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ON EIGHT DIMENSIONAL QUASISPIN

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ABSTRACT

The present work formulates the exact solution of the pairing problem in the Ω(spatial)-ST scheme in terms of the matrix elements of the pair creation and annihilation operators coupled to zero spatial angular momentum quantum number. This makes it possible to study the pairing interaction with different strengths for the S=0 (T=1) and S=1 (T=0) pairs, as well as for mixed configurations of several single particle levels. The mathematical formulation of the problem has involved the study of an orthogonal group in eight dimensions, the so-called quasispin group. The representations are broken down according to $O(8) \supset O(6) \sim SU(4) \supset SU(2) \times SU(2)$, where SU(4) is the usual Wigner supermultiplet. In addition, the basis is chosen such that the number operator of the system is diagonal. By using the Wigner-Eckart theorem, the matrix elements of the pair operators are split into segments of reduced matrix elements. They are those connecting O(8) with SU(4) and those connecting SU(4) with the SU(2)-spin space and SU(2)-isospin space groups. The Wigner coefficients involved in these segments are calculated and tabulated for the representations needed for seniority v=0 and v=1. In order to do this, the state functions built from O(8) to O(6) and SU(4) to SU(2) X SU(2) are studied. With this general formulation a sample calculation has been carried out for the case of a pairing interaction of variable strength connecting two single particle levels similar to those of the s-d shell. The calculation is done for nucleon numbers of 4, 6, 8, and 10 and states with seniority v=0. The results show that (1) the pairing interaction is very effective compared with the particle energy and wins out in the competition with the single particle excitations for all but the weakest pairing strengths, and (2) the pairing interaction tends to make more stable those states built from the largest possible number of α-like groupings of four particles.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Concluded)

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

The quasispin method was first applied to problems in nuclear physics by Kerman[1] in his treatment of the pairing interaction. Quasispin operators are built from pair creation and annihilation operators involving nucleon pairs coupled to zero angular momentum. In the j-j coupling scheme of the shell model, involving configurations of identical nucleons (neutrons only or protons only), the quasispin operators have the commutation properties of conventional (three-dimensional) angular momentum operators. Generalization of the quasispin method to configurations with both neutrons and protons leads to more complicated operators and requires more complicated mathematical tools.

Conventional nuclear spectroscopy done in the spirit of Racah is based on group chains starting with the unitary groups in $(2j+1)$ (or $2\ell+1$) dimensions and involving the sympletic (or orthogonal) subgroups in these dimensions. The infinitesimal operators which generate these groups can be built from operators which preserve the total nucleon number. As a result the spectroscopic problem for each configuration must be solved separately for each nucleon number. An alternate group chain discovered more recently be Helmers[2] is based on the direct product of the symplectic (or orthogonal) group with the appropriate quasispin group; which leads to two parallel subgroup chains (1) that based on the symplectic (or orthogonal) group which contains the quantum numbers associated with the space variables, and (2) that based on the quasispin group in which the more trivial quantum numbers such as the total nucleon number N and the isospin T are associated with the lowest subgroups of the chain. It is the latter group chain which includes among its infinitesimal operators the pair-creation and annihilation operators and is therefore most directly applicable to the study of a nuclear pairing interaction.

For configurations of both protons and neutrons there are two basic coupling schemes, namely the JT and LST schemes. The quasispin groups appropriate to the two schemes have been identified by Flowers and Szpikowski[3] and others as rotational groups in abstract spaces of 5 and 8 dimensions, respectively. For the J-T scheme, Ginocchio[5] has calculated Wigner coefficients involving the four dimensional (spinor) representation of O(5) to extract the N-T dependence of the fractional parentage coefficients in the seniority scheme; and Hecht[6] has extensively worked out the exact solutions of the pairing Hamiltonian and provided the general algebraic expressions of the matrix elements of the infinitesimal operators of O(5) for states with reduced isospin $t = 0$, $1/2$, and 1, in a scheme in which both nucleon number N and isospin T are good quantum numbers. Hecht[7] has also used the quasispin technique in the study of the N-T dependence of one- and two-body operators in the seniority scheme. In the LST scheme, there are six pair creation operators coupled to $L = 0$ (with $S = 1$, $T = 0$ or $S = 0$, $T = 1$) and a similar set of six pair annihilation operators. Flowers and Szpikowski have shown that these 12 operators together with the number operators and the 15 operators which are the SU(4) super multiplet operators of Wigner form the infinitesimal generators for a orthogonal group in eight dimensions. They have also evaluated the eigenvalue of the pairing Hamiltonian for the pure configuration ℓ^N in terms of the Casimir operators of O(8) and SU(4).

The present work formulates exact solutions of the pairing Hamiltonian in the LST scheme in terms of the matrix elements of the L = 0 pair creation and annihilation operators. This makes it possible to study the pairing interaction with <u>different</u> strengths for the S = 1 (T = 0) and S = 0 (T = 1) pairs, as well as for mixed configurations of several single particle levels. It is thus possible to study the competition between pairing effects and single particle excitations. Although the LST scheme may be a good zeroth approximation for light nuclei, a two body interaction approximated by a simple pairing interaction is not sufficient to describe the excitation spectra and binding energies of such nuclei. The present work is therefore intended mainly as a model study to further elucidate the properties of pairing interactions.

In the present work the group chain of $O(8) \supset O(7) \supset O(6)$ is used, where O(6) can be identified with SU(4) and the representations of the Wigner super multiplets, while the four numbers needed to specify the irreducible representations of the rank 4 group O(8) can be identified with the quantum numbers of the seniority scheme. Although the irreducible representation labels of both O(8) and O(6) thus have ready physical significance the quantum numbers of O(7) have no easily identifed physical meaning. Even worse, the nucleon number operator is in general not diagonal in a scheme based on the group chain $O(8) \supset O(7) \supset O(6)$. To make the nucleon number, N, a good quantum number it is necessary to find specific linear combinations of the O(7) representations allowed by the O(8) and O(6) quantum numbers. A similar problem occurs when the Wigner supermultiplet representations of O(6) are further reduced to S and T. In the canonical group chain $O(6) \supset O(5) \supset O(3) \supset O(2)$ only one of the quantun numbers, <u>either</u> S <u>or</u> T, can be identified with the irreducible representation of O(3). Because of these difficulties it has not beenpossible to give a completely general algebraic expression for the matrix elements of the pair operators, valid for all irreducible representations. However, if the seniority number v is restricted to 0 or 1 the single quantum number N is sufficient to completely specify the states of the O(8) to O(6) chain, while the states of the possible O(6) representations for these cases are fully identified by S and T only. Since the seniority v gives the number of unpaired nucleons (entirely free of L = 0 coupled pairs), states of lowest seniority such as v = 0 and v = 1 are precisely those of greatest interest in problems dominated by a pairing interaction.

By using the Wigner Eckart theorem the matrix elements of the pair creation and amihilation operators can be split into a few generalized Wigner coefficients corresponding to the physically significant segments of the above group chains. The whole set of Wigner coefficients needed for v = 0 and v = 1 are worked out in the following chapters. This makes it possible to write the matrix elements of a charge independent pairing Hamiltonian in general algebraic form for these cases.

A few numerical examples are worked out for the v = 0 cases involving two single particle levels to demonstrate the competition between pairing effect and single particle excitations.

Since the matrix elements of the infinitesimal operators for the chain $O(8) \supset O(7) \supset O(6)$ are needed for this investigations, a full discussion of the general group chain $O(n) \supset O(n-1) \supset \ldots$ and the matrix elements of its infinitesmal operators is included as a supplement at the end.

While this work was in progress an investigation by Richardson has appeared[8] dealing with the problem of finding solutions for a charge spin independent pairing Hamiltonian. However, Richardson's approach is very different from the present one. It requires the solution of a system of coupled algebraic equations with subsidiary conditions. Furthermore, at the moment it is restricted to the special case of seniority zero. With general expressions for the matrix elements of the pair operators, the approach used in this investigation, it is easier to consider perturbation treatments for the weak or strong pairing limits for $S = 0$ or $S = 1$ pairs.

CHAPTER II

THE QUASISPIN FORMALISM

A. THE THREE-DIMENSIONAL QUASISPIN

The quasispin formalism was applied by Kerman to a pure configuration of N identical particles j^N with the pairing Hamiltonian

$$H = -G \sum_{m',m>0} (-)^{2j-m-m'} a^+_{jm}\, a^+_{j-m}\, a_{j-m'}\, a_{jm'} \qquad (2.A.1)$$

Where Condon and Shortley phase conventions for the second quantization fermion operator of creation a^+_{jm} and amnihilation a_{jm} are used.

The seniority v state functions which diagonalize H are of the following general form

$$A_+ A_+ \ldots A_+ a^+_{jm_1} a^+_{jm_2} \ldots a^+_{jm_v} \mid 0 > \qquad (2.A.2)$$

Where A_+ is a pair creation operator which creates a pair of nucleons coupled to total angular momentum zero (see (2.A.3) below); and all the other v single nucleon creation operators can not be coupled into pairs of total angular momentum zero. (If the set m values $m_1, m_2, \ldots, m_v$ include the possibility $m_\alpha = -m_\beta$, it is necessary to take linear combinations of the state functions of the above form such that the resultant function is free of pair coupled to angular momentum zero.)

The number of A_+ in a semiority v state can be 0,1,2,... up to $1/2(2j+1-2v) = j + 1/2 - v$.

The set of pairing operators

$$A_+ = \sum_{m>0} (-)^{j-m} a^+_{jm} a^+_{j-m}$$

$$A_- = \sum_{m>0} (-)^{j-m} a_{j-m} a_{jm}$$

$$A_0 = \frac{1}{2} \sum_{m>0} (a^+_{jm} a_{jm} + a^+_{j-m} a_{j-m} - 1) \qquad (2.A.3)$$

satisfy the commutation relations of a three dimensional orthogonal group

$$[A_+, A_-] = 2A_0$$

$$[A_0, A_+] = A_+$$

$$[A_0, A_-] = -A_- \qquad (2.A.4)$$

with these relations, the pairing Hamiltonian can be written as

$$H = -GA_+A_- = -G(\underline{A}^2 - A_0^2 + A_0) \qquad (2.A.5)$$

The eigenvalue of A_0 is related to the number of particles, N, in the configuration j^N; the eigenvalue $A(A+1)$ of $\underline{A}^2$ is related to the maximum possible value of A_0 for the seniority v.

$$A_0 = \frac{1}{2}(N - j - \frac{1}{2})$$

$$A = \frac{1}{2}(N_{max} - j - \frac{1}{2}) = \frac{1}{2}(j + \frac{1}{2} - v) \qquad (2.A.6)$$

Putting (2.A.6) into (2.A.5), the energy of interaction is

$$E = -\frac{1}{4}G(N - v)(2j + 3 - N - v) \qquad (2.A.7)$$

B. THE EIGHT-DIMENSIONAL QUASISPIN GROUP

There are two ways of extending the above quasispin operators for configurations of both neutrons and protons, that is for particles with spatial, spin, and isospin variables. In one scheme the single nucleon orbital and spin angular momenta are coupled to j, and the single nucleon states are labeled by $n\ell sjm_j$, tm_t. (The isospin quantum numbers are $t = 1/2$, and in the common nuclear physics convention $m_t = 1/2$ for the neutron, $m_t = -1/2$ for the proton.) In this scheme the quasispin operators are based on the pair creation operator which create pairs of nucleons coupled to $J = 0$ and $T = 1$. These lead to a family of 10 operators, the generalization of (2.A.3); they are the infinitesimal operators which generate a group O(5). In the other scheme, the one of interest in this investigation, the single nucleon state, are labeled by $n\ell m_\ell$, sm_s, tm_t. The quasispin operators are now built from pair creation operators which create pairs coupled to total orbital angular momentum $L = 0$. This leads to a family of 28 operators, and it has been shown by Flowers and Szpikowski that these generate a group O(8). The properties of these operators is reviewed in the following paragraphs based mainly on their paper.[3]

B.1. The 28 Quasispin Operators, Their Origin

The 28 quasispin operators can be classified as follows:

(a) Pair creation operators: There are six types of pair creation operators coupled to total $L = 0$. They are those which

(1) Create a spin up protron and a spin down proton.

$$Q_+^{p-p} = \sum_m (-)^{\ell-m} a^+_{\ell m\uparrow p} a^+_{\ell -m\downarrow p}$$

(2) Create a spin up proton and spin up neutron

$$Q_+^{pn} = \sum_m (-)^{\ell-m} a^+_{\ell m\uparrow p} a^+_{\ell -m\uparrow n}$$

(3) Create a spin up proton and a spin down neutron

$$Q_+^{p-n} = \sum_m (-)^{\ell-m} a^+_{\ell m\uparrow p} a^+_{\ell -m\downarrow n}$$

(4) Create a spin up neutron and a spin down neutron

$$Q_+^{n-n} = \sum_m (-)^{\ell-m} a^+_{\ell m\uparrow n} a^+_{\ell -m\downarrow n}$$

(5) Create a spin down proton and a spin up neutron

$$Q_+^{-pn} = \sum_m (-)^{\ell-m} a^+_{\ell m\downarrow p} a^+_{\ell -m\uparrow n}$$

(6) Create a spin down proton and a spin down neutron

$$Q_+^{-p-n} = \sum_m (-)^{\ell-m} a^+_{\ell m\downarrow p} a^+_{\ell -m\downarrow n}$$

Due to the anticommutation properties of the nucleon creation operators these are the only six independent types all the others like $Q^{\alpha\alpha} = 0$ and $Q^{\alpha\beta} = -Q^{\alpha\beta}$ when $\alpha \neq \beta$. The $(-)^{\ell-m}$ appears through the angular momentum coupling coefficient. In the Condon and Shortley phase convention:

$$\langle \ell m \ell -m | 00 \rangle = (-)^{\ell-m} \frac{1}{\sqrt{2\ell+1)}}$$

(b) Pair annihilation operators: There are six types of pair annihilation operators corresponding to the six types of pair creation operators. They are defined as their Hermitian conjugates

$$Q_-^{\alpha\beta} = (Q_+^{\beta\alpha})^+ \qquad (2.B.1)$$

(c) Number-conserving operators: The commutations of pair creation and pair annihilation operators lead to operators of the general type $Q_o = \sum a^+a$ which do not change the number of particles in a system. For ready physical interpretation they are divided into four sets:

(1) Number operators

$$Q_o^{pp}, \; Q_o^{nn}, \; Q_o^{-p-p}, \; Q_o^{-n-n}$$

(2) Spin-exchange operators

$$Q_o^{p-p}, \; Q_o^{n-n}, \; Q_o^{-pp}, \; Q_o^{-nn}$$

(3) Change-exchange operators

$$Q_o^{pn}, \; Q_o^{np}, \; Q_o^{-p-n}, \; Q_o^{-n-p}$$

(4) Space-exchange operators

$$Q_o^{p-n}, \; Q_o^{n-p}, \; Q_o^{-pn}, \; Q_o^{-np}$$

For example

$$Q_0^{p-n} = \sum_m a^+_{\ell m\uparrow p}\, a^+_{\ell m\downarrow n}$$

Note that $(-)^{\ell-m}$ does not appear, because $a_{\ell m\ldots}$ has the same transformation properties in the angular momentum space as $(-)^{\ell-m}|\ell\text{-}m\rangle$.

B.2. The 28 Quasispin Operators, Their Relations To The Infinitesimal Generators of O(8).

(a) Reclassification of the quasispin operator: The above 28 quasispin operators which have been presented according to their simple physical meaning will be regrouped into more convenient form.

The six pair creation operators are rewritten as

$$^{2S+1,\ 2T+1}A^+(M_s,M_T) = \sqrt{\ell+\tfrac{1}{2}}\sum_{m,m_t,m_s} \langle \ell m\ \ell\text{-}m|00\rangle \langle \tfrac{1}{2}m_t\tfrac{1}{2}m_t'|TM_T\rangle \times$$

$$\langle \tfrac{1}{2}\,m_s\,\tfrac{1}{2}\,m_s'\,|SM_s\rangle \times a^+_{\ell m m_t m_s}\, a^+_{\ell\text{-}m\ m_t'\ m_s'} \qquad (2.B.2)$$

Through the restrictions of the Pauli exclusion principle, the pair creation operators can be coupled either to total spin S = 1 and total isospin T = 0; or to total spin S = 0 and total isospin T = 1. They are

$$^{13}A^+(0,M_T) \qquad M_T = -1,\ 0,\ 1$$

and

$$^{31}A^+(M_s,0) \qquad M_s = -1,\ 0,\ 1$$

Their relations to the Q_+ operators are

$$^{13}A^{+}(0,1) = Q_{+}^{n-n} \qquad ^{31}A(1,0) = Q_{+}^{np}$$

$$^{13}A^{+}(0,0) = \frac{1}{\sqrt{2}}(Q_{+}^{p-n} + Q_{+}^{n-p}) \qquad ^{31}A(0,0) = \frac{1}{\sqrt{2}}(Q_{+}^{n-p} + Q_{+}^{-np})$$

$$^{13}A\ (0,-1) = Q_{+}^{p-p} \qquad ^{31}A(-1,0) = Q_{+}^{-n-p} \tag{2.B.3}$$

Similiarly, the six annihilation operators are

$$^{13}A(0,1) = Q_{-}^{-nn} \qquad ^{31}A(1,0) = Q_{-}^{pn}$$

$$^{13}A(0,0) = \frac{1}{\sqrt{2}}(Q_{-}^{-np} + Q_{-}^{-pn}) \qquad ^{31}A(0,0) = \frac{1}{\sqrt{2}}(Q_{-}^{-pn} + Q_{-}^{p-n})$$

$$^{13}A(0,-1) = Q_{-}^{-pp} \qquad ^{31}A(-1,0) = Q_{-}^{-p-n} \tag{2.B.4}$$

The 4 number operators, Q_o can be grouped into

$$Q_o = -\frac{1}{2}(Q_o^{nn} + Q_o^{-n-n} + Q_o^{pp} + Q_o^{-p-p}) + \Omega = -\frac{1}{2}N_{op} + \Omega$$

$$S_o = \frac{1}{2}(Q_o^{nn} + Q_o^{pp} - Q_o^{-n-n} - Q_o^{-p-p})$$

$$T_o = \frac{1}{2}(Q_o^{nn} + Q_o^{-n-n} - Q_o^{pp} - Q_o^{-p-p})$$

$$E_{oo} = \frac{1}{2}(Q_o^{nn} + Q_o^{-p-p} - Q_o^{pp} - Q_o^{-n-n}) \tag{2.B.5}$$

where N_{op} is the number operator which gives the total number of nucleons; and Ω is the spatial degeneracy of the single level, it is, for example, $2\ell+1$ if the spatical scheme is the ordinary orbital angular momentum scheme.

The 12 Q_o exchange operators can be grouped into

$$S_+ = \frac{1}{\sqrt{2}}(Q_o^{n-n} + Q_o^{p-p}) \qquad S_- = \frac{1}{\sqrt{2}}(Q_o^{-nn} + Q_o^{-pp})$$

$$T_+ = \frac{1}{\sqrt{2}}(Q_o^{np} + Q_o^{-n-p}) \qquad T_- = \frac{1}{\sqrt{2}}(Q_o^{pn} + Q_o^{-p-n})$$

$$E_{10} = \frac{1}{\sqrt{2}}(Q_o^{n-n} - Q_o^{p-p}) \qquad E_{-10} = \frac{1}{\sqrt{2}}(Q_o^{-nn} - Q_o^{-pp})$$

$$E_{01} = \frac{1}{\sqrt{2}}(Q_o^{np} - Q_o^{-n-p}) \qquad E_{0-1} = \frac{1}{\sqrt{2}}(Q_o^{pn} - Q_o^{-p-n})$$

$$E_{1-1} = Q_o^{p-n} \qquad E_{-11} = Q_o^{-np}$$

$$E_{11} = Q_o^{n-p} \qquad E_{1-1} = Q^{-p-n} \qquad (2.B.6)$$

where $E_{\alpha\beta}$ corresponds to an operator which steps up the m_s to $m_s+\alpha$ and m_t to $m_t+\beta$.

