Structure of the combinatorial generalization of hypergeometric functions for $SU(n)$ states. II

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(Received 8 August 1972)

In the construction of the general $SU(5)$ states, the action of each individual lowering operator (raised to a power) operating on the semimaximal state leads to an operator-valued polynomial which is shown to belong to the class of generalized hypergeometric functions in the sense of Gel'fand (namely, they are Radon transform of linear forms). Three new functions are found at the $SU(5)$ level and their content in terms of known lower-hierarchy functions are explicitly exhibited. The structure of the general $SU(n)$ states due to the combined action of all lowering operators is quite complicated, but the action of each individual lowering operator taken one at a time may still be manageable for higher $n$, and, in the spirit of boson operator formalism, this may be one systematic way of producing high-hierarchy generalized hypergeometric functions.

I. INTRODUCTION

Previous work$^1$-$^4$ shows that the combinatorics of the boson operator formalism in the construction of the $SU(n)$ states provides a natural scheme for the appearance of certain generalized hypergeometric functions. We recall that a general state is obtained by operating an appropriate string of lowering operators $L_i^n$ (raised to a power) on the so-called semimaximal state, the latter being expressed as products of certain (anti-symmetrized) creation operators acting on the vacuum state. As a result of pushing the lowering operators through the creation operators, the nonvanishing commutators thus yield an operator-valued polynomial (operating on the vacuum). For the $SU(3)$ state, this operator-valued polynomial is simply expressed as the Gauss hypergeometric function $\sum_1^3 F (a,b;c;x)$, as pointed out by Baird and Biedenharn,$^1$ namely,

\[ |\text{general } SU(3) \text{ state} | = \text{const} \times \sum_1^3 F_1 (a,b;c;x) \text{ (0)}. \tag{1} \]

Or, symbolically, the relevant ingredient reads

\[ SU(3): (L_{ij})^n [a] \rightarrow \text{Gauss } \sum_1^3 F_1, \tag{2} \]

where each factor of $a$ in the bracket stands for an anti-symmetrized $(a_{12345})^6$ that the lowering operator has to negotiate with.

What is the generalization of the statement (1)? It was found$^3$-$^4$ that a general $SU(4)$ state which is obtained via a product of three lowering operators $(L_{ij})^n, (L_{kl})^n, (L_{mj})^n$ does not have a simple form, but may be regarded as folded products of known functions. In other words, at the $SU(4)$ level the action of each individual lowering operator still yields a recognizable function, namely to either invent new names for these generalized hypergeometric functions if one adopts the viewpoint that the boson operator formalism is a good way of generating (hopefully systematically) such functions, or alternatively one may try to exhibit the inner structure thereof in terms of known functions.

In this paper, we examine the structure of the general $SU(5)$ states, obtained by pushing through a set of six lowering operators $L_{12}, L_{13}, L_{14}, L_{15}, L_{24}, L_{25}, L_{35}$ (each raised to a power). Their individual action can be summarized as follows (the details are given in Sec. III):

\[
\begin{array}{ccc}
SU(5): & \text{resulting} & \text{Gel'fand} \\
& \text{operator} & \text{N-fold} & \text{criterion:} \\
& & \text{sum} & \text{Radon} & \text{content} \\
& & & \text{transform} & \text{of linear} \\
\end{array}
\]

<table>
<thead>
<tr>
<th>SU(5): operator</th>
<th>resulting N-fold sum</th>
<th>content</th>
<th>Gel'fand criterion: Radon transform of linear forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(L_{12})^n[a]$</td>
<td>3 $\sum_1^3 F_2$</td>
<td>Appell $F_2 \times 3 F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{13})^n[a]$</td>
<td>3 $\sum_1^3 F_2$</td>
<td>Appell $F_2 \times 3 F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{14})^n[a]$</td>
<td>1 $\sum_1^3 F_2$</td>
<td>Appell $F_1 \times 3 F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{15})^n[a]$</td>
<td>8 $\sum_1^3 F_2$</td>
<td>Appell $F_1 \times 3 F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{23})^n[a]$</td>
<td>2 $\sum_1^3 F_2$</td>
<td>Appell $F_1 \times 3 F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{24})^n[a]$</td>
<td>6 $\sum_1^3 F_2$</td>
<td>Appell $F_1 \times 3 F_2$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

The following remarks are obvious at the $SU(5)$ level:

(a) The operator $(L_{ij})^n[a]$ yields the Gauss $F_1$ function. This result is analogous to the action of $(L_{12})^n[aa]$ at the $SU(3)$ level, or that of $(L_{12})^n[aa]$ at the $SU(4)$ level.

