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HALF-SPACE MULTI-GROUP TRANSPORT THEORY

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ABSTRACT

A method for solving various half-space multi-group transport problems for the case of a symmetric transfer matrix is explained. This method is based on the full-range completeness and orthogonality properties of the infinite medium eigenfunctions. First, the albedo problem is considered. A system of Fredholm integral equations is derived for the emergent distribution of the albedo problem, and it is shown that this system has a unique solution. Then by using the full-range eigenfunction completeness, the inside angular distribution is obtained from the emergent distribution. Finally the Milne problem and the half-space Green's function problem are solved in terms of the emergent distribution of the albedo problem and the infinite medium eigenfunctions.

I. INTRODUCTION

In recent years much effort has been given to solving the energy-dependent Boltzmann equation. Various approximations have been used. The most rewarding approximation to date has been the multi-group technique, and often the diffusion theory approximation is employed to simplify further the calculations. However, there is a definite need for exact solutions of the multi-group transport equations, since these solutions serve as a standard against which one can compare the approximate results.

Recently, the solution of the infinite medium Green's function has been obtained explicitly for the two-group[1] and N-group[2,3] cases. Several two-group half-space problems have been investigated,[4,5,6] and in a paper by Siewert and Zweifel[7] a special N-group Milne problem for radiative transfer was solved. The general case of N-group half-space problems with symmetric transfer matrix was studied by Leonard and Gerziger[3]; they proved full and half-range completeness of the N-group transport equation eigenfunctions. In all these works, the solution of a half-space transport problem is expanded in terms of the eigenfunctions and then a set of equations for the expansion coefficients is derived.

In this paper we consider also N-group half-space problems for a symmetric transfer matrix. This form of $\underset{\sim}{C}$ is not so restrictive as it may appear at first glance. For instance, all two-group problems (see Appendix II) and the N-group equations for thermal neutrons may be transformed into such a case (see Appendix I and Ref. 3). This symmetric $\underset{\sim}{C}$ also appears in the special astrophysical situation of radiative transfer with local thermodynamic equilibrium, the picket-fence model for the absorption coefficient, and isotropic scattering.[7]

In our approach, we do not need half-range completeness property of the eigenfunctions. We solve half-space transport problems in two steps. First, the emergent distribution is calculated and then the distribution inside the medium is evaluated by using the full-range completeness and orthogonality properties of the N-group eigenfunctions. These eigensolutions to the N-group isotropic transport equation and their full-range completeness theorem have been known for several years,[8] while their orthogonality relations have recently been obtained by Yoshimura.[2]

Section II briefly summarizes the N-group eigenfunctions and their full-range orthogonality relations as described by Yoshimura.[2] In addition it is shown for symmetric $\underset{\sim}{C}$ that the discrete eigenvalues are real or purely imaginary. In Section III a system of Fredholm equations is obtained which uniquely determines the emergent distribution for the albedo problem. It is shown that the uniqueness of solution of this system of Fredholm equations also implies half-range completeness of the eigenfunctions. Finally in Section IV, the emergent distributions of the Milne's and Green's function problems are expressed in terms of the emergent albedo problem distribution and the complete solutions obtained from the full-range completeness and orthogonality properties.

II. INFINITE MEDIUM EIGENFUNCTIONS

The linear Boltzmann equation for N energy groups in plane geometry and with isotropic scattering can be written in the form[4]

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\psi}(x,\mu) + \underset{\approx}{\Sigma}\, \underset{\sim}{\psi}(x,\mu) = \underset{\approx}{C} \int_{-1}^{1} d\mu' \, \underset{\sim}{\psi}(x,\mu') \quad . \tag{2.1}$$

The vector $\underset{\sim}{\psi}(x,\mu)$ is an N-component vector, of which the i-th component, $\psi_i(x,\mu)$, is the angular flux of the i-th group. The components of the diagonal matrix, $\underset{\sim}{\Sigma}$ are $\sigma_i \delta_{ij}$ where σ_i is the total interaction cross section for the i-th group. The elements, C_{ij}, of the transfer matrix, $\underset{\sim}{C}$, describe the transfer of neutrons from the j-th group to the i-th group. In some problems, for instance thermal neutron transport theory, $\underset{\sim}{C}$ can be written as a product of diagonal matrices, $\underset{\sim}{D}_i$, and a symmetric matrix $\underset{\sim}{A}$ (see Appendix I), as

$$\underset{\sim}{C} = \underset{\sim}{D}_1 \underset{\sim}{A} \underset{\sim}{D}_2 \quad . \tag{2.2}$$

Equation (2.1) can then be so transformed that the elements of the transformed $\underset{\sim}{\Sigma}$ matrix are ordered as

$$\sigma_1 \geq \sigma_2 \geq \cdot \cdot \cdot \geq \sigma_N \quad , \tag{2.3}$$

and the new $\underset{\sim}{C}$ matrix is symmetric (Appendix II). It will be assumed for the remainder of the paper that the transport equation has this special form of an ordered $\underset{\sim}{\Sigma}$ matrix and symmetric $\underset{\sim}{C}$ matrix. Finally, by measuring distance in units of the smallest mean free path we can set $\sigma_N = 1$.

