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Axial Quasi-Brillouin Streams

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A nonlaminar model for axially uniform electron streams is postulated and conditions for which the model is consistent with physical requirements are derived. The character of the current density profiles permitted by the model are discussed.

IN a previous note an analysis was presented of a nonlaminar model for a rectilinear quasi-Brillouin flow.[1] In this note a corresponding analysis is developed of a nonlaminar model for an axial quasi-Brillouin flow.

The hydrodynamic description of a dc particle flow is given in terms of the following three conservation equations[2]:

$$\nabla\cdot\rho\bar{v}=0, \tag{1}$$

$$-\eta[\bar{E}+\bar{v}\times\bar{B}]=(\bar{v}\cdot\nabla)\bar{v}+\rho^{-1}\nabla\cdot\rho\tilde{P}, \tag{2}$$

$$2\eta\phi=\bar{v}\cdot\bar{v}+\langle\bar{u}\cdot\bar{u}\rangle. \tag{3}$$

The average electric field, magnetic field, velocity, and space-charge density associated with the electron velocity distribution in a given volume element are represented by $\bar{E}$, $\bar{B}$, $\bar{v}$, and ρ, respectively. The magnitude of the electron charge-to-mass ratio is η, and ϕ is the electrostatic potential corresponding to $\bar{E}$. The excess-over-average velocity $\bar{u}$ is the difference between the velocity of a particular electron and the local average velocity $\bar{v}$. The enclosures $\langle\ \rangle$ denote the average over the velocity distribution of the enclosed quantity. The symmetrical tensor $\tilde{P}$ is defined by its elements $\langle u_\alpha u_\beta\rangle$, where α and β form all combinations in pairs of the cylindrical coordinates r, θ, z.

As written, Eq. (3) is predicated on the assumption that the electron velocity distribution is a distribution in direction only, i.e., all electrons in a given volume element have the same magnitude of velocity. This assumption is justified provided it is acceptable to consider the electrons to have originated on an unipotential cathode with negligible initial velocities.

As was done in the previous note, a model for the electron flow will be assumed, and then the conditions for which the model is consistent with physical laws will be determined. The electron flow is assumed to be one in which a uniform axial magnetic field maintains an axial and circumferential uniformity, i.e., $\partial/\partial z=\partial/\partial\theta=0$. Subject to consistency requirements the nonlaminar flow is assumed to have a constant space-charge density, and the average stream velocity is assumed to be described by the equation

$$\bar{v}=[v^2(0)+\alpha r+\beta r^2]^{\frac{1}{2}}\bar{z}+r\Omega\bar{\theta}, \tag{4}$$

where α, β, Ω are as yet unspecified constants, and $\bar{z}$ $\bar{\theta}$ are unit vectors. The assumed form of the axial velocity is a generalization of the form of the axial velocity corresponding to a laminar flow under the prescribed conditions.

The model is not yet completely specified in that additional information is needed to solve Eqs. (1)–(3). An additional specification of particular interest which completes the model is the requirement that $\langle u_z^2\rangle$ be constant. This assumption is not physically unreasonable on supposing that $v^2(0)\gg\langle u_z^2\rangle$ in most cases of interest, and on recognizing that the axial and transverse motion of electrons in the model are not coupled.

Under the specified conditions, the solution of Eqs. (1)–(3) is straightforward. It is found that

$$2\langle u_r^2\rangle=\langle u_\theta^2\rangle=[\omega_p^2+2\Omega(\Omega-\omega_c)]r(r-b), \tag{5}$$

$$\langle u_z^2\rangle=\langle u_r u_\theta\rangle=\langle u_r u_z\rangle=\langle u_\theta r_z\rangle=0, \tag{6}$$

$$2\eta\phi=v^2(0)+\omega_p^2r^2/2, \tag{7}$$

$$v^2(r)=v^2(0)+[\omega_p^2+2\Omega(\Omega-\omega_c)]3br/2 -[\omega_p^2+\Omega(4\Omega-3\omega_c)]r^2, \tag{8}$$

$$\omega_p^2+2\Omega(\Omega-\omega_c)\leq 0. \tag{9}$$

The plasma frequency ω_p and the cyclotron frequency ω_c are defined as usual, and b is the stream radius.

These results are obtained with the aid of the boundary condition that $\bar{u}$, and hence all its averages, is identically zero on $r=0$, b.

The inequality, Eq. (9), follows from Eq. (5) and the fact that $\langle u_r^2\rangle$ and $\langle u_\theta^2\rangle$ are inherently positive quantities. Equation (9) can be conveniently interpreted with the aid of Fig. 1. In the coordinates of this figure, Eq. (9) graphs as the interior of a circle of radius $\frac{1}{2}$. The circle itself is the locus of Eq. (9) with the equal sign used, and corresponds to all laminar flows meeting the prescribed conditions; the two nodes marked B correspond to laminar axial Brillouin flow. The minimum value of $\omega_p^2+2\Omega(\Omega-\omega_c)$, obtained at the center of the circle, is $-\frac{1}{2}$. From the circle it can be seen that

$$\Omega\leq\omega_c, \tag{10}$$

$$2\omega_p^2\leq-\omega_c^2/2. \tag{11}$$

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[1] M. H. Miller, J. Appl. Phys. **32**, 1791 (1961).
[2] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Cases* (Cambridge University Press, New York, 1952).