(b) The operator $(L_{12})^n[aaa]$ yields $F_1$, the Appell function of the first kind (in 2-variables).

(c) The operator $(L_{12})^n[aaaaa]$ yields $F_1$, the Lauricella function of the fourth kind in 6-variables.

For higher-rank $SU(n)$ states ($n > 5$), it turns out that our present repertory of generalized hypergeometric functions clearly is not adequate to accommodate even the action of each individual lowering operator. One has

\[
\begin{array}{ccc}
SU(4): & \text{resulting} & \text{Gel'fand} \\
& \text{operator} & \text{N-fold} & \text{criterion:} \\
& & \text{sum} & \text{Radon} & \text{content} \\
& & & \text{transform} & \text{of linear} \\
\end{array}
\]

<table>
<thead>
<tr>
<th>SU(4): operator</th>
<th>resulting N-fold sum</th>
<th>content</th>
<th>Gel'fand criterion: Radon transform of linear forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(L_{12})^n[aa]$</td>
<td>2 $\sum_1^3 F_2$</td>
<td>Appell $F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{12})^n[a]$</td>
<td>1 $\sum_1^3 F_2$</td>
<td>Gauss $F_2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$(L_{12})^n[aa]$</td>
<td>3 $\sum_1^3 F_2$</td>
<td>Lauricella $F_2[3]$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

For higher-rank $SU(n)$ states ($n > 5$), it turns out that our present repertory of generalized hypergeometric functions clearly is not adequate to accommodate even the action of each individual lowering operator. One has
II. GENERAL SU(5) STATES

As is well known, a general SU(5) state may be constructed by applying a set of appropriate lowering operators to the semimaximal state.

\[ |\text{general SU}(5) \text{ state} \rangle = (L^\downarrow)^{m_{15}}(L^\downarrow)^{m_{25}}(L^\downarrow)^{m_{35}}(L^\downarrow)^{m_{45}}(0) \]

\[ \times (m_{14} m_{24} m_{34} m_{44} 0) \]

\[ \times (m_{13} m_{23} m_{33} m_{12} m_{22}) \]

\[ \times (m_{11} m_{11}) \]

\[ = \text{const} (L^\uparrow)^{m_{15}}(L^\uparrow)^{m_{25}}(L^\uparrow)^{m_{35}}(L^\uparrow)^{m_{45}}(L^\uparrow)^{m_{14}} \]

\[ \times (a_{1234} a_{1235} a_{123} a_{125} a_{12}) \]

\[ \times (a_{13} a_{14} a_{15})^4 \]

\[ \times (0) \]

(5)

The set of lowering operators \( L^\downarrow \) are defined in Ref. 6. Those with \( i < j \sim 3 \) appeared in the discussion of \( SU(4) \) case. 3,4 \( L^\downarrow \) reads explicitly

\[ L^\downarrow_4 = \delta_{12} \delta_{13} E_{41} + \delta_{14} E_{21} + \delta_{14} E_{31} + E_{23} E_{31} \]

(6)

The exponents \( n_{ij}, \nu_{ij} \) in Eq. (5) are shorthand notations as before,4 namely

\[ n_{ij} = m_{ij} - m_{ij-1}, \quad \nu_{ij} = m_{ij} - m_{i-1,j+1}. \]

(7)

III. ACTION OF EACH INDIVIDUAL LOWERING OPERATOR

By a straightforward calculation, the action of each \( (L^\downarrow)^{n} \) operator on the relevant set of creation operators turns out to be as follows:

**Step 1, \( L^\downarrow_4 \):**

\[ A = (a_{1234} a_{1235} a_{123} a_{125} a_{12}) \]

\[ \times (a_{13} a_{14} a_{15})^4 \]

\[ \times (0) \]

\[ \times \sum_{k_1, k_2} \frac{(-n_{14} + k_1 + k_2)_{a_{12}} (-n_{35} a_{13} (-s_1 - 1)_{a_{12}}}{(-s_3 - 2)_{k_2}} \]

\[ \times \frac{u_1 u_2 u_3}{k_1! k_2! k_3!} |0\rangle, \]

(9a)

\[ \text{const}(w_0)^{n_{14}} A \]

\[ \times \sum_{k_1, k_2} \frac{(-n_{14} + k_1 + k_2)_{a_{12}} (-n_{35} a_{13} (-s_1 - 1)_{a_{12}}}{(-s_3 - 2)_{k_2}} \]

\[ \times \frac{u_1 u_2 u_3}{k_1! k_2! k_3!} |0\rangle, \]