Using the analogy of the one-group problem,[9] one seeks a set of eigenfunction solutions, $\underset{\sim}{\psi}(\nu,x,\mu)$, to Eq. (2.1) of the form

$$\underset{\sim}{\psi}(\nu,x,\mu) = e^{-x/\nu} \underset{\sim}{\varphi}(\nu,\mu) \quad . \tag{2.4}$$

Substituting Eq. (2.4) into Eq. (2.1) the self-adjoint equation for the eigenvectors, $\underset{\sim}{\phi}(\nu,\mu)$, is obtained:

$$\left(\underset{\sim}{\Sigma} - \frac{\mu}{\nu}\underset{\sim}{E}\right)\underset{\sim}{\phi}(\nu,\mu) = \underset{\sim}{C}\int_{-1}^{1} d\mu'\, \underset{\sim}{\phi}(\nu,\mu') \quad , \tag{2.5}$$

where $\underset{\sim}{E}$ is the unit matrix. The explicit form of these eigenfunctions has been obtained by several authors.[1,2,3,8] We will use, with slight changes, the notation of Yoshimura.[2]

The eigenvector spectrum is divided into two regions.

(a) Region I: $\nu \notin (-1,1)$

In this region there may exist an even number, say 2M, of discrete eigenvectors, which in component form are written as

$$\phi_i(\nu_{os},\mu) = \frac{\nu_{os}\, b_i(\nu_{os})}{\sigma_i \nu_{os} - \mu} \quad , \quad i = 1 \sim N \quad , \tag{2.6}$$

where $\underset{\sim}{b}(\nu_{os})$ is a well defined vector.[2] It can be shown that if ν_{os} is an eigenvalue then also $-\nu_{os}$ and ν^*_{os} (complex conjugate) are eigenvalues with

$$\underset{\sim}{b}(\nu_{os}) = \underset{\sim}{b}(-\nu_{os}) = b^*(\nu_{os}) \quad . \tag{2.7}$$

For our case of symmetric $\underset{\sim}{C}$, the discrete eigenvalues, ν_{os}, are either real or imaginary—never complex. To see this, multiply Eq. (2.5) by $\tilde{\underset{\sim}{\phi}}^*(\nu,\mu)$

and integrate over μ. (Here the superscript tilde denotes the transpose.) In this way one obtains the equation

$$\frac{1}{\nu_{os}}\int_{-1}^{1} d\mu\,\mu\, \tilde{\underset{\sim}{\varphi}}^{*}(\nu_{os},\mu)\,\underset{\sim}{\varphi}(\nu_{os},\mu) = \int_{-1}^{1} d\mu\, \tilde{\underset{\sim}{\varphi}}^{*}(\nu_{os},\mu)\,\underset{\sim}{\Sigma}\,\underset{\sim}{\varphi}(\nu_{os},\mu)$$
$$-\int_{-1}^{1} d\mu\, \tilde{\underset{\sim}{\varphi}}^{*}(\nu_{os},\mu)\,\underset{\sim}{C}\int_{-1}^{1} d\mu'\,\varphi(\nu_{os},\mu'). \qquad (2.8)$$

Since $\underset{\sim}{\Sigma}$ is diagonal and $\underset{\sim}{C}$ is symmetric, the right-hand side of Eq. (2.8) is real since it is a sum of products of complex conjugate terms. The integral on the left-hand side of Eq. (2.8), which in view of Eq. (2.6), can be written as

$$\frac{1}{\nu_{os}\nu_{os}^{*}}\int_{-1}^{1} d\mu\,\mu\, \tilde{\underset{\sim}{\varphi}}^{*}(\nu_{os},\mu)\,\underset{\sim}{\varphi}(\nu_{os},\mu) = \sum_{i=1}^{N} b_i(\nu_{os})\,b_i^{*}(\nu_{os})\int_{-1}^{1}\frac{\mu\, d\mu}{(\nu_{os}\sigma_i-\mu)(\nu_{os}^{*}\sigma_i-\mu)} \qquad (2.9)$$

is also real.

If the above integral (2.9) is not zero, it follows then that the eigenvalue, ν_{os}, must be real! It will now be shown that this integral can vanish only for purely imaginary eigenvalues.

Let us assume, for the sake of the argument, that ν_{os} is complex and $\mathrm{Re}\{\nu_{os}\} > 0$. It can easily be verified that in this case

$$0 < (\nu_{os}\sigma_i-\mu)(\nu_{os}^{*}\sigma_i-\mu) < (\nu_{os}\sigma_i+\mu)(\nu_{os}^{*}\sigma_i+\mu) \quad ,\mu>0,\ i=1\sim N. \qquad (2.10)$$

Hence, each integral in the sum on the right-hand side of Eq. (2.9) is strictly positive, and since at least one of the terms $b_i(\nu_{os})\, b_i^*(\nu_{os})$ is also strictly positive in view of Eq. (2.7), the sum is strictly positive for $\mathrm{Re}\{\nu_{os}\} > 0$. Similarly, it can be proved that for $\mathrm{Re}\{\nu_{os}\} < 0$ the sum is strictly negative. Thus the integral (2.9) never vanishes if $\mathrm{Re}\{\nu_{os}\} \neq 0$.

However, if ν_{os} is purely imaginary, we have

$$(\nu_{os}\sigma_i - \mu)(\nu_{os}^*\sigma_i - \mu) = (\nu_{os}\sigma_i + \mu)(\nu_{os}^*\sigma_i + \mu), \tag{2.11}$$

and each integral in the right-hand side of Eq. (2.10) is zero. Thus we conclude, the discrete eigenvalues, ν_{os}, lie on only the real or imaginary axis.