(b) Identification of the quasispin group: The group O(8) is generated by 28 infinitesmal Cartesian orthogonal generaters.

$$J_{mn} = -i\left(x_m\frac{\partial}{\partial x_n} - x_n\frac{\partial}{\partial x_m}\right) \qquad (2.B.7)$$

where $m,n = 1,2,\ldots 8$ and $m < n$

The commutation relations are

$$[J_{pq}, J_{rs}] = I(\delta_{sp}J_{rq} + \delta_{rq}J_{sp} - \delta_{rp}J_{sq} - \delta_{sq}J_{rp}) \qquad (2.B.8)$$

Since Q_o, T_o, E_{oo}, S_o form a set of mutually commuting operators they can be identified with the set of operators J_{78}, J_{56}, J_{34}, J_{12}. From the commutation table given by Flowers and Szpikowski, the remaining operators are identified in terms of the operators J_{mn} as follows:

$$S_+ = \frac{1}{\sqrt{2}}(J_{13}+iJ_{23}) \qquad S_- = \frac{1}{\sqrt{2}}(J_{13}-iJ_{23})$$

$$T_+ = \frac{1}{\sqrt{2}}(J_{45}+iJ_{46}) \qquad T_- = \frac{1}{\sqrt{2}}(J_{45}-iJ_{46})$$

$$E_{10} = \frac{i}{\sqrt{2}}(J_{14}+iJ_{24}) \qquad E_{-10} = \frac{-i}{\sqrt{2}}(J_{14}-iJ_{24})$$

$$E_{01} = \frac{-i}{\sqrt{2}}(J_{35}+iJ_{36}) \qquad E_{0-1} = \frac{i}{\sqrt{2}}(J_{35}-iJ_{36})$$

$$E_{1-1} = \frac{-(J_{15}+J_{25}) + i(J_{16}+iJ_{26})}{2} \qquad E_{-11} = \frac{-(J_{15}-iJ_{25}) - i(J_{16}-iJ_{26})}{2}$$

$$E_{11} = \frac{(J_{15}+iJ_{25}) + i(J_{16}+iJ_{26})}{2} \qquad E_{-1-1} = \frac{(J_{15}-iJ_{25}) - i(J_{16}-iJ_{26})}{2}$$

$${}^{13}A^+(0,1) = \frac{(J_{57}+iJ_{67}) - i(J_{58}+iJ_{68})}{2} \qquad A(0,1) = \frac{(J_{57}-iJ_{67}) + i(J_{58}-iJ_{68})}{2}$$

$${}^{13}A^+(0,0) = \frac{-i}{\sqrt{2}}(J_{47}-iJ_{48}) \qquad {}^{13}A(0,0) = \frac{i}{\sqrt{2}}(J_{47}-iJ_{48})$$

$${}^{13}A^+(0,-1) = \frac{(J_{57}-iJ_{67}) - i(J_{58}-iJ_{68})}{2} \qquad {}^{13}A(0-1) = \frac{(J_{57}+iJ_{67}) + i(J_{58}+iJ_{68})}{2}$$

$${}^{31}A^+(1,0) = \frac{i(J_{17}+iJ_{27}) + (J_{18}+iJ_{28})}{2} \qquad {}^{31}A(1,0) = \frac{-i(J_{17}-iJ_{27}) + (J_{18}-iJ_{28})}{2}$$

$${}^{31}A^+(0,0) = \frac{-(J_{37}-iJ_{38})}{\sqrt{2}} \qquad {}^{31}A(0,0) = \frac{-(J_{37}+iJ_{38})}{\sqrt{2}}$$

$${}^{31}A^+(-1,0) = \frac{i(J_{17}-iJ_{27}) + (J_{18}-iJ_{28})}{2} \qquad {}^{31}A(-1,0) = \frac{-i(J_{17}+iJ_{27}) + (J_{18}+iJ_{28})}{2}$$

(2.B.9)

From these relations it can be seen that the set of operators $\underline{S}$, $\underline{T}$, $E_{\alpha\beta}$ form a subgroup O(6) in the 1,2,...6 subspace of O(8). Within this O(6), the $\underline{S}$ operators form a subgroup O(3) in the 1,2,3 space and the $\underline{T}$ operators form a subgroup O(3) in the 4,5,6 space. The $\underline{S}$ and T commute. The E_{10}, E_{-10} and S_o form a subgroup in 1,2,4 space; and the E_{01}, E_{0-1}, and T_o form a subgroup in 3,5,6 space; however the $\underline{S}$ operators do not commute with $E_{0,\pm 1}$; nor the $\underline{T}$ operators with $E_{\pm 1,0}$. It is quite clear from such a representation that O(6) has subgroups formed by the direct products O(3) x O(3). The most meaningufl of these are the $\underline{S}$ and $\underline{T}$ operators. Further, $^{13}A^{+}(0,0)$, $^{13}A(0,0)$ and Q_0 form a group O(3) in 4,7,8 space, and $^{31}A^{+}(0,0)$, $^{31}A(00)$ and Q_0 form a group O(3) in 3,7,8 space; but of all these operators only Q_0 commutes with $\underline{S}$ and $\underline{T}$, and it is related to the number operator.

B.3. The General Properties Of The O(8) $\supset$ O(6) $\supset$ O(3) x O(3) Quasispin Group Chain

The n dimensional orthogonal groups are generated by a set of $\frac{n(n-1)}{2}$ operators $J_{\alpha\beta}$. These operators are governed by the commutation rule (2.B.8). The rank of the group, which is the number of mutually commuting operators and also the number of integers or half integers required to specify the representation, is k for n = 2k and n = 2k+1. For example 4 numbers are necessary to specify the irreducible representations of O(8), whereas only 3 are required for O(6) and O(7).

(a) The irreducible representations of O(6): The irreducible representation O(6) are specified by the highest weights, that is by the largest possible eigenvalues of the 3 commuting operators T_0, E_{00}, S_0. Since O(6) is isomorphic with SU(4), the irreducible representations can also be labeled according to the notation standard for the special unitary group associated with the 4-dimensional spin-isospin space. It is convenient to include the number operator (or

Q_o) which commtes with the 15 operators $\underline{S}$, $\underline{T}$, $E_{\alpha\beta}$, and together with them generates a group U(4), so that the irreducible representations can be specified by the symmetries of n-nucleon spin-isospin functions. These symmetries are characterized by Young tableaux or partition numbers $[f_1 f_2 f_3 f_4]$ on N objects where λ_i are integers such that

$$f_1 + f_2 + f_3 + f_4 = N$$

with

$$\Omega \geq f_1 \geq f_2 \geq f_3 \geq f_4 \geq 0$$

The partition number f_i specifies the length of the ith row of the Young tableau. Since the number of single nucleon spatial state is Ω while the number of spin-isospin states is four, the Young tableaux are restricted to have at most Ω columns and 4 rows. Since there can be at most f_1 neutrons with spin up, and subject to this restriction at most f_2 additional neutrons with spind down, etc..., the largest possible eigenvalues of the 3 commuting operators T_o, E_{oo}, S_o are specified by the partition numbers f_i. Independent of the ordering of the single particle states, the highest weights which define the O(6) irreducible representations are thus characterized by

$$P = \tfrac{1}{2}(f_1 + f_2 - f_3 - f_4)$$

$$P' = \tfrac{1}{2}(f_1 - f_2 + f_3 - f_4)$$

$$P'' = \tfrac{1}{2}(f_1 - f_2 - f_3 + f_4) \qquad (2.B.10)$$

These P, P', P'' are the Wigner super multiplet quantum numbers.

(b) The irreducible representation of O(8): The irreducible representation of O(8) are specified by the highest weights, defined by the largest possible eigenvalues of the 4 commuting operators Q_o, T_o, E_{oo}, S_o. In a state of speicifed seniority v, there must be at least v nucleons (the number entirely free of pairs coupled to L = 0). The highest possible eigenvalues of the generator $Q_o = \Omega - \frac{1}{2} N_{op}$. is therefore

$$Q = \Omega - \frac{1}{2} v$$

Subject to the restriction to the highest possible Q_o, the highest eigenvalues of T_o, E_{oo}, and S_o are therefore speicifed by the symmetry of a v nucleon spin-isospin function characterized by partition numbers $[\mu_1 \mu_2 \mu_3 \mu_4]$ on v objects, where

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = v$$

and where the μ_i are again integers, satisfying

$$\frac{\Omega}{2} \geq \mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq 0$$

Among the set of v nucleons corresponding to the highest weight state there can again be at most μ_1 neutrons with spin up, and subject to this restriction at most μ_2 additional neutrons with spin down, etc.... The highest eigenvalues of T_o, E_{oo}, S_o in the highest weight state of O(8) are therefore specified by

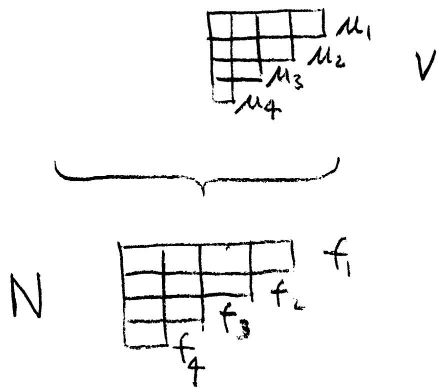
$$p = \frac{1}{2}(\mu_1 + \mu_2 - \mu_3 - \mu_4)$$

$$p' = \frac{1}{2}(\mu_1 - \mu_2 + \mu_3 - \mu_4)$$

$$p'' = \frac{1}{2}(\mu_1 - \mu_2 - \mu_3 + \mu_4) \qquad (2.B.11)$$

The p, p', p" are called reduced supermultiplet quantum numbers. They are supermultiplet number of the v nucleons free of pairs coupled to L = 0. In analogy with (2.A.2) the state function for such a representation can be written as

$$A^+A^+\ldots A^+\underbrace{a^+a^+\ldots a^+}_{v}|0\rangle$$



(c) Decomposition of O(8) into O(6): For a specific v, p, p', p", the O(8) irreducible representation is denoted by $(\Omega - \frac{1}{2}v, p, p', p'')$, which decomposes into different irreducible representation of the subgroup O(6) characterized by (P,P',P"). This decomposition is due to the fact that the $\frac{(n-v)}{2}$ operators A^+ with O(6) irreducible tensor character (100) can be coupled to the O(6) representations (p,p',p") in many different ways.

For fixed $(\Omega - \frac{1}{2}v, p, p', p'')$ and (P, P', P") the decomposition from O(8) to O(6) is not unique. For the $O(8) \supset O(7) \supset O(6)$ chain it can be seen that three other quantum numbers which specify the representations of O(7) are needed to identify the "parent" of O(6). The physical content of these three quantum number in not so easily determined.

A similar problem occurs in the decomposition of O(6) into the direct product O(3) x O(3) of the spin and isospin groups. Once again the set of (P, P', P") and ($S\ M_S$, TM_T) are not sufficient to give a unique decomposition. Two other quantum numbers are needed to completely specify a state. Although Moshinsky and Nagel[9] have succeeded in finding the needed operators, they are of third and fourth degree in the infinitesmal operators of O(6) and the physical content of these two operators is again obscure.

The search for such missing operators, though noble, may thus not lead to any practical results. The present work tries to do without them, at the cost of limiting the scope of interest to include only states of low seniority. Since these are precisely the states of greatest interest for problems in pairing theory, this is not a severe restriction.

In the most general case a state vector for a specific O(8) irreducible representation would be completely specified by 16 quantum numbers.

$$|(Q = \Omega - \frac{v}{2},\ p, p', p''),\ (\alpha_1\alpha_2\alpha_3),\ (PP'P''),\ SM_STM_T,\ \omega\phi\rangle$$

Where, for example, ω and ϕ could be chosen as the eigenvalues of Moshinsky and Nagel's third and fourth degree O(6) operators, while $(\alpha_1\alpha_2\alpha_3)$ could be chosen as the representation labels of the group O(7) in the group chain $O(8) \supset O(7) \supset O(6)$. The general branching law for the group chain $O(n) \supset O(n-1) \supset \ldots$ (which is discussed in detail in the supplement to this thesis) requires

$$Q \geq \alpha_1 \geq p \qquad p \geq \alpha_2 \geq p' \qquad p' \geq \alpha_3 \geq p''$$

$$\alpha_1 \geq P \qquad P \geq \alpha_2 \geq P' \qquad P' > \alpha_3 > |P''|$$

If the seniority v is restricted to be either 0 or 1, p p' p'' are restricted to the values 000 or $\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}$, respectively. In these special cases the remaining quantum numbers are therefore also severely restricted as indicated in the following table.

	For v = 0	For v = 1
$O(8) = (Qpp'p'')$	$(\Omega 000)$	$(\Omega - \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$
$O(7) = (\alpha_1\alpha_2\alpha_3)$	$(\alpha 00)$	$(\alpha'\,\frac{1}{2}\,\frac{1}{2})$
$O(6) = (PP'P'')$	$(n00)$	$(n - \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$
	α = integer	$\alpha' = \frac{1}{2}$ integer
	$\Omega \geq \alpha \geq n$	$\Omega - \frac{1}{2} \geq \alpha' \geq n - \frac{1}{2}$

In both cases therefore only a single O(7) quantum number is needed to completely specify the states. However the quantum numbers α have no ready physical significance. Even worse, a state of definite α is <u>not</u> a state of a definite number of nucleons, N. The number operator is in general not diagonal in the $O(8) \supset O(7) \supset O(6)$ scheme. (The highest weight state is an exception.) Since it can be shown that the number of distinct eigenvalues of the number operator is equal to Ω - n+1, which is equal to the number of distinct values of α (or α') in the above two cases, the number operator itself can be used, in place of an operator whose eigenvalues determine α, as the additional operator which make the decomposition of O(8) into O(6) unique in these two cases. Restriction of v to either 0 or 1 implies a restriction to the O(6) representations (noo) or $(n - \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$ and $(n - \frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2})$. It will be shown in the next chapter that the decomposition of O(6) into O(3) x O(3) is unique in those special cases, so that the quantum number SM_S TM_T are sufficient to completely specify the states of O(6).

B.4. Matrix Element Of Operators In The O(8) ⊃ O(6) ⊃ O(3) x O(3) Scheme

A general irreducible tensor operator in the O(8) ⊃ O(6) ⊃ O(3) x O(3) chain can be identified as $T\begin{pmatrix} Qpp'p'' \\ PP'P'' \\ SM_S TM_T \end{pmatrix}\begin{pmatrix}\alpha_1\alpha_2\alpha_3\end{pmatrix}_{(\omega\phi)}$. Its matrix elements can be calculated by the Wigner Eckart theorem. The matrix elements of interest in their work are those of the infinitesmal generators of O(8), or their linear combinations which have the tensor character $T\begin{pmatrix} 1100 \\ PP'0 \\ SM_S TM_T \end{pmatrix}^N$

where (PP'0) = (100) or (110)

$(SM_S TM_T)$ = $(1\,M_S 00)$ or $(001M_T)$

N = ±2 (Pair creation or annihilation operator)

or 0 (number preserving operator).

A table of the 28 operators and their tensor properties with respect to the group chain spaces are included.

TABLE 2.1

TENSOR PROPERTIES OF THE O(8) QUASISPIN OPERATORS

Generators	P	P'	P"	S	M_S	T	M_T	N
Q_0	0	0	0	0	0	0	0	0
T_0	1	1	0	0	0	1	0	0
E_{00}	1	1	0	1	0	1	0	0
S_0	1	1	0	1	0	0	0	0
S_+	1	1	0	1	1	0	0	0
T_+	1	1	0	0	0	1	1	0
E_{10}	1	1	0	1	1	1	0	0
E_{01}	1	1	0	1	0	1	1	0
E_{1-1}	1	1	0	1	1	1	-1	0
E_{11}	1	1	0	1	1	1	1	0
${}^{31}A^+(10)$	1	0	0	1	1	0	0	2
${}^{31}A^+(00)$	1	0	0	1	0	0	0	2
${}^{31}A^+(-10)$	1	0	0	1	-1	0	0	2
${}^{13}A^+(01)$	1	0	0	0	0	1	1	2
${}^{13}A^+(00)$	1	0	0	0	0	1	0	2
${}^{13}A^+(0-1)$	1	0	0	0	0	1	-1	2

Properties of their conjugates and phase relations will be discussed later.

CHAPTER III

THE SU(4) PART OF THE PAIRING PROBLEM

A. INTRODUCTION

In calculating the matrix elements of the pair creation operators, the Wigner Eckart theorem will be applied to each segment of the quasispin group chain $O(8) \supset O(6) \supset [O(3) \times O(3)]$. In this chapter the $O(6) \supset [O(3) \times O(3)]$ part of the chain and its Wigner coefficients are taken up first.

$O(6)$ is generated by the 15 infinitesimal operators $\underline{S}$, $\underline{T}$, $E_{\alpha\beta}$. By choosing the group chain $O(6) \supset O(5) \supset O(4) \supset O(3) \supset O(2)$, a canonical way of specifying a state function is obtained; that is, the state function of a given irreducible representation of $O(6)$ are completely specified by the irreducible representation labels of the subgroups in the chain. A state function can be denoted by

$$\left| \begin{matrix} m_{61} & m_{62} & m_{63} \\ m_{51} & m_{52} & \\ m_{41} & m_{42} & \\ m_{31} & & \\ m_{21} & & \end{matrix} \right/$$

where the m_{nk} are the irreducible representation labels for $O(n)$. In the notation of Chapter II $(m_{61}m_{62}m_{63}) = (P, P', P'')$. (The group chain $O(n) \supset O(n-1) \supset \ldots$ is studied in detail in the supplement to the thesis where the possible values branching laws for m_{nk} are discussed.) Besides the three $O(6)$ labels, six additional quantum numbers are needed. However, if m_{62} and $|m_{63}|$ are fixed at either 0 or $\frac{1}{2}$, m_{52} and $|m_{42}|$ are also fixed, so that only four additional quantum numbers are needed to specify a state. Although the specification through the above group chain is natural in the mathematical sense, it is not the physically relevant one, since S and T appear on a very different footing and S^2 and T^2 are not simultaneously diagonal. Either S M_S or TM_T can be identified with m_{31}, m_{21}; but if SM_S are chosen, a linear combination of the states with different m_{5k}, m_{4k} are neede to diagonalize T^2 and T_0. The physically relevant way of specifying the state function therefore requires the group chain $O(6) \supset [O(3) \times O(3)]$. In the special cases mentioned above the quantum numbers of $O(3) \times O(3)$ are sufficient, since SM_S TM_T completely specify the states.

Since O(6) and SU(4), as well as O(3) and SU(2), have Lie algebras of the same structure, it will be convenient to substitute $SU(4) \supset [SU(2) \times SU(2)]$ for $O(6) \supset [O(3) \times O(3)]$ in the following sections. The quantum numbers S and T are then more easily recognized.

Since the calculation of the $SU(4) \supset [SU(2) \times SU(2)]$ Wigner coefficients needed for the matrix elements of the pair creation operators in rather lenghty, a brief outline of Chapter III and the general method of attack will be given here first. A review of the general properties of SU(4) is given in section B where the step-down (up) operators are discussed. The explicit constructions of the state functions, in terms of repeated application of normalized step-down operators, are given in section C and D for the SU(4) representations of primary interest in the pairing calculations. After a brief review of the general properties of SU(4) Wigner coefficient in section E, matrix elements of the infinitesimal operators are calculated in section F and expressed in terms of SU(4) Wigner coefficients. With these matrix elements of the infinitesimal operators, recursion relations are set up from which further SU(4) Wigner coefficients are calculated. The simplest such coefficients are those involving the coupling of one-particle states to the representations of interest. These are calculated in section G. Finally the SU(4) Wigner coefficients involving the coupling of pairs are calculated from those involving a coupling of one particle state by a build up process illustrated in section H. The final results are shown in Tables 3.12-3.19.

B. PROPERTIES OF SU(4)

B.1. Generators and Their Commutation Relations

The SU(4) basis is generated by the four state functions in spin and isospin space

$$
\begin{aligned}
|1\rangle &= |\uparrow\uparrow\rangle \\
|2\rangle &= |\uparrow\downarrow\rangle \\
|3\rangle &= |\downarrow\uparrow\rangle \\
|4\rangle &= |\downarrow\downarrow\rangle
\end{aligned}
\tag{3.B.1}
$$

where the first arrow indicates spin and the second arrow indicates isospin. The generator of SU(4) are the number conserving operators:

$$
A_{ij} = \sum_{\ell m} a^{\dagger}_{\ell m i}\, a_{\ell m j}
\tag{3.B.2}
$$

The i, j correspond to $|i\rangle$ and $|j\rangle$, for example

$$A_{14} = \sum_{\ell m} a^{\dagger}_{\ell m \uparrow\uparrow} a_{\ell m \downarrow\downarrow}$$

Recombining these generators into Wigner supermultiplet operators

$$\begin{aligned}
S_0 &= \tfrac{1}{2}(A_{11} + A_{22} - A_{33} - A_{44}) \\
T_0 &= \tfrac{1}{2}(A_{11} - A_{22} + A_{33} - A_{44}) \\
E_{00} &= \tfrac{1}{2}(A_{11} - A_{22} - A_{33} + A_{44}) \\
S_+ &= \frac{A_{13} + A_{24}}{\sqrt{2}} & S_- &= \frac{A_{31} + A_{42}}{\sqrt{2}} \\
T_+ &= \frac{A_{12} + A_{34}}{\sqrt{2}} & T_- &= \frac{A_{12} - A_{34}}{\sqrt{2}} \\
E_{10} &= \frac{A_{13} - A_{24}}{\sqrt{2}} & E_{-10} &= \frac{A_{31} - A_{42}}{\sqrt{2}} \\
E_{01} &= \frac{A_{12} - A_{34}}{\sqrt{2}} & E_{0-1} &= \frac{A_{21} - A_{43}}{\sqrt{2}} \\
E_{11} &= A_{14} & E_{-1-1} &= A_{41} \\
E_{1-1} &= A_{23} & E_{-11} &= A_{32}
\end{aligned} \tag{3.B.3}$$

where

$$A_{ij} = A^{\dagger}_{ji} \tag{3.B.4}$$

and

$$E^{\dagger}_{\alpha\beta} = E_{-\alpha-\beta}$$

The commutation relations of the A can be worked out through the simple anticommutation relations for a^+ and a. They are

$$[A_{ij}, A_{kl}] = A_{il}\,\delta_{kj} - A_{kj}\,\delta_{il} \tag{3.B.5}$$

With the aid of these relations, the commutation relations for the Wigner supermultiplet operators are as follows

$$[\vec{T}, \vec{S}] = 0$$

$$[S_0, E_{\alpha\beta}] = \alpha E_{\alpha\beta} \qquad [T_0, E_{\alpha\beta}] = \beta E_{\alpha\beta}$$

$$[E_{00}, E_{10}] = S_+ \qquad [E_{00}, E_{01}] = T_+ \qquad [E_{00}, E_{11}] = 0$$

$$[E_{00}, S_+] = E_{10} \qquad [E_{00}, T_+] = E_{01}$$

$$[S_+, E_{-1\alpha}] = E_{0\alpha} \qquad [S_+, E_{0\alpha}] = -E_{1\alpha} \qquad [S_+, E_{1\alpha}] = 0$$

$$[T_+, E_{\alpha-1}] = E_{\alpha 0} \qquad [T_+, E_{\alpha 0}] = -E_{\alpha 1} \qquad [T_+, E_{\alpha 1}] = 0$$

$$[E_{10}, E_{-10}] = S_0 \qquad [E_{01}, E_{0-1}] = T_0$$

$$[E_{11}, E_{-1-1}] = S_0 + T_0 \qquad [E_{1-1}, E_{-11}] = S_0 - T_0$$

$$[E_{-10}, E_{11}] = T_+ \qquad [E_{-10}, E_{1-1}] = -T_-$$
$$[E_{0-1}, E_{11}] = S_+ \qquad [E_{0-1}, E_{-11}] = -S_-$$

(3.B.6)

B.2. Step Up and Step Down Operators

Within SU(4), there are some irreducible representation for which the state function are uniquely determined by SM_STM_T such representations are the ones of primary interest in this investigation. Their state functions are denoted by

$$\left| \begin{matrix} [f] \\ (S\, m_S)(T\, m_T) \end{matrix} \right\rangle$$

(3.B.7)

Where [f] characterize the SU(4) irreducible representations ([f] is a short hand notation for $[f_1f_2f_3f_4]$ discussed in Chapter II. For SU(4), unlike U(4), $f_4 = 0$. Since the Kronecker product of the U(4) representation [kkkk] with $[f_1f_2f_30]$ leads to the single representation $[f_1+k, f_2+k, f_3+k, k]$ and since [kkkk] is invariant under unitary transformations with determinant 1, the irreducible representations of SU(4), unlike those of U(4), are restricted to those with $f_4 = 0$.) The relation between the standard O(6) notation (PP'P") and the SU(4) notation [f] is given by (2.B.10). To avoid confusion in the following sections O(6) quantum numbers will always be placed in parenthesis while SU(4) quantum numbers are always enclosed by square brackets.

