Classical SU(3) gauge field equations*

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Admissible forms of the static solutions to the SU(3) gauge field equation are examined. It is shown that by a proper choice of the form of solutions which extricate the SU(3) indices, the set of nonlinear partial differential equations is reducible to nonlinear ordinary differential equations for the radial functions.

I. INTRODUCTION

In a previous paper coauthored by one of us, 1 some static solutions of the classical SU(2) isotopic gauge field2 equations were discussed. A crucial feature is the fact that by a judicious choice of the form of solutions which properly extricate the isotopic indices, the set of nonlinear partial differential equations is reducible to nonlinear ordinary differential equations for the radial functions.

The purpose of the present note is to examine the static case of SU(3) unitary gauge field3 equations. We find that the above feature also holds for the SU(3) gauge field equations.

II. LOCAL SU(3) GAUGE FIELD

As is well known, the number of the gauge field components is equal to the dimension of the regular representation of the underlying group. For SU(3), this number is eight. The octet gauge field may be arranged in terms of 3 x 3 matrices.

\[ \lambda_{A} = \frac{\kappa \epsilon}{C} \lambda_{A} \]  

where \( \kappa \) is a scale factor and \( \lambda_{A} \) are the set of eight 3 x 3 Gell-Mann matrices4 satisfying the commutation relation

\[ [\lambda_{A}, \lambda_{B}] = \frac{\kappa \epsilon}{C} \lambda_{A} \lambda_{B} \]  

Let

\[ F_{\mu\nu} = \kappa \epsilon \lambda_{A} \lambda_{A} \]  

where

\[ f^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu} - \kappa \epsilon g_{\mu\nu} c^{\rho}_{\lambda} c^{\rho}_{\rho} \]  

in which \( \epsilon \) is the coupling constant and the comma with respect to \( \mu, \nu \) is a short-hand notation for the differentiation

\[ c^{\lambda}_{\mu\nu} = \frac{\partial}{\partial x^{\mu}} c^{\lambda}_{\nu} \]  

Away from sources, we take as the free Lagrangian

\[ \mathcal{L}_{0} = -\frac{1}{4} f_{\mu\nu} f^{\lambda}_{\mu\nu} \]  

The equation of motion reads

\[ f_{\mu\nu, \lambda} = -\kappa \epsilon g_{\mu\nu} c^{\rho}_{\lambda} c^{\rho}_{\rho} \]  

Substitution of Eq. (4) then gives

\[ c^{\lambda}_{\mu\nu} + \kappa \epsilon g_{\mu\nu} (2 c^{\rho}_{\mu\nu} - c^{\rho}_{\nu\mu}) c^{\rho}_{\lambda} - (\kappa \epsilon^{2}) g_{\mu\nu} c^{\rho}_{\lambda} c^{\rho}_{\rho} c^{\rho}_{\rho} c^{\rho}_{\rho} = 0. \]  

From here on, we choose the scale so that

\[ \kappa \epsilon = 1. \]  

III. STATIC CASE

We shall primarily be interested in the static situation where

\[ c^{\lambda}_{\mu\nu} = 0, \]  

Thus the only nonvanishing components are the spatial ones.

Furthermore, the subsidiary condition holds:

\[ c^{\lambda}_{\rho\rho} = 0. \]  

Instead of the label \( A = 1, \ldots, 8 \), we find it convenient to adopt double indices \( i, j \), each running from 1 to 3. The traceless condition would give us still eight independent components \( D_{i}^{\mu} \). Under such a correspondence, we make the following transcription.

For the indices:

\[ A \rightarrow i, B \rightarrow pq, D \rightarrow rs, E \rightarrow uv, F \rightarrow xy; \]  

for the field

\[ c^{\lambda}_{\mu\nu} \rightarrow D_{i}^{\mu} ; \]  

and for the structure constants

\[ g_{ABD} \rightarrow g_{im, pq, rs} ; \]  

Explicitly, the structure constants read

\[ g_{im, pq, rs} = \delta_{im} \delta_{pq} \delta_{rs} - \delta_{ip} \delta_{mr} \delta_{pq}. \]  

Equation (8) becomes then for the static case

\[ D_{i}^{\mu} + g_{im, pq, rs} (2 D_{i}^{rs} - D_{i}^{pq}) D_{j}^{pq} \]

\[ - g_{im, pq, rs} g_{pq, rs} D_{i}^{rs} D_{j}^{pq} = 0, \]

where the comma with respect to the spatial components \( i \) or \( j \) only denotes a differentiation.

IV. FORM OF STATIC SOLUTIONS

We look for the static solutions to the field equations (16) in the following form:

\[ D_{i}^{\mu} = (\epsilon_{ij} x_{i} x_{j} + \epsilon_{imj} x_{i} x_{j}) f + \epsilon_{imj} \epsilon_{i} \epsilon_{m} x_{i} x_{j}, \]  

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where \( f \) and \( h \) are two functions of the radial distance \( r \) alone, \( r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \).

It should be emphasized that for Eq. (16) to be satisfied for every internal index, severe restrictions must prevail on the admissible forms of the solutions. It is rather remarkable that the choice (17) indeed gives a consistent solution. In other words, the symmetric combination \( \epsilon \sigma x \) (in front of \( f \)) and the antisymmetric combination \( \epsilon \sigma x \) (in front of \( h \)) become jointly preserved under the operations indicated on the left-hand side of (16). After a straightforward calculation, their coefficients can be collected. The vanishing of these coefficients gives a pair of coupled nonlinear ordinary differential equations for \( f \) and \( h \). The result is

\[
f'' + 6r^{-1}f' - 14fh + 7r^2f^2 - r^4f^3 = 0, \tag{18a}
\]

\[
h'' + 4r^{-1}h' + 7r^2f^2 - 3h^2 - 7r^4f^2h + r^2h^3 = 0. \tag{18b}
\]

Or, in terms of a \((F, H)\) pair defined as

\[
F = r^3f, \tag{19a}
\]

\[
H = 1 - r^2h, \tag{19b}
\]

Eq. (18) reads

\[
F'' + r^{-2}F(-13 + 7H^2 - F^2) = 0, \tag{20a}
\]

\[
H'' + r^{-2}[4 - (5 + 7F^2)H + H^3] = 0. \tag{20b}
\]

We shall not attempt to discuss the solutions to Eqs. (20) here except by making the following obvious remarks.

(i) Equations (20) possess three real singular points located at

\[
\left( \begin{array}{c} F \\ H \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \text{ and } \left( \begin{array}{c} 0 \\ \frac{1}{2}(-1 \pm \sqrt{17}) \end{array} \right). \tag{21}
\]

(ii) In (17), the \( f \) part solutions cannot exist by itself, while the \( h \) part can, i.e., when \( h = 0, f = 0 \); however, when \( f = 0, h \) has nonzero solutions.

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\footnote{C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).}

\footnote{For extensions of Yang-Mills formalism to higher rank groups, see e.g., R. Utiyama, Phys. Rev. 101, 1597 (1956); Y. Ne'eman, Nucl. Phys. 26, 222 (1961); and others.}

\footnote{M. Gell-Mann, Caltech Report No. CTSL-20 (1961); Phys. Rev. 125, 1067 (1962); see collection in M. Gell-Mann and Y. Ne'eman, The Eightfold Way (Benjamin, New York, 1964).}