(b) Region II

This region is divided in N subintervals, ν_j, $j = 1 \sim N$, such that for $\nu \in \nu_j$, $\frac{1}{\sigma_{j-1}} < |\nu| < \frac{1}{\sigma_j}$. For the j-th sub-interval, there are (N-j+1) linearly independent eigenvectors, $\underset{\sim}{\phi}{}_j^m(\nu,\mu)$, whose i-th component has the form

$$\left[\underset{\sim}{\phi}{}_j^m(\nu,\mu)\right]_i = P\frac{\nu}{\sigma_i\nu-\mu}\left[\underset{\sim}{b}{}_j^m(\nu)\right]_i + \delta(\sigma_i\nu-\mu)\left[\underset{\sim}{\lambda}{}_j^m(\nu)\right]_i \quad ,$$

$$m = j \sim N,\quad j = 1\sim N. \tag{2.12}$$

where P indicates the Cauchy principle value is to be used when these functions are integrated. The vectors $\underset{\sim}{b}{}_j^m(\nu)$ and $\underset{\sim}{\lambda}{}_j^m(\nu)$ are also defined by Yoshimura.[2]

From the eigenvalue equation (2.5), one finds that the eigenvectors are orthogonal in the following sense:

$$\int_{-1}^{1} d\mu\, \mu\, \tilde{\underset{\sim}{\Phi}}(\nu,\mu)\, \underset{\sim}{\Phi}(\nu',\mu) = 0 \quad \text{if} \quad \nu' \neq \nu \ . \tag{2.13}$$

Moreover, it is possible to choose particular linear combinations of eigenvectors for the independent eigenvectors of each subinterval, ν_j, such that all the "continuum" eigenvectors are orthogonal in the following sense

$$\int_{-1}^{1} d\mu\, \mu\, \tilde{\underset{\sim}{\Phi}}{}_j^m(\pm\nu,\mu)\, \underset{\sim}{\Phi}{}_j^{m'}(\pm\nu',\mu) = \pm N_j^m(\nu)\, \delta_{mm'}\, \delta(\nu-\nu') \ . \tag{2.14}$$

Similarly for the "discrete" eigenvalues, we have

$$\int_{-1}^{1} d\mu\, \mu\, \tilde{\underset{\sim}{\Phi}}(\nu_{0s},\mu)\, \underset{\sim}{\Phi}(\nu_{0s'},\mu) = \pm N_s\, \delta_{ss'} \ , \quad s = 1 \sim M \ . \tag{2.15}$$

The functions N_s, and $N_j^m(\nu)$ are given by Yoshimura.[2]

Finally, there is one more relationship between the eigenvectors which we will need later. From Yoshimura's work[2] the functions $\underset{\sim}{b}{}_j^m(\nu)$ and $\underset{\sim}{\lambda}{}_j^m(\nu)$ are even functions of ν, and it follows

$$\underset{\sim}{\Phi}(-\nu,\mu) = \underset{\sim}{\Phi}(\nu,-\mu) \ . \tag{2.16}$$

III. THE ALBEDO PROBLEM

In this section we will consider the albedo problem for a half-space. This problem will be shown to be important because the solutions of all other half-space problems can be expressed in terms of the albedo solution.

(a) Emergent Distribution

Let us now consider the albedo problem for which the incident neutron beam belongs solely to the i-th energy group. In this case the angular flux will be denoted by $\underset{\sim}{\psi}_i(0,\mu_0;x,\mu)$. It is the solution of Eq. (2.1) with the boundary conditions

$$\text{(i)} \quad \underset{\sim}{\psi}_i(0,\mu_0;0,\mu) = \underset{\sim}{e}_i\,\delta(\mu-\mu_0)\,,\ \mu>0\,,\ \mu_0>0\ , \tag{3.1}$$

$$\text{(ii)} \quad \lim_{x\to\infty} \underset{\sim}{\psi}_i(0,\mu_0;x,\mu) = \underset{\sim}{0} \quad , \tag{3.2}$$

where $\underset{\sim}{e}_i$ is a vector, all of whose components are zero except the i-th, which is unity. Since our eigenfunctions are complete,[2,8] the solution for this albedo problem can be expanded in terms of the eigenfunctions which satisfy the boundary conditions at infinity:

$$\underset{\sim}{\psi}_i(0,\mu_0;x,\mu) = \sum_{s=1}^{M} \alpha(\nu_{0s})\,\underset{\sim}{\phi}(\nu_{0s},\mu)\,e^{-x/\nu_{0s}} + \sum_{j=1}^{N}\int_{\eta_{j-1}}^{\eta_j} d\nu \left\{\sum_{m=j}^{N} A_j^m(\nu)\,\underset{\sim}{\phi}_j^m(\nu,\mu)\,e^{-x/\nu}\right\},\quad i=1\sim N, \tag{3.3}$$

where $\eta_j = 1/\sigma_j$, $j = 1 \sim N$, and $\eta_0 = 0$.