In order to shorten some notations which appear quite often later, let

$$\left| \begin{matrix} [f] \\ \{S,T\} \end{matrix} \right\rangle \equiv \left| \begin{matrix} [f] \\ (SS)(TT) \end{matrix} \right\rangle \tag{3.B.8}$$

A step operator is defined by

$$O_{\alpha\beta} \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle = N^{[f]}_{\alpha\beta}(ST) \left| \begin{matrix} [f] \\ \{S+\alpha, T+\beta\} \end{matrix} \right\rangle \tag{3.B.9}$$

where

$$N^{[f]}_{\alpha\beta}(ST) = \sqrt{\left\langle \begin{matrix} [f] \\ \{ST\} \end{matrix} \right| O_{-\alpha-\beta} O_{\alpha\beta} \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle} \tag{3.B.10}$$

and

$$N^{[f]}_{\alpha\beta}(ST) = N^{[f]}_{-\alpha-\beta}(S+\alpha, T+\beta) \tag{3.B.11}$$

The choice of the positive sign specifies the phase convention used in this investigation since

$$S_+ \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle = T_+ \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle = 0 \tag{3.B.12}$$

Therefore, the $O_{\alpha\beta}$ must satisfy

$$[T_+, O_{\alpha\beta}] \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle = 0 \tag{3.B.13}$$

and

$$[S_+, O_{\alpha\beta}] \left| \begin{matrix} [f] \\ \{ST\} \end{matrix} \right\rangle = 0$$

To construct $O_{\alpha\beta}$, one starts with $E_{\alpha\beta}$ and with (3.B.6) more terms are added until a closed form is obtained

$$O_{11} = E_{11}$$

$$O_{01} = E_{01} + S_- E_{11} \frac{1}{S_0+1}$$

$$O_{10} = E_{10} + T_- E_{11} \frac{1}{T_0+1}$$

$$O_{00} = E_{00} + S_- E_{10} \frac{1}{S_0+1} + T_- E_{01} \frac{1}{T_0+1} + S_- T_- E_{11} \frac{1}{(S_0+1)(T_0+1)}$$

$$O_{1-1} = E_{1-1} - T_- E_{10} \frac{1}{T_0} - T_-^2 E_{11} \frac{1}{T_0(2T_0+1)}$$

$$O_{-11} = E_{-11} - S_- E_{01} \frac{1}{S_0} - S_-^2 E_{11} \frac{1}{S_0(2S_0+1)}$$

$$O_{0-1} = E_{0-1} + S_- E_{1-1} \frac{1}{(S_0+1)} - T_- E_{00} \frac{1}{T_0} - T_-^2 E_{01} \frac{1}{T_0(2T_0+1)}$$
$$- S_- T_- E_{10} \frac{1}{T_0(S_0+1)} - T_-^2 S_- E_{11} \frac{1}{(S_0+1)T_0(2T_0+1)}$$

$$O_{-10} = E_{-10} + T_- E_{-11} \frac{1}{(T_0+1)} - S_- E_{00} \frac{1}{S_0} - S_-^2 E_{10} \frac{1}{S_0(2S_0+1)}$$
$$- S_- T_- E_{01} \frac{1}{S_0(T_0+1)} - T_- S_-^2 E_{11} \frac{1}{(T_0+1)S_0(2S_0+1)}$$

$$O_{-1-1} = E_{-1-1} - T_- E_{-10} \frac{1}{T_0} - S_- E_{0-1} \frac{1}{S_0} - T_-^2 E_{-11} \frac{1}{T_0(2T_0+1)}$$
$$- S_-^2 E_{1-1} \frac{1}{S_0(2S_0+1)} + S_- T_- E_{00} \frac{1}{S_0 T_0} + S_-^2 T_- E_{10} \times$$
$$\frac{1}{S_0(2S_0+1)T_0} + S_- T_-^2 E_{01} \frac{1}{S_0 T_0(2T_0+1)} + S_-^2 T_-^2 E_{11} \times$$
$$\frac{1}{S_0 T_0 (2S_0+1)(2T_0+1)} \qquad (3.B.14)$$

The $O_{\alpha\beta}$ are not unique. For example, if $O_{\alpha\beta}$ satisfies (3.B.12) so does $O_{\alpha\beta} + T_+$, and many others. However the most simple form of $O_{\alpha\beta}$ have been chosen in (3.B.14). There are all together nine step operators in terms of nine E generators. Sometimes it is more convenient to use E. The E generators can also be expressed in terms of the O operators.

$$
\begin{aligned}
E_{11} &= O_{11} \\
E_{01} &= O_{01} - S_- O_{11} \frac{1}{(S_0+1)} \\
E_{10} &= O_{10} - T_- O_{11} \frac{1}{(T_0+1)} \\
E_{00} &= O_{00} - S_- O_{10} \frac{1}{(S_0+1)} - T_- O_{01} \frac{1}{(T_0+1)} + S_- T_- O_{11} \frac{1}{(S_0+1)(T_0+1)} \\
E_{1-1} &= O_{1-1} + T_- O_{10} \frac{1}{T_0} - T_-^2 O_{11} \frac{1}{(T_0+1)(2T_0+1)} \\
E_{-11} &= O_{-11} + S_- O_{01} \frac{1}{S_0} - S_-^2 O_{11} \frac{1}{(S_0+1)(2S_0+1)} \\
E_{0-1} &= O_{0-1} + T_- O_{00} \frac{1}{T_0} - S_- O_{1-1} \frac{1}{(S_0+1)} - T_- S_- O_{10} \frac{1}{T_0(S_0+1)} \\
&\quad - T_-^2 O_{01} \frac{1}{(2T_0+1)(T_0+1)} + S_- T_-^2 O_{11} \frac{1}{(S_0+1)(T_0+1)(2T_0+1)} \\
E_{-10} &= O_{-10} + S_- O_{00} \frac{1}{S_0} - T_- O_{-11} \frac{1}{(T_0+1)} - T_- S_- O_{01} \frac{1}{S_0(T_0+1)} \\
&\quad - S_-^2 O_{10} \frac{1}{(2S_0+1)(S_0+1)} + T_- S_-^2 O_{11} \frac{1}{(T_0+1)(S_0+1)(2S_0+1)} \\
E_{-1-1} &= O_{-1-1} + S_- O_{0-1} \frac{1}{S_0} + T_- O_{-10} \frac{1}{T_0} + T_- S_- O_{00} \frac{1}{S_0 T_0} \\
&\quad - S_-^2 O_{1-1} \frac{1}{(S_0+1)(2S_0+1)} + T_-^2 O_{-11} \frac{1}{(T_0+1)(2T_0+1)} \\
&\quad - T_-^2 S_- O_{01} \frac{1}{S_0(2T_0+1)(T_0+1)} - T_- S_-^2 O_{10} \frac{1}{T_0(2S_0+1)(S_0+1)} \\
&\quad + T_-^2 S_-^2 O_{11} \frac{1}{(T_0+1)(S_0+1)(2S_0+1)(2T_0+1)}
\end{aligned}
\tag{3.B.15}
$$

B.3. Casimir Operators

The quadratic invariant of O(n) is

$$C_n = \sum_{\alpha<\beta} J_{\alpha\beta}^2$$

Since O(6) ~ SU(4), putting the quadratic invariant in terms of E, $\vec{S}$, $\vec{T}$ operators, gives

$$\begin{aligned} C = {} & 2E_{-1-1}E_{11} + 2E_{-10}E_{10} + 2E_{0-1}E_{01} + 2E_{-11}E_{1-1} \\ & + 2S_-S_+ + 2T_-T_+ + S_0^2 + T_0^2 + E_{00}^2 \\ & + 4S_0 + 2T_0 \end{aligned} \qquad (3.B.16)$$

B.4. Tensor Character of the Generators

The fifteen $J_{\alpha\beta}$ in O(6) transform according to the representation (110), and their linear combinations transform according to [211] in SU(4). The branching of [211] is to nine states of S = 1 and T = 1, and six states of S = 1 T = 0 and S = 0 T = 1. State functions of [211] can thus be uniquely expressed by $\vec{S}$ and $\vec{T}$.

The 15 infinitesimal operators of SU(4) can be classified as ineducible tensor operators of SU(4) tensor character [211] through the commutator equations.

$$\left[E_{\alpha\beta}, T^{[211]}_{(S\alpha')(T\beta')}\right] = \sum_{S'T'} \left\langle {[211] \atop (S',\alpha+\alpha')(T',\beta+\beta')} \middle| E_{\alpha\beta} \middle| {[211] \atop (S\alpha')(T\beta')} \right\rangle \times T^{[211]}_{(S',\alpha+\alpha')(T',\beta+\beta')}$$

(3.B.17)

And the analogous well known commutator equations involving $\vec{S}$ and $\vec{T}$ in place of $E_{\alpha\beta}$ above. In terms of the latter the components of $\vec{S}$ and $\vec{T}$, using the standard SU(2) phases, can be identified as

$$\begin{array}{ll} T^{[211]}_{(00)(11)} = -T_+ & T^{[211]}_{(11)(00)} = -S_+ \\ T^{[211]}_{(00)(10)} = T_0 & T^{[211]}_{(10)(00)} = S_0 \\ T^{[211]}_{(00)(1-1)} = T_- & T^{[211]}_{(1-1)(00)} = S_- \end{array}$$

(3.B.18)

In order to investigate the tensor properties of the remaining nine operators, a few matrix elements must be worked out. Note that

$$E_{\alpha\beta}\left| {[211] \atop \{11\}} \right\rangle = 0 \qquad \text{if either } \alpha=1 \text{ or } \beta=1$$

The normalization constant of

$$\mathcal{O}_{0-1}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right\rangle = N_{0-1}^{[211]}\{11\}\left|\begin{matrix}[211]\\ \{10\}\end{matrix}\right\rangle$$

is thus

$$N_{0-1}^{[211]}\{11\} = \sqrt{\left\langle\begin{matrix}[211]\\ \{11\}\end{matrix}\right|\mathcal{O}_{01}\mathcal{O}_{0-1}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right\rangle}$$
$$= \sqrt{\left\langle\begin{matrix}[211]\\ \{11\}\end{matrix}\right|E_{01}E_{0-1}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right\rangle} = 1$$

Therefore

$$E_{0-1}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right\rangle = \left|\begin{matrix}[211]\\ \{10\}\end{matrix}\right\rangle$$

$$E_{-10}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right. = \left|\begin{matrix}[211]\\ \{01\}\end{matrix}\right\rangle$$

$$E_{-1-1}\left|\begin{matrix}[211]\\ \{11\}\end{matrix}\right\rangle = \left|\begin{matrix}[211]\\ (10)(00)\end{matrix}\right\rangle + \left|\begin{matrix}[211]\\ (00)(10)\end{matrix}\right\rangle \tag{3.B.19}$$

Similarly by using

$$\left\langle \begin{matrix}[211]\\ \{10\}\end{matrix} \middle| O_{10}\, O_{-10} \middle| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle = 0$$

$$\left\langle \begin{matrix}[211]\\ \{10\}\end{matrix} \middle| O_{1-1}\, O_{-11} \middle| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle = 0$$

$$\left\langle \begin{matrix}[211]\\ \{11\}\end{matrix} \middle| O_{01}\, O_{0-1} \middle| \begin{matrix}[211]\\ \{11\}\end{matrix} \right\rangle = 1$$

and

$$\left\langle \begin{matrix}[211]\\ \{10\}\end{matrix} \middle| O_{0-1}\, O_{01} \middle| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle = 1$$

then

$$\begin{aligned}
E_{00} \left| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle &= - \left| \begin{matrix}[211]\\ (11)(10)\end{matrix} \right\rangle \\
E_{01} \left| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle &= \left| \begin{matrix}[211]\\ \{11\}\end{matrix} \right\rangle \\
E_{-11} \left| \begin{matrix}[211]\\ \{10\}\end{matrix} \right\rangle &= \left| \begin{matrix}[211]\\ (10)(11)\end{matrix} \right\rangle \\
E_{00} \left| \begin{matrix}[211]\\ \{01\}\end{matrix} \right\rangle &= - \left| \begin{matrix}[211]\\ (10)(11)\end{matrix} \right\rangle \\
E_{10} \left| \begin{matrix}[211]\\ \{01\}\end{matrix} \right\rangle &= \left| \begin{matrix}[211]\\ \{11\}\end{matrix} \right\rangle \\
E_{1-1} \left| \begin{matrix}[211]\\ \{01\}\end{matrix} \right\rangle &= \left| \begin{matrix}[211]\\ (11)(10)\end{matrix} \right\rangle
\end{aligned} \qquad (3.B.20)$$

with (3.B.17), (3.B.18), (3.B.19), (3.B.20), and (3.B.6), the tensor classfication is

$$E_{11} = -T^{[211]}_{(11)(11)} \qquad E_{10} = T^{[211]}_{(11)(10)} \qquad E_{1-1} = T^{[211]}_{(11)(1-1)}$$

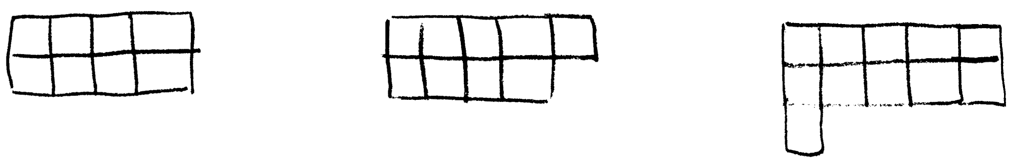
$$E_{01} = T^{[211]}_{(10)(11)} \qquad E_{00} = -T^{[211]}_{(10)(10)} \qquad E_{0-1} = -T^{[211]}_{(10)(1-1)}$$

$$E_{-11} = T^{[211]}_{(1-1)(11)} \qquad E_{-10} = -T^{[211]}_{(1-1)(10)} \qquad E_{-1-1} = -T^{[211]}_{(1-1)(1-1)}$$

(3.B.21)

C. CONSTRUCTION OF WAVE FUNCTION OF [nno] IN SU(4)

As discussed previously, the Wigner supermultiplet state of primary interest in this work are given by the O(6) representations (noo) $(n + \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(n + \frac{1}{2}, \frac{1}{2} - \frac{1}{2})$. In terms of the SU(4) quantum numbers these are the representations [nno], [n+1,n,o] and [n+1, n+1,1], respectively. Their Young tableanx are



This section is devoted for the construction of [nno]; the construction of [n+1, n, o] will be carried out in section D.

C.1. Branching Law for [nno]

It has been mentioned that these wave functions can be specified uniquely by SM_S, TM_T. But what is the complete set of S and T present in [nno]. The answer to the branching problem has been given by Racah[10] in general algebraic form for any irreducible representation of SU(4). Using Racah's general procedure the reduction of the representation of SU(4) into the representations [ST] of SU(2) x SU(2) can be carried out. The general branching law is given in Table 3.1.

TABLE 3.1

BRANCHING FORMULA FOR [nno] → [S,T]

[S,T]					
[n,0]					
[n-1,1]					
[n-2,2]	[n-2,0]				
[n-3,3]	[n-3,1]				
[n-4,4]	[n-4,2]	[n-4,0]			
⋮	⋮	⋮	...	[n-2i,0]	...
			...	⋮	
[3,n-3]	[3,n-5]	[3,n-7]	...	⋮	
[2,n-2]	[2,n-4]	[2,n-6]	...	[2,n-2i-2]	...
[1,n-1]	[1,n-3]	[1,n-5]	...	[1 n-2i-1]	...
[0,n]	[0,n-2]	[0,n-4]	...	[0,n-2i]	...

where only S and T are indicated. In fact for each S there are (2S+1) states and each T there are 2T+1 states so that there are (2S+1)(2T+1) states in [S,T]. The first column starts with s = n, the second column starts with s = n-2, and the ith column starts with s = n-2i+2.

Stepping down to the next row in each column S decreases by 1 whereas T increases by 1. The sum of S and T of any member in the same column is the same, for example it is n for the first column and n-2 for the second column, and 1(0) for the last colun if n is odd (even).

C.2. Casimir Invariant

The quadratic Casimir operator of (3.B.16) has eigenvalues which depend only on n. Therefore

$$\left\langle \begin{matrix}[nno]\\ \{ST\}\end{matrix} \right| C \left| \begin{matrix}[nno]\\ \{ST\}\end{matrix} \right\rangle = \left\langle \begin{matrix}[nno]\\ \{no\}\end{matrix} \right| C \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle = C^{[nno]}$$

(3.C.1)

All the E operators of (3.B.16) give zero when operating on the highest weight state $\left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle$ except E_{oo}. However,

$$O_{-10} \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle = 0$$

since {n-1,0} does not exist. Thus

$$\left\langle \begin{matrix}[nno]\\ \{no\}\end{matrix} \right| O_{10} O_{-10} \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle = 0$$

which together with (3.B.14) and (3.B.6) implies

$$\left\langle \begin{matrix}[nno]\\ \{no\}\end{matrix} \right| E_{00}E_{00} \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle = 0$$

Hence

$$\left\langle \begin{matrix}[nno]\\ \{no\}\end{matrix} \right| C \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle = \left\langle \begin{matrix}[nno]\\ \{no\}\end{matrix} \right| S_0^2 + T_0^2 + 4S_0 + 2T_0 \left| \begin{matrix}[nno]\\ \{no\}\end{matrix} \right\rangle$$

and

$$C^{[nno]} = n^2 + 4n$$

Using the well known properties of the operators $S_\pm$, $T_\pm$

$$\left\langle \begin{matrix}[nno]\\ \{ST\}\end{matrix} \right| 2E_{1-1}E_{11} + 2E_{-10}E_{10} + 2E_{0-1}E_{01} + 2E_{-11}E_{1-1} + E_{00}^2 \left| \begin{matrix}[nno]\\ \{ST\}\end{matrix} \right\rangle$$

$$= n^2 + 4n - S^2 - T^2 - 4S - 2T \tag{3.C.3}$$

which is a useful relation for the calculation of the normalization coefficients of the O operators.

C.3. Matrix Element $< \begin{matrix}[nno]\\ \{ST\end{matrix} \mid E_{-\alpha-\beta}\, E_{\alpha\beta} \mid \begin{matrix}[nno]\\ \{ST\}\end{matrix} >$ and Normalization Constants

In order to evaluate the normalization constants of the step operators, the matrix element of the type

$$\left\langle \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \middle| E_{-\alpha-\beta} E_{\alpha\beta} \middle| \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \right\rangle = \overline{E_{-\alpha-\beta} E_{\alpha\beta}}$$

must be evaluated first. There are altogether five independent types, namely 11, -11, 1o, o1, oo for α and β, respectively.

From the branching formula, it can be seen that two neighbor states differ from each other by more than one unit step, that is $|\Delta S|+|\Delta T = 2$.

