Classical $SU(3)$ gauge field equations*

Alfred C. T. Wu

Department of Physics, University of Michigan, Ann Arbor, Michigan 48104

Tai Tsun Wu

Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138
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Admissible forms of the static solutions to the $SU(3)$ gauge field equation are examined. It is shown that by a proper choice of the form of solutions which extricate the $SU(3)$ indices, the set of nonlinear *partial* differential equations is reducible to nonlinear *ordinary* differential equations for the radial functions.

I. INTRODUCTION

In a previous paper coauthored by one of us,[1] some static solutions of the classical $SU(2)$ isotopic gauge field[2] equations were discussed. A crucial feature is the fact that by a judicious choice of the form of solutions which properly extricate the isotopic indices, the set of nonlinear *partial* differential equations is reducible to nonlinear *ordinary* differential equations for the radial functions.

The purpose of the present note is to examine the static case of $SU(3)$ unitary gauge field[3] equations. We find that the above feature also holds for the $SU(3)$ gauge field equations.

II. LOCAL $SU(3)$ GAUGE FIELD

As is well known, the number of the gauge field components is equal to the dimension of the regular representation of the underlying group. For $SU(3)$, this number is eight. The octet gauge field may be arranged in terms of 3×3 matrices.

$$C_\mu = \kappa c_\mu^A \lambda_A, \qquad \text{summed over } A = 1, .., 8, \tag{1}$$

where κ is a scale factor and λ_A are the set of eight 3×3 Gell–Mann matrices[4] satisfying the commutation relation

$$[\lambda_A, \lambda_B] = i g_{AB}^C \lambda_C. \tag{2}$$

Let

$$F_{\mu\nu} = \kappa f_{\mu\nu}^A \lambda_A, \tag{3}$$

where

$$f_{\mu\nu}^A = c_{\mu,\nu}^A - c_{\nu,\mu}^A - \kappa\epsilon g_{BE}^A c_\mu^D c_\nu^E, \tag{4}$$

in which ϵ is the coupling constant and the comma with respect to μ, ν is a short-hand notation for the differentiation

$$c_{\mu,\nu}^A \equiv \partial_\nu c_\mu^A. \tag{5}$$

Away from sources, we take as the free Lagrangian

$$\mathcal{L}_0 = -\tfrac{1}{4} f_{\mu\nu A} f_{\mu\nu}^A. \tag{6}$$

The equation of motion reads

$$f_{\mu\nu,\nu A} = -\kappa\epsilon g_{AB}^D c_\nu^B f_{\mu\nu D}. \tag{7}$$

Substitution of Eq. (4) then gives

$$c_{\mu,\nu\nu}^A + \kappa\epsilon g_{BD}^A (2c_{\mu,\nu}^D - c_{\nu,\mu}^D) c_\nu^B - (\kappa\epsilon)^2 g_{BD}^A g_{EF}^D c_\nu^B c_\mu^E c_\nu^F = 0. \tag{8}$$

From here on, we choose the scale so that

$$\kappa\epsilon = 1. \tag{9}$$

III. STATIC CASE

We shall primarily be interested in the static situation where

$$c_4^A = 0, \tag{10a}$$

$$c_{i,4}^A = 0. \tag{10b}$$

Thus the only nonvanishing components are the spatial ones.

Furthermore, the subsidiary condition holds:

$$c_{\mu,\mu}^A = 0. \tag{11}$$

Instead of the label $A = 1, \ldots, 8$, we find it convenient to adopt double indices lm, each running from 1 to 3. The traceless condition would give us still eight independent components D_μ^{lm}. Under such a correspondence, we make the following transcription.

For the indices:

$$A \to lm, B \to pq, D \to rs, E \to uv, F \to xy; \tag{12}$$

for the field

$$c_\mu^A \to D_\mu^{lm}; \tag{13}$$

and for the structure constants

$$g_{ABD} \to \mathcal{G}_{lm,pq,rs}. \tag{14}$$

Explicitly, the structure constants read

$$\mathcal{G}_{lm,pq,rs} = \delta_{mp}\delta_{qr}\delta_{sl} - \delta_{lq}\delta_{mr}\delta_{ps}. \tag{15}$$

Equation (8) becomes then for the static case

$$D_{i,jj}^{lm} + \mathcal{G}_{pq,rs}^{lm}(2D_{i,j}^{rs} - D_{j,i}^{rs})D_j^{pq} - \mathcal{G}_{pq,rs}^{lm}\mathcal{G}_{uv,xy}^{rs} D_j^{pq} D_i^{uv} D_j^{xy} = 0, \tag{16}$$

where the comma with respect to the spatial components i or j *only* denotes a differentiation.

IV. FORM OF STATIC SOLUTIONS

We look for the static solutions to the field equations (16) in the following form:

$$D_i^{lm} = (\epsilon_{ilj} x_j x_m + \epsilon_{imj} x_j x_l) f + \epsilon_{lmk}\epsilon_{kin} x_n h, \tag{17}$$

where f and h are two functions of the radial distance r alone, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

It should be emphasized that for Eq. (16) to be satisfied for every internal index, severe restrictions must prevail on the admissible forms of the solutions. It is rather remarkable that the choice (17) indeed gives a consistent solution. In other words, the symmetric combination ϵxx (in front of f) and the antisymmetric combination $\epsilon\epsilon x$ (in front of h) become jointly preserved under the operations indicated on the left-hand side of (16). After a straightforward calculation, their coefficients can be collected. The vanishing of these coefficients gives a pair of coupled nonlinear ordinary differential equations for f and h. The result is

$$f'' + 6r^{-1}f' - 14fh + 7r^2fh^2 - r^4f^3 = 0, \tag{18a}$$

$$h'' + 4r^{-1}h' + 7r^2f^2 - 3h^2 - 7r^4f^2h + r^2h^3 = 0. \tag{18b}$$

Or, in terms of a (F, H) pair defined as

$$F \equiv r^3 f, \tag{19a}$$

$$H \equiv 1 - r^2 h, \tag{19b}$$

Eq. (18) reads

$$F'' + r^{-2}F(-13 + 7H^2 - F^2) = 0, \tag{20a}$$

$$H'' + r^{-2}[4 - (5 + 7F^2)H + H^3] = 0. \tag{20b}$$

We shall not attempt to discuss the solutions to Eqs. (20) here except by making the following obvious remarks.

(i) Equations (20) possess three real singular points located at

$$\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \frac{1}{2}(-1 \pm \sqrt{17}) \end{pmatrix}. \tag{21}$$

(ii) In (17), the f part solutions cannot exist by itself, while the h part can, i.e., when $h = 0, f = 0$; however, when $f = 0$, h has nonzero solutions.

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