We will assume that all the ν_{os} are real. Clearly eigenfunctions with imaginary eigenvalues cannot satisfy our infinity boundary condition. Setting x = 0, and using the full range orthogonality relations plus boundary condition (3.1), we obtain the expansion coefficients as

$$\alpha(\nu_{os}) = \frac{\mu_0}{N_s} \tilde{\underset{\sim}{\Phi}}(\nu_{os}, \mu_0) \underset{\sim}{e}_i - \frac{1}{N_s} \int_0^1 d\mu\mu \, \tilde{\underset{\sim}{\Phi}}(\nu_{os}, -\mu) \underset{\sim}{\Psi}_i(0,\mu_0; 0, -\mu), \quad (3.4)$$

and

$$A_j^m(\nu) = \frac{\mu_0}{N_j^m(\nu)} \tilde{\underset{\sim}{\Phi}}{}_j^m(\nu, \mu_0) \underset{\sim}{e}_i - \frac{1}{N_j^m(\nu)} \int_0^1 d\mu\mu \, \tilde{\underset{\sim}{\Phi}}{}_j^m(\nu, -\mu) \underset{\sim}{\Psi}_i(0,\mu_0; 0, -\mu). \quad (3.5)$$

Substituting these coefficients into Eq. (3.3) with x = 0, we obtain the following inhomogeneous Fredholm equation for the emergent distribution:

$$\underset{\sim}{\Psi}_i(0,\mu_0; 0, -\mu) = \underset{\sim}{F}(\mu) \underset{\sim}{e}_i - \int_0^1 d\mu'\mu' \, \underset{\sim}{K}(\mu', \mu) \underset{\sim}{\Psi}_i(0,\mu_0; 0, -\mu'). \quad (3.6)$$

Here we have defined the matrices

$$\underset{\sim}{F}(\mu) = \mu_0 \sum_{s=1}^{M} \frac{1}{N_s} \underset{\sim}{\Phi}(\nu_{0s}, -\mu)\, \underset{\sim}{\tilde{\Phi}}(\nu_{0s}, \mu_0)$$
$$+ \mu_0 \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu \left\{ \sum_{m=j}^{N} \frac{1}{N_j^m(\nu)} \underset{\sim}{\Phi}_j^m(\nu, -\mu)\, \underset{\sim}{\tilde{\Phi}}_j^m(\nu, \mu_0) \right\} , \tag{3.7}$$

and

$$\underset{\sim}{K}(\mu', \mu) = \sum_{s=1}^{M} \frac{1}{N_s} \underset{\sim}{\Phi}(\nu_{0s}, -\mu)\, \underset{\sim}{\tilde{\Phi}}(\nu_{0s}, -\mu')$$
$$+ \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu \left\{ \sum_{m=j}^{N} \frac{1}{N_j^m(\nu)} \underset{\sim}{\Phi}_j^m(\nu, -\mu)\, \underset{\sim}{\tilde{\Phi}}_j^m(\nu, -\mu') \right\} . \tag{3.8}$$

It can be verified that $\underset{\sim}{K}(\mu', \mu)$ and $\underset{\sim}{F}(\mu)$ are continuous functions of their arguments.

One can also obtain a singular integral equation for $\underset{\sim}{\psi}_i(0, \mu_0; 0, -\mu)$ by considering the incident distribution as given by Eqs. (3.3), (3.4), and (3.5); explicitly

$$\delta(\mu - \mu_0)\underset{\sim}{e}_i = \underset{\sim}{F}(-\mu)\underset{\sim}{e}_i - \int_0^1 d\mu' \mu' \underset{\sim}{K}(\mu', -\mu)\underset{\sim}{\psi}_i(0, \mu_0; 0, -\mu') . \tag{3.9}$$

Either Eqs. (3.6) and/or (3.9) may be used to determine the emergent distribution. Case has obtained the same pair of equations expressed in terms of the infinite medium Green's function,[10] by using a different approach. When explicit expressions for the Green's functions are substituted into his equations, Eqs. (3.6) and (3.9) are obtained.

In the one-speed case, the singular integral equation (3.9) and the Fredholm equation (3.6) may be solved together in closed form.[10] However, for the multi-group situation no closed-form solutions have been obtained and to determine the emergent distribution numerical procedures must be used.

It will be shown that the emergent distribution is uniquely determined by the system of Fredholm integral equations (3.6) alone, and this system of equations can be solved by standard numerical techniques.

Once Eq. (3.6) has been solved for $\underset{\sim}{\psi}_i(0,\mu_0;0,-\mu)$, $\mu > 0$, the expansion coefficients can be completely determined from Eqs. (3.4) and (3.5). Then Eq. (3.3) gives the complete solution for the albedo problem.

(b) Uniqueness of Solution of Fredholm Equation

To show that our Fredholm equation has a unique solution, we consider the homogeneous equation

$$\underset{\sim}{\psi}_i'(0,\mu_0;0,-\mu) = -\int_0^1 d\mu'\mu' \, \underset{\sim}{K}(\mu',\mu) \, \underset{\sim}{\psi}_i'(0,\mu_0;0,-\mu'), \quad \mu > 0. \tag{3.10}$$

Defining

$$\underset{\sim}{\chi}(\mu) = \sqrt{\mu} \; \underset{\sim}{\psi}_i'(0,\mu_0;0,-\mu), \tag{3.11}$$

$$\underset{\sim}{D}(\mu',\mu) = \sqrt{\mu\mu'} \; \underset{\sim}{K}(\mu',\mu), \tag{3.12}$$

we have

$$\underset{\sim}{X}(\mu) = -\int_0^1 \underset{\sim}{D}(\mu',\mu)\,\underset{\sim}{X}(\mu')\,d\mu' . \qquad (3.13)$$

