1528 (1960); M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962); D. F. DuBois and V. Gilinsky, Phys. Rev. **133**, A1308 and A1317 (1964). For measurements see for example: P. W. Chan and R. A. Nodwell, Phys. Rev Letters **16**, 122 (1966); S. A. Ramsden and W. E. R. Davies, *ibid.* **16**, 303 (1966); O. A. Anderson, *ibid.* **16**, 978 (1966); and S. A. Ramsden, P. K. John, B. Kronast, and R. Benesch, *ibid.* **19**, 688 (1967).

Using (13), (14a), and (14b), we find that

$$\left(\frac{\partial}{\partial t} + D(1)\right) \delta g_1^A(1, t) + \Omega^{AA}(1, 2)[\langle g_1^A(1, t)\rangle\, \delta g_1^A(2, t) + \delta g_1^A(1, t)\langle g_1^A(2, t)\rangle] + \Omega^{AB}(1, 2)[\langle g_1^A(1, t)\rangle\, \delta g_2^B(2, t) + \delta g_1^A(1, t)\langle g_1^B(2, t)\rangle] = -\Omega^{AA}(1, 2)\, \delta G_2^{AA}(1, 2, t) - \Omega^{AB}(1, 2)\, \delta G_2^{AB}(1, 2, t) \equiv S^A(1, t) - \nu\, \delta g_1^A(1, t). \tag{15}$$

Except in the limit $\nu \to 0$, the function $S^A(1, t)$ is somewhat ill-defined; hence all results will be evaluated in this limit. Thus the term containing ν is to be regarded as of purely formal significance—introduced solely for the purpose of simplifying certain steps in the subsequent analysis. It is tempting, however, to interpret ν as a collision frequency.

If the averaged densities can be assumed to be functions of velocity only, i.e., $\langle g_1^A(1, t)\rangle = f^A(\mathbf{v})$, then the Fourier transform of Eq. (15) appears as

$$\left(\frac{\partial}{\partial t} + \nu - i\mathbf{k}\cdot\mathbf{v}\right) \delta g_1^A(\mathbf{k}, \mathbf{v}, t) + iU^{AA}(\mathbf{k}, \mathbf{v}) \cdot \int d^3v'\, \delta g_1^A(\mathbf{k}, \mathbf{v}', t) + iU^{AB}(\mathbf{k}, \mathbf{v}) \cdot \int d^3v'\, \delta g_1^B(\mathbf{k}, \mathbf{v}', t) = S^A(\mathbf{k}, \mathbf{v}, t), \tag{16}$$

where

$$U^{AB}(\mathbf{k}, \mathbf{v}) = \frac{V^{AB}(k)\mathbf{k}\cdot\nabla_v f^A(\mathbf{v})}{M_A}. \tag{17}$$

Now note that

$$\Psi(\mathbf{k}, \mathbf{v}, t) = (2\pi)^3\, \delta(\mathbf{k}) \begin{pmatrix} f^E(\mathbf{v}) \\ f^I(\mathbf{v}) \end{pmatrix} + \begin{pmatrix} \delta g_1^E(\mathbf{k}, \mathbf{v}, t) \\ \delta g_1^I(\mathbf{k}, \mathbf{v}, t) \end{pmatrix}, \tag{18}$$

and may be taken to be

$$\Psi(\mathbf{k}, \mathbf{v}, t) = \begin{pmatrix} \delta g_1^E(\mathbf{k}, \mathbf{v}, t) \\ \delta g_1^I(\mathbf{k}, \mathbf{v}, t) \end{pmatrix} \tag{19}$$

if we are interested in correlations characterized by $k \neq 0$ only.

Define the operator matrix

$$M \equiv \begin{pmatrix} \nu - i\mathbf{k}\cdot\mathbf{v} + iU^{EE}\int d^3v' & iU^{EI}\int d^3v' \\ iU^{IE}\int d^3v' & \nu - i\mathbf{k}\cdot\mathbf{v} + iU^{II}\int d^3v' \end{pmatrix} \tag{20}$$

and source matrix

$$S(\mathbf{k}, \mathbf{v}, t) \equiv \begin{pmatrix} S^E(\mathbf{k}, \mathbf{v}, t) \\ S^I(\mathbf{k}, \mathbf{v}, t) \end{pmatrix}. \tag{21}$$

Then Eqs. (16) are summarized by

$$\left(\frac{\partial}{\partial t} + M\right)\Psi(\mathbf{k}, \mathbf{v}, t) = S(\mathbf{k}, \mathbf{v}, t). \tag{22}$$

