Burgers' Turbulence Models

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A detailed investigation of the stability of the solutions and the growth of secondary solutions beyond the critical points is carried out for the Burgers' model equations. It is found that the transitions at some critical points are very much like the intuitive description given by Landau; however, the possibility of finite jumps is also encountered.

I. INTRODUCTION

The problem of the transition from a laminar to a turbulent flow, as the Reynolds number is increased, is one of considerable interest. Various theories have been proposed.[1] Owing to the mathematical difficulties, not much calculation has been carried out to verify these theories. Here a *branching theory* is verified for some model equations proposed by Burgers.

The branching theory of the transition to turbulent flow is well described by Landau.[2] His theory is briefly reproduced here.

If one investigates the stability of the laminar solution by the linearized Navier–Stokes equations for small perturbations, one finds that the laminar flow is stable at low Reynolds number, but becomes unstable beyond certain critical Reynolds number $R_{cr}^{(1)}$. At a slightly higher Reynolds number $R \gtrsim R_{cr}^{(1)}$, an undamped sinusoidal oscillation of frequency $\omega_1(R)$ which depends on R is superimposed on the laminar solution. The equation for the growth of the amplitude $|A|$ is

$$\frac{d\,|A|^2}{dt} = \alpha\,|A|^2 - \beta\,|A|^4, \qquad (1)$$

with $\alpha \propto R - R_{cr}^{(1)}$. Here β is assumed to be positive, otherwise the solution goes to infinity at a finite time. (It will be shown in Sec. II, however, that β is not always positive.) The solution of (1) is

$$|A|^2 = \frac{\alpha\,|A(0)|^2 \exp(\alpha t)}{\alpha - \beta\,|A(0)|^2\,[1 - \exp(\alpha t)]}. \qquad (2)$$

As $t \to \infty$, $|A|^2 \to 0$ for $R < R_{cr}^{(1)}$ and $|A|^2 \to \alpha/\beta$ for $R > R_{cr}^{(1)}$. Thus, the amplitude of oscillation $|A| \propto [R - R_{cr}^{(1)}]^{1/2}$, i.e., is proportional to the square root of the difference between the actual and the critical Reynolds number. The transition is continuous, i.e., for an infinitesimal increase of the Reynolds number beyond $R_{cr}^{(1)}$, there is an infinitesimal increase in $|A|$. The phase of the oscillation is arbitrary. Its value is determined by the initial phase of the disturbance, hence, there is one degree of freedom. If one further increases R, the solution is no longer simple sinusoidal, but is of the form

$$v = \sum_{n=-\infty}^{+\infty} A_n \exp(-\,in\Phi_1),$$

where $\Phi_1 = \omega_1 t + \beta_1$, and β_1 is the arbitrary phase depending on initial conditions. One may write a set of linearized Navier–Stokes equations for small perturbations of this solution, and one finds that this solution becomes unstable at some larger critical Reynolds number $R_{cr}^{(2)}$. Slightly beyond $R_{cr}^{(2)}$, the solution is sinusoidal with two fundamental frequencies $\omega_1(R)$ and $\omega_2(R)$, and there are two arbitrary phases. One may proceed in this manner and the solution will quickly have many frequencies and the same number of arbitrary phases. The flow becomes "turbulent."

A rigorous mathematical description of this program is one remaining to be given. An investigation in this direction was recently made by Velte in the case of rotating coaxial cylindrical flow,[3] and it was verified experimentally by Donnelly *et al.*,[4] that in this case the flow beyond the critical Reynolds number of the laminar flow was indeed of the form described by Landau. However, no time-dependent description was given by Velte. In what follows, a time-dependent description will be given for two sets of model equations due to Burgers,[5] in the neighborhood of some critical points analogous to the critical Reynolds number. It will be shown that a

[1] See, for example, S. Pai, *Viscous Flow Theory—Turbulent Flow* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1956), pp. 6 and 7.
[2] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press Ltd., London, 1959), pp. 103–107.
[3] W. Velte, Arch. Ratl. Mech. Anal. **22**, 1 (1966).
[4] R. J. Donnelly, K. W. Schwarz, and P. H. Roberts, Proc. Roy. Soc. (London) **A283**, 531 (1965).
[5] J. M. Burgers, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. (Amsterdam) **17**, 1 (1939); or *Advances in Applied Mechanics*, R. V. Mises and Th. V. Kármán, Eds. (Academic Press Inc., New York, 1948), Vol. 1, p. 171.

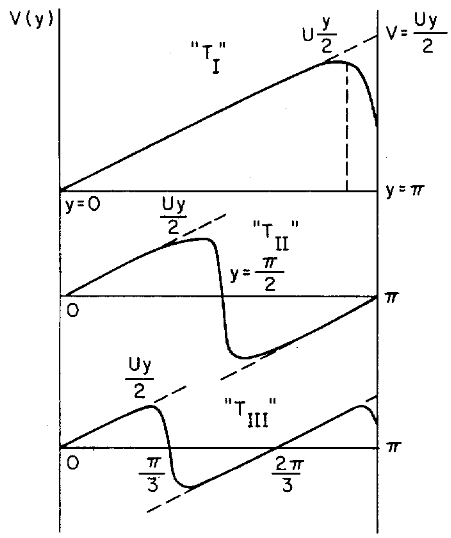

FIG. 1. Velocities as functions of y for T_I, T_{II}, and T_{III}.

situation similar to that described by Landau, namely, a continuous transition, occurs at some critical points; while at some other critical points, an infinitesimal change of the analogous Reynolds number beyond the critical point induces a finite change in the solution. The mathematical method used will be the generalized Bogoliubov–Mitropolsky perturbation method.[6]

The one-dimensional Burgers' equations are considered first. A brief summary of the necessary results obtained by Burgers in Ref. 5 will first be given. Section IIA and the first part of Sec. IIIA up to Eq. (19) will be restatements of Burgers' results. The time-dependent descriptions will be carried out after investigating the stability of the solutions. The investigation on the two-dimensional model is carried out along similar lines.