Therefore

$$O_{\alpha\beta} \left| \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \right\rangle = 0$$

for either $|\alpha| = 1 \quad \beta = 0$
or $\alpha = 0 \quad |\beta| = 1$

This implies

$$\left\langle \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \middle| O_{-\alpha 0} O_{\alpha 0} \middle| \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \right\rangle = 0$$

and

for $\alpha = \pm 1$

$$\left\langle \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \middle| O_{0-\alpha} O_{0\alpha} \middle| \begin{matrix}[n n o] \\ \{ST\}\end{matrix} \right\rangle = 0$$

These lead to the 4 equations

$$\overline{E_{-10}E_{10}} = \overline{E_{-1-1}E_{11}}\,\frac{1}{(T+1)}$$

$$\overline{E_{0-1}E_{01}} = \overline{E_{-1-1}E_{11}}\,\frac{1}{(S+1)}$$

$$\overline{E_{-10}E_{10}} + \overline{E_{0-1}E_{01}}\,\frac{1}{S(T+1)} - \overline{E_{-11}E_{1-1}}\,\frac{1}{(T+1)} - \overline{E_{00}E_{00}}\,\frac{1}{S} + \frac{T(S+1)}{(T+1)} = 0$$

$$\overline{E_{-10}E_{10}}\,\frac{1}{T(S+1)} + \overline{E_{0-1}E_{01}} - \overline{E_{-11}E_{1-1}}\,\frac{1}{(S+1)} - \overline{E_{00}E_{00}}\,\frac{1}{T} + T = 0$$

Solving these in terms of $\overline{E_{00}E_{00}}$, one gets

$$\overline{E_{-1-1}E_{11}} = \overline{E_{00}E_{00}}\,(S+1)(T+1)$$

$$\overline{E_{-10}E_{10}} = \overline{E_{00}E_{00}}\,(S+1)$$

$$\overline{E_{0-1}E_{01}} = \overline{E_{00}E_{00}}\,(T+1)$$

$$\overline{E_{-11}E_{1-1}} = \overline{E_{00}E_{00}}\,(T+1)(S+1) + T(S+1)$$

$$\overline{E_{1-1}E_{-11}} = \overline{E_{00}E_{00}}\,(T+1)(S+1) + S(T+1)$$

(3.C.4)

Putting (3.C.4) into (3.C.3)

$$\overline{E_{00}E_{00}} = \frac{(n-S-T)(n+S+T+4)}{(2S+3)(2T+3)}$$

(3.C.5)

The normalization constant as defined in (3.B.10) can be determined by using (3.C.4) and (3.C.5)

$$N_{11}^{[nno]}(S,T) = \sqrt{\frac{(S+1)(T+1)(n-S-T)(n+S+T+4)}{(2S+3)(2T+3)}}$$

$$N_{1-1}^{[nno]}(S,T) = \sqrt{\frac{(S+1)(T)(n+S-T+3)(n+T-S+1)}{(2S+3)(2T+1)}}$$

(3.C.6)

Normalization constants N_{10} and N_{01} do not appear simply because the unit step operators do not give states which exist in [nno]. Once the above normalization constants are known, one can start from the highest state $\left|\begin{matrix}[nno]\\ \{no\}\end{matrix}\right>$ and step down to any state $\left|\begin{matrix}[nno]\\ \{ST\}\end{matrix}\right>$ by operating with O_{-11} and O_{-1-1} a suitable number of times. By operating on such state with T_- and S_- a suitable number of times, these states can be further reduced to $\left|\begin{matrix}[nno]\\ (SM_S)(TM_T)\end{matrix}\right>$, with the use of the well known SU(2) normalization constants.

D. CONSTRUCTION OF WAVE FUNCTIONS OF [n+1 no] in SU(4)

D.1. Branching Law For [n+1 no]

The branching law which shows how the SU(4) representations [n+1,n,0] are reduced to representations of SU(2) x SU(2) is shown in Table 3.2

TABLE 3.2

THE BRANCHING FORMULA FOR [n+1 no] → [ST]

$[S, T]$					
$[n+\frac{1}{2}, \frac{1}{2}]$					
$[n-\frac{1}{2}, \frac{3}{2}]$	$[n-\frac{1}{2}, \frac{1}{2}]$				
$[n-\frac{3}{2}, \frac{5}{2}]$	$[n-\frac{3}{2}, \frac{3}{2}]$	$[n-\frac{3}{2}, \frac{1}{2}]$			
$[n-\frac{5}{2}, \frac{7}{2}]$	$[n-\frac{5}{2}, \frac{5}{2}]$	$[n-\frac{5}{2}, \frac{3}{2}]$	$[n-\frac{5}{2}, \frac{1}{2}]$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$[\frac{3}{2}, n-\frac{1}{2}]$	$[\frac{3}{2}, n-\frac{3}{2}]$	$[\frac{3}{2}, n-\frac{5}{2}]$	$[\frac{3}{2}, n-\frac{7}{2}]$	---	
$[\frac{1}{2}, n+\frac{1}{2}]$	$[\frac{1}{2}, n-\frac{1}{2}]$	$[\frac{1}{2}, n-\frac{3}{2}]$	$[\frac{1}{2}, n-\frac{5}{2}]$	---	$[\frac{1}{2}, \frac{1}{2}]$

Again only S and T are indicated. Each column starts with $T = \frac{1}{2}$ and changes T by 1 and S by -1 for every downward step, so that the sum of S and T of any member in the same column is the same. The sum is (n+1) for the first column, n for the second column,... n+2-i for the ith column down to the last column for which the sum is 1. All S and T values are half integers.

D.2 Casimir Invariant of [n+1, n, 0] in SU(4)

$$\mathbb{C}^{[n+1,n,0]} = \left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| \mathbb{C} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix}[n+1\ n\ 0]\\ \{n+\frac{1}{2}\ \frac{1}{2}\}\end{matrix} \middle| \mathbb{C} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{n+\frac{1}{2}, \frac{1}{2}\}\end{matrix} \right\rangle$$

where the C operator has been given in (3.B.16). Since n + $\frac{1}{2}$ is the highest possible value of S

$$E_{\alpha 1} \left| \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right\rangle = 0 \qquad \text{for } \alpha = \pm 1, 0$$

and

$$E_{10} \left| \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right\rangle = 0$$

Also, since the lowest possible value of T in $\frac{1}{2}$,

$$O_{\alpha -1} \left| \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right\rangle = 0$$

and

$$\left\langle \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right| O_{01} \, O_{0-1} \left| \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right\rangle = 0$$

with these relations

$$\left\langle \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right| E_{00} \, E_{00} \left| \begin{matrix} [n+1 \; n \; 0] \\ \{n+\frac{1}{2} \; \frac{1}{2}\} \end{matrix} \right\rangle = \frac{1}{4}$$

The eigenvalue of the quadratic Casimir invariant in therefore

$$C^{[n+1\ n\ o]} = n^2 + 5n + \frac{15}{4}$$

(3.D.1)

Combining this with (2.B.16) one is led to the relation

$$\left\langle {[n+1\ no] \atop \{ST\}} \middle| 2E_{-1-1}E_{11} + 2E_{-10}E_{10} + 2E_{0-1}E_{01} + 2E_{1-1}E_{-11} + E_{00}E_{00} \middle| {[n+1\ no] \atop \{ST\}} \right\rangle$$
$$= n^2 + 5n + \frac{15}{4} - S^2 - T^2 - 4S - 2T$$

(3.D.2)

D.3. Matrix Elements $< {[n+1\ no] \atop \{ST\}} |E_{-\alpha-\beta}\, E_{\alpha\beta}| {[n+1\ no] \atop \{ST\}} >$ and Normalization Constants.

Since the unit step operators $O_{\alpha o}$ and $O_{o\alpha}$ do not give zero when acting on states of the representation [n+1, n, 0], the construction of the state functions for this representation is much more complicated than in the case of the representation [nno]. However, by using

$$\left\langle {[n+1\ no] \atop \{ST\}} \middle| O_{-\alpha-\beta} O_{\alpha\beta} \middle| {[n+1\ no] \atop \{ST\}} \right\rangle = \left\langle {[n+1\ no] \atop \{S+\alpha,\ T+\beta\}} \middle| O_{\alpha\beta} O_{-\alpha-\beta} \middle| {[n+1\ no] \atop \{S+\alpha,\ T+\beta\}} \right\rangle$$

(3.D.3)

One can get four equations, from the $\alpha\beta$ values 11, 10, 01, and 1-1. These four equations combined with (2.D.2) are sufficient to calculate the five needed matrix elements. However, equation (2.D.3) are recursion equations relating matrix elements of states in the kth and $(k+\alpha+\beta)^{th}$ columns of Table 3.2, so that the needed matrix elements must be evaluated through rather complicated recursion techniques. One possible recursive process is illustrated in the next paragraphs. First all matrix elements are evaluated for the first column, then the 2nd column, the third column, etc.

(a) The first column: E_{11}, E_{10} and E_{01} give zero when operating on a wave function of the first column. Since E_{00} commutes with E_{-11} which is equal to 0_{-11} in the first column, $\overline{E_{00}E_{00}}$ has the value $\frac{1}{4}$ for all states in the first column. Knowing $\overline{E_{00}E_{00}}$, $\overline{E_{1-1}E_{-11}}$ is obtained from (3.D.2)

$$\left\langle \begin{matrix}[n+1\ n\,0]\\ \{S,T\}\end{matrix} \right| E_{-11}E_{1-1} \left| \begin{matrix}[n+1\ n\,0]\\ \{S,T\}\end{matrix} \right\rangle_{1st\,col.} = (S+\tfrac{1}{2})(T-\tfrac{1}{2})$$

$$\left\langle \begin{matrix}[n+1\ n\,0]\\ \{S,T\}\end{matrix} \right| E_{0-1}E_{01} \left| \begin{matrix}[n+1\ n\,0]\\ \{ST\}\end{matrix} \right\rangle_{1st\,col.}$$

$$= \left\langle \begin{matrix}[n+1\ n\,0]\\ \{S,T\}\end{matrix} \right| E_{-10}E_{10} \left| \begin{matrix}[n+1\ n\,0]\\ \{ST\}\end{matrix} \right\rangle_{1st\,col.} = 0$$

$$\left\langle \begin{matrix}[n+1\ n\,0]\\ \{S,T\}\end{matrix} \right| E_{-1-1}E_{11} \left| \begin{matrix}[n+1\ n\,0]\\ \{ST\}\end{matrix} \right\rangle_{1st\,col.} = 0$$

and

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right| O_{-11}\, O_{1-1} \left| \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right\rangle_{1st.\ col.} = (S+\tfrac{1}{2})(T-\tfrac{1}{2})$$

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right| O_{10}\, O_{-10} \left| \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right\rangle_{1st.\ col.} = \frac{(S-\frac{1}{2})(S+T+1)}{2S\,(T+1)}$$

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right| O_{01}\, O_{0-1} \left| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle_{1st.\ col.} = \frac{(T-\frac{1}{2})(S+T+1)}{2T\,(S+1)}$$

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S,T\} \end{matrix} \right| O_{11}\, O_{-1-1} \left| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle_{1st\ col.} = \frac{(S-\frac{1}{2})(T-\frac{1}{2})(S+T+1)}{ST}$$

(3.D.4)

(b) The second column: By using (3.D.3) and (3.D.4)

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right| O_{0-1}\, O_{01} \left| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S,T+1\} \end{matrix} \right| O_{01}\, O_{0-1} \left| \begin{matrix} [n+1\ n\ 0] \\ \{S,T+1\} \end{matrix} \right\rangle_{1st.\ col.} = \frac{(T+\frac{1}{2})(S+T+2)}{2(T+1)(S+1)}$$

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right| O_{-10}\, O_{10} \left| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle_{2nd\ col}$$

$$= \left\langle \begin{matrix} [n+1\ n\ 0] \\ \{S+1\ T\} \end{matrix} \right| O_{10}\, O_{-10} \left| \begin{matrix} [n+1\ n\ 0] \\ \{S+1\ T\} \end{matrix} \right\rangle_{1st\ col.} = \frac{(S+\frac{1}{2})(S+T+2)}{2(T+1)(S+1)}$$

In the 2nd column S+T = n, therefore

$$E_{11}\left|\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right\rangle = 0$$

then

$$\left\langle\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right| E_{0-1}E_{01}\left|\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right\rangle_{2nd\ col.} = \frac{(T+\frac{1}{2})(S+T+2)}{2(S+1)(T+1)}$$

$$\left\langle\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right| E_{-10}E_{10}\left|\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right\rangle_{2nd\ col.} = \frac{(S+\frac{1}{2})(S+T+2)}{2(S+1)(T+1)}$$

(3.D.5)

and

$$\left\langle\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right| O_{-11}O_{1-1}\left|\begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix}\right\rangle_{2nd\ col}$$

$$= \left\langle\begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T-1\}\end{matrix}\right| O_{1-1}O_{-11}\left|\begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T-1\}\end{matrix}\right\rangle_{2nd\ col}$$

(3.D.6)

with

$$X_{ST} = \left\langle\begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix}\right| E_{-11}E_{1-1}\left|\begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix}\right\rangle_{2nd\ col}$$

Equation (2.D.6) can then be written as a recursion equation

$$X_{ST} = X_{S+1\,T-1} + \frac{(n+2)(n+\frac{3}{2})(S-T+1)}{2(S+1)(S+2)T(T+1)} + (S-T+2)$$

(3.D.7)

with the initial term

$$X_{n-\frac{1}{2}\,\frac{1}{2}} = \frac{2n(n+2)}{3(n+\frac{1}{2})}$$

which is derived from

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{n-\frac{1}{2}\ \frac{1}{2}\}\end{matrix} \middle| O_{-11}\, O_{1-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{n-\frac{1}{2}\ \frac{1}{2}\}\end{matrix} \right\rangle = 0$$

The recursion gives

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix} \middle| E_{-11}\, E_{1-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix} \right\rangle_{\text{2nd col.}} = \frac{(S+\frac{1}{2})(T+\frac{1}{2})(S+T+2)}{(S+1)(T+1)} + (S+\frac{1}{2})(T-\frac{1}{2})$$

(3.D.8)

The only matrix element left is that for $E_{00}E_{00}$ which can be obtained from (3.D.2)

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S,T\}\end{matrix} \middle| E_{00}\, E_{00} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \right\rangle = \frac{1}{4} + \frac{(S+T+2)}{2(S+1)(T+1)}$$

Once the above five type of E operators are worked out, one gets, by re-grouping the terms

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| O_{10}O_{-10} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle_{2nd\ col} = \frac{(S-\frac{1}{2})(S+1)}{2S(T+1)^2}$$

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| O_{01}O_{0-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle_{2nd\ col.} = \frac{(T-\frac{1}{2})(T+1)}{2T(S+1)^2}$$

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| O_{11}O_{-1-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle_{2nd\ col.} = \frac{(S-\frac{1}{2})(T-\frac{1}{2})(S+T+1)}{ST}$$

(c) The kth column: The general procedure to work out the matrix elements for the kth column is as follows:

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| O_{-1-1}O_{11} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle_{k^{th}\ col.} = \left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T+1\}\end{matrix} \middle| O_{11}O_{-1-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T+1\}\end{matrix} \right\rangle_{(k-2)^{th}\ col}$$

Since the matrix element of the $(k-2)^{th}$ column is known one immediately obtains

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \middle| E_{-1-1}E_{11} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{ST\}\end{matrix} \right\rangle$$

with this knowledge and the knowledge of the matrix elements for the $(k-1)^{th}$ column

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \middle| O_{-10}\,O_{10} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \right\rangle_{k^{th}\,col}$$

$$= \left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T\}\end{matrix} \middle| O_{10}\,O_{-10} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T\}\end{matrix} \right\rangle_{(k-1)^{th}\,col.}$$

One obtains

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \middle| E_{-10}\,E_{10} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \right\rangle_{k^{th}\,col.}$$

and similarly

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \middle| E_{0-1}\,E_{01} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \right\rangle_{k^{th}\,col.}$$

Having determined the above three matrix elements, one gets to

$$\left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \middle| O_{-11}\,O_{1-1} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S\,T\}\end{matrix} \right\rangle_{k^{th}\,col.}$$

$$= \left\langle \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T-1\}\end{matrix} \middle| O_{1-1}\,O_{-11} \middle| \begin{matrix}[n+1\ n\ 0]\\ \{S+1\ T-1\}\end{matrix} \right\rangle_{k^{th}\,col.}$$

A recursion similar to (3.D.7) is then set up.

An initial condition is derived from

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{n+\frac{3}{2}-k, \frac{1}{2}\} \end{matrix} \middle| O_{-11} O_{1-1} \middle| \begin{matrix} [n+1\ n\ 0] \\ \{n+\frac{3}{2}-k, \frac{1}{2}\} \end{matrix} \right\rangle = 0$$

and

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \middle| E_{-11} E_{1-1} \middle| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle_{k^{th}\,col}$$

is then determined. By using these 4 known matrix elements and (3.D.2)

$$\left\langle \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \middle| E_{00} E_{00} \middle| \begin{matrix} [n+1\ n\ 0] \\ \{ST\} \end{matrix} \right\rangle_{k^{th}\,col}$$

is determined. By regrouping the five EE matrix elements, the five OO matrix elements are obtained from them. These correspond to the five normalization constants N. Table 3.3 gives the results. The results have a different algebraic form for the columns with S+T = even, and S+T = odd.

TABLE 3.3

MATRIX ELEMENT $\left\langle \begin{matrix}[n+1\ n\ o]\\ \{ST\}\end{matrix} \right| E_{-\alpha-\beta}\, E_{\alpha\beta} \left| \begin{matrix}[n+1\ n\ o]\\ \{ST\}\end{matrix} \right\rangle$ AND

NORMALIZATION CONSTANT OF O OPERATOR IN [n+1 n o]

	$(-1)^{S+T+n+1} = 1$	$(-1)^{S+T+n+1} = -1$
$\overline{E_{-1-1}E_{11}}$	$\frac{(2S+1)(2T+1)(n+S+T+5)(n-S-T+1)}{16(S+1)(T+1)}$	$\frac{(2S+1)(2T+1)(n+S+T+4)(n-S-T)}{16(S+1)(T+1)}$
$\overline{E_{-10}E_{10}}$	$\frac{(2S+1)(n-S-T+1)(n+S+T)}{8(S+1)(T+1)}$	$\frac{(2S+1)(n+S+T+4)(n-S-T+1)}{8(S+1)(T+1)}$
$\overline{E_{-11}E_{1-1}}$	$(S+\frac{1}{2})(T-\frac{1}{2})+\frac{(2S+1)(2T+1)(n+S+T+3)(n-S-T+1)}{16(S+1)(T+1)}$	$(S+\frac{1}{2})(T-\frac{1}{2})+\frac{(2S+1)(2T+1)(n-S-T+2)(n+S+T+4)}{16(S+1)(T+1)}$
$\overline{E_{0-1}E_{01}}$	$\frac{(2T+1)(n-S-T+1)(n+S+T+4)}{8(S+1)(T+1)}$	$\frac{(2T+1)(n-S-T+1)(n+S+T+4)}{8(S+1)(T+1)}$
$\overline{E_{00}E_{00}}$	$\frac{1}{4}+\frac{(n-S-T+1)(n+S+T+4)}{4(S+1)(T+1)}$	$\frac{1}{4}+\frac{(n-S-T+1)(n+S+T+4)}{4(S+1)(T+1)}$
$\overline{O_{11}O_{-1-1}}$	$\frac{(2S-1)(2T-1)(n+S+T+3)(n-S-T+3)}{16ST}$	$\frac{(2S-1)(2T-1)(n+S+T+2)(n-S-T+2)}{16ST}$
$\overline{O_{10}O_{-10}}$	$\frac{(2S-1)(n-S+T+3)(n+S+T+3)}{16S(T+1)^2}$	$\frac{(2S-1)(n+S-T+2)(n-S-T+2)}{16S(T+1)^2}$
$\overline{O_{01}O_{0-1}}$	$\frac{(2T-1)(n-T+S+3)(n+S+T+3)}{16T(S+1)^2}$	$\frac{(2T-1)(n+T-S+2)(n-S-T+2)}{16T(S+1)^2}$

$$\overline{A_{-\alpha-\beta}A_{\alpha\beta}} = \left\langle \begin{matrix}[n+1\ n\ o]\\ \{ST\}\end{matrix} \right| A_{-\alpha-\beta}A_{\alpha\beta} \left| \begin{matrix}[n+1\ n\ o]\\ \{ST\}\end{matrix} \right\rangle$$

E. PRELIMINARY REMARKS ON THE SU(4) WIGNER COEFFICIENTS

With the normalization constants given in (3.C.6) and Table 3.3 the full set of states for the representations [n,n,o] and [n+1,n,o] can be constructed in terms of repeated application of the O operators. With these construction and (3.B.15), matrix elements of the E operators can be calculated. Since these matrix elements form the starting point of the calculation of SU(4) Wigner coefficients and since they are themselves best expressed in terms of SU(4) Wigner coefficients, the general properties of the Wigner coefficients are discussed next.

Before proceeding to the calculation of the Wigner coefficients, some preliminary remarks are needed on the phase conventions, orthogonality, and symmetry properties of the SU(4) Wigner coefficients, as well as the generalization of the Wigner Eckart theorem.

E.1. Definitions, Orthogonality, Phase Convention

Since the only representations of interest in this work are those in which S and T fully specify the states of the representations, only SU(4) Wigner coefficients involving such representations will be discussed. (Quantum number such as ω and ϕ are not needed and will be suppressed.) The full SU(4) Wigner coefficient can be considered as the scalar product of a coupled function with the product of uncoupled functions. It can be written as

$$\left\langle \begin{matrix}[f^1] & [f^2] \\ (S_1 m_{S_1})(T_1 m_{T_1}) & (S_2 m_{S_2})(T_2 m_{T_2})\end{matrix} \middle| \begin{matrix}[f^3] \\ (S_3 m_{S_3})(T_3 m_{T_3})\end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix}[f^1] & [f^2] \\ S_1 T_1 & S_2 T_2\end{matrix} \middle\| \begin{matrix}[f^3] \\ S_3 T_3\end{matrix} \right\rangle \langle S_1 m_{S_1} S_2 m_{S_2} | S_3 m_{S_3} \rangle \langle T_1 m_{T_1} T_2 m_{T_2} | T_3 m_{T_3} \rangle$$

(3.E.1)

that is it can be factored into a reduced SU(4) [SU(2) x SU(2)] Wigner coefficient (denoted by double bar) and two ordinary SU(2) Wigner coefficients in S and T. The latter carry the entire dependence on the magnetic quantum number M_S and M_T.