By Fourier transformation in time, we obtain

$$(M + i\omega I)\Psi(\mathbf{k}, \mathbf{v}, \omega) = S(\mathbf{k}, \mathbf{v}, \omega). \tag{23}$$

Defining

$$\psi(\mathbf{k}, \omega) \equiv \int d^3v\, \Psi(\mathbf{k}, \mathbf{v}, \omega), \tag{24}$$

it is a straightforward matter to show that

$$\Psi(\mathbf{k}, \mathbf{v}, \omega) = \Gamma(\mathbf{k}, \mathbf{v}, \omega)\psi(\mathbf{k}, \omega) + i\zeta(\mathbf{k}, \mathbf{v}, \omega), \tag{25}$$

where the matrix Γ is given by

$$(\mathbf{k}\cdot\mathbf{v} - \omega + i\nu)\Gamma = \begin{pmatrix} U^{EE} & U^{EI} \\ U^{IE} & U^{II} \end{pmatrix}, \tag{26}$$

and

$$(\mathbf{k}\cdot\mathbf{v} - \omega + i\nu)\zeta = S. \tag{27}$$

We may now further show that

$$\psi(\mathbf{k}, \omega) = Q(\mathbf{k}, \omega)s(\mathbf{k}, \omega), \tag{28}$$

where

$$i\Delta Q = \begin{pmatrix} 1 + \alpha^{II} & -\alpha^{EI} \\ -\alpha^{IE} & 1 + \alpha^{EE} \end{pmatrix}, \tag{29}$$

$$\Delta = 1 + \alpha^{EE} + \alpha^{II}, \tag{30}$$

$$\alpha^{AB} = -\int d^3v\, \Gamma^{AB}, \tag{31}$$

and where

$$s(\mathbf{k}, \omega) = -\int d^3v\, \zeta(\mathbf{k}, \mathbf{v}, \omega). \tag{32}$$

Entering (28) into (25) yields

$$\Psi = \Gamma Q s + i\zeta. \tag{33}$$

Equations (28), (32), and (33) together imply that

$$\int d^3v\ \Gamma = I + iQ^{-1}, \tag{34}$$

where I is the two-dimensional unit matrix. We may further compact Eq. (33) by writing

$$\Psi(\mathbf{k}, \mathbf{v}, \omega) = \int d^3v'\ H(\mathbf{v}, \mathbf{v}')\zeta(\mathbf{k}, \mathbf{v}', \omega), \tag{35}$$

where the matrix H is given by

$$H(\mathbf{v}, \mathbf{v}') = iI\ \delta(\mathbf{v} - \mathbf{v}') - \Gamma(\mathbf{k}, \mathbf{v}, \omega)Q(\mathbf{k}, \omega). \tag{36}$$

It is important to note that

$$\int H(\mathbf{v}, \mathbf{v}')\ d^3v = -Q, \tag{37}$$

and hence that Eq. (28) is readily recaptured from Eq. (35).

Up to this point our main preoccupation has been with formulism and notation. No approximations of any kind have been introduced. We now turn to the task of calculating the correlation matrix; which, by virtue of Eqs. (4) and (35) is readily displayed as

$$\Lambda(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega) = \int d^3v'''\ d^3v''\ H^*(\mathbf{v}', \mathbf{v}''')\langle\zeta^*(\mathbf{k}, \mathbf{v}''', \omega)\tilde{\zeta}(\mathbf{k}, \mathbf{v}'', \omega)\rangle\tilde{H}(\mathbf{v}, \mathbf{v}'')$$
$$= \int \frac{d^3v'''\ d^3v''\ H^*(\mathbf{v}', \mathbf{v}''')\langle S^*(\mathbf{k}, \mathbf{v}''', \omega)\tilde{S}(\mathbf{k}, \mathbf{v}'', \omega)\rangle\tilde{H}(\mathbf{v}, \mathbf{v}'')}{(\mathbf{k}\cdot\mathbf{v}''' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v}'' - \omega + i\nu)}. \tag{38}$$