(9b)

where

\[ \text{const} = [\nu_{14}/(\nu_{14} - n_{14})] [(s_4 + 1)/(s_4 + 1 - n_{24})], \]

(10)

\[ s_4 = \nu_{24} + \nu_{34} + n_{35} + n_{45} - k_3, \]

(11)

\[ w_0 = a_{14}/a_{12}, \quad u_1 = a_{12} a_{145}/a_{12} a_{15}, \]

\[ u_2 = a_{14} a_{1234}/a_{14} a_{1235}, \]

(12)

As a generalized hypergeometric series in three variables, the expression (9a) does not seem to be a known function. Alternatively, Eq. (9b) shows that it may be written as a folded product of an Appell \( F_2 \) function (in two variables) with a \( 3F_2 \) function (in one variable).

**Step 2, \( L^\downarrow_4 \):**

\[ B = (a_{1234} a_{1234} a_{123} a_{125} a_{12} a_{12}) \]

\[ \times (a_{13} a_{14} a_{15}) \]

\[ \times (0) \]

\[ \times \sum_{k_1, k_2} \frac{(-n_{14} + k_1 + k_2)_{a_{12}} (-n_{35} a_{13} (-s_1 - 1)_{a_{12}}}{(-s_3 - 2)_{k_2}} \]

\[ \times \frac{u_1 u_2 u_3}{k_1! k_2! k_3!} |0\rangle, \]

(14a)

\[ \text{const}(w_0)^{n_{14}} B \]

\[ \times \sum_{k_1, k_2, k_3} \frac{(-n_{14} + k_1 + k_2)_{a_{12}} (-n_{35} a_{13} (-s_1 - 1)_{a_{12}}}{(-s_3 - 2)_{k_2}} \]

\[ \times \frac{u_1 u_2 u_3}{k_1! k_2! k_3!} |0\rangle, \]

(14b)

where

\[ \text{const} = [\nu_{24}/(\nu_{24} - n_{24})] [(s_4 + 1)/(s_4 + 1 - n_{24})], \]

(15)

\[ s_4 = \nu_{24} + \nu_{34} + n_{35} + n_{45} - k_3, \]

(16)

\[ u_0 = a_{14}/a_{12}, \quad u_1 = a_{12} a_{145}/a_{12} a_{15}, \]

(17)

The expression (14a) in three variables does not seem
to be a known function, but Eq. (14b) shows that it has the structure of a folded product of Appell $F_2$ function

Step 3, $L_3^2$:

$$C = (a_{1235})^{14} r_{12}^{-14} (a_{123})^{12} C_{12},$$  \hspace{1cm} (18)

$$\left( L_3^2 \right)^{s=4} C = \text{const} \left( \frac{a_{124}}{a_{123}} \right)^{n_{14}} \times \frac{(-n_{14} + k_3 + l_2 + l_3)_{14}}{(1 + v_{14} - n_{14} - k_3 + l_2 + l_3)_{14} \cdot k_1} \cdot 0 \right)$$  \hspace{1cm} (19a)

$$D = \left( a_{124} \right)^{n_{14} + k_3 + l_2 + l_3} (a_{123})^{n_{14} + k_3 + l_2 + l_3} \frac{1 + v_{14} - n_{14} - k_3 + l_2 + l_3}{0 \cdot k_1} \cdot 0$$  \hspace{1cm} (20)

$$\left( L_3^2 \right)^{s=4} D = \text{const} \left( v_0 \right)^{n_{14} D} \sum_{a_4} \left( -1 \right)^{a_4 + a_5} \frac{(-1)_{a_4 + a_5} \cdot (-n_{14} + k_3 + l_2 + l_3)_{a_4}}{(1 + v_{14} - n_{14} + k_3 + l_2 + l_3)_{a_4} \cdot k_1} \cdot 0 \right)$$  \hspace{1cm} (21)

$$\left( L_3^2 \right)^{s=4} D = \text{const} \left( v_0 \right)^{n_{14} D} \sum_{a_4} \left( -1 \right)^{a_4 + a_5} \frac{(-1)_{a_4 + a_5} \cdot (-n_{14} + k_3 + l_2 + l_3)_{a_4}}{(1 + v_{14} - n_{14} + k_3 + l_2 + l_3)_{a_4} \cdot k_1} \cdot 0 \right)$$  \hspace{1cm} (22)

where

$$\text{const} = n_{14} / (v_{14} - n_{14} - k_3 + l_2 + l_3)_{a_4}$$  \hspace{1cm} (23)

$$w_{4} = a_{123}^{124} / a_{123}^{124} a_{123}^{124}$$  \hspace{1cm} (24)