From the orthogonality of the full Wigner coefficients, it can be seen that the reduced Wigner coefficients satisfy the orthogonality rule

$$\sum_{S_1 T_1 S_2 T_2} \left\langle \begin{matrix}[f^1] & [f^2] \\ S_1 T_1 & S_2 T_2\end{matrix} \middle\| \begin{matrix}[f] \\ ST\end{matrix} \right\rangle_\rho \left\langle \begin{matrix}[f^1] & [f^2] \\ S_1 T_1 & S_2 T_2\end{matrix} \middle\| \begin{matrix}[f''] \\ ST\end{matrix} \right\rangle_{\rho'}$$

$$= \delta_{[f][f'']} \delta_{\rho\rho'}$$

(3.E.2)

The indices ρ and ρ' are needed only in those cases where the products $[f_1] \times [f_2]$ are not simply reducible. For example [n, n-1, 0] x [211] contains [n, n-1, 0] twice. However, this is the only case where an index ρ is needed in the present work. All other products of actual interest in this investigation are simply reducible. Even for the case [n,n-1,o] x [211] $\rightarrow$ [n,n-1,o] the only SU(4) Wigner coefficients needed are those which give the matrix elements of the infinitesimal operators.

If the orthogonality role (3.E.2) is considered as one involving sums over row indices, there is a second orthogonality relation involving sums over column indices. This is more complicated but is not needed in the present work.

The over all phase of SU(4) Wigner coefficients is fixed by a generalized Condon and Shortley phase convention. The coefficients can be chosen to be real, and the leading coefficients connecting the state of the highest S_1 and highest T_1 with the restriction of the highest S_1 to the state of the highest S_3 and T_3 with the restriction ofthe highest S_3 is chosen to be positive. This is sufficient to determine the phase of all the reduced SU(4) Wigner coefficients of actual interest here. The only exception occurs when $[f_1] = [211]$ and $[f] = [f_1]$. In this case all three values S T = 10, 01, and 11 can connect the highest weight states. Here the phase are fixed by the further restriction

$$\left\langle \begin{matrix} [f] & [211] \\ ST & 10 \end{matrix} \middle\| \begin{matrix} [f] \\ ST \end{matrix} \right\rangle > 0 \tag{3.E.3}$$

and

$$\left\langle \begin{matrix}[f] \\ ST\end{matrix} \;\begin{matrix}[211] \\ 01\end{matrix} \right\| \left. \begin{matrix}[f] \\ ST\end{matrix} \right\rangle > 0$$

which is a natural choice since it follows from

$$S_0 \left| \begin{matrix}[f] \\ (S m_S)(T m_T)\end{matrix} \right\rangle = m_S \left| \begin{matrix}[f] \\ (S m_S)(T m_T)\end{matrix} \right\rangle$$

and

$$T_0 \left| \begin{matrix}[f] \\ (S m_S)(T m_T)\end{matrix} \right\rangle = m_T \left| \begin{matrix}[f] \\ (S m_S)(T m_T)\end{matrix} \right\rangle$$

E.2. Conjugation Properties

The representation $[f_1 f_2 f_3]$ and its conjugate $[\tilde{f}] = [f_1, f_1-f_3, f_1-f_2]$ are intimately connected. The basis vectors of an irreducible representation and their conjugates are thus also simply related.

The conjugation operator K applied to the infinitesimal operators has the simple properties.

$$\begin{aligned} K J_{ij} K^{-1} &= -J_{ij} \\ K E_{ab} K^{-1} &= -E_{-a-b} \\ K T_0 K^{-1} &= -T_0 \qquad K T_\pm K^{-1} = -T_\mp \\ K S_0 K^{-1} &= -S_0 \qquad K S_\pm K^{-1} = -S_\mp \end{aligned} \tag{3.E.4}$$

For states which can be fully specified by $SM_S TM_T$

$$K\left|[f](SM_S)(TM_T)\right\rangle = (-)^{\eta+S-M_S+T-M_T}\left|[\tilde{f}](S-M_S)(T-M_T)\right\rangle$$

(3.E.5)

The (SM_S) and (TM_T)-dependent factors are chosen according to the usual angular momentum phase conventions associated with the spin and isospin group. The η factor carries the dependence on [f] and any additional S, T-dependence.

Starting with the assumption that the one particle and one hole state are related by the simple conjugation relation.

$$\left|[100](\tfrac{1}{2}m_s)(\tfrac{1}{2}m_T)\right\rangle^{*} = (-)^{\frac{1}{2}-m_s+\frac{1}{2}-m_T}\left|\,\boxed{}\!\!\boxed{}\,(\tfrac{1}{2}-m_s)(\tfrac{1}{2}-m_T)\right\rangle$$

(3.E.6)

The two particle (or hole) states [110] can be built from the one participle states above and the Wigner coefficients

$$\left\langle \begin{matrix}[100] & [100] \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[110] \\ 10\end{matrix}\right\rangle = 1$$

$$\left\langle \begin{matrix}[100] & [100] \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[110] \\ 01\end{matrix}\right\rangle = 1$$

$$\left\langle \begin{matrix}[111] & [111] \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[110] \\ 10\end{matrix}\right\rangle = 1$$

$$\left\langle \begin{matrix}[111] & [111] \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[110] \\ 01\end{matrix}\right\rangle = -1$$

(2.E.7)

These are derived by simple recursion techniques and the knowledge of the matrix elements of the infinitesimal operator (see Section 3C and 3D).

Thus the conjugation properties of the states for [110] follow from those of (3.E.6). Similarly, states of [220],...[nno] etc. can be obtained by separated coupling with [110], so that their conugation properties follow from those of [110]. By such a build up process the conjugation relations for all representations of interest have been determined. The phase factors η are as follows:

TABLE 3.4

PHASE RELATIONS BETWEEN A REPRESENTATION AND ITS CONJUGATE

$[f]$	$(-)^{\eta}$
$[n\,n\,0]$	$(-)^{T} = (-)^{n-S}$
$[n\,n\,1]$	$(-)^{n-(S+\frac{1}{2})}$
$[n+1\,n\,0]$	$(-)^{(n+1)-(S+\frac{1}{2})}$
$[2\,1\,1]$	$(-)^{1+S-T}$

E.3 Symmetry Properties of Wigner Coefficients

With these phase relations between the vector and its conjugate few symmetry properties of Wigner coefficients can be derived.

$$(1)\ \left\langle \begin{matrix} [\tilde{f}^1] \\ S_1T_1 \end{matrix}\ \begin{matrix} [\tilde{f}^2] \\ S_2T_2 \end{matrix} \middle\| \begin{matrix} [\tilde{f}^3] \\ S_3T_3 \end{matrix} \right\rangle = (-)^{\eta_1+\eta_2-\eta_3} \left\langle \begin{matrix} [f^1] \\ S_1T_1 \end{matrix}\ \begin{matrix} [f^2] \\ S_2T_2 \end{matrix} \middle\| \begin{matrix} [f^3] \\ S_3T_3 \end{matrix} \right\rangle$$

(3.E.8)

Proof:

$$\sum_{\substack{S_1 T_1 S_2 T_2\\ M_{S_1} M_{T_1} M_{S_2} M_{T_2}}} \left\langle \begin{matrix}[f^1]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \;\; \begin{matrix}[f^2]\\(S_2 M_{S_2})(T_2 M_{T_2})\end{matrix} \middle| \begin{matrix}[f^3]\\(S_3 M_{S_3})(T_3 M_{T_3})\end{matrix} \right\rangle \times$$

$$\left| \begin{matrix}[f^1]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \right\rangle \left| \begin{matrix}[f^2]\\(S_2 M_{S_2})(T_2 M_{T_2})\end{matrix} \right\rangle$$

$$= \left| \begin{matrix}[f^3]\\(S_3 M_{S_3})(T_3 M_{T_3})\end{matrix} \right\rangle$$

Taking complex conjugates

$$\sum_{\substack{S_1 T_1 S_2 T_2\\ M_{S_1} M_{T_1} M_{S_2} M_{T_2}}} (-)^{[S_1]+[S_2]+[T_1]+[T_2]+\eta_1+\eta_2} \left\langle \begin{matrix}[f^1]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \;\; \begin{matrix}[f^2]\\(S_2 M_{S_2})(T_2 M_{T_2})\end{matrix} \middle| \begin{matrix}[f^3]\\(S_3 M_{S_3})(T_3 M_{T_3})\end{matrix} \right\rangle$$

$$\times \left| \begin{matrix}[\tilde{f}^1]\\(S_1 -M_{S_1})(T_1 -M_{T_1})\end{matrix} \right\rangle \left| \begin{matrix}[\tilde{f}^2]\\(S_2 -M_{S_2})(T_2 -M_{T_2})\end{matrix} \right\rangle = \left| \begin{matrix}[\tilde{f}^3]\\(S_3 -M_{S_3})(T_3 -M_{T_3})\end{matrix} \right\rangle (-)^{[S_3]+[T_3]+\eta_3}$$

Where $[S_i] = S_i - M_{S_i}$, $[T_i] = T_i - M_{T_i}$

Taking the scalar product with the uncoupled states $[\tilde{f}_1]$, $[\tilde{f}_2]$ from the left

$$\left\langle \begin{matrix}[f^1]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \;\; \begin{matrix}[f^2]\\(S_2 M_{S_2})(T_2 M_{T_2})\end{matrix} \middle| \begin{matrix}[f^3]\\(S_3 M_{S_3})(T_3 M_{T_3})\end{matrix} \right\rangle (-)^{[S_1]+[S_2]+[T_1]+[T_2]+\eta_1+\eta_2}$$

$$= \left\langle \begin{matrix}[\tilde{f}^1]\\(S_1 -M_{S_1})(T_1 -M_{T_1})\end{matrix} \;\; \begin{matrix}[\tilde{f}^2]\\(S_2 -M_{S_2})(T_2 -M_{T_2})\end{matrix} \middle| \begin{matrix}[\tilde{f}^3]\\(S_3 -M_{S_3})(T_3 -M_{T_3})\end{matrix} \right\rangle (-)^{[S_3]+[T_3]+\eta_3}$$

With (3.E.1) to express the full Wigner coefficients in terms of reduced Wigner coefficients and maing use of the well known symmetry properties of the ordinary SU(2) Wigner coefficients, one then obtain (3.E.8).

$$(2)\quad \left\langle \begin{matrix}[f^1] & [f^2] \\ S_1T_1 & S_2T_2\end{matrix} \middle\| \begin{matrix}[f^3] \\ S_3T_3\end{matrix}\right\rangle = (-)^{S_1+S_2-S_3+T_1+T_2-T_3+\eta_2+\sigma} \times$$

$$\sqrt{\frac{\dim[f^3]\,(2S_1+1)(2T_1+1)}{\dim[f^1]\,(2S_3+1)(2T_3+1)}}\left\langle \begin{matrix}[f^3] & [\tilde{f}_2] \\ S_3\,T_3 & S_2T_2\end{matrix} \middle\| \begin{matrix}[f_1] \\ S_1T_1\end{matrix}\right\rangle$$

(3.E.9)

Where dim [f] stands for the dimension of the irreducible representation $[f_1,f_2,f_3]$

$$\dim[f_1,f_2,f_3] = \frac{(f_1+3)(f_2+2)(f_3+1)(f_1-f_2+1)(f_1-f_3+2)(f_2-f_3+1)}{12}$$

(3.E.10)

The phase factor σ is chosen to be consistent with the phase convention for the Wigner coefficients. The proof of this relation follows from standard techniques (see, for example de Swart's approach to the symmetry properties of SU(3) Wigner coefficients[11])

$$(3)\quad \left\langle \begin{matrix}[f^1] & [f^2] \\ S_1T_1 & S_2T_2\end{matrix} \middle\| \begin{matrix}[f^3] \\ S_3T_3\end{matrix}\right\rangle = (-)^{S_1+S_2-S_3+T_1+T_2-T_3+\sigma-\eta_3+\eta_1} \times$$

$$\sqrt{\frac{\dim[f^3]\,(2S_1+1)(2T_1+1)}{\dim[f^1]\,(2S_3+1)(2T_3+1)}}\left\langle \begin{matrix}[\widetilde{f^3}] & [f^2] \\ S_3T_3 & S_2T_2\end{matrix} \middle\| \begin{matrix}[\widetilde{f^1}] \\ S_1T_1\end{matrix}\right\rangle$$

(3.E.11)

The proof of this relation follows from the combination of (3.E.8) and (3.E.9). A Table of σ for some of the case of interest is listed here

TABLE 3.5a

PHASE FACTOR σ FOR SOME SIMPLE REPRESENTATION
WITH $[f^2] = [110]$

$[f^1]$	$[f^3]$	σ
$[n\ n\ 0]$	$[n{+}1\ \ n{+}1\ \ 0]$	0
$[n\ n\ 0]$	$[n{-}1\ \ n{-}1\ \ 0]$	0
$[n{+}1\ \ n\ \ 0]$	$[n{+}2\ \ n{+}1\ \ 0]$	0
$[n{+}1\ \ n\ \ 0]$	$[n{+}1\ \ n{+}1\ \ 1]$	1
$[n{+}1\ \ n\ \ 0]$	$[n\ \ n{-}1\ \ 0]$	0
$[n\ \ n\ \ 1]$	$[n{+}1\ \ n{+}1\ \ 1]$	0
$[n\ \ n\ \ 1]$	$[n\ \ n{-}1\ \ 0]$	1
$[n\ \ n\ \ 1]$	$[n{-}1\ \ n{-}1\ \ 1]$	0

TABLE 3.5b

PHASE FACTOR σ FOR SOME SIMPLE REPRESENTATION
WITH $[f^2] = [100]$

$[f^1]$	$[f^3]$	σ
$[n\ n\ 0]$	$[n{+}1\ \ n\ \ 0]$	0
$[n\ n\ 0]$	$[n\ \ n\ \ 1]$	1
$[n{+}1\ \ n\ \ 0]$	$[n{+}1\ \ n{+}1\ \ 0]$	1
$[n\ \ n\ \ 1]$	$[n{-}1\ \ n{-}1\ \ 0]$	0

E.4. Matrix Elements of Tensor Operators, The Wigner Eckard Theorem

The matrix elements of an SU(4) tensor operator $T^{[f]}_{(s\alpha)(T\beta)}$ can be split into appropriate Wigner coefficients and SM_STM_T-independent reduced matrix elements. In general

$$\left\langle {[f''] \atop (S''M_S+\alpha)(T''M_T+\beta)} \middle| T^{[f']}_{(S'\alpha)(T'\beta)} \middle| {[f] \atop (SM_S)(TM_T)} \right\rangle =$$

$$\sum_\rho \langle [f''] \| T^{[f']} \| [f] \rangle_\rho \left\langle {[f] \atop ST} \; {[f'] \atop S'T'} \middle\| {[f''] \atop S''T''} \right\rangle_\rho \langle SM_S S'\alpha | S''M_S+\alpha \rangle \langle TM_T T'\beta | T''M_T+\beta \rangle$$

(3.E.12)

The indices ρ, however, are again needed only for those cases where the product [f] x [f'] is <u>not</u> simply reducible. The only such case needed in this work involves the representation [n,n-1,0] and the operator [211] for which there are in general two independent coefficients, with $\rho = 1$ and 2. The [211] operators arise since the infinitesimal operators which generate SU(4) have irreducible tensor character [211] as indicated in (3.B.21). The reduced matrix elements of the infinitesimal operator, however, are such that they are diagonal in [f] and can be taken to define one of the two independent coefficients of 3.E.12, say the one with $\rho = 1$. With $T^{[211]} = E$, therefore, one has

$$\langle [n,n-1,0] \| E \| [n,n-1,0] \rangle_{\rho=2} = 0$$

and only the Wigner coefficients with $\rho = 1$ are needed.

F. THE WIGNER COEFFICIENTS $\left\langle \begin{matrix}[nno]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[nno]\\ ST\end{matrix}\right\rangle$ and $\left\langle \begin{matrix}[n,n-1,o]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n,n-1o]\\ ST\end{matrix}\right\rangle$ $\rho = 1$

The E operatos are written in terms of the O operators in (3.B.15), and the tensor properties of E have been identified in (3.B.21). Further since the normalization constants have been given by (3.C.6) and Table 3.3, the Wigner coefficients can be evaluated by using (3.E.12) with [f'] = [211], [f"] = [f] = [nno]. In this simple case no index ρ is needed since the product [nno] x [211] is simply reducible.

F.1. Wigner Coefficients $\left\langle \begin{matrix}[nno]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[nno]\\ ST\end{matrix}\right\rangle$

The unit step operators do not appear for the [nno] representation

$$E_{-1-1} = O_{-1-1} + T_- S_- O_{00}\frac{1}{S_0 T_0} - S_-^2 O_{1-1}\frac{1}{(S_0+1)(2S_0+1)} - T_-^2 O_{-11}\frac{1}{(T_0+1)(2T_0+1)}$$
$$+ T_-^2 S_-^2 O_{11}\frac{1}{(T_0+1)(S_0+1)(2S_0+1)(2T_0+1)}$$

(3.F.1)

From (3.C.6) and (3.F.1), one gets

$$\left\langle \begin{matrix}[nno]\\ \{S-1\ T-1\}\end{matrix} \middle| E_{-1-1} \middle| \begin{matrix}[nno]\\ \{ST\}\end{matrix}\right\rangle = \left\langle \begin{matrix}[nno]\\ \{S-1\ T-1\}\end{matrix} \middle| O_{-1-1} \middle| \begin{matrix}[nno]\\ \{ST\}\end{matrix}\right\rangle$$
$$= \sqrt{\frac{ST(n-S+T+2)(n+S+T+2)}{(2S+1)(2T+1)}}$$

(3.F.2)

From (3.B.21) and (3.B.22)

$$\left\langle \begin{matrix}[nn0]\\ \{S-1\ T-1\}\end{matrix} \middle| E_{-1-1} \middle| \begin{matrix}[nn0]\\ \{ST\}\end{matrix} \right\rangle = -\left\langle \begin{matrix}[nn0]\\ \{S-1\ T-1\}\end{matrix} \middle| T^{[211]}_{(1-1)(1-1)} \middle| \begin{matrix}[nn0]\\ \{ST\}\end{matrix} \right\rangle$$

$$= -\left\langle \begin{matrix}[nn0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[nn0]\\ S-1\ T-1\end{matrix} \right\rangle \langle SS;\, 1-1 \,|\, S-1\ S-1\rangle \times$$

$$\langle TT;\, 1-1 \,|\, T-1\ T-1\rangle \langle [nn0] \| T \| [nn0]\rangle$$

$$= -\sqrt{\frac{(2S-1)(2T-1)}{(2S+1)(2T+1)}} \langle [nn0] \| T \| [nn0]\rangle \left\langle \begin{matrix}[nn0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[nn0]\\ S-1\ T-1\end{matrix} \right\rangle$$

Putting back into (3.F.2)

$$\left\langle \begin{matrix}[nn0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[nn0]\\ S-1\ T-1\end{matrix} \right\rangle \langle [nn0] \| T \| [nn0]\rangle = -\sqrt{\frac{ST(n-S-T+2)(n+S+T+2)}{(2S-1)(2T-1)}}$$

By shifting indices, and using

$$\langle [nn0] \| T \| [nn0]\rangle = \sqrt{n^2+4n}$$

one obtains

$$\left\langle \begin{matrix} [nno] & [211] \\ S+1\ T+1 & 11 \end{matrix} \middle\| \begin{matrix} [nno] \\ ST \end{matrix} \right\rangle = -\sqrt{\frac{ST(n-S-T+2)(n+S+T+2)}{(n^2+4n)(2S+1)(2T+1)}}$$

Similarly all the other Wigner coefficients can be evaluated. They are tabulated in Table 3.6.

TABLE 3.6

WIGNER COEFFICIENTS FOR THE COUPLING
[nno] x [211] → [nno]

S_1	T_1	S_2T_2	$\left\langle \begin{matrix} [nno] & [211] \\ S_1T_1 & S_2T_2 \end{matrix} \middle\| \begin{matrix} [nno] \\ ST \end{matrix} \right\rangle$
$S+1$	$T+1$	11	$\sqrt{\frac{(S+1)(T+1)(n-S-T)(n+S+T+4)}{(2S+1)(2T+1)(n^2+4n)}}$
$S+1$	$T-1$	11	$\sqrt{\frac{T(S+1)(n+T-S+1)(n+S-T+3)}{(2S+1)(2T+1)(n^2+4n)}}$
$S-1$	$T+1$	11	$\sqrt{\frac{S(T+1)(n+S-T+1)(n+T-S+3)}{(2S+1)(2T+1)(n^2+4n)}}$
$S-1$	$T-1$	11	$-\sqrt{\frac{ST(n-S-T+2)(n+S+T+2)}{(2S+1)(2T+1)(n^2+4n)}}$
S	T	10	$\sqrt{\frac{S(S+1)}{n^2+4n}}$
S	T	01	$\sqrt{\frac{T(T+1)}{n^2+4n}}$

F.2. The Wigner Coefficients $\left\langle \begin{matrix}[n\ n\text{-}1\ o] \\ S_1T_1\end{matrix} \begin{matrix}[211] \\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n\ n\text{-}1\ o] \\ ST\end{matrix} \right\rangle\ \rho = 1$

Since we are interested in the matrix elements of the infinitesimal operators, only SU(4) Wigner coefficients of type $\rho = 1$ are considered for the coupling [n, n-1, o] x [211] → [n, n-1, o]. Henceforth this will be understood and the index $\rho = 1$ will not be explicitly indicated.