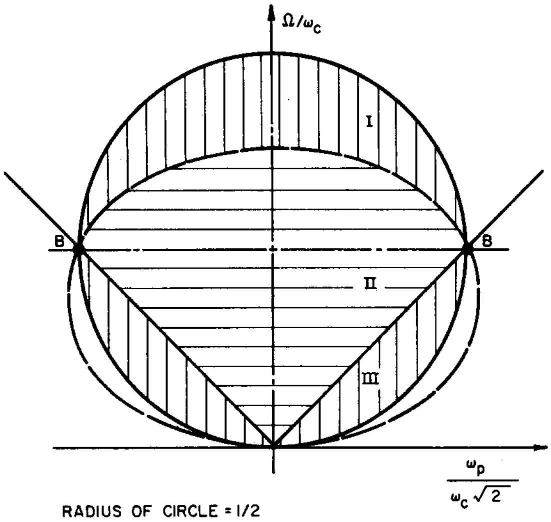

FIG. 1. Consistency diagram for nonlaminar flow model.

It is of some interest to study the various possible velocity profiles (and hence current density profiles). As a matter of convenience this is done by considering $v^2(r)$.

Consideration of Eqs. (8) and (9) show that $v^2(r)$ starts from the origin with a slope which is less than or equal to zero. The curvature of the $v^2(r)$ profile is determined by the coefficient of the r^2 term in Eq. (8). The zero curvature locus, plotted in Fig. 1, is an ellipse passing through the Brillouin points. All points

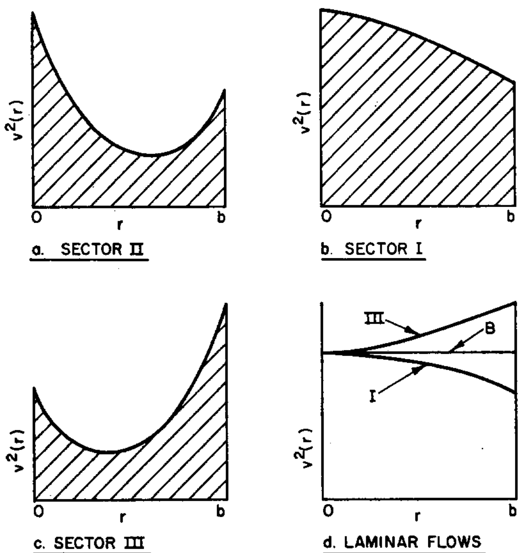

FIG. 2. $v^2(r)$ profiles. Roman numerals refer to Fig. 1.

within the intersection of the circle and ellipse have a positive curvature; the segment of the circle outside the ellipse corresponds to negative curvature. The two lines passing through the Brillouin points are the locus of $\omega_p^2-2\Omega^2=0$. It happens that $v^2(b)=v^2(0)+(\omega_p^2-2\Omega^2)/2$. Hence, points below the lines correspond to flows with $v^2(b)>v^2(0)$, and points above the lines to flows with $v^2(0)>v^2(b)$. Finally, it can be noted that Fig. 1 has mirror symmetry about the vertical axis.

Sketches of profiles of $v^2(r)$ between $r=0$ and $r=b$ for streams located in various sectors of Fig. 1 are shown in Fig. 2. As noted before, the current density profiles are proportional to $v(r)$, and hence have the same general character as is indicated by Fig. 2.

As a measure of the influence of nonlaminarity, use can be made of the fraction, denoted by R, of the total stream energy involved in the excess-over-average motion;

$$R=\frac{\int_0^b (\langle u_r^2+u_\theta^2\rangle)rdr}{\int_0^b 2\eta\phi rdr}=-\frac{\omega_p^2+2\Omega(\Omega-\omega_c)}{\omega_p^2+4v^2(0)/b^2}. \quad (12)$$

R has a maximum value of $\omega_c^2b^2/8v^2(0)$, which it attains at the center of the consistency circle. This introduces another restriction on the model, since $R\leq 1$. However, the restriction is not a disabling one. The maximum value of R can be interpreted physically as being half of the ratio of tangential energy to axial energy at the beam boundary of a laminar Brillouin flow. In usual circumstances it can be expected that the maximum value of R will not only be less than one, but it will be considerably less than one. This implies that, for the usual range of parameters involved in axial flows, any point within the circle of Fig. 1 is accessible with only a small percentage of nonlaminarity (as measured by R).

The nonlaminar model described is rather idealized to be a quantitative model for uniform axial flows. However, it can serve as a guide in characterizing and ameliorating the consequences of nonlaminarity. For example, Fig. 1 is in agreement with the intuitive attitude that laminar axial Brillouin flow is in a practical sense difficult to attain. A possible alternative to designing for laminar Brillouin flow is to accept current density profiles of the type suggested by Fig. 2(b). This means moving the flow into Sector I of Fig. 1, i.e., increasing the angular velocity over the Brillouin value, possibly by means of a "bucking" magnetic field through the cathode on which the flow originates.