For the purpose of the subsequent steps, it will be convenient to rewrite (21) with the aid of the identity

$$w_{4} = \sum_{k_4} \frac{(-k_4)_{k_4}}{k_4!} \left( a_{124}^{124} a_{123}^{124} \right)^{k_4} \cdot 0$$  \hspace{1cm} (25)

This has the effect of simplifying the expressions (23), (29), and (33) in not having to include the factor $(a_{124}^{124})^{k_4}$ (which does not commute with $L_3^2$ nor $a_{124}$ with $L_3^2$).

Step 4, $L_3^2$:

$$D = \left( a_{124}^{124} a_{123}^{123} \right)^{n_{14} + k_3 + l_2 + l_3} \frac{1 + v_{14} - n_{14} - k_3 + l_2 + l_3}{0 \cdot k_1} \cdot 0$$  \hspace{1cm} (26)

The expression (24a) does not seem to correspond to a known function. On the other hand, Eq. (24b) shows that it has the following structure: Appell function in $v_{41}, v_{52}$, Lauricella $F_{3(3)}$ in $v_{41}, v_{52}, v_{53}$, and Lauricella $F_{4(3)}$ in $v_{41}, v_{52}, v_{53}, v_{54}, v_{55}, v_{56}$, the last which is a generalization of the Appell $F_3$ function makes its first appearance at the $SU(5)$ level.

Step 5, $L_3^2$:

$$D = \left( a_{124}^{124} a_{123}^{123} \right)^{n_{14} + k_3 + l_2 + l_3} \frac{1 + v_{14} - n_{14} - k_3 + l_2 + l_3}{0 \cdot k_1} \cdot 0$$  \hspace{1cm} (27)

$$\left( L_3^2 \right)^{s=4} E = \text{const} \left( v_0 \right)^{n_{14} E} \sum_{r_1} \left( -n_{23} \right)_{r_1} \frac{(-n_{14} - k_3 + l_2 + l_3 + \sigma_4 + \sigma_5)_{r_1}}{(1 + v_{14} - n_{14} + l_2 + l_3 - \sigma_5)_{r_1}}$$  \hspace{1cm} (28)

In Eq. (24b), we have for $j = 1, 2, 3$

$$\hat{s}_j = a_{j} + a_{5}, \quad \hat{v}_j = v_{j} + v_{j} \cdot i$$  \hspace{1cm} (29)

\[ \times (-n_{35} + h_2 + l_1 + k_5 + a_5 + a_7 r_2 r_1 r_2! | 0) \]

\[ = \text{const}(\mu_0)^{a_{31}} E \times _{F_1}(-n_{23}; -n_{34} - h_2 + l_3 + k_5 + a_4 + a_6, -n_{35} + h_2 + l_1 + k_5 + a_5 + a_7; \mu_1, \mu_2) | 0) . \]

\[ \text{IV. GEL'FAND CRITERION: RADON TRANSFORM OF LINEAR FORMS} \]

One class of generalized hypergeometric functions has the property that they are Radon transforms of linear forms. It so happens that all the known low-hierarchy functions such as Gauss, Appell, and Lauricella functions satisfy this Gel'fand criterion. From the expression (4), the simple functions associated with the action of each individual operator \((L_0^1)^n, (L_0^2)^n, (L_0^3)^n\) obviously have this property. For the others, it is not apparent from their contents as folded products of simple functions. In general, the Gel'fand criterion which holds for each constituent may not be preserved under folded multiplication. However, it is rather remarkable that the functions associated with the action of each operator \((L_0^1)^n, (L_0^2)^n, (L_0^3)^n\), at the SU(5) level still satisfy the Gel'fand criterion. The proof of this statement, which consists of using well-known integral representation for each constituent and a simple change of variables, is left for the reader.

ACKNOWLEDGMENT

The paper was written while the second-named author (A. Wu) was at the University of Michigan Cyclotron Laboratory. He wishes to thank Professor W. C. Parkinson for the hospitality extended to him.

\[ x_1 = a_1 a_2 / a_1 a_15, \quad x_2 = a_1 a_2 / a_2 a_{14}, \quad x_3 = a_1 a_2 / a_2 a_{13}, \quad x_4 = a_1 a_2 / a_2 a_{145}, \quad x_5 = a_1 a_2 a_2 a_135, \quad x_6 = a_1 a_2 a_2 a_{134}. \]