The E operators can be written in terms of O operators, for example

$$
\begin{aligned}
E_{-1-1} = {} & O_{-1-1} + S_- O_{0-1}\frac{1}{S_0} + T_- O_{-10}\frac{1}{T_0} + T_- S_- O_{00}\frac{1}{S_0 T_0} \\
& - S_-^2 O_{1-1}\frac{1}{(S_0+1)(2S_0+1)} - T_-^2 O_{-11}\frac{1}{(T_0+1)(2T_0+1)} \\
& + T_-^2 S_- O_{01}\frac{1}{S_0(2T_0+1)(T_0+1)} - T_- S_-^2 O_{10}\frac{1}{T_0(2S_0+1)(S_0+1)} \\
& + T_-^2 S_-^2 O_{11}\frac{1}{(T_0+1)(S_0+1)(2S_0+1)(2T_0+1)}
\end{aligned}
\tag{3.B.15}
$$

From Table 3.3 and (3.B.15)

$$
\left\langle \begin{matrix}[n\ n\text{-}1\ o] \\ \{S-1\ T-1\}\end{matrix} \middle| E_{-1-1} \middle| \begin{matrix}[n\ n\text{-}1\ o] \\ \{ST\}\end{matrix} \right\rangle = \left\langle \begin{matrix}[n\ n\text{-}1\ o] \\ \{S-1\ T-1\}\end{matrix} \middle| O_{-1-1} \middle| \begin{matrix}[n\ n\text{-}1\ o] \\ \{ST\}\end{matrix} \right\rangle
$$

There are two cases

(a) For n-S-T = even:

$$\left\langle \begin{matrix}[n\ n-1\ 0]\\ \{S-1\ T-1\}\end{matrix} \middle| E_{-1-1} \middle| \begin{matrix}[n\ n-1\ 0]\\ \{S\,T\}\end{matrix} \right\rangle = -\left\langle \begin{matrix}[n\ n-1\ 0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[n\ n-1\ 0]\\ S-1\ T-1\end{matrix} \right\rangle$$

$$\langle S\,S\,1\,-1 | S-1\,S-1\rangle \langle T\,T\,1\,-1 | T-1\,T-1\rangle$$

$$\langle [n\ n-1\ 0] \| T \| [n\ n-1\ 0] \rangle$$

$$= -\sqrt{\frac{(2S-1)(2T-1)}{(2S+1)(2T+1)}}\ \langle [n\ n-1\ 0] \| T \| [n\ n-1\ 0] \rangle \left\langle \begin{matrix}[n\ n-1\ 0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[n\ n-1\ 0]\\ S-1\ T-1\end{matrix} \right\rangle$$

$$= \sqrt{\frac{(2S-1)(2T-1)(n+S+T+2)(n-S-T+2)}{16ST}}$$

Then

$$\langle [n\ n-1\ 0] \| T \| [n\ n-1\ 0] \rangle \left\langle \begin{matrix}[n\ n-1\ 0]\\ ST\end{matrix}\ \begin{matrix}[211]\\ 11\end{matrix} \middle\| \begin{matrix}[n\ n-1\ 0]\\ S-1\ T-1\end{matrix} \right\rangle$$

$$= -\sqrt{\frac{(2S+1)(2T+1)(n+S+T+2)(n-S-T+2)}{16ST}}$$

(b) For $n-S-T$ = odd:

$$\left\langle \begin{matrix}[n\ n-1\ 0]\\ \{S-1\ T-1\}\end{matrix} \middle| E_{-1-1} \middle| \begin{matrix}[n\ n-1\ 0]\\ \{S\,T\}\end{matrix} \right\rangle = -\sqrt{\frac{(2S-1)(2T-1)(n+S+T+1)(n-S-T+1)}{16ST}}$$

Then

$$\langle [n\ n\text{-}1\ o] \| T \| [n\ n\text{-}1\ o] \rangle \left\langle \begin{matrix} [n\ n\text{-}1\ o] \\ ST \end{matrix} \begin{matrix} [211] \\ 11 \end{matrix} \middle\| \begin{matrix} [n\ n\text{-}1\ o] \\ S\text{-}1\ T\text{-}1 \end{matrix} \right\rangle$$

$$= - \sqrt{\frac{(2S+1)(2T+1)(n+S+T+1)(n-S-T+1)}{16ST}}$$

With $< [n\ n\text{-}1\ o] \| T \| [n\ n\text{-}1\ o] > \; = \sqrt{n^2+3n-\frac{1}{4}}$ this gives the desired Wigner coefficients for the two cases. The other Wigner coefficients are calculated in a similar way. The results are tabulated in Table 3.7.

G. WIGNER COEFFICIENTS $< \begin{matrix} [f^1] \\ S_1T_1 \end{matrix} \begin{matrix} [100] \\ \frac{1}{2}\,\frac{1}{2} \end{matrix} \| \begin{matrix} [f] \\ ST \end{matrix} >$; COUPLING WITH ONE PARTICLE REPRESENTATION

With the knowledge of the Wigner coefficients of Tables 3.6 and 3.7, the matrix elements of the infinitesimal operators can be used to calculate further Wigner coefficients by recursion techniques. The SU(4) Wigner coefficients needed for the pairing calculations are those in which the SU(4) irreducible representations [nno], [n, n-1, o] or [nn1] are coupled with the 2 particle representations [110] to representation of the preceeding three types. These are most easily calculated by a build up process from the simpler Wigner coefficients involving the coupling with 1-particle representation [100], therefore these latter are calculated first.

G.1. The Wigner Coefficients $< \begin{matrix} [n\ n\text{-}1\ o] \\ S_1T_1 \end{matrix} \begin{matrix} [100] \\ \frac{1}{2}\,\frac{1}{2} \end{matrix} \| \begin{matrix} [nno] \\ ST \end{matrix} >$

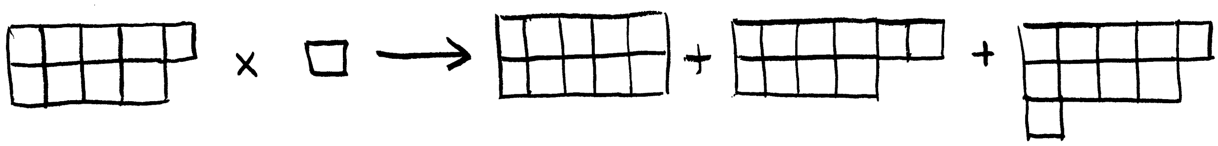
The coupling of [n n-1 o] with [100] yields the representations [nno], [n+1, n-1, o] and [n, n-1, 1].

TABLE 3.7

WIGNER COEFFICIENT $\left\langle \begin{matrix}[n\ n{-}1\ 0]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n\ n{-}1\ 0]\\ ST\end{matrix} \right\rangle_{\rho=1}$

S_1	T_1	S_2T_2	$(-1)^{n+S+T}=1$	$(-)^{n+S+T}=-1$
$S+1$	$T+1$	11	$-\sqrt{\dfrac{(2T+3)(2S+3)(n-S-T)(n+S+T+4)}{4(S+1)(T+1)(4n^2+12n-1)}}$	$-\sqrt{\dfrac{(2T+3)(2S+3)(n-S-T-1)(n+S+T+3)}{4(S+1)(T+1)(4n^2+12n-1)}}$
$S+1$	T	11	$-\sqrt{\dfrac{(2S+3)(n-S-T)(n+S-T+2)}{4(S+1)T(T+1)(4n^2+12n-1)}}$	$-\sqrt{\dfrac{(2S+3)(n-S+T+1)(n+S+T+3)}{4(S+1)T(T+1)(4n^2+12n-1)}}$
$S+1$	$T-1$	11	$\sqrt{\dfrac{(2S+3)(2T-1)(n-S+T)(n+S-T+2)}{4T(S+1)(4n^2+12n-1)}}$	$\sqrt{\dfrac{(2S+3)(2T-1)(n+S-T+3)(n-S+T+1)}{4T(S+1)(4n^2+12n-1)}}$
S	$T+1$	11	$-\sqrt{\dfrac{(2T+3)(n-S-T)(n-S+T+2)}{4S(S+1)(T+1)(4n^2+12n-1)}}$	$-\sqrt{\dfrac{(2T+3)(n+S-T+1)(n+S+T+3)}{4S(S+1)(T+1)(4n^2+12n-1)}}$
S	T	11	$\dfrac{-[(n+\frac{3}{2})+2(S+\frac{1}{2})(T+\frac{1}{2})]}{\sqrt{4ST(S+1)(T+1)(4n^2+12n-1)}}$	$\dfrac{-[(n+\frac{3}{2})-2(S+\frac{1}{2})(T+\frac{1}{2})]}{\sqrt{4ST(S+1)(T+1)(4n^2+12n-1)}}$
S	$T-1$	11	$\sqrt{\dfrac{(2T-1)(n+S-T+2)(n+S+T+2)}{4S(S+1)T(4n^2+12n-1)}}$	$\sqrt{\dfrac{(2T-1)(n-S-T+1)(n-S+T+1)}{4S(S+1)T(4n^2+12n-1)}}$
$S-1$	$T+1$	11	$\sqrt{\dfrac{(2S-1)(2T+3)(n-S+T+2)(n+S-T)}{4S(T+1)(4n^2+12n-1)}}$	$\sqrt{\dfrac{(2S-1)(2T+3)(n+S-T+1)(n-S+T+3)}{4S(T+1)(4n^2+12n-1)}}$
$S-1$	T	11	$\sqrt{\dfrac{(2S-1)(n-S+T+2)(n+S+T+2)}{4ST(T+1)(4n^2+12n-1)}}$	$\sqrt{\dfrac{(2S-1)(n-S-T+1)(n+S-T+1)}{4ST(T+1)(4n^2+12n-1)}}$
$S-1$	$T-1$	11	$-\sqrt{\dfrac{(n+S+T+2)(n-S-T+2)(2S-1)(2T-1)}{4ST(4n^2+12n-1)}}$	$-\sqrt{\dfrac{(n-S-T+1)(n+S+T+1)(2S-1)(2T-1)}{4ST(4n^2+12n-1)}}$
S'	T'	10	$2\sqrt{\dfrac{S(S+1)}{(4n^2+12n-1)}}\,\delta_{SS'}\delta_{TT'}$	$2\sqrt{\dfrac{S(S+1)}{(4n^2+12n-1)}}\,\delta_{SS'}\delta_{TT'}$
S'	T'	01	$2\sqrt{\dfrac{T(T+1)}{(4n^2+12n-1)}}\,\delta_{SS'}\delta_{TT'}$	$2\sqrt{\dfrac{T(T+1)}{(4n^2+12n-1)}}\,\delta_{SS'}\delta_{TT'}$

$$\left\langle \begin{matrix}[n\,n\,1]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n\,n\,1]\\ S_3T_3\end{matrix} \right\rangle_{\rho} = (-)^{S_3-S_1+1+S_2-T_2} \left\langle \begin{matrix}[n\ n{-}1\ 0]\\ S_1T_1\end{matrix}\ \begin{matrix}[211]\\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n\ n{-}1\ 0]\\ S_3T_3\end{matrix} \right\rangle_{\rho}$$



[n n-1 o] [100] [nno] [n+1 n-1 o] [n n-1 1]

The Wigner coefficients of interest are those involving the coupling to [nno]. These are calculated in the present paragraph.

Expanding a wave function of [nno] in terms of the base vectors of [n n-1 o] x [100], one gets

$$\left| \begin{matrix} [nno] \\ (SM_S)(TM_T) \end{matrix} \right\rangle = \sum_{\substack{M_{S_1} M_{S_2} S_1 \\ M_{T_1} M_{T_2} T_1}} \left\langle \begin{matrix} [n\ n\text{-}1\ o] \\ (S_1M_{S_1})(T_1M_{T_1}) \end{matrix} \begin{matrix} [100] \\ (\tfrac{1}{2}M_{S_2})(\tfrac{1}{2}M_{T_2}) \end{matrix} \right| \left. \begin{matrix} [n\ n\ o] \\ (SM_S)(TM_T) \end{matrix} \right\rangle \left| \begin{matrix} [n\ n\text{-}1\ o] \\ (S_1M_{S_1})(T_1M_{T_1}) \end{matrix} \right\rangle \left| \begin{matrix} [100] \\ (\tfrac{1}{2}M_{S_2})(\tfrac{1}{2}M_{T_2}) \end{matrix} \right\rangle \tag{3.G.1}$$

By operating on (3.G.1) with $E_{\alpha\beta}$ recursion relations for the SU_4 Wigner coefficients are obtained. As a special case E_{11} is chosen to illustrate the method.

$$\sum_{S'T'} \left| \begin{matrix}[n\,n\,0]\\(S' M_S+1)(T' M_T+1)\end{matrix}\right\rangle \left\langle \begin{matrix}[n\,n\,0]\\(S' M_S+1)(T' M_T+1)\end{matrix}\right| E_{11} \left| \begin{matrix}[n\,n\,0]\\(S M_S)(T M_T)\end{matrix}\right\rangle$$

$$= \sum_{\substack{M_{S_1} M_{S_2} S_1 S_1' \\ M_{T_1} M_{T_2} T_1 T_1'}} \left\langle \begin{matrix}[n\,n{-}1\,0]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix} \right| \left. \begin{matrix}[n\,n\,0]\\(S M_S)(T M_T)\end{matrix}\right\rangle \times$$
$$\left\langle \begin{matrix}[n\,n{-}1\,0]\\(S' M_S+1)(T' M_T+1)\end{matrix}\right| E_{11} \left| \begin{matrix}[n\,n\,0]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix}\right\rangle \left| \begin{matrix}[n\,n{-}1\,0]\\(S_1' M_{S_1}+1)(T_1' M_{T_1}+1)\end{matrix}\right\rangle \left| \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix}\right\rangle$$

$$= \sum_{\substack{M_{S_1} M_{S_2} S_1 \\ M_{T_1} M_{T_2} T_1}} \left\langle \begin{matrix}[n\,n{-}1\,0]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix} \right| \left. \begin{matrix}[n\,n\,0]\\(S M_S)(T M_T)\end{matrix}\right\rangle$$
$$\left\langle \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_{S_2}+1)(\tfrac{1}{2} M_{T_2}+1)\end{matrix}\right| E_{11} \left| \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_S)(\tfrac{1}{2} M_T)\end{matrix}\right\rangle \left| \begin{matrix}[n\,n{-}1\,0]\\(S_1 M_{S_1})(T_1 M_{T_1})\end{matrix}\right\rangle \left| \begin{matrix}[1\,0\,0]\\(\tfrac{1}{2} M_{S_2}+1)(\tfrac{1}{2} M_{T_2}+1)\end{matrix}\right\rangle$$

By shifting variables and using the orthogonality of the base vectors, one gets the recursion formula

$$\sum_{S'T'} \left\langle \begin{matrix}[n\ n-1\ 0] \\ (S_1 M_{S_1})(T_1 M_{T_1})\end{matrix}\ \begin{matrix}[100]\\ (\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix} \middle| \begin{matrix}[n\ n\ 0]\\ (S'\ M_S+1)(T'\ M_T+1)\end{matrix}\right\rangle \times$$

$$\left\langle \begin{matrix}[n\ n\ 0]\\ (S'\ M_S+1)(T'\ M_T+1)\end{matrix} \middle| E_{11} \middle| \begin{matrix}[n\ n\ 0]\\ (S M_S)(T M_T)\end{matrix}\right\rangle$$

$$= \sum_{S'T'} \left\langle \begin{matrix}[n\ n-1\ 0]\\ (S' M_S-1)(T_1' M_{T_1}-1)\end{matrix}\ \begin{matrix}[100]\\ (\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix} \middle| \begin{matrix}[n\ n\ 0]\\ (S M_S)(T M_T)\end{matrix}\right\rangle \left\langle \begin{matrix}[n\ n-1\ 0]\\ (S_1 M_{S_1})(T_1 M_{T_1})\end{matrix} \middle| E_{11} \middle| \begin{matrix}[n\ n-1\ 0]\\ (S_1' M_S-1)(T_1' M_{T_1}-1)\end{matrix}\right\rangle$$

$$\left\langle \begin{matrix}[n\ n-1\ 0]\\ (S_1 M_{S_1})(T_1 M_{T_1})\end{matrix}\ \begin{matrix}[100]\\ (\tfrac{1}{2} M_{S_2}-1)(\tfrac{1}{2} M_{T_2}-1)\end{matrix} \middle| \begin{matrix}[n\ n\ 0]\\ (S M_S)(T M_T)\end{matrix}\right\rangle \left\langle \begin{matrix}[100]\\ (\tfrac{1}{2} M_{S_2})(\tfrac{1}{2} M_{T_2})\end{matrix} \middle| E_{11} \middle| \begin{matrix}[100]\\ (\tfrac{1}{2} M_{S_2}-1)(\tfrac{1}{2} M_{T_2}-1)\end{matrix}\right\rangle$$

(3.G.2)

From the restrictions on the possible S+T values of [nno], [S+T) must have the same parity as n.

Example: The coefficients $\left\langle \begin{matrix}[n\ n-1\ o] \\ S+\frac{1}{2}\ \ T+\frac{1}{2}\end{matrix}\ \begin{matrix}[100]\\ \frac{1}{2}\ \frac{1}{2}\end{matrix} \middle\| \begin{matrix}[nno]\\ ST\end{matrix} \right\rangle$

The procedure is to relate these by repeated recursion to the Wigner coefficient with $\left\langle \begin{matrix}[n\ n-1\ o] \\ n-\frac{1}{2}\ \ \frac{1}{2}\end{matrix}\ \begin{matrix}[100]\\ \frac{1}{2}\ \frac{1}{2}\end{matrix} \middle\| \begin{matrix}[nno]\\ no\end{matrix} \right\rangle$. The latter can be set equal to 1 since it corresponds to a 1 x 1 unitary transformation matrix.

Although the recursion formula (3.G.2) will in general contain a large number of terms, it collapses to a two-term recursion formula, with the choice of quantum numbers $M_S = S \quad M_T = T;\ M_{S_2} = M_{T_2} = -\frac{1}{2}$. With this choice (3.G.2) reduced to

$$\frac{\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \{S+\frac{1}{2}\ T+\frac{1}{2}\}\end{matrix}\ \begin{matrix}[1\,0\,0] \\ (\frac{1}{2}\ {-\frac{1}{2}})(\frac{1}{2}\ {-\frac{1}{2}})\end{matrix} \middle| \begin{matrix}[n\ n\ 0] \\ \{S\ T\}\end{matrix} \right\rangle}{\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \{S+\frac{3}{2}\ T+\frac{3}{2}\}\end{matrix}\ \begin{matrix}[1\,0\,0] \\ (\frac{1}{2}\ {-\frac{1}{2}})(\frac{1}{2}\ {-\frac{1}{2}})\end{matrix} \middle| \begin{matrix}[n\ n\ 0] \\ \{S+1\ T+1\}\end{matrix} \right\rangle} = \frac{\left\langle \begin{matrix}[n\ n\ 0] \\ \{S+1\ T+1\}\end{matrix} \middle| E_{11} \middle| \begin{matrix}[n\ n\ 0] \\ \{S\ T\}\end{matrix} \right\rangle}{\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \{S+\frac{3}{2}\ T+\frac{3}{2}\}\end{matrix} \middle| E_{11} \middle| \begin{matrix}[n\ n{-}1\ 0] \\ \{S+\frac{1}{2}\ T+\frac{1}{2}\}\end{matrix} \right\rangle} = \sqrt{\frac{n-S-T}{n-S-T-2}}$$

(3.G.3)

The repeat application of this recursion formula is illustrated by the following diagram

$$\begin{tikzcd}
\left\{\frac{n+S-T-1}{2},\ \frac{n-S+T-1}{2}\right\} \arrow[d, Rightarrow] & \cdots \arrow[l] & \arrow[l, "E_{11}"'] \arrow[d, Rightarrow] \{S+\frac{3}{2}\ T+\frac{3}{2}\} \arrow[d, Rightarrow, "\frac{1}{2}\frac{1}{2}"] & \{S+\frac{1}{2},\ T+\frac{1}{2}\} \arrow[l, "E_{11}"'] \arrow[d, Rightarrow, "\frac{1}{2}\frac{1}{2}"] \\
\left\{\frac{n+S-T-2}{2},\ \frac{n-S+T-2}{2}\right\} & \cdots \arrow[l] & \arrow[l, "E_{11}"'] \{S+1\ T+1\} & \{S,T\} \arrow[l, "E_{11}"']
\end{tikzcd}$$

(3.G.4)

The top row illustrates the $\{S_1T_1\}$ values of the wave function [n n-1 o] and the bottom row indicates the $\{S,T\}$ values of the wave function [nn0]. The only intermediate S_2T_2 values possible are always $\frac{1}{2}\,\frac{1}{2}$. Every application of E_{11} brings an increment of both S and T by 1. After