Let us assume a nontrivial solution exists. Multiplying Eq. (3.13) by $\underset{\sim}{X}^*(\mu)$, integrating over μ, and substituting explicitly for $\underset{\sim}{D}(\mu',\mu)$ from Eqs. (3.12) and (3.8), one obtains

$$\int_0^1 d\mu\,\tilde{\underset{\sim}{X}}^*(\mu)\,\underset{\sim}{X}(\mu) = -\sum_{s=1}^{M}\frac{1}{N_s}\int_0^1 d\mu\sqrt{\mu}\left[\widetilde{\underset{\sim}{\phi}(\nu_{0s},-\mu)\underset{\sim}{X}^*(\mu)}\right]\int_0^1 d\mu'\sqrt{\mu'}\,\tilde{\phi}(\nu_{0s},-\mu')\,\underset{\sim}{X}(\mu')$$

$$-\sum_{j=1}^{N}\int_{\eta_{j-1}}^{\eta_j} d\nu\left\{\sum_{m=j}^{N}\frac{1}{N_j^m(\nu)}\int_0^1 d\mu\sqrt{\mu}\left[\tilde{\underset{\sim}{\phi}}{}_j^m(\nu,-\mu)\,\underset{\sim}{X}^*(\mu)\right]\int_0^1 d\mu'\sqrt{\mu'}\,\underset{\sim}{\phi}{}_j^m(\nu,-\mu')\,\underset{\sim}{X}(\mu')\right\}. \qquad (3.14)$$

Since all the eigenvalues are real, $\underset{\sim}{\phi}(\nu,\mu)$ is also real and hence both sides of Eq. (3.14) are composed of terms which are products of complex conjugates. Thus we have the contradiction that the right-hand side of Eq. (3.14) must be real and negative, while the left-hand side is real and strictly positive. Hence $\underset{\sim}{X}(\mu)$ must be identically zero, or equivalently, the homogeneous equation (3.10) has only the null vector as a solution.

Because a system of integral equations may be transformed into a single integral equation,[11] it follows from the known properties of Fredholm integral equations that the solutions of Eq. (3.6) exists and is unique since the homogeneous equation has only the trivial zero solution.[12]

An immediate consequence of the uniqueness of solution of Eq. (3.6) is that the coefficients in the eigenvector expansion in Eq. (3.3) are also uniquely determined. This in turn implies that the eigenvectors $\underset{\sim}{\phi}(\nu,\mu)$, $\nu > 0$ are half-range complete in the sense of Case.[9]

IV. SOLUTIONS OF TYPICAL HALF-SPACE PROBLEMS

By using the results of the previous section it will be shown how the emergent distribution for various half-space problems may be expressed in terms of the emergent distributions of the albedo problems, $\underset{\sim}{\psi}_i(0,\mu_o;0,-\mu)$, $i = 1 \sim N$.

(a) Generalized Milne Problem

For every positive eigenvalue $\nu \in (0,1)$ or $\nu = \nu_{os}$, $s = 1 \sim M$ we define a Milne problem $\underset{\sim}{\psi}_\nu(x,\mu)$ by Eq. (2.1) and the following boundary conditions,

(i) $$\underset{\sim}{\psi}_\nu(0,\mu) = \underset{\sim}{0} \quad , \mu > 0, \tag{4.1}$$

(ii) $$\lim_{x\to\infty} \underset{\sim}{\psi}_\nu(x,\mu) = \underset{\sim}{\phi}(-\nu,\mu)\, e^{-x/\nu} . \tag{4.2}$$

where $\underset{\sim}{\phi}(-\nu,\mu)$ may be any of the eigenvectors—regular or singular.

First let us determine the emergent distribution, $\underset{\sim}{\psi}_\nu(0,-\mu)$. Consider a solution of the transport equation, $\underset{\sim}{\psi}(x,\mu)$, defined as

$$\underset{\sim}{\psi}(x,\mu) = \underset{\sim}{\psi}_\nu(x,\mu) + \underset{\sim}{\psi}_a(x,\mu) , \tag{4.3}$$

where $\underset{\sim}{\psi}_a(x,\mu)$ is also a solution of the transport equation with the boundary conditions

(i) $$\underset{\sim}{\psi}_a(0,\mu) = \underset{\sim}{\phi}(-\nu,\mu) \quad , \mu > 0 , \tag{4.4}$$

(ii) $$\lim_{x\to\infty} \underset{\sim}{\psi}_a(x,\mu) = \underset{\sim}{0} . \tag{4.5}$$

Therefore from (4.3), $\underset{\sim}{\psi}(x,\mu)$ must have the boundary conditions

(i) $$\underset{\sim}{\psi}(0,-\mu) = \underset{\sim}{\phi}(-\nu,\mu) \quad , \mu > 0 \quad , \tag{4.6}$$

(ii) $$\lim_{x\to\infty} \underset{\sim}{\psi}(x,\mu) = \underset{\sim}{\phi}(-\nu,\mu) e^{x/\nu} \quad . \tag{4.7}$$

Clearly the unique solution for $\underset{\sim}{\psi}(x,\mu)$ is

$$\underset{\sim}{\psi}(x,\mu) = \underset{\sim}{\phi}(-\nu,\mu) e^{x/\nu} \quad . \tag{4.8}$$

Eqs. (4.3) and (4.8) then yield for x = 0,

$$\underset{\sim}{\phi}(-\nu,-\mu) = \underset{\sim}{\psi}_\nu(0,-\mu) + \underset{\sim}{\psi}_a(0,-\mu) \quad , \quad -1 \le \mu \le 1 \quad . \tag{4.9}$$

Using the results of the previous section the reflected distribution $\underset{\sim}{\psi}_a(0,-\mu)$, $\mu > 0$, can be expressed in terms of the incident distribution, $\underset{\sim}{\phi}(-\nu,\mu)$ as

$$\underset{\sim}{\psi}_a(0,-\mu) = \sum_{i=1}^{N} \int_0^1 d\mu' \left[\underset{\sim}{\phi}(-\nu,\mu')\right]_i \underset{\sim}{\psi}_i(0,\mu';0,-\mu) \quad , \mu > 0 \ . \tag{4.10}$$