II. "ONE DIMENSIONAL" BURGERS' MODEL EQUATIONS

A. Preliminary Considerations

To investigate the effects of the nonlinear terms of the Navier–Stokes equations, Burgers studied two sets of simpler model equations similar in form to the Navier–Stokes equations. The one-dimensional equations are as follows:

$$\frac{dU}{dt} = P - \nu U - \frac{1}{\pi}\int_0^\pi v^2\,dy,^7$$
$$\frac{\partial v}{\partial t} = Uv + \nu\frac{\partial^2 v}{\partial y^2} - \frac{\partial}{\partial y}(v^2). \tag{3}$$

In these equations, U depends on t only, and is independent of y, while $v = v(y, t)$ is subjected to the boundary conditions

$$v(y = 0, t) = v(y = \pi, t) = 0.$$

An obvious solution to Eq. (3) is $U = P/\nu$, $v = 0$, which will be called the *laminar solution*.

Further stationary solutions may be obtained by equating the right sides of Eq. (3) to zero. Making the substitution $\eta = -(2/U)(dv/dy)$ in the second equation of Eq. (3), and integrating, we obtain

$$v = \pm\left(\frac{U\nu}{2}\right)^{1/2}[C - \eta - \ln(1 + \eta)]^{1/2}, \tag{4}$$

where C is an arbitrary constant of integration to be determined by initial conditions. The expression on the right side of Eq. (4) has exactly two zeroes for a fixed C, say η_1 and η_2. These two values correspond to η at $y = 0$, π/m, where m is an integer. Integrating Eq. (4) to-and-fro between the two zeroes, one obtains solutions with no node, one node, two nodes, etc., as shown in Fig. 1.[8] Let us call these *turbulent solutions* T_I, T_{II}, T_{III}, etc. The shape of T_I between $y = 0$ and π is similar to that of T_{II} between $y = 0$ and $\pi/2$ or $y = \pi/2$ and π. This is obvious, since the same function is integrated.

Let us expand v into a sine series in y,

$$v = \sum_{n=1}^{\infty} \xi_n \sin ny,$$

then the boundary conditions are automatically satisfied, and Eq. (3) becomes

$$\frac{du}{dt} = P - \nu U - \frac{1}{2}\sum_{n=1}^{\infty}\xi_n^2$$
$$\frac{d\xi_n}{dt} = (U - n^2\nu)\xi_n = n\left(\frac{1}{2}\sum_{k=1}^{n-1}\xi_k\xi_{n-k} - \sum_{k=1}^{\infty}\xi_k\xi_{n+k}\right); \quad (n = 1, 2, \cdots). \tag{5}$$

It is easily seen that T_I is obtained by solving the right sides of Eq. (5) with ξ_i, $(i = 1, 2, \cdots)$ in general all nonzero; T_{II} is obtained by setting $\xi_{2k-1} = 0$, $\xi_{2k} \neq 0$, $(k = 1, 2, \cdots)$; T_{III} is obtained by setting $\xi_{3k-2} = \xi_{3k-1} = 0$, $\xi_{3k} \neq 0$, $(k = 1, 2, \cdots)$, and so on. In particular, in the neighborhood of $P = n^2\nu^2 (n = 1, 2, \cdots)$ the solution T_n is obtained by substituting $P = n^2\nu^2 + \epsilon p$, $\xi_{nk-i} = 0$, $(k = 1, 2, \cdots; i = 1, 2, \cdots n - 1)$; $\xi_{nk} = \epsilon^{k/2}\eta_{nk}$ $(k = 1, 2, \cdots)$ and $U = n^2\nu + \epsilon u$ into (5), and solving the resulting equations in successive powers

[6] K. M. Case, Progr. Theoret. Phys. Suppl. No. 37, 1 (1966).
[7] A factor of π is inserted to facilitate calculations.

[8] Burgers has shown in Ref. 5 that for a given finite P, only a finite number of solutions are allowable.

of ϵ. In fact, as we will see later, such a substitution allows us even to obtain the time-dependent solution at $P = \nu^2 + \epsilon p$. Owing to instability of the other solutions, such a perturbation cannot be applied to obtain the time-dependent solutions at the other values of P.

B. Truncation of the Sine Series

The exact forms of the turbulent solutions are not known. However, we may truncate the sine series, and hope that the result will be a good approximation at sufficiently low values of P. It will be shown later that in order to obtain the correct solution T_I in the neighborhood of $P = \nu^2$, at least two terms have to be taken (one term will not give the correct solution even to the first order). Similarly, $2n$ terms will give the stationary solution T_n correctly in the neighborhood of $P = n^2\nu^2$. This gives us a rough idea of how many terms we have to take for a particular value of P. In any case, the truncated equations are of interest by themselves, since they could be considered model equations individually. Let us note that all solutions, including the time-dependent ones, must be bounded as $t \to \infty$. This is easily seen from the energy equation[9] which is obtained by multiplying the equation for dU/dt by U, and those for $d\xi_i/dt$ $(i=1, 2, \cdots, n)$ by $\xi_i/2$, respectively, and adding all the resulting equations; the final equation is of the form:

$$\frac{d}{dt}\left(\frac{U^2}{2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\xi_i^2}{2}\right) = PU - \nu U^2 - \sum_{k=1}^{n}\frac{k^2}{2}\nu\xi_k^2. \quad (6)$$