$(\frac{n}{2} - \frac{S}{2} - \frac{T}{2} - 1)$ successive applications of such recursion, the $\{S_1T_1\}$, $\{ST\}$ values reached are such that $S_1 + T_1 = n-1$ and $S + T = n - 2$; that is, both correspond to states in the second column of Table 3.2 and Table 3.1. The result of the successive application of (3.G.3) gives

$$\left\langle \begin{matrix}[n\; n{-}1\; 0] \\ \{S+\frac{1}{2}\; T+\frac{1}{2}\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\, -\frac{1}{2})(\frac{1}{2}\, -\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\, n\, 0] \\ \{ST\}\end{matrix} \right\rangle$$

$$= \sqrt{\frac{n-S-T}{2}} \left\langle \begin{matrix}[n\;\; n{-}1\;\; 0] \\ \left\{\frac{n+S-T-1}{2}\;\; \frac{n-S+T-1}{2}\right\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\, -\frac{1}{2})(\frac{1}{2}\, -\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\, n\, 0] \\ \left\{\frac{n+S-T-2}{2}\;\; \frac{n-S+T-2}{2}\right\}\end{matrix} \right\rangle \tag{3.G.5}$$

In order to relate the above coefficient to one with $S+T = n$ and $S_1 + T_1 = n$; that is to ST values corresponding to the first columns, a recursion formula based on the E_{01} operator has to be used. This recursion is illustrated by the diagram

$$\begin{tikzcd}
\left\{\frac{n+S-T-1}{2},\ \frac{n-S+T+1}{2}\right\} \arrow[d, Rightarrow, "\frac{1}{2}\frac{1}{2}"] & \left\{\frac{n+S-T-1}{2},\ \frac{n-S+T-1}{2}\right\} \arrow[l, "E_{01}"'] \arrow[d, Rightarrow, "\frac{1}{2}\frac{1}{2}"] \\
\left\{\frac{n+S-T}{2},\ \frac{n-S+T}{2}\right\} & \left\{\frac{n+S-T-2}{2},\ \frac{n-S+T-2}{2}\right\} \arrow[l, "E_{01}"']
\end{tikzcd} \tag{3.G.6}$$

and is given by

$$\frac{\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \left\{\frac{n+S-T-1}{2}\ \ \frac{n-S+T-1}{2}\right\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\,-\frac{1}{2})(\frac{1}{2}\,-\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\ n{-}1\ 0] \\ \left\{\frac{n+S-T-2}{2}\ \ \frac{n-S+T-2}{2}\right\}\end{matrix}\right\rangle}{\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \left\{\frac{n+S-T-1}{2}\ \ \frac{n-S+T+1}{2}\right\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\,-\frac{1}{2})(\frac{1}{2}\,-\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\ n{-}1\ 0] \\ \left(\frac{n+S-T}{2}\ \ \frac{n+S-T-2}{2}\right)\left(\frac{n-S+T}{2}\ \ \frac{n-S+T}{2}\right)\end{matrix}\right\rangle} = -\sqrt{2} \tag{3.G.7}$$

With the use of appropriate ordinary Clebsch-Gordan coefficients and (3.G.5) and (3.G.7)

$$\left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \{S+\frac{1}{2}\ T+\frac{1}{2}\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\,-\frac{1}{2})(\frac{1}{2}\,-\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\ n\ 0] \\ \{S\,T\}\end{matrix}\right\rangle$$

$$= -\sqrt{\frac{n-S-T}{n+S-T}} \left\langle \begin{matrix}[n\ n{-}1\ 0] \\ \left\{\frac{n+S-T-1}{2}\ \ \frac{n-S+T+1}{2}\right\}\end{matrix} \begin{matrix}[100] \\ (\frac{1}{2}\,\frac{1}{2})(\frac{1}{2}\,-\frac{1}{2})\end{matrix} \middle| \begin{matrix}[n\ n{-}1\ 0] \\ \left\{\frac{n+S-T}{2},\ \frac{n-S+T}{2}\right\}\end{matrix}\right\rangle \tag{3.G.8}$$

The coefficient with S+T = n can be related to that with S = n, T = 0 by a further repeated application of a recursion formula obtained by operation with E_{-11}, illustrated by

$$\begin{array}{ccc} \{S-\frac{3}{2},\ T+\frac{3}{2}\} & \xleftarrow{E_{-11}} & \{S-\frac{1}{2}\ \ T+\frac{1}{2}\} \\ \Downarrow & & \Downarrow \\ \{S-1\ \ T+1\} & \xleftarrow{E_{-11}} & \{S,\ T\} \end{array} \qquad \text{with } S+T=n$$

$$\frac{\left\langle \begin{matrix} [n\ n{-}1\ 0] \\ \{S-\frac{1}{2}\ T+\frac{1}{2}\} \end{matrix}\ \begin{matrix} [1\,0\,0] \\ (\frac{1}{2}\frac{1}{2})(\frac{1}{2}-\frac{1}{2}) \end{matrix} \middle| \begin{matrix} [n\ n\ 0] \\ \{S\,T\} \end{matrix} \right\rangle}{\left\langle \begin{matrix} [n\ n{-}1\ 0] \\ \{S-\frac{3}{2}\ T+\frac{3}{2}\} \end{matrix}\ \begin{matrix} [1\,0\,0] \\ (\frac{1}{2}\frac{1}{2})(\frac{1}{2}-\frac{1}{2}) \end{matrix} \middle| \begin{matrix} [n\ n\ 0] \\ \{S-1\ T+1\} \end{matrix} \right\rangle} = \frac{\left\langle \begin{matrix} [n\ n\ 0] \\ \{S-1\ T+1\} \end{matrix} \middle| E_{-11} \middle| \begin{matrix} [n\ n\ 0] \\ \{S,T\} \end{matrix} \right\rangle}{\left\langle \begin{matrix} [n\ n{-}1\ 0] \\ \{S-\frac{3}{2}\ T+\frac{3}{2}\} \end{matrix} \middle| E_{-11} \middle| \begin{matrix} [n\ n{-}1\ 0] \\ \{S-\frac{1}{2}\ T+\frac{1}{2}\} \end{matrix} \right\rangle}$$

$$= \sqrt{\frac{S}{S-1}}$$

Then

$$\frac{\left\langle \begin{matrix} [n\ n{-}1\ 0] \\ \{S-1\ T+\frac{1}{2}\} \end{matrix}\ \begin{matrix} [1\,0\,0] \\ (\frac{1}{2}\frac{1}{2})(\frac{1}{2}-\frac{1}{2}) \end{matrix} \middle| \begin{matrix} [n\ n\ 0] \\ \{S\,T\} \end{matrix} \right\rangle}{\left\langle \begin{matrix} [n\ n{-}1\ 0] \\ \{n+\frac{1}{2}\ \frac{1}{2}\} \end{matrix}\ \begin{matrix} [1\,0\,0] \\ (\frac{1}{2}\frac{1}{2})(\frac{1}{2}-\frac{1}{2}) \end{matrix} \middle| \begin{matrix} [n\ n\ 0] \\ \{n\ 0\} \end{matrix} \right\rangle} = \sqrt{\frac{S}{n}}$$

(3.G.9)

With (3.G.8) and (3.G.9), and use of ordinary Clebsch-Gordan coefficients such as $\langle S + \frac{1}{2}\ S + \frac{1}{2}\ \frac{1}{2} - \frac{1}{2} \mid SS \rangle = \sqrt{\frac{S+1}{2S+1}}$, the reduced SU(4) Wigner coefficient becomes

$$\left\langle \begin{matrix} [n\ n\text{-}1\ 0] \\ S+\frac{1}{2}\ T+\frac{1}{2} \end{matrix}\ \begin{matrix} [100] \\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\| \begin{matrix} [n\ n\ 0] \\ S\,T \end{matrix} \right\rangle = -\sqrt{\frac{(n-S-T)(S+1)(T+1)}{n(2S+1)(2T+1)}} \qquad (3.G.10)$$

By similar recursion techniques, the other coefficients can be calculated. The results are given in Table 3.8

TABLE 3.8

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\ n\text{-}1\ 0] \\ S_1\ T_1 \end{matrix}\ \begin{matrix} [100] \\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\| \begin{matrix} [n\ n\ 0] \\ S\,T \end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix} [n\ n\text{-}1\ 0] \\ S_1\ T_1 \end{matrix}\ \begin{matrix} [100] \\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\| \begin{matrix} [n\ n\ 0] \\ S\,T \end{matrix} \right\rangle$
$S+\frac{1}{2}$	$T+\frac{1}{2}$	$-\sqrt{\frac{(n-S-T)(S+1)(T+1)}{n(2S+1)(2T+1)}}$
$S+\frac{1}{2}$	$T-\frac{1}{2}$	$\sqrt{\frac{(n+T-S+1)\,T\,(S+1)}{n(2S+1)(2T+1)}}$
$S-\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n+S-T+1)\,S\,(T+1)}{n(2S+1)(2T+1)}}$
$S-\frac{1}{2}$	$T-\frac{1}{2}$	$-\sqrt{\frac{(n+S+T+2)\,S\,T}{n(2S+1)(2T+1)}}$

The Wigner coefficient $\left\langle \begin{matrix}[nno] \\ S_1T_1\end{matrix} \quad \begin{matrix}[100] \\ \frac{1}{2}\,\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[nn1] \\ ST\end{matrix} \right\rangle$ can be obtained by using (3.E.11) with the knowledge of Table 3.8. The results are given in Table 3.9.

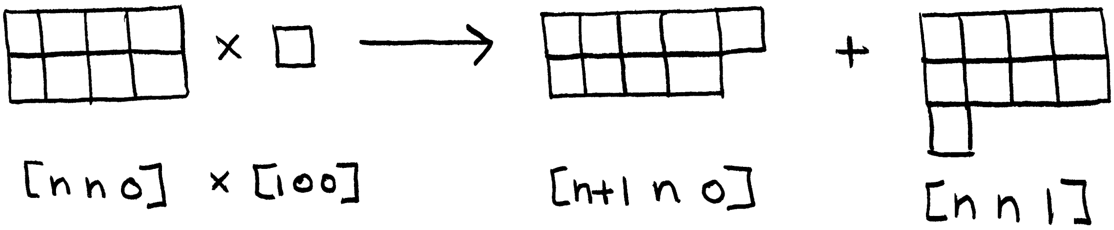
TABLE 3.9

WIGNER COEFFICIENTS $\left\langle \begin{matrix}[nno] \\ S_1T_1\end{matrix} \quad \begin{matrix}[100] \\ \frac{1}{2}\,\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[nn1] \\ ST\end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix}[nno] \\ S_1T_1\end{matrix} \quad \begin{matrix}[100] \\ \frac{1}{2}\,\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[nn1] \\ ST\end{matrix} \right\rangle$
$S+\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n+S+T+3)}{2(n+2)}}$
$S+\frac{1}{2}$	$T-\frac{1}{2}$	$\sqrt{\frac{n+S-T+2}{2(n+2)}}$
$S-\frac{1}{2}$	$T+\frac{1}{2}$	$-\sqrt{\frac{n-S+T+2}{2(n+2)}}$
$S-\frac{1}{2}$	$T-\frac{1}{2}$	$-\sqrt{\frac{n-S-T+1}{2(n+2)}}$

G.2. The Wigner Coefficients $\left\langle \begin{matrix}[nno] \\ S_1T_1\end{matrix} \quad \begin{matrix}[100] \\ \frac{1}{2}\,\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n+1\ n\ o] \\ S\ T\end{matrix} \right\rangle$

The (coupling) of [nno] with [100] yields the representations [n+1 n o] and (nn1) both of which are of interest in the work.



The coefficients of the type

$$\left\langle \begin{matrix} [nn0] \\ S_1T_1 \end{matrix} \quad \begin{matrix} [100] \\ \frac{1}{2}\frac{1}{2} \end{matrix} \;\middle\|\; \begin{matrix} [n+1\ n\ 0] \\ S\ T \end{matrix} \right\rangle$$

can be calculated by recursion formula analagous to (3.G.2). And $\left\langle \begin{matrix} [nn1] \\ S_1T_1 \end{matrix} \; \begin{matrix} [100] \\ \frac{1}{2}\frac{1}{2} \end{matrix} \;\middle\|\; \begin{matrix} [n-1\ n-1\ 0 \\ S\ T \end{matrix} \right\rangle$ can be obtained by (3.E.11). Their results are given in Tables 3.10 and Table 3.11.

H. WIGNER COEFFICIENTS $\left\langle \begin{matrix} [f^1] \\ S_1T_1 \end{matrix} \; \begin{matrix} [110] \\ S_2T_2 \end{matrix} \;\middle\|\; \begin{matrix} [f^3] \\ S_3T_3 \end{matrix} \right\rangle$; COUPLING WITH TWO PARTICLE REPRESENTATION

In general the $[f^1] \times [f^2] \rightarrow [f^3]$ coupling coefficients can be generated by recursion formulae similar to (3.G.2). In practice, at least one of the $[f]$, say $[f^2]$, is simple, so that the recursion is manageable. However, if a simple $[f^2]$ and its coupling coefficient with respect to $[f^1]$ and $[f^3]$ are known, a coupling coefficients involving a more complicate $[f^2]$ with the same $[f^1]$ and $[f^3]$ can be obtained through a build up process using techniques similar to those employed in the recoupling of angular momentum.

A recoupling process for three representations $[f_1]$, $[f_2]$, $[f_3]$ is illustrated in Figure 1 by a diagram generalized from those introduced by

TABLE 3.10

WIGNER COEFFICIENTS $\left\langle \begin{matrix}[n\,n\,0] & [1\,0\,0] \\ S_1 T_1 & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n+1\ n\ 0] \\ S\,T\end{matrix} \right\rangle$

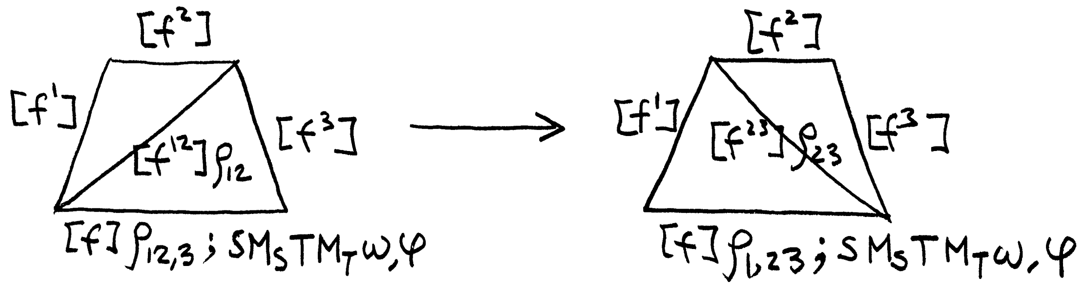
S_1	T_1	$\left\langle \begin{matrix}[n\,n\,0] & [1\,0\,0] \\ S_1 T_1 & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n+1\ n\ 0] \\ S\,T\end{matrix} \right\rangle$
$S+\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n-S-T+1)}{2(n+2)}}$
$S+\frac{1}{2}$	$T-\frac{1}{2}$	$\sqrt{\frac{(n-S+T+2)}{2(n+2)}}$
$S-\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n+S-T+2)}{2(n+2)}}$
$S-\frac{1}{2}$	$T-\frac{1}{2}$	$\sqrt{\frac{(n+S+T+3)}{2(n+2)}}$

TABLE 3.11

WIGNER COEFFICIENTS $\left\langle \begin{matrix}[n\,n\,1] & [1\,0\,0] \\ S_1 T_1 & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n-1\ n-1\ 0] \\ S\,T\end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix}[n\,n\,1] & [1\,0\,0] \\ S_1 T_1 & \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n-1\ n-1\ 0] \\ S\,T\end{matrix} \right\rangle$
$S+\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n+S+T+3)(S+1)(T+1)}{(n+3)(2S+1)(2T+1)}}$
$S+\frac{1}{2}$	$T-\frac{1}{2}$	$-\sqrt{\frac{(n+S-T+2)T(S+1)}{(n+3)(2S+1)(2T+1)}}$
$S-\frac{1}{2}$	$T+\frac{1}{2}$	$\sqrt{\frac{(n-S+T+2)S(T+1)}{(n+3)(2S+1)(2T+1)}}$
$S-\frac{1}{2}$	$T-\frac{1}{2}$	$\sqrt{\frac{(n-S-T+1)ST}{(n+3)(2S+1)(2T+1)}}$

French[12] for the recoupling of ordinary angular momentum



The recoupling process involves a transformation from a scheme in which the intermediate coupling involves $[f^1] \times [f^2]$ to one in which the intermediate coupling is $[f^2] \times [f^3]$.

Since the product representations may occur more than once, the ρ indicates which type occurs. The two coupling systems illustrated in the above figure are connected by a unitary transformation whose matrix elements are the generalized SU(4) Racah coefficients or U-coefficients.

$$
\begin{aligned}
&\left|\left(\left\{\left([f^1][f^2]\right)[f^{12}]\rho_{12}\right\}[f^3]\right)[f]\,\rho_{12,3}\,;\,SM_S, TM_T, \omega\varphi\right\rangle \\
&= \sum_{[f^{23}]\,\rho_{23}\,\rho_{1,23}} \left|\left([f^1]\left\{\left([f^2][f^3]\right)[f^{23}]\rho_{23}\right\}\right)[f]\,\rho_{1,23}\,;\,SM_S TM_T, \omega\varphi\right\rangle \times \\
&\qquad U\begin{pmatrix} [f^1] & [f^2] & ; & [f^{12}] & \rho_{12} & \rho_{12,3} \\ [f^3] & [f] & ; & [f^{23}] & \rho_{23} & \rho_{1,23} \end{pmatrix}
\end{aligned}
$$

(3.H.1)

Similar to the SU(2) U-functions which are independent of the magnetic quantum numbers, the SU(4) U-functions are independent of the SM_S TM_T $\omega\phi$ quantum numbers, and are real. They satisfy the orthogonality relation

$$\sum_{\alpha} U\left(\begin{matrix}---\alpha\\---\mu\end{matrix}\right) U\left(\begin{matrix}---\alpha\\---\mu'\end{matrix}\right) = \delta_{\mu\mu'}$$

$$\sum_{\mu} U\left(\begin{matrix}---\alpha\\---\mu\end{matrix}\right) U\left(\begin{matrix}---\alpha'\\---\mu\end{matrix}\right) = \delta_{\alpha\alpha'}$$

(3.H.2)

where α is a short band notation for $[f^{12'}]\ \rho_{12}\ \rho_{12,3}$ and μ is a short band notation for $[f^{23'}]\ \rho_{23}\ \rho_{1,23}$.

Through the decoupling process as in the case of SU(2), the U function can be related to this Wigner coefficients by

$$U\left(\begin{matrix}[f^1]\ [f^2]\ ;\ [f^{12}]\ \rho_{12}\ \rho_{12,3}\\ [f^{3'}]\ [f]\ ;\ [f^{23}]\ \rho_{23}\ \rho_{1,23}\end{matrix}\right)$$

$$= \sum_{\substack{\varepsilon_{12}\,\varepsilon_{23}\\ \varepsilon_1\,\varepsilon_2\,\varepsilon_3}} \langle [f^1]\varepsilon_1 ; [f^2]\varepsilon_2 \| [f^{12}]\varepsilon_{12}\rangle_{\rho_{12}} \langle [f^{12}]\varepsilon_{12} ; [f^3]\varepsilon_3 \| [f]\varepsilon\rangle_{\rho_{12,3}}$$
$$\langle [f^2]\varepsilon_2 ; [f^3]\varepsilon_3 \| [f^{23}]\varepsilon_{23}\rangle_{\rho_{23}} \langle [f^1]\varepsilon_1 ; [f^{23}]\varepsilon_{23} \| [f]\varepsilon\rangle_{\rho_{1,23}}$$
$$U(S_1S_2SS_3;S_{12}S_{23})\,U(T_1T_2TT_3;T_{12}T_{23})$$

(3.H.3)

where ε_i is a short band notation for $T_i\ S_i\ \omega_i\ \phi_i$. The sums over M_S and M_T have been performed and give the ordinary spin and isospin SU(2) U-functions. By using the orthogonality of the Wigner coefficients, a "reverse" formula of (3.H.1) is obtained

$$\sum_{\rho_{1,23}} \langle [f^1]\varepsilon_1 ; [f^{23}]\varepsilon_{23} \| [f]\varepsilon \rangle_{\rho_{1,23}} U\begin{pmatrix} [f^1] & [f^2] ; [f^{12}] & \rho_{12} & \rho_{12,3} \\ [f^3] & [f] ; [f^{23}] & \rho_{23} & \rho_{1,23} \end{pmatrix}$$

$$= \sum_{\varepsilon_2 \varepsilon_3 \varepsilon_{12}} \langle [f^1]\varepsilon_1 ; [f^2]\varepsilon_2 \| [f^{12}]\varepsilon_{12} \rangle_{\rho_{12}} \langle [f^{12}]\varepsilon_{12} ; [f^3]\varepsilon_3 \| [f]\varepsilon \rangle_{\rho_{12,3}}$$

$$\langle [f^2]\varepsilon_2 ; [f^3]\varepsilon_3 \| [f^{23}]\varepsilon_{23} \rangle_{\rho_{23}} U(S_1 S_2 S S_3 ; S_{12} S_{23})$$

$$U(T_1 T_2 T T_3 ; T_{12} T_{23})$$

(3.H.4)

This is the basic equation for the build-up process. Although the quantum numbers ρ have been retained in the above, the build-up process is used in this investigation only in relating simple cases where none of the quantum numbers ρ are needed. Also the labels ε are here fully specified by S T.
The two body operator of tensor character [110] can be built from the coupling of two one body operators, it is obvious that $[f_2]$ and $[f_3]$ are chosen as [100], and (3.H.4) is simplified into

$$\left\langle \begin{matrix}[f'] \\ S_1T_1\end{matrix} ; \begin{matrix}[110] \\ S_2T_2\end{matrix} \middle\| \begin{matrix}[f] \\ ST\end{matrix} \right\rangle U\begin{pmatrix}[f'] & [100] ; & [f^{12}] \\ [100] & [f] ; & [110]\end{pmatrix}$$

$$= \sum_{S_{12}, T_{12}} \left\langle \begin{matrix}[f'] \\ S_1T_1\end{matrix} \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[f^{12}] \\ S_{12}T_{12}\end{matrix} \right\rangle \left\langle \begin{matrix}[f^{12}] \\ S_{12}T_{12}\end{matrix} \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[f] \\ ST\end{matrix} \right\rangle$$

$$U(S_1 \tfrac{1}{2} S \tfrac{1}{2} ; S_{12} S_{23}) \, U(T_1 \tfrac{1}{2} T \tfrac{1}{2} ; T_{12} T_{23})$$

(3.H.5)

where in (3.H.4) has been replaced by S_2T_2, and from (3.E.7) the coefficients $< \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \| \begin{matrix}[110] \\ S_2T_2\end{matrix} >$ have been replaced by 1 for both the cases $S_2 = 1$ $T_2 = 0$ and $S_2 = 0$ $T_2 = 1$. The coefficients on the right hand side of (3.H.5) are known. The SU(4) U-function on the left can be treated as an normalization constant Θ.