Thus the emergent distribution for the Milne problem becomes, in view of Eqs. (4.9) and (2.16)

$$\underset{\sim}{\psi}_\nu(0,-\mu) = \underset{\sim}{\phi}(\nu,\mu) - \sum_{i=1}^{N} \int_0^1 d\mu' \left[\underset{\sim}{\phi}(-\nu,\mu')\right]_i \underset{\sim}{\psi}_i(0,\mu';0,-\mu) \ . \tag{4.11}$$

Finally to obtain the complete solution for the generalized Milne problem we use the following expansion:

$$\underset{\sim}{\psi}_\nu(x,\mu) = \underset{\sim}{\phi}(-\nu,\mu)e^{x/\nu} + \sum_{s=1}^{M} \alpha(\nu_{0s})\, \underset{\sim}{\phi}(\nu_{0s},\mu)e^{-x/\nu_{0s}}$$
$$+ \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu \left\{ \sum_{m=j}^{N} A_j^m(\nu)\, \underset{\sim}{\phi}_j^m(\nu,\mu) \right\} e^{-x/\nu} \ . \tag{4.12}$$

The expansion coefficients are obtained by applying full-range orthogonality relations and Eq. (4.11). Explicitly they are

$$\alpha(\nu_{0s}) = -\frac{1}{N_s}\int_0^1 d\mu\, \tilde{\phi}(\nu_{0s},-\mu)\left[\mu\, \underset{\sim}{\phi}(\nu,\mu) - \sum_{i=1}^{N}\int_0^1 d\mu' \left[\underset{\sim}{\phi}(-\nu,\mu')\right]_i \underset{\sim}{\psi}_i(0,\mu';0,-\mu)\right], \tag{4.13}$$

and

$$A_j^m(\nu) = -\frac{1}{N_j^m(\nu)} \int_0^1 d\mu \, \tilde{\underset{\sim}{\phi}}_j^m(\nu,-\mu) \Big[\mu \underset{\sim}{\phi}(\nu,\mu) - \sum_{i=1}^{N} \int_0^1 d\mu' [\underset{\sim}{\phi}(-\nu,\mu')]_i \underset{\sim}{\psi}_i(0,\mu';0,-\mu) \, . \tag{4.14}$$

(b) Half Space Green's Function

In a manner similar to that used for the Milne problem, the emergent distribution for the half-space Green's function can be expressed in terms of the emergent albedo problem distributions. The half-space Green's function, with the source neutrons belonging to the i-th group, $\underset{\sim}{G}_i(x_o,\mu_o;x,\mu)$, is defined by the equation

$$\left(\mu \frac{\partial}{\partial x} \underset{\sim}{E} + \underset{\sim}{\Sigma}\right) \underset{\sim}{G}_i(x_0,\mu_0;0,\mu) = \underset{\sim}{C} \int_{-1}^{1} d\mu' \underset{\sim}{G}_i(x_0,\mu_0;x,\mu') + \delta(\mu_0-\mu)\delta(x-x_0)\underset{\sim}{e}_i \, , \quad x_0 > 0 \tag{4.15}$$

with the boundary conditions

(i) $$\underset{\sim}{G}_i(x_0,\mu_0;0,\mu) = \underset{\sim}{0} \quad , \mu > 0 \, , \tag{4.16}$$

(ii) $$\lim_{x\to\infty} \underset{\sim}{G}_i(x_0,\mu_0;x,\mu) = \underset{\sim}{0} \quad . \tag{4.17}$$

To determine this function, we will assume the infinite medium Green's function, $\underset{\sim}{G}_i^{\infty}(x_o,\mu_o;x,\mu)$, which also satisfies Eq. (4.15) is known.[2] This infinite medium Green's function can be expressed in terms of the half-space Green's function as

$$\underset{\sim}{G}_i^{\infty}(x_0,\mu_0;x,\mu) = \underset{\sim}{G}_i(x_0,\mu_0;x,\mu) + \underset{\sim}{\psi}_a(x,\mu) \quad , x_0 > 0 , \tag{4.18}$$

where $\underset{\sim}{\psi}_a(x,\mu)$ is an albedo problem solution satisfying Eq. (2.1) with boundary condition

(i) $$\underset{\sim}{\psi}_a(0,\mu) = \underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu) \quad , \mu > 0 , \tag{4.19}$$

(ii) $$\lim_{x\to\infty} \underset{\sim}{\psi}_a(x,\mu) = 0 . \tag{4.20}$$

Expressing the emergent distribution for this albedo problem in terms of the known incident distribution and the vectors $\underset{\sim}{\psi}_i(0,\mu_o;0,-\mu)$, Eq. (4.18) yields

$$\underset{\sim}{G}_i(x_0,\mu_0;0,-\mu) = \underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,-\mu) - \sum_{j=1}^{N}\int_0^1 d\mu' \left[\underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu')\right]_j \underset{\sim}{\psi}_j(0,\mu';0,-\mu) . \tag{4.21}$$