Recalling the definition of S in Eq. (15), we see that all that we have accomplished so far is the derivation of an explicit, rigorous, and compact relation between the desired second-order correlation function and a certain fourth-order correlation function. The first approximation is now introduced in the form of a stochastic assumption[3] concerning the form of this fourth-order correlation function. According to Eq. (9), it is $T^{-1}\Lambda$ in the limit of large T that is related to observables, so consider

$$\frac{1}{T}\langle S^*(\mathbf{k}, \mathbf{v}''', \omega)\tilde{S}(\mathbf{k}, \mathbf{v}'', \omega)\rangle = \frac{1}{T}\int_{-T/2}^{T/2} dt_1$$
$$\cdot\int_{-T/2}^{T/2} dt_2\, e^{i\omega t_1 - i\omega t_2}\langle S^*(\mathbf{k}, \mathbf{v}''', t_1)\tilde{S}(\mathbf{k}, \mathbf{v}'', t_2)\rangle. \tag{39}$$

We introduce an approximation by assuming that

$$\langle S^*(\mathbf{k}, \mathbf{v}''', t_1)\tilde{S}(\mathbf{k}, \mathbf{v}'', t_2)\rangle = g(t_1 - t_2)D(\mathbf{k}, \mathbf{v}''', \mathbf{v}''). \tag{40}$$

A Markoffian description of the system is achieved[3] if we choose

$$g(t_1 - t_2) = \delta(t_1 - t_2). \tag{41}$$

A tractible, non-Markoffian description useful for "modeling" calculations of plasma fluctuations is obtained if we assume a Gaussian or Lorentzian form for $g(t)$. Since the present work is preliminary, for the time being we content ourselves with Eq. (41), where we find that

$$\frac{1}{T}\langle S^*(\mathbf{k}, \mathbf{v}''', \omega)\tilde{S}(\mathbf{k}, \mathbf{v}'', \omega)\rangle = D(\mathbf{k}, \mathbf{v}''', \mathbf{v}''). \tag{42}$$

Equation (38) for the correlation matrix now reads

$$\frac{1}{T}\Lambda(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega)$$
$$= \int \frac{d^3v'''\ d^3v''\ H^*(\mathbf{v}', \mathbf{v}''')\ D(\mathbf{k}, \mathbf{v}''', \mathbf{v}'')H(\mathbf{v}, \mathbf{v}'')}{(\mathbf{k}\cdot\mathbf{v}''' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v}'' - \omega + i\nu)}. \tag{43}$$

Defining

$$\Lambda(\mathbf{k}, \omega) \equiv \int d^3v'\ d^3v\ \frac{1}{T}\Lambda(\mathbf{k}, \mathbf{v}', \mathbf{v}, \omega), \tag{44}$$

and recalling Eq. (37), we find that

$$\Lambda(\mathbf{k}, \omega) = Q^*$$
$$\cdot\int \frac{d^3v'''\ d^3v''\ D(\mathbf{k}, \mathbf{v}''', \mathbf{v}'')}{(\mathbf{k}\cdot\mathbf{v}''' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v}'' - \omega + i\nu)}\tilde{Q}. \tag{45}$$

Now defining

$$\Omega(\mathbf{k}, \mathbf{v}', \mathbf{v}) \equiv \langle\Psi^*(\mathbf{k}, \mathbf{v}', t = 0)\Psi(\mathbf{k}, \mathbf{v}, t = 0)\rangle, \tag{46}$$

and making use of the solutions of Eq. (22) in the time domain, we find that

$$\Omega(\mathbf{k}, \mathbf{v}', \mathbf{v}) = \int_0^\infty dx\ \exp\left[-xM^*(\mathbf{k}, \mathbf{v}')\right]$$
$$\cdot D(\mathbf{k}, \mathbf{v}', \mathbf{v})\ \exp\left[-x\tilde{M}(\mathbf{k}, \mathbf{v})\right]. \tag{47}$$

This equation is readily solved to obtain

$$D(\mathbf{k}, \mathbf{v}', \mathbf{v}) = M^*(\mathbf{k}, \mathbf{v}')\Omega(\mathbf{k}, \mathbf{v}', \mathbf{v}) + \Omega(\mathbf{k}, \mathbf{v}', \mathbf{v})\tilde{M}(\mathbf{k}, \mathbf{v}). \tag{48}$$

[3] The many papers on the subject by Melvin Lax are clarifying here, among which a few are: Rev. Mod. Phys. **32**, 25 (1960); Phys. Rev. **145**, 110 (1966); and Rev. Mod. Phys. **38**, 541 (1966).

This result, together with Eq. (45), provides a relation between the desired correlation function and the static correlation functions which make up the elements of the matrix Ω. This relation is subject to approximation only by the implications of Eqs. (40) and (41). Of course, such a relation merely represents a formal solution for time behavior in terms of initial values. Some possible implications of this result are discussed in the next section.