The expression

$$\left(U^2 + \sum_{i=1}^{n}\frac{\xi_i^2}{2}\right)$$

may be considered as the square of the magnitude of the total velocity, v^2. Then Eq. (6) states that $\frac{1}{2}(dv^2/dt)$ is negative outside the ellipse

$$PU - \nu U^2 - \sum_{k=1}^{n}\frac{k^2}{2}\nu\xi_k^2 = 0$$

or

$$\left(\frac{U - (P/2\nu)}{P/2\nu}\right)^2 + \sum_{k=1}^{n}\frac{\xi_k^2}{(P/\sqrt{2}\,n\nu)^2} = 1.$$

It is obvious that all solutions will eventually go into the ellipse

$$U^2 + \sum_{i=1}^{n}\frac{\xi_i^2}{2} = \left(\frac{P}{\nu}\right)^2,$$

and hence are bounded.

[9] In Ref. 5, Eq. (7.3), Burgers has obtained the general untruncated energy equation.

TABLE I. Stability of L, T_I, and T_{II}.

P	L	T_I	T_{II}
$P < \nu^2$	stable	...	...
$P = \nu^2$	start to be unstable	both ($\pm$) coincide with L	...
$\nu^2 < P < 4\nu^2$	unstable	both solutions stable	...
$4\nu^2 < P < \frac{17}{2}\nu^2$	unstable	both solutions stable	both unstable
$P > \frac{17}{2}\nu^2$	unstable	...	T_{II-} is stable and coincides with T_I at $P = \frac{17}{2}\nu^2$. T_{II+} is unstable.

Detailed studies are carried out for the cases of two terms ξ_1, ξ_2; and three terms ξ_1, ξ_2, ξ_3.

C. Two-Term Case

When only ξ_1 and ξ_2 are taken, Eq. (5) becomes

$$\frac{dU}{dt} = P - \nu U - \tfrac{1}{2}\xi_1^2 - \tfrac{1}{2}\xi_2^2,$$
$$\frac{d\xi_1}{dt} = (U - \nu)\xi_1 + \xi_1\xi_2, \quad (7)$$
$$\frac{d\xi_2}{dt} = (U - 4\nu)\xi_2 - \xi_1^2.$$

The stationary solutions are:

(1) Laminar : $U = P/\nu$, $\xi_1 = \xi_2 = 0$;

(2) T_I : $U = \dfrac{2P + 3\nu^2}{5\nu}$,

$$\xi_1 = \pm\left(-\frac{(2P - 17\nu^2)(2P - 2\nu^2)}{25\nu^2}\right)^{1/2},$$

$$\xi_2 = -\frac{2P - 2\nu^2}{5\nu};$$

(3) T_{II} : $U = 4\nu$, $\xi_1 = 0$,

$$\xi_2 = \pm[2(P - 4\nu^2)]^{1/2}.$$

Note that at $P = \nu^2$, the laminar solution and T_I coincide, and at $P = \frac{17}{2}\nu^2$, T_I and one of T_{II} {namely, $\xi_2 = -[2(P - 4\nu^2)]^{1/2}$} coincide. These two points are critical points.

The stability of the solutions is found by setting the linearized small perturbation equations for Eq. (7) and by straightforward application of the Hurwitz criterion of stability for the resulting eigenvalue equations (i.e., the conditions for all eigenvalues to have negative real parts so that all solutions decay in time). The results are given in Table I.

Therefore, the stability changes exactly at the critical points.

Now the quasilinear perturbation theory is ap-

TABLE II. Asymptotic values of u, ξ_1, ξ_2 at $P = \nu^2 + \epsilon p$.

	$p < o$	$p > o$
u	$\frac{p}{\nu}$	$\frac{2p}{5\nu}$
ξ_1	0	$\pm\left[\frac{6p}{5}\right]^{1/2}$
ξ_2	0	$-\frac{2p}{5\nu}$

plied to Eq. (7). First of all, consider the lower critical point, $P = \nu^2$. Let

$$P = \nu^2 + \epsilon p, \quad U = \nu + \epsilon u,$$
$$\xi_1 = \epsilon^{1/2}\eta_1, \quad \xi_2 = \epsilon\eta_2.$$

The particular powers in ϵ for U, ξ_1, ξ_2 are suggested by the exact stationary solution T_I. Substituting these expressions in Eq. (7), we have

$$\begin{aligned} \frac{du}{dt} &= p - \nu u - \tfrac{1}{2}\eta_1^2 - \tfrac{1}{2}\epsilon\eta_2^2, \\ \frac{d\eta_1}{dt} &= \epsilon(u\eta_1 + \eta_2\eta_1), \\ \frac{d\eta_2}{dt} &= -3\nu\eta_2 - \eta_1^2 + \epsilon(u\eta_2). \end{aligned} \tag{8}$$

To zeroth order, disregarding terms of order ϵ, we have

$$\begin{aligned} u &= x_0 \exp(-\nu t) + \frac{p}{\nu} - \frac{x_1^2}{2\nu}, \\ \eta_1 &= x_1 = \text{const}, \\ \eta_2 &= x_2 \exp(-3\nu t) - \frac{x_1^2}{3\nu}. \end{aligned} \tag{8'}$$

Now let x_0, x_1, and x_2 slowly vary with time. After substituting Eq. (8′) in Eq. (8) and applying the barring operation,[6] which discards the appropriate terms that decay as $\exp(-\nu t)$ and $\exp(-3\nu t)$, we have

$$\begin{aligned} \frac{dx_0}{dt} &= \epsilon\left(\frac{p}{\nu^2} - \frac{8}{9}\frac{x_1^2}{\nu^2}\right)x_1^2 \exp(\nu t), \\ \frac{dx_1}{dt} &= \epsilon x_1\left(\frac{p}{\nu} - \frac{5}{6}\frac{x_1^2}{\nu}\right), \\ \frac{dx_2}{dt} &= \epsilon\left(\frac{p}{\nu} - \frac{7x_1^2}{6\nu}\right)\frac{x_1^2}{3\nu}\exp(3\nu t). \end{aligned} \tag{8''}$$