Since the angular flux for the half-space Green's function is now known at x = 0 for all μ, the complete solution can be found by using the full-range completeness and orthogonality theorems. Explicitly

$$\underset{\sim}{G}_i(x_0,\mu_0;x,\mu) = \underset{\sim}{G}_i^{\infty}(x_0,\mu_0;x,\mu) + \sum_{s=1}^{M} \alpha(\nu_{0s})\, \underset{\sim}{\varphi}(\nu_{0s},\mu)\, e^{-x/\nu_{0s}}$$

$$+ \sum_{j=1}^{N} \int_{\eta_{j-1}}^{\eta_j} d\nu \left\{ \sum_{m=j}^{N} A_j^m(\nu)\, \underset{\sim}{\varphi}_j^m(\nu,\mu) \right\} e^{-x/\nu} \tag{4.22}$$

where

$$\alpha(\nu_{0s}) = -\frac{1}{N_s} \int_0^1 d\mu\, \mu\, \underset{\sim}{\tilde{\varphi}}(\nu_{0s},\mu)\, \underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu)$$

$$+ \frac{1}{N_s} \int_0^1 d\mu\, \mu\, \underset{\sim}{\tilde{\varphi}}(\nu_{0s},-\mu) \sum_{k=1}^{N} \int_0^1 d\mu' \left[\underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu')\right]_k \underset{\sim}{\psi}_k(0,\mu';0,-\mu), \tag{4.23}$$

$$A_j^m(\nu) = -\frac{1}{N_j^m(\nu)} \int_0^1 d\mu\, \mu\, \underset{\sim}{\tilde{\varphi}}_j^m(\nu,\mu)\, \underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu)$$

$$+ \frac{1}{N_j^m(\nu)} \int_0^1 d\mu\, \mu\, \underset{\sim}{\tilde{\varphi}}_j^m(\nu,-\mu) \sum_{k=1}^{N} \int_0^1 d\mu' \left[\underset{\sim}{G}_i^{\infty}(x_0,\mu_0;0,\mu')\right]_k \underset{\sim}{\psi}_k(0,\mu';0,-\mu) . \tag{4.24}$$

V. SUMMARY

It has been shown that the solutions of all multi-group half-space problems involving a symmetric transfer matrix can be expressed in terms of the emergent albedo problem distribution and the infinite medium eigenfunctions.

This emergent albedo distribution is uniquely determined by the Fredholm equation (3.6), which can be solved by standard numerical procedures.

In this paper, attention was restricted to those cases which could be transformed such that the transfer matrix was symmetric. This assumption was necessary to prove that (i) the eigenvalues of the transport equation are real or imaginary, and (ii) the emergent albedo distribution is uniquely determined by Eq. (3.6). In a future paper, this restriction will be relaxed and the case of a general transfer matrix will be discussed.

APPENDIX I. THERMAL REACTOR MODEL

The linear Boltzmann equation for a homogeneous nonmultiplying medium in plane geometry and with isotropic scattering may be written as

$$\left(\mu\frac{\partial}{\partial x} + \Sigma(E)\right)\psi(x,\mu,E) = \int_{-1}^{1} d\mu' \int_{0}^{\infty} \Sigma_s(E'\rightarrow E)\,\psi(x,\mu',E') \quad , \tag{A-1}$$

where $\psi(x,\mu,E)$ is the angular flux, and $\Sigma(E)$ and $\Sigma_s(E'\rightarrow E)$ are the total and differential scattering cross sections, respectively.

Using the usual multi-group technique,[13] the energy variable is split into N regions and integrating Eq. (A-1) over the i-th region we obtain the i-th multi-group equation

$$\left(\mu\frac{\partial}{\partial x} + \sigma_i\right)\psi_i(x,\mu) = \sum_{j=1}^{N} c_{ij}\int_{-1}^{1} d\mu'\,\psi_j(x,\mu') \quad , \tag{A-2}$$

where we define

$$\psi_i(x,\mu) = \int_{\Delta E_i} dE\,\psi(x,\mu,E) \quad , \tag{A-3}$$

$$\sigma_i = \frac{1}{\psi_i(x,\mu)}\int_{\Delta E_i} dE\,\Sigma(E)\,\psi(x,\mu,E) \quad , \tag{A-4}$$

$$c_{ij} = \frac{1}{\psi_j(x,\mu)}\int_{\Delta E_i} dE \int_{\Delta E_j} dE'\,\Sigma_s(E'\rightarrow E)\,\psi(x,\mu,E') \quad . \tag{A-5}$$

To make the multi-group constants σ_i and C_{ij} independent of x and μ, it is usual to assume that the energy dependence of the angular flux is separable. Further for a system in thermal equilibrium a good first approximation is to assume this energy dependence is Maxwellian with some effective temperature, T. With these assumptions the multi-group group parameters are given by

$$C_{ij} = \alpha_j \int_{\Delta E_i} dE \int_{\Delta E_j} dE' \Sigma_s(E' \rightarrow E)\, M(E', T) \quad , \tag{A-6}$$

$$\sigma_i = \alpha_i \int_{\Delta E_i} dE\, \Sigma(E)\, M(E,T) \quad , \tag{A-7}$$

$$\frac{1}{\alpha_i} = \int_{\Delta E_i} dE\, M(E,T) \quad . \tag{A-8}$$

The cross section $\Sigma_s(E' \rightarrow E)$ must obey the detailed balance relation[14]

$$\Sigma_s(E' \rightarrow E)\, M(E', T) = \Sigma_s(E \rightarrow E')\, M(E,T) \quad , \tag{A-9}$$

or

$$\alpha_i C_{ij} = \alpha_j C_{ji} \quad . \tag{A-10}$$

Finally, defining the symmetric matrix $\underset{\sim}{A}$ as

$$[\underset{\sim}{A}]_{ij} = \frac{1}{\alpha_j} C_{ij} \tag{A-11}$$

the transfer matrix can be written in the special form

$$\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{D} \quad , \tag{A-12}$$

where $\underset{\sim}{D}$ is a diagonal matrix with elements, $\alpha_i > 0$.