IV. APPLICATIONS AND DISCUSSION

As mentioned earlier, the principal application of the above analysis to the interpretation of experiment is to photon scattering. In order to be explicit and to compare it with the work of others,[2] we first make use of an approximate solution to the equations for the relevant static correlation functions which yields the conventional formula for the photon-scattering cross section. We define a doublet density as

$$f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t) \equiv \langle g^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t)\rangle$$

$$= \left\langle \sum_{\sigma}^{N_A} \sum_{\alpha}^{N_B}{}' \delta[\mathbf{x}' - \mathbf{x}^\sigma(t)] \cdot \delta[\mathbf{v}' - \mathbf{v}^\sigma(t)]\ \delta[\mathbf{x} - \mathbf{x}^\alpha(t)]\ \delta[\mathbf{v} - \mathbf{v}^\alpha(t)] \right\rangle, \quad (49)$$

where (as before) the prime on the double sum means delete the terms for $\sigma = \alpha$ if $A = B$. Note that if we label the elements of the matrix Ω by Ω_{AB}, we have

$$\Omega_{AB}(\mathbf{k}, \mathbf{v}', \mathbf{v}) = F^{AB}(\mathbf{k}, \mathbf{v}', \mathbf{v}) + \delta_{AB} V\ \delta(\mathbf{v}' - \mathbf{v}) f_1^A(\mathbf{v}), \quad (50)$$

where

$$F^{AB}(\mathbf{k}, \mathbf{v}', \mathbf{v}) \equiv \int d^3x\ d^3x'\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, 0), \quad (51)$$

and where singlet densities have been assumed independent of space and time. The quantity V appearing in Eq. (50) is the system volume. An equation which describes f_2^{AB} (ignoring electromagnetic fields) is

$$\left(\frac{\partial}{\partial t} + v_i' \frac{\partial}{\partial x_i'} + v_i \frac{\partial}{\partial x_i}\right) f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t) - \frac{1}{M_A} \frac{\partial V^{AB}(|\mathbf{x}' - \mathbf{x}|)}{\partial x_i'} \frac{\partial f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t)}{\partial v_i'} - \frac{1}{M_B} \frac{\partial V^{AB}(|\mathbf{x}' - \mathbf{x}|)}{\partial x_i} \frac{\partial f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, t)}{\partial v_i} = \frac{1}{M_A} \sum_C \int d^3x''\ d^3v'' \frac{\partial V^{AC}(|\mathbf{x}' - \mathbf{x}''|)}{\partial x_i'} \cdot \frac{\partial f_3^{ABC}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, \mathbf{x}'', \mathbf{v}'', t)}{\partial v_i'} + \frac{1}{M_B} \sum_C \int d^3x''\ d^3v'' \frac{\partial V^{BC}(|\mathbf{x} - \mathbf{x}''|)}{\partial x_i} \frac{\partial F_3^{ABC}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, \mathbf{x}'', \mathbf{v}'', t)}{\partial v_i}. \quad (52)$$

We now examine the steady-state versions of these equations in accordance with the following approximations and/or assumptions[4]:

(i) $$f_3^{ABC}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}, \mathbf{x}'', \mathbf{v}'', 0) = f_1^A(\mathbf{v}') f_2^{BC}(\mathbf{x}, \mathbf{v}, \mathbf{x}'', \mathbf{v}'') + f_1^B(\mathbf{v}) f_2^{AC}(\mathbf{x}', \mathbf{v}', \mathbf{x}'', \mathbf{v}'') + f_1^C(\mathbf{v}'') f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}); \quad (53)$$

(ii) All singlet densities are functions of velocity only; (iii) All doublet densities are functions of inter-distances in configuration space only, i.e.,

$$f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}) = f_2^{AB}(|\mathbf{x}' - \mathbf{x}|, \mathbf{v}'\ \mathbf{v}); \quad (54)$$

(iv) the doublet densities in the terms on the left-hand side of Eq. (52) containing the potentials are approximated as, e.g.,

$$f_2^{AB}(\mathbf{x}', \mathbf{v}', \mathbf{x}, \mathbf{v}) \simeq f_1^A(\mathbf{v}') f_1^B(\mathbf{v}). \quad (55)$$

[4] The work of Rosenbluth and Rostoker mentioned in Ref. 2 is particularly relevant here.