Solving for the second equation of Eq. (8″), we obtain

$$x_1 = \pm\left(\frac{x_1^2(0)p \exp[(2p/\nu)t]}{\frac{5}{6}x_1^2(0)\{\exp[(2p/\nu)t] - 1\} + p}\right)^{1/2}, \tag{9}$$

where $x_1(0) = x_1]_{t=0}$, and the sign of x_1 is the same as that of $x_1(0)$. This dependence on the initial conditions can be considered to be an extra degree of freedom analogous to that mentioned in Sec. I.

From Eq. (8′) and Eq. (9), we may readily find the limiting values of u, ξ_1, ξ_2 as $t \to \infty$. The limiting values are listed in Table II.

From this we clearly see that the solution approaches the laminar solution or T_I, around the critical point, depending on whether one decreases p or increases p. The sign depending on the initial condition is a sort of arbitrariness; and if P/ν^2 is considered to be the analog to R, the Reynolds number, then ξ_1, constituting the first order term of v, varies as $[R - R_{cr}^{(1)}]^{1/2}$, exactly analogous to the case described by Landau.

Now, let us consider the second critical point at $P = \frac{17}{2}\nu^2$: Let

$$P = \tfrac{17}{2}\nu^2 + \epsilon p, \qquad U = 4\nu + \epsilon u,$$
$$\xi_1 = \epsilon^{1/2}\eta_1, \qquad \xi_2 = -3\nu + \epsilon\eta_2,$$

then,

$$\begin{aligned} \frac{du}{dt} &= p - \nu u + 3\nu\eta_2 - \tfrac{1}{2}\eta_1^2 - \epsilon\frac{\eta_2^2}{2}, \\ \frac{d\eta_1}{dt} &= \epsilon(u\eta_1 + \eta_1\eta_2), \\ \frac{d\eta_2}{dt} &= -3\nu u - \eta_1^2 + \epsilon u\eta_2. \end{aligned} \tag{10}$$

By the transformation:

$$u = \frac{1 + (-35)^{1/2}}{6}x_0 + \frac{1 - (-35)^{1/2}}{6}x_2, \tag{11}$$
$$\eta_2 = x_0 + x_2,$$

Eq. (10) is put into canonical form, and to zeroth order, neglecting the ϵ terms,

$$\begin{aligned} x_0 &= \frac{6}{(-35)^{1/2}[1 + (-35)^{1/2}]\nu}\left(p - \frac{2 + (-35)^{1/2}}{6}x_1^2\right) + y_0 \exp\left(-\frac{1 + i(35)^{1/2}}{2}\nu t\right), \\ x_2 &= -\frac{6}{(-35)^{1/2}[1 - (-35)^{1/2}]\nu}\left(p - \frac{2 - (-35)^{1/2}}{6}x_1^2\right) + y_2 \exp\left(-\frac{1 - i(35)^{1/2}}{2}\nu t\right), \\ x_1 &= y_1. \end{aligned} \tag{12}$$

TABLE III. Asymptotic values of u, η_1, η_2 at $P = \frac{17}{2}\nu + \epsilon p$.

	$p < 0$	$p > 0$
u	$\frac{2p}{5\nu}$	0
η_1	$\pm\left[-\frac{6p}{5}\right]^{1/2}$	0
η_2	$-\frac{2p}{5\nu}$	$-\frac{p}{3\nu}$

Repeating the quasilinearizing procedure as before, we have the equation for dy_i/dt; in particular,

$$\frac{dy_1}{dt} = \epsilon \frac{y_1}{18\nu}(-6p - 5y_1^2). \quad (13)$$

The solution for Eq. (13) is

$$y_1 = -\left(\frac{6py_1^2(0)\exp[-(2p/3\nu)t]}{6p + 5y_1^2(0)[1 - \exp[-(2p/3\nu)t]]}\right)^{1/2}. \quad (14)$$

Again the asymptotic value of u, ξ_1, ξ_2 are readily found from Eqs. (11), (12), and (14) and are listed in Table III.

Hence, the solution approaches T_I or T_{II}, depending on whether one decreases or increases p.

D. Three-Term Case

The case when ξ_1, ξ_2, ξ_3 are taken is much more tedious. The procedure is the same: The stationary solutions are found, their stability investigated, the critical points are located, and finally the time-dependent perturbation theory is applied at the critical points. The results are stated briefly.

The equations are

$$\begin{aligned}\frac{dU}{dt} &= P - \nu U - \tfrac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2),\\ \frac{d\xi_1}{dt} &= (U - \nu)\xi_1 - (-\xi_1\xi_2 - \xi_2\xi_3),\\ \frac{d\xi_2}{dt} &= (U - 4\nu)\xi_2 - 2(\tfrac{1}{2}\xi_1^2 - \xi_1\xi_3),\\ \frac{d\xi_3}{dt} &= (U - 9\nu)\xi_3 - 3\xi_1\xi_2.\end{aligned} \quad (15)$$

The stationary solutions are:

(1) Laminar solution:

$$\xi_1 = \xi_2 = \xi_3 = 0,\ U = \frac{P}{\nu};$$

(2) $$T_I : \xi_1^2 = \frac{(U - 4\nu)(U - 9\nu)[-(U - 9\nu) \pm [(U - 9\nu)^2 - 12(U - \nu)(U - 9\nu)]^{1/2}}{12(U - 9\nu) \mp 6[(U - 9\nu)^2 - 12(U - \nu)(U - 9\nu)]^{1/2}},$$

$$\xi_2 = \frac{-(U - 9\nu) \pm [(U - 9\nu)^2 - 12(U - \nu)(U - 9\nu)]^{1/2}}{6},$$

$$\xi_3 = \frac{3\xi_1\xi_2}{U - 9\nu},$$

while U is related to P by the equation

$$(U - 9\nu)[67\nu U - (27\nu^2 + 24P)]^2 = (-11U + 3\nu)[25\nu U + (3\nu^2 - 12P)]^2. \quad (16)$$

The graph of Eq. (16) is shown in Fig. 2. The solutions with (+) sign in the numerator will be called T_{I+} and the solutions with (−) sign in the numerator will be called T_{I-}. T_{I-} exist in the range $\frac{13}{5}\nu > U > \nu$ and $9\nu > U > 4\nu$; but T_{I+} exist only for $9\nu > U > 4\nu$.

(3) $T_{II} : \xi_1 = \xi_3 = 0$, $U = 4\nu$,

$$\xi_2 = \pm[2(P - 4\nu^2)]^{1/2};$$

(4) $T_{III} : \xi_1 = \xi_2 = 0$, $U = 9\nu$,

$$\xi_3 = \pm[2(P - 9\nu^2)]^{1/2}.$$

The stability of the solutions is as shown in Table IV.

Again there are two critical points of interest:

(1) At $P = \nu^2$, the laminar solution agrees with the two T_{I-} that begin to exist. When P is decreased slightly, the solution approaches the laminar solution asymptotically, and when P is increased

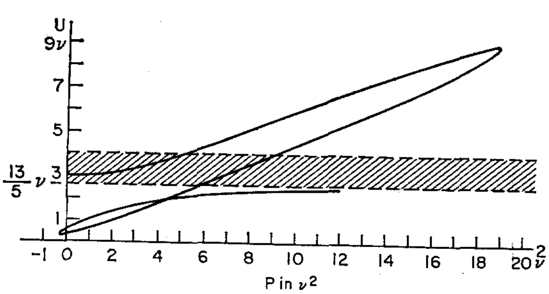

FIG. 2. Graph of U vs P for Eq. (16).

TABLE IV. Stability of solutions L, T_I, T_{II}, T_{III}.

	Laminar	T_I	T_{II}	T_{III}
$0 < P < \nu^2$	1 stable	...	...	...
$\nu^2 \leq P < \frac{17}{9}\nu^2$	1 unstable	(i) 2 stable T_{I-}; $L \to T_I$ for $P > \nu^2$	...	...
$\frac{17}{9}\nu^2 \leq P < 4.0\nu^2$	1 unstable	(i) 2 stable T_{I-}. (ii) 2 more stable T_{I-}.	... 2 unstable	... ...
$4.0\nu^2 < P < 4.3\nu^2$	1 unstable	as above		
$P \approx 4.3\nu^2$	1 unstable	(i) and (ii) coincide	2 unstable	...
$4.3\nu^2 < P < 5.2\nu^2$	1 unstable	(i) 2 stable T_{I-}. (ii) 2 stable T_{I-}.	2 unstable	...
$5.2\nu^2 < P < 5.9\nu^2$	1 unstable	(i) 2 T_{I-} (probably stable). (ii) 2 T_{I-} (become unstable somewhere). (iii) 4 more unstable solutions, 2 T_{I-} and 2 T_{I+}.	(i) one becomes stable, transition with T_I (iii) at $P = 5.2\nu^2$. (ii) one remains unstable	...
$5.9\nu^2 < P < 9\nu^2$	1 unstable	(i) 2 T_{I-} (stability unknown). (ii) no longer exist. (iii) 4 unstable.	(i) 1 stable (ii) 1 unstable	...
$9\nu^2 < P < 9.2\nu^2$	1 unstable	(i) 2 T_{I-} (stability unknown). (ii) 4 unstable	as above	2 unstable
$9.2\nu^2 < P < 19\nu^2$	1 unstable	(i) 2 T_{I-} (stability unknown) (iii) 4 unstable (iv) 4 more unstable solutions; coincide with (iii) at $P = 19\nu^2$.	as above as above	as above as above
$19\nu^2 < P < 19\nu^2 + \epsilon$	1 unstable	as above		
$19\nu^2 + \epsilon < P < \infty$	1 unstable	(i) 2 T_{I-} which begins to be unstable somewhere. (iii) and (iv) no longer exist.	as above	as above

slightly, the solution approaches T_{I-}. It should be noted that in the neighborhood of $P = \nu^2$, ξ_1 and ξ_2 agrees with those in (c) when only two terms ξ_1 and ξ_2 are taken. In fact, it can be easily verified that around $P = \nu^2 + \epsilon p$, the method of truncation gives T_I correctly for the ξ's taken.