APPENDIX II

In certain physical models the transfer matrix may be written as $\underset{\sim}{C} = \underset{\sim}{D}_1 \underset{\sim}{A} \underset{\sim}{D}_2$ where $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_2$ are diagonal matrices with strictly positive diagonal elements and A is a positive symmetric matrix. The elements of the $\underset{\sim}{\Sigma}$ matrix generally will not be ordered but will be arranged as

$$\sigma_k \geq \sigma_\ell \geq \cdot \cdot \cdot \geq \sigma_m > 0 \quad , \quad 1 \leq k, \ell, \cdots, m \leq N . \tag{B-1}$$

It is possible to transform Eq. (2.1) into a form which has a purely symmetric transfer matrix and an ordered matrix. First, we construct a permutation matrix, $\underset{\sim}{P}$, such that

$$\begin{aligned} [\underset{\sim}{P}]_{1k} &= 1 \; ; \; [\underset{\sim}{P}]_{1i} = 0 \; , \; i \neq k \\ [\underset{\sim}{P}]_{2\ell} &= 1 \; ; \; [\underset{\sim}{P}]_{2i} = 0 \; , \; i \neq \ell \\ &\vdots \\ [\underset{\sim}{P}]_{Nm} &= 1 \; ; \; [\underset{\sim}{P}]_{Ni} = 0 \; , \; i \neq m \quad . \end{aligned} \tag{B-2}$$

By multiplying Eq. (2.1) from the left by $\underset{\sim}{P}$, one obtains

$$\left[\mu \frac{\partial}{\partial x} \underset{\sim}{E} + \underset{\sim}{\Sigma}'\right] \underset{\sim}{\psi}'(x,\mu) = \underset{\sim}{D}_1' \underset{\sim}{A}' \underset{\sim}{D}_2' \int_{-1}^{1} d\mu \, \underset{\sim}{\psi}'(x,\mu) \quad , \tag{B-3}$$

where

$$\underset{\sim}{\psi}'(x,\mu) = \underset{\sim}{P}\,\underset{\sim}{\psi}(x,\mu)\,,$$

$$\underset{\sim}{\Sigma}' = \underset{\sim}{P}\,\underset{\sim}{\Sigma}\,\underset{\sim}{P}^{-1}$$

$$\underset{\sim}{A}' = \underset{\sim}{P}\,\underset{\sim}{A}\,\underset{\sim}{P}^{-1}$$

$$\underset{\sim}{D}_i' = \underset{\sim}{P}\,\underset{\sim}{D}_i\,\underset{\sim}{P}^{-1}\qquad,\quad i=1,2\qquad. \tag{B.4}$$

Since $\underset{\sim}{P}^{-1} = \tilde{\underset{\sim}{P}}$ it can be shown by inspection that $\underset{\sim}{\Sigma}'$ is a diagonal matrix with ordered elements

$$\sigma_1' \geq \sigma_2' \geq \cdots \geq \sigma_N' \qquad. \tag{B.5}$$

Further $\underset{\sim}{D}_1'$ and $\underset{\sim}{D}_2'$ are diagonal matrices with positive diagonal elements, and $\underset{\sim}{A}'$ is symmetric.

Now we define the diagonal matrices $\underset{\sim}{D}_i^{1/2}$ and $\underset{\sim}{D}_i^{-1/2}$ as

$$\left[\underset{\sim}{D}_i^{\pm 1/2}\right]_{jk} = \left\{\left[\underset{\sim}{D}_i\right]_{jk}\right\}^{\pm 1/2}\,,\quad i=1,2\qquad. \tag{B.6}$$

Multiplying Eq. (B.3) from the left by $\underset{\sim}{D}_1^{-1/2}\,\underset{\sim}{D}_2^{1/2}$ we have

$$\left[\mu\frac{\partial}{\partial x}\underset{\sim}{E} + \underset{\sim}{\Sigma}'\right]\underset{\sim}{\psi}''(x,\mu) = \underset{\sim}{A}''\int_{-1}^{1}d\mu\,\underset{\sim}{\psi}''(x,\mu)\qquad, \tag{B.7}$$

where

$$\underset{\sim}{\Psi}''(x,\mu) = \underset{\sim}{D}_1^{-1/2}\underset{\sim}{D}_2^{1/2}\underset{\sim}{\Psi}''(x,\mu),$$

$$\underset{\sim}{A}'' = \underset{\sim}{D}_1^{1/2}\underset{\sim}{D}_2^{1/2}\underset{\sim}{A}'\underset{\sim}{D}_1^{1/2}\underset{\sim}{D}_2^{1/2} = \tilde{\underset{\sim}{A}}''. \tag{B-8}$$

For the two-group model, there exists a transformation, $\underset{\sim}{S}$, which will symmetrize any strictly positive $\underset{\sim}{C}$ matrix and leave $\underset{\sim}{\Sigma}$ diagonal, namely

$$\underset{\sim}{S} = \begin{bmatrix} 0 & \sqrt{C_{12}} \\ \sqrt{C_{21}} & 0 \end{bmatrix}. \tag{B-9}$$

On the other hand, if one or both off-diagonal elements are zero, the resulting multi-group equations can be solved consecutively by applying one-speed theory.

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