The resulting equations are then Fourier transformed in accordance with Eq. (51), and then compared with Eq. (48). It is then found that

$$D_{11}(\mathbf{k}, \mathbf{v}', \mathbf{v}) \simeq 2\nu V f_1^E(\mathbf{v})\ \delta(\mathbf{v} - \mathbf{v}') + 2\nu F^{EE}(\mathbf{k}, \mathbf{v}', \mathbf{v}) + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v}) V\ \delta(\mathbf{v}' - \mathbf{v}) f_1^E(\mathbf{v}), \quad (56a)$$

$$D_{12}(\mathbf{k}, \mathbf{v}', \mathbf{v}) \simeq 2\nu F^{EI}(\mathbf{k}, \mathbf{v}', \mathbf{v}), \quad (56b)$$

$$D_{21}(\mathbf{k}, \mathbf{v}', \mathbf{v}) \simeq 2\nu F^{IE}(\mathbf{k}, \mathbf{v}', \mathbf{v}), \quad (56c)$$

$$D_{22}(\mathbf{k}, \mathbf{v}', \mathbf{v}) \simeq 2\nu V f_1^I(\mathbf{v})\ \delta(\mathbf{v}' - \mathbf{v}) + 2\nu F^{II}(\mathbf{k}, \mathbf{v}', \mathbf{v}) + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v}) V\ \delta(\mathbf{v}' - \mathbf{v}) f_1^I(\mathbf{v}). \quad (56d)$$

Entering these results into Eq. (45) and taking the limit as $\nu \to 0$, we obtain [recalling Eq. (29)]

$$\Lambda_{11}(k, \omega) = \frac{2\pi N^E}{k} \left|\frac{1 + \alpha^{II}}{\Delta}\right|^2 \int d^3v\ \delta\!\left(\hat{\mathbf{k}}\cdot\mathbf{v} - \frac{\omega}{k}\right) M^E(\mathbf{v}) + \frac{2\pi N^I}{k} \left|\frac{\alpha^{EI}}{\Delta}\right|^2 \int d^3v\ \delta\!\left(\hat{\mathbf{k}}\cdot\mathbf{v} - \frac{\omega}{k}\right) M^I(\mathbf{v}), \quad (57)$$

where N^E and N^I are the numbers of electrons and ions in the system, $M^E(\mathbf{v})$ and $M^I(\mathbf{v})$ are velocity distributions for electrons and ions normalized to unity, and $\hat{\mathbf{k}} \equiv \mathbf{k}/k$. Equation (57) is the result obtained previously by a variety of arguments.[2]

Alternatively, we may enter (29) into (45) and, making use of the fact that

$$D^\dagger(\mathbf{k}, \mathbf{v}', \mathbf{v}) = D(\mathbf{k}, \mathbf{v}, \mathbf{v}'), \tag{58}$$

obtain directly

$$\Lambda_{11}(k, \omega) = \left|\frac{1 + \alpha^{II}}{\Delta}\right|^2 \cdot \int \frac{d^3v'\, d^3v\, D_{11}(\mathbf{k}, \mathbf{v}', \mathbf{v})}{(\mathbf{k}\cdot\mathbf{v}' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v} - \omega + i\nu)} + \left|\frac{\alpha^{EI}}{\Delta}\right|^2 \cdot \int \frac{d^3v'\, d^3v\, D_{22}(\mathbf{k}, \mathbf{v}', \mathbf{v})}{(\mathbf{k}\cdot\mathbf{v}' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v} - \omega + i\nu)} - \frac{2}{|\Delta|^2} \operatorname{Re}\left[(1 + \alpha^{II})^* \alpha^{EI} \cdot \int \frac{d^3v'\, d^3v\, D_{12}(\mathbf{k}, \mathbf{v}', \mathbf{v})}{(\mathbf{k}\cdot\mathbf{v}' - \omega - i\nu)(\mathbf{k}\cdot\mathbf{v} - \omega + i\nu)}\right]. \tag{59}$$

This equation reduces immediately to Eq. (57) after use of Eqs. (56a), (56b), and (56d) and taking the limit $\nu \rightarrow 0$. But Eqs. (56) are subject to certain approximations, discussed above, which in fact may, or may not be implied by the relation (59). The question is thereby raised as to whether or not the result contained in Eq. (59) has a greater validity and a wider range of applicability than the explicit solution exhibited in Eq. (57). Further study of this matter is required and is being undertaken.

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