(2) The situation is different at the second critical point

$$P = \frac{259 - 5(205)^{1/2}}{6}\nu^2 \approx 5.2\nu^2.$$

Here, a "T_I" agrees with a T_{II} at $U = 4\nu$,

$$\xi_1 = \xi_3 = 0,\ \xi_2 = \frac{5 - (205)^{1/2}}{6}\nu \approx 1.55\nu.$$

However, this T_I branches out into two unstable T_I, namely, T_{I-}, when P is increased, and T_I does not exist when P is decreased; while the T_{II-} is stable for $P > 5.2\nu^2$, but is unstable for $P < 5.2\nu^2$. So for $P \lesssim 5.2\nu^2$, the perturbation theory would clearly break down. If one applies the time-dependent perturbation theory at this point, the equation corresponding to Eq. (1) for the amplitudes has negative β and $\alpha \sim -p$. It is clear from Eq. (2) that when α is negative, i.e., when $p > 0$, $|A|^2 \to 0$, and the whole solution approaches T_{II}, but when α is positive, or $p < 0$, $|A|^2 \to \infty$ as

$$t \to \frac{1}{\alpha} \ln \frac{\alpha - \beta\,|A(0)|^2}{-\beta\,|A(0)|^2}.$$

But formally, as $t \to \infty$, $|A|^2 \to \alpha/\beta$ so that $|A|$ is imaginary. This shows that there is no stable solution in that neighborhood when P is slightly decreased. Since it was shown that the solutions are bounded, the asymptotic solution must be some function finitely different from the stationary solution at the critical point. This suggests the possibility of a transition to turbulence by a sudden finite change, in addition to continuous changes. That is, it is possible that an infinitesimal change of the Reynolds number may cause a finite change in the solution. In the future, we will call such a transition a finite jump.

E. The Untruncated Infinite Set of Equations

The perturbation method may be used on the untruncated system of equations (5) at least at one critical point, namely, $P = \nu^2$. For $P < \nu^2$, the laminar solution is stable. From previous experience, we set

$$P = \nu^2 + \epsilon p,\ U = \nu + \epsilon u,$$

$$\xi_1 = \epsilon^{i/2}\eta_i,\ I = 1, 2, 3 \cdots.$$

The equation for η_1 is just the same as that in the case where the sine series is truncated to two terms

ξ_1 and ξ_2. Once the asymptotic form of ξ_1 is found, the other ξ's may be successively deduced, and one readily sees that the solution approaches the laminar solution for $P \gtrsim \nu^2$, and approaches T_I for $P \lesssim \nu^2$.

Application of this perturbation method to further critical points is of interest. We may hope that there are continuous transitions to other turbulent solutions. The fact that there exists no such transition is partially revealed in the case when three terms are taken, and in general is easily seen from Fig. 1. For a continuous change from T_I to T_{II}, the node of the velocity distribution must go continuously from $y = \pi$ to $y = \pi/2$ (or from 0 to $\pi/2$), as P is increased. However, from the method of integration of Eq. (4), the node can only exist at $y = \pi/m$. Therefore, a continuous transition from T_I to T_{II} is impossible. This leads to the belief that all T_n, $n \geq 2$ are unstable.

A necessary condition for T_{II} to be stable is that the determinant

$$-\begin{vmatrix} -(U^{(0)}-\nu)-\xi_2^{(0)} & -(\xi_2^{(0)}+\xi_4^{(0)}) & -(\xi_4^{(0)}+\xi_6^{(0)}) & \cdots \\ 3\xi_2^{(0)} \quad 3\xi_4^{(0)} & -(U^{(0)}-9\nu)-3\xi_6^{(0)} & & \cdots \\ 5\xi_4^{(0)} - 5\xi_6^{(0)} & \cdots & & \cdots \\ \vdots \quad\quad \vdots & \vdots & & \vdots \end{vmatrix} > 0. \tag{17}$$

The determinant (17) is simply the determinant of the coefficients of the linearized perturbed equations of (5). The superscript (0) denotes stationary values. It is easily seen that the above determinant being equal to zero is also a sufficient condition for T_{II} coinciding with T_I. The fact that there exists no such transition means that the determinant is never zero, and it is readily shown that T_{II} is unstable at $P = 4\nu^2$, with (17) negative. By continuity, T_{II} is unstable.

If T_{II} is always unstable, all other higher T's cannot be stable, since they contain extra modes of excitation in addition to all those of T_{II}, which are those modes in T_{III}, for example, that keep $v(y = 0) = v(y = 2\pi/3) = 0$.

III. TWO-DIMENSIONAL BURGERS' EQUATIONS

A. The Untruncated Equations

The two-dimensional Burgers' equations are of more interest, since here one finds the possibility of periodic solutions, and the arbitrariness of phases as suggested in Ref. 2. But here one again finds the additional possibility of a finite jump at a critical point. The equations are

$$\frac{dU}{dt} = P - \nu U - \frac{1}{\pi}\int_0^\pi dy\,(v^2 + \omega^2),$$

$$\frac{\partial v}{\partial t} = U(v - \omega) + \nu\frac{\partial^2 v}{\partial y^2} - \frac{\partial}{\partial y}(v^2 - \omega^2), \tag{18}$$

$$\frac{\partial \omega}{\partial t} = U(v + \omega) + \nu\frac{\partial^2 \omega}{\partial y^2} + \frac{\partial}{\partial y}(2\,v\omega),$$

subjected to the boundary conditions

$$v(y = 0) = v(y = \pi)$$
$$= \omega(y = 0) = \omega(y = \pi) = 0.$$

Making the substitution $V = v + i\omega$, and representing V by a Fourier sine series in y,

$$V = \sum_{n=1}^{\infty} \xi_n \sin ny,$$

where the ξ_n's are complex, Eqs. (18) are transformed into

$$\frac{dU}{dt} = P - \nu U - \frac{1}{2}\sum_{n=1}^{\infty}\xi_n\xi_n^*$$

$$\frac{d\xi_n}{dt} = [(1 + i)U - n^2\nu]\xi_n$$
$$+ n\sum_{k=1}^{\infty}\xi_k^*\xi_{n+k}^* - \frac{n}{2}\sum_{k=1}^{n-1}\xi_k^*\xi_{n-k}^*, \tag{19}$$

where * denotes complex conjugation. Again there exists a trivial solution

$$U = P/\nu, \quad \xi_i = 0$$

(laminar solution). This solution is stable for $P < \nu^2$. The critical point is $P = \nu^2$. It is tempting to apply a time-dependent perturbation method near this point. Making the substitution

$$P = \nu^2 + \epsilon p,\; U = \nu + \epsilon u,$$
$$\xi_n = \epsilon^{n/2}\eta_n,\; n = 1, 2, \cdots,$$

and after cancelling appropriate powers of ϵ, and then keeping terms up to order ϵ, the following equations are obtained:

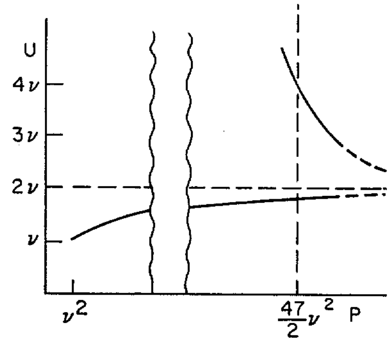

FIG. 3. Graph of U vs P for Eq. (26).

$$\frac{du}{dt} = p - \nu u - \tfrac{1}{2}\eta_1^*\eta_1 - \epsilon\tfrac{1}{2}\eta_2^*\eta_2,$$

$$\frac{d\eta_1}{dt} = i\nu\eta_1 + \epsilon[(1+i)u\eta_1 + \eta_1^*\eta_2^*] + O(\epsilon^2), \tag{20}$$

$$\frac{d\eta_n}{dt} = [(1+i)\nu - n^2\nu]\eta_n - \frac{n}{2}\sum_{k=1}\eta_{n-k}^*\eta_k^* + \epsilon[(1+i)u\eta_n + n\eta_{n+1}^*\eta_1^*] + O(\epsilon^2); n = 2, 3 \cdots.$$

To zeroth order,

$$u = \rho_0 e^{-\nu t} + \frac{1}{\nu}\left(p - \frac{|\rho_1|^2}{2}\right),$$

$$\eta_1 = \rho_1 \exp(i\nu t), \tag{21}$$

$$\eta_2 = \rho_2 \exp[(-3+i)\nu t] - \frac{(1+i)\rho_1^{*2}}{6\nu}\exp(-i2\nu t)$$

for $n = 2, 3, \cdots$; η_n is readily found once we find η_i for $i = 1, 2, \cdots (n-1)$; in particular, when

$$t \to \infty,\ \eta_n \sim \sum_{k=-n}^{+n} A_k(\rho_1, \cdots, \rho_{n-1}) \exp(ik\nu t).$$

Substituting Eq. (21) in the second equation of (20), letting the ρ's slowly vary with time, we have

$$\frac{d\rho_1}{dt} = \epsilon\frac{\rho_1}{\nu}\left((1+i)p - \frac{2+i}{3}|\rho_1|^2\right). \tag{22}$$

Multiplying Eq. (22) by ρ_1^*, and the corresponding equation for $d\rho_1^*/dt$ by ρ_1, and adding, we get

$$\frac{d|\rho_1|^2}{dt} = \epsilon\frac{2|\rho_1|^2}{\nu}(p - \tfrac{2}{3}|\rho_1|^2). \tag{23}$$

Solving Eq. (23),

$$|\rho_1| = \left(\frac{p|\rho_1(0)|^2 \exp[(2p/\nu)t]}{p + \frac{2}{3}|\rho_1(0)|^2\{\exp[(2p/\nu)t] - 1\}}\right)^{1/2}, \tag{24}$$

where $|\rho_1(0)|$ is $[|\rho_1|]_{t=0}$. Note that the phase of ρ_1 is arbitrary, and that the asymptotic value of $|\rho_1|$ is independent of $|\rho_1(0)|$; if $p < 0$, $|\rho_1|_{t\to\infty} = 0$. From the zeroth-order solutions for η_n, and Eq. (24), the asymptotic values of $|\rho_n|$ and u can be found. Thus, one reaches the interesting result that as $P > \nu^2$, the solution approaches a *periodic turbulent solution*; and the amplitude of oscillation is proportional to $(p)^{1/2}$ the analogous difference of the actual and the critical Reynolds number. The periodic solution has one arbitrary phase, as mentioned. Note that the periodic solution is found without previous knowledge of its existence.

B. The Truncated Series

Only the case of two terms is considered. The equations are

$$\frac{dU}{dt} = P - \nu U - \tfrac{1}{2}(\xi_1^*\xi_1 + \xi_2^*\xi_2),$$

$$\frac{d\xi_1}{dt} = [(1+i)U - \nu]\xi_1 + \xi_1^*\xi_2^*, \tag{25}$$

$$\frac{d\xi_2}{dt} = [(1+i)U - 4\nu]\xi_2 - \xi_1^{*2}.$$

The following solutions are obtained:

(i) laminar solution: $U = P/\nu$, $\xi_1 = \xi_2 = 0$;
(ii) first periodic turbulent solution T_I

$$\xi_1 = \rho_1 \exp[i(\omega t + \varphi_1)] \quad \text{where} \quad \omega = \frac{-U\nu}{U - 2\nu},$$

$$\rho_1^2 = -(U-\nu)(U-4\nu)\left(1 + \frac{U^2}{(U-2\nu)^2}\right),$$

$$\xi_2 = \rho_2 \exp[i(-2\omega t + \varphi_2)],$$

$$\rho_2^2 = (U-\nu)^2\left(1 + \frac{U^2}{(U-2\nu)^2}\right),$$

$$\tan(2\varphi_1 + \varphi_2) = -\frac{U}{U - 2\nu},$$

so that there is only one arbitrary phase, and U is related to P by the expression

$$P - \nu U - \tfrac{3}{2}\nu(U-\nu)\left(1 + \frac{U^2}{(U-2\nu)^2}\right) = 0. \tag{26}$$

The graph of Eq. (26) is roughly shown in Fig. 3.

(iii) Second periodic turbulent solution T_{II}:

$$U = 4\nu,$$

$$\xi_1 = 0,$$

$$\xi_2 = \rho_2 \exp[i(4\nu t + \varphi_2)],$$

where

$$|\rho_2| = [2(P - 4\nu^2)]^{1/2}.$$

TABLE V. Stability of L, T_I, T_{II}.

U	Laminar	T_I	T_{II}
$0 < U < \nu$	stable	...	...
$\nu < U \lesssim 2.533\nu$	unstable	stable (continuous transition from L at $U = \nu$)	...
$2.533\nu \lesssim U < 4\nu$	unstable	unstable	...
$4\nu < U$	unstable	...	exist but unstable

The stability of the solutions is shown in Table V. Two critical points are present: $U = \nu$ and $U \cong 2.533\nu$. The time-dependent development at $U = \nu$ is just as before. The time-dependent calculation for $U \sim 2.533\nu$ is briefly stated. Let

$$P = P_0 + \epsilon p, \text{ [where } P_0$$

and $U^{(0)} = 2.533\nu$ are related by Eq. (26)],

$$u \cong 2.533\nu + \epsilon u, \omega_1 \cong -4.61\nu,$$

$$\rho_1 \approx 7.081\nu + \epsilon\eta_1,$$

$$\rho_2 \approx 7.336\nu + \epsilon\eta_2,$$

$$2\varphi_1 + \varphi_2 = (2\varphi_{10} + \varphi_{20}) + \epsilon x \qquad \text{where } \tan(2\varphi_{10} + \varphi_{20}) \approx -4.61;$$

substituting these into Eqs. (25), the equations for du/dt, $d\eta_1/dt$, $d\eta_2/dt$, and dx/dt are obtained. These equations are put into canonical form by the transformation

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -0.445 & 0.456 - i0.325 & -0.380 - i0.170 & i0.561 \\ -0.445 & 0.456 + i0.325 & -0.380 + i0.170 & -i0.561 \\ -0.0876 & -0.357 - i0.107 & 0.352 - i0.0961 & 0.0716 - i0.846 \\ -0.0876 & -0.357 + i0.107 & 0.352 + i0.0961 & 0.0716 + i0.846 \end{pmatrix} \begin{pmatrix} u \\ \eta_1 \\ \eta_2 \\ x \end{pmatrix}. \tag{27}$$

After a straightforward but tedious calculation, we have

$$\frac{dy_0}{dt} = i7.96\nu y_0 - 0.445p + \epsilon(a_{ij}y_iy_j), \qquad i, j = 0, 1, 2, 3, \tag{28}$$

where a_{ij} are complex numbers,

$$\frac{dy_2}{dt} = (-0.394\nu + i16.7\nu)y_2 - 0.0876p + O(\epsilon);$$

dy_1/dt and dy_3/dt are just the complex conjugates of dy_0/dt and dy_2/dt, respectively. Thus, to zeroth order,

$$y_0 = x_0 \exp(i7.96\nu t) + \frac{0.445p}{i7.96\nu},$$
$$y_2 = x_2 \exp[(-0.394\nu + i16.7\nu)t] + \frac{0.0876p}{(-0.394 + i16.69)\nu}. \tag{29}$$

After quasilinearization of Eqs. (28), we have

$$\frac{dx_0}{dt} = \epsilon(-0.00287 - i0.117)\frac{p}{\nu}x_0, \tag{30}$$

and a similar equation for x_2. Thus, for $p < 0$, x_0 grows exponentially, while for $p > 0$, $x_0 \to 0$ as $t \to \infty$, so that $y_0 \to 0.445p/i7.96\nu$ and the solution approaches T_I as expected. Hence, there is again the possibility of a finite jump. Note that the particular form of Eq. (30) is a result of the assumption that $\rho_i = \rho_i^{(0)} + \epsilon\eta_i$. Hence, if T_I becomes unstable at nonzero values of ξ_i, $i = 1, 2, 3 \cdots$, then perturbation will lead to equations of the type (30) and a finite jump will occur at the corresponding critical point. (Actually, we have another unmentioned critical point similar in nature to this in the one-dimensional case, with the series truncated to 3-terms. Referring to Table IV and Fig. 2, clearly there is such a critical point for T_{I-} between $13\nu/5 > U > \nu$, since T_{I-} is unstable near $U \lesssim 13\nu/5$. Further, one can easily show without truncation of the infinite series that for the one-dimensional case, if T_I is unstable beyond a certain P_e where all the Fourier coefficients ξ_{ie} are nonzero, then a finite jump will occur.) It is also interesting to note that the transition from the laminar solution to T_I involves the increase of a spatial degree of freedom (namely, the extra y dependence), while at the second critical point, no extra spatial degree of freedom is available, and the solution could only be stabilized by a finite jump.

IV. SUMMARY

The time-dependent perturbation method works well whenever there are stable solutions in the neighborhood of the critical points; the failure of the method indicates the absence of stable solutions near these points.

For the Burgers' one-dimensional equations, there is a continuous transition from the "laminar" to the turbulent solutions at the lowest critical point $P = \nu^2$. But it is believed that the transition at the second critical point is a finite jump. The eventual instability of "T_I" remains to be proved in order to confirm this finite jump, in general.

For the Burgers' two-dimensional equations, a periodic solution is obtained with an arbitrary phase. The "laminar" solution transits into the periodic solutions continuously at $P = \nu^2$. The second transition is again believed to be a finite jump.

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