Example: Calculation of coefficients for, [n n-1 0] x [110] → [n+1 n 0]

$$\left\langle \begin{matrix}[n\,n{-}1\,0] \\ S\,T{-}1\end{matrix} \; \begin{matrix}[100] \\ 0\,1\end{matrix} \middle\| \begin{matrix}[n{+}1\,n\,0] \\ S\,T\end{matrix} \right\rangle \mathcal{U}\begin{pmatrix}[n\,n{-}1\,0] & [100] & ; [n\,n\,0] \\ [100] & [n{+}1\,n\,0] & ; [110]\end{pmatrix}$$

$$= \left\langle \begin{matrix}[n\,n{-}1\,0] \\ S\;T{-}1\end{matrix} \; \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n\,n\,0] \\ S{+}\frac{1}{2}\;T{-}\frac{1}{2}\end{matrix} \right\rangle \left\langle \begin{matrix}[n\,n\,0] \\ S{+}\frac{1}{2}\;T{-}\frac{1}{2}\end{matrix} \; \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n{+}1\,n\,0] \\ S\,T\end{matrix} \right\rangle$$

$$\mathcal{U}(S\tfrac{1}{2}S\tfrac{1}{2};S{+}\tfrac{1}{2}\,0)\,\mathcal{U}(T{-}1\tfrac{1}{2}\,T\tfrac{1}{2};T{-}\tfrac{1}{2}\,1)$$

$$+ \left\langle \begin{matrix}[n\,n{-}1\,0] \\ S\;T{-}1\end{matrix} \; \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n\,n\,0] \\ S{-}\frac{1}{2}\;T{-}\frac{1}{2}\end{matrix} \right\rangle \left\langle \begin{matrix}[n\,n\,0] \\ S{-}\frac{1}{2}\;T{-}\frac{1}{2}\end{matrix} \; \begin{matrix}[100] \\ \frac{1}{2}\frac{1}{2}\end{matrix} \middle\| \begin{matrix}[n{+}1\,n\,0] \\ S\,T\end{matrix} \right\rangle$$

$$\mathcal{U}(S\tfrac{1}{2}S\tfrac{1}{2};S{-}\tfrac{1}{2}\,0)\,\mathcal{U}(T{-}1\tfrac{1}{2}\,T\tfrac{1}{2};T{-}\tfrac{1}{2}\,1)$$

$$= -\frac{1}{4}\sqrt{\frac{(2T-1)(n+S+T+2)(n-S+T+2)}{2n\,(n+2)T}} \qquad \text{for the case } (-)^{n+S+T} = 1$$

$$= -\frac{1}{4}\sqrt{\frac{(2T-1)(n+S+T+3)(n-S+T+1)}{2n\,(n+2)T}} \qquad \text{for the case } (-)^{n+S+T} = -1$$

Similarly for the other possible values of S_1T_1, S_2T_2. If all the terms of

$$\left\langle \begin{matrix}[n\,n{-}1\,0] \\ S_1T_1\end{matrix} \; \begin{matrix}[110] \\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n{+}1\,n\,0] \\ S\,T\end{matrix} \right\rangle \mathcal{U}\begin{pmatrix}[n\,n{-}1\,0] & [100] & [n\,n\,0] \\ [100] & [n{+}1\,n\,0] & [110]\end{pmatrix}$$

are calculated, the magnitude of U is determined by the restriction that

$$\sum_{S_1S_2T_1T_2} \left| \left\langle \begin{matrix}[n\,n{-}1\,0] \\ S_1T_1\end{matrix} \; \begin{matrix}[100] \\ S_2T_2\end{matrix} \middle\| \begin{matrix}[n{+}1\,n\,0] \\ S\,T\end{matrix} \right\rangle \right|^2 = 1$$

By using the phase convention that when $S_1 = n - \frac{1}{2}$ $T_1 = \frac{1}{2}$ and $S = n + \frac{1}{2}$ $T = \frac{1}{2}$, the Wigner coefficient is positive, the sign of U can be found out also, and all the Wigner coefficients of $[n\ n\text{-}1\ 0] \times [110] \rightarrow [n+1\ n\ 0]$ are determined.

Similarly, the Wigner coefficients for

$$[n\ n\ 1] \times [110] \rightarrow [n\ n\text{-}1\ 0]$$
$$[n\ n\ 1] \times [110] \rightarrow [n\text{-}1\ n\text{-}1\ 1]$$
$$[n\ n\text{-}1\ 0] \times [110] \rightarrow [n\ n\ 1]$$

are calculated, and the coefficients for

$$[n\text{-}1\ n\text{-}1\ 1] \times [110] \rightarrow [n\ n\ 1]$$

and

$$[n+1\ n\ 0] \times [110] \rightarrow [n\ n\text{-}1\ 0]$$

are derived from the conjugate relation of

$$[n\ n\text{-}1\ 0] \times [110] \rightarrow [n+1\ n\ 0]$$
$$[n\ n\ 1] \times [110] \rightarrow [n\text{-}1\ n\text{-}1\ 1]$$

respectively, using (3.E.8), (3.E.11) and shifting the n index by our unit. These are the SU(4) Wigner coefficients needed for the calculation of the matrix elements of the pair creation and annihilation operators. They are all tabulated in the following tables.

TABLE 3.12

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\,n\,0] \\ S_1 T_1 \end{matrix} \quad \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{+}1\; n{+}1\; 0] \\ S \quad T \end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix} [n\,n\,0] \\ S_1 T_1 \end{matrix} \; \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{+}1\; n{+}1\; 0] \\ S\;T \end{matrix} \right\rangle$
$S+1$	T	$-\sqrt{\dfrac{(S+1)(n-S+T+2)(n-S-T+1)}{2(n+1)(n+2)(2S+1)}}$
S	$T+1$	$-\sqrt{\dfrac{(T+1)(n-T+S+2)(n-S-T+1)}{2(n+1)(n+2)(2T+1)}}$
S	$T-1$	$\sqrt{\dfrac{T(n+S+T+3)(n+T-S+2)}{2(n+1)(n+2)(2T+1)}}$
$S-1$	T	$\sqrt{\dfrac{S(n+S+T+3)(n+S-T+2)}{2(n+1)(n+2)(2S+1)}}$

TABLE 3.13

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\,n\,0] \\ S_1 T_1 \end{matrix} \quad \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{-}1\; n{-}1\; 0] \\ S\,T \end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix} [n\,n\,0] \\ S_1 T_1 \end{matrix} \; \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{-}1\; n{-}1\; 0] \\ S\;T \end{matrix} \right\rangle$
$S+1$	T	$\sqrt{\dfrac{(n+S-T+2)(n+S+T+3)(S+1)}{2(n+2)(n+3)(2S+1)}}$
S	$T+1$	$-\sqrt{\dfrac{(n-S+T+2)(n+S+T+3)(T+1)}{2(n+2)(n+3)(2T+1)}}$
S	$T-1$	$\sqrt{\dfrac{(n-S-T+1)(n+S-T+2)T}{2(n+2)(n+3)(2T+1)}}$
$S-1$	T	$-\sqrt{\dfrac{(n-S-T+1)(n-S+T+2)S}{2(n+2)(n+3)(2S+1)}}$

TABLE 3.14

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\ n\text{-}1\ 0] \\ S_1 T_1 \end{matrix} \begin{matrix} [110] \\ S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\text{+}1\ n\ 0] \\ S T \end{matrix} \right\rangle$

$S_1 \quad T_1$	$\left\langle \begin{matrix} [n\ n\text{-}1\ 0] \\ S_1 T_1 \end{matrix} \begin{matrix} [110] \\ S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\text{+}1\ n\ 0] \\ S T \end{matrix} \right\rangle$ $(-1)^{n+S+T} = 1$	$(-)^{n+S+T} = -1$
$S+1 \quad T$	$-\frac{1}{2}\sqrt{\frac{(n-S-T)(n-S+T+2)(2S+3)}{2n(n+2)(S+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n-S-T+1)(2S+3)}{2n(n+2)(S+1)}}$
$S_2=1$ $S \quad T$	$-\frac{1}{2}\sqrt{\frac{(n+T-S+2)(n-T+S+2)}{2n(n+2)S(S+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n+S+T+3)(n-S-T+1)}{2n(n+2)S(S+1)}}$
$S-1 \quad T$	$\frac{1}{2}\sqrt{\frac{(2S-1)(n+S+T+2)(n+S-T+2)}{2n(n+2)S}}$	$\frac{1}{2}\sqrt{\frac{(2S-1)(n+S+T+3)(n+S-T+1)}{2n(n+2)S}}$
$S \quad T+1$	$-\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S-T+2)(2T+3)}{2n(n+2)(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n-S-T+1)(n+S-T+1)(2T+3)}{2n(n+2)(T+1)}}$
$T_2=1$ $S \quad T$	$-\frac{1}{2}\sqrt{\frac{(n+S-T+2)(n-S+T+2)}{2n(n+2)T(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n+S+T+3)(n-S-T+1)}{2n(n+2)T(T+1)}}$
$S \quad T-1$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n+S+T+2)(n-S+T+2)}{2n(n+2)T}}$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n+T-S+1)(n+S+T+3)}{2n(n+2)T}}$

TABLE 3.15

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\,n\text{-}1\,0] & [1\,1\,0] \\ S_1T_1 & S_2T_2 \end{matrix} \middle\| \begin{matrix} [n\,n\,1] \\ ST \end{matrix} \right\rangle$

S	T	$\left\langle \begin{matrix} [n\,n\text{-}1\,0] & [1\,1\,0] \\ S_1T_1 & S_2T_2 \end{matrix} \middle\| \begin{matrix} [n\,n\,1] \\ ST \end{matrix} \right\rangle$ $(-1)^{n+S+T} = 1$	$(-)^{n+S+T} = -1$
$S+1$	T	$-\frac{1}{4}\sqrt{\frac{(2S+3)(n+S-T+2)(n-S-T)}{(n+1)(n+2)(S+1)}}$	$-\frac{1}{4}\sqrt{\frac{(2S+3)(n-S+T+1)(n+S+T+3)}{(n+1)(n+2)(S+1)}}$
S ($S_2=1$)	T	$\frac{1}{4}\sqrt{\frac{S+1}{(n+1)(n+2)S}}(n+T-S+2)$ $+\frac{1}{4}\sqrt{\frac{S}{(n+1)(n+2)(S+1)}}(n+S-T+2)$	$\frac{1}{4}\sqrt{\frac{S}{(n+1)(n+2)(S+1)}}(n+S+T+3)$ $+\frac{1}{4}\sqrt{\frac{S+1}{(n+1)(n+2)S}}(n-S-T+1)$
$S-1$	T	$-\frac{1}{4}\sqrt{\frac{(2S-1)(n+S+T+2)(n-S+T+2)}{(n+1)(n+2)S}}$	$-\frac{1}{4}\sqrt{\frac{(2S-1)(n+S-T+1)(n-S-T+1)}{(n+1)(n+2)S}}$
S	$T+1$	$\frac{1}{4}\sqrt{\frac{(2T+3)(n-S-T)(n+T-S+2)}{(n+1)(n+2)(T+1)}}$	$-\frac{1}{4}\sqrt{\frac{(2T+3)(n+S-T+1)(n+S+T+3)}{(n+1)(n+2)(T+1)}}$
S	T ($T_2=1$)	$-\frac{1}{4}\sqrt{\frac{T+1}{(n+1)(n+2)T}}(n+S-T+2)$ $-\frac{1}{4}\sqrt{\frac{T}{(n+1)(n+2)(T+1)}}(n+T-S+2)$	$\frac{1}{4}\sqrt{\frac{T}{(n+1)(n+2)(T+1)}}(n+S+T+3)$ $\frac{1}{4}\sqrt{\frac{T+1}{(n+1)(n+2)T}}(n-S-T+1)$
S	$T-1$	$\frac{1}{4}\sqrt{\frac{(2T-1)(n+S+T+2)(n+S-T+2)}{(n+1)(n+2)T}}$	$-\frac{1}{4}\sqrt{\frac{(2T-1)(n+T-S+1)(n-S-T+1)}{(n+1)(n+2)T}}$

TABLE 3.16

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\,n\,1] & [1\,1\,0] \\ S_1 T_1 & S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\,n{-}1\,0] \\ S\,T \end{matrix} \right\rangle$

	$\left\langle \begin{matrix} [n\,n\,1] & [1\,1\,0] \\ S_1 T_1 & S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\,n{-}1\,0] \\ S\,T \end{matrix} \right\rangle$	
S T	$(-1)^{n+S+T} = 1$	$(-1)^{n+S+T} = -1$
$S+1$ T	$\frac{1}{4}\sqrt{\frac{(n+S-T+2)(n-S-T)(2S+3)}{(n+1)(n+2)(S+1)}}$	$\frac{1}{4}\sqrt{\frac{(n+S+T+3)(n+T-S+1)(2S+3)}{(n+1)(n+2)(S+1)}}$
S T $S_2=1$	$\frac{1}{4}\sqrt{\frac{S}{(n+1)(n+2)(S+1)}}(n-S-T)$ $+\frac{1}{4}\sqrt{\frac{S+1}{(n+1)(n+2)S}}(n+S+T+2)$	$\frac{1}{4}\sqrt{\frac{S+1}{(n+1)(n+2)S}}(n+S-T+1)$ $\frac{1}{4}\sqrt{\frac{S}{(n+1)(n+2)(S+1)}}(n+T-S+1)$
$S-1$ T	$\frac{1}{4}\sqrt{\frac{(n-S+T+2)(n+S+T+2)(2S-1)}{(n+1)(n+2)S}}$	$\frac{1}{4}\sqrt{\frac{(n-S-T+1)(n+S-T+1)(2S-1)}{(n+1)(n+2)S}}$
S $T+1$	$-\frac{1}{4}\sqrt{\frac{(n-S+T+2)(n-S-T)(2T+3)}{(n+1)(n+2)(T+1)}}$	$\frac{1}{4}\sqrt{\frac{(n+S+T+3)(n+S-T+1)(2T+3)}{(n+1)(n+2)(T+1)}}$
S T $T_2=1$	$\frac{1}{4}\sqrt{\frac{T}{(n+1)(n+2)(T+1)}}(n-S-T)$ $+\frac{1}{4}\sqrt{\frac{T+1}{(n+1)(n+2)T}}(n+S+T+2)$	$-\frac{1}{4}\sqrt{\frac{T+1}{(n+1)(n+2)T}}(n-S+T+1)$ $-\frac{1}{4}\sqrt{\frac{T}{(n+1)(n+2)(T+1)}}(n+S-T+1)$
S $T-1$	$-\frac{1}{4}\sqrt{\frac{(n+S-T+2)(n+S+T+2)(2T-1)}{(n+1)(n+2)T}}$	$\frac{1}{4}\sqrt{\frac{(n-S-T+1)(n-S+T+1)(2T-1)}{(n+1)(n+2)T}}$

TABLE 3.17

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n\,n\,1] \\ S_1 T_1 \end{matrix} \; \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{-}1\; n{-}1\; 1] \\ S T \end{matrix} \right\rangle$

S_1	T_1	$\left\langle \begin{matrix} [n\,n\,1] \\ S_1 T_1 \end{matrix} \; \begin{matrix} [1\,1\,0] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n{-}1\; n{-}1\; 1] \\ S T \end{matrix} \right\rangle$ $(-1)^{n-1+S+T} = 1$	$(-1)^{n-1+S+T} = -1$
$S+1$	T	$\frac{1}{2}\sqrt{\frac{(n-T+S+1)(n+S+T+3)(2S+3)}{2(n+1)(n+3)(S+1)}}$	$\frac{1}{2}\sqrt{\frac{(n-T+S+2)(n+S+T+2)(2S+3)}{2(n+1)(n+3)(S+1)}}$
S	T	$\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)S}{2(n+1)(n+3)(S+1)}}$	$\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)S}{2(n+1)(n+3)(S+1)}}$
$S_2=1$		$-\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)(S+1)}{2(n+1)(n+3)S}}$	$-\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)(S+1)}{2(n+1)(n+3)S}}$
$S-1$	T	$-\frac{1}{2}\sqrt{\frac{(2S-1)(n+T-S+1)(n-S-T+1)}{2(n+1)(n+3)S}}$	$-\frac{1}{2}\sqrt{\frac{(2S-1)(n+T-S+2)(n-S-T)}{2(n+1)(n+3)S}}$
S	$T+1$	$-\frac{1}{2}\sqrt{\frac{(n+S+T+3)(n-S+T+1)(2T+3)}{2(n+1)(n+3)(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n-S+T+2)(n+S+T+2)(2T+3)}{2(n+1)(n+3)(T+1)}}$
S	T	$-\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)(T+1)}{2(n+1)(n+3)T}}$	$\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)T}{2(n+1)(n+3)(T+1)}}$
$T_2=1$		$+\frac{1}{2}\sqrt{\frac{(n+S-T+1)(n-S+T+1)T}{2(n+1)(n+3)(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)(T+1)}{2(n+1)(n+3)T}}$
S	$T-1$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n+S-T+1)(n-S-T+1)}{2(n+1)(n+3)T}}$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n+S-T+2)(n-S-T)}{2(n+1)(n+3)T}}$

TABLE 3.18

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n+1\ n\ 0] & [110] \\ S_1 T_1 & S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\ n-1\ 0] \\ S\,T \end{matrix} \right\rangle$

	$\left\langle \begin{matrix} [n+1\ n\ 0] & [110] \\ S_1 T_1 & S_2 T_2 \end{matrix} \middle\| \begin{matrix} [n\ n-1\ 0] \\ S\,T \end{matrix} \right\rangle$	
S_1 T_1	$(-)^{n+S+T} = 1$	$(-)^{n+S+T} = -1$
$S+1$ T	$\frac{1}{2}\sqrt{\frac{(2S+3)(n+S+T+4)(n+S-T+2)}{2(n+2)(n+4)(S+1)}}$	$\frac{1}{2}\sqrt{\frac{(2S+3)(n+S+T+3)(n+S-T+3)}{2(n+2)(n+4)(S+1)}}$
$S_2=1$ S T	$\frac{1}{2}\sqrt{\frac{(n-S+T+2)(n+S-T+2)}{2(n+2)(n+4)S(S+1)}}$	$\frac{1}{2}\sqrt{\frac{(n+T+S+3)(n-T-S+1)}{2(n+2)(n+4)S(S+1)}}$
$S-1$ T	$-\frac{1}{2}\sqrt{\frac{(2S-1)(n-S-T+2)(n-S+T+2)}{2(n+2)(n+4)S}}$	$-\frac{1}{2}\sqrt{\frac{(2S-1)(n-S-T+1)(n-S+T+3)}{2(n+2)(n+4)S}}$
S $T+1$	$-\frac{1}{2}\sqrt{\frac{(2T+3)(n+T-S+2)(n+S+T+4)}{2(T+1)(n+2)(n+4)}}$	$-\frac{1}{2}\sqrt{\frac{(2T+3)(n+S+T+3)(n-S+T+3)}{2(T+1)(n+2)(n+4)}}$
$T_2=1$ S T	$-\frac{1}{2}\sqrt{\frac{(n+S-T+2)(n-S+T+2)}{2(n+2)(n+4)T(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(n+S-T+3)(n-S-T+1)}{2(n+2)(n+4)T(T+1)}}$
S $T-1$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n-S-T+2)(n+S-T+2)}{2(n+2)(n+4)T}}$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n-S-T+1)(n+S-T+3)}{2(n+2)(n+4)T}}$

TABLE 3.19

WIGNER COEFFICIENTS $\left\langle \begin{matrix} [n-1\ n-1\ 1] & [1\,1\,0] \\ S_1 T_1 & S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n\,n\,1] \\ S T \end{matrix} \right\rangle$

	$\left\langle \begin{matrix} [n-1\ n-1\ 1] & [1\,1\,0] \\ S_1 T_1 & S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n\,n\,1] \\ S T \end{matrix} \right\rangle$	
$S_1 \quad T_1$	$(-1)^{n+S+T} = 1$	$(-)^{n+S+T} = -1$
$S+1 \quad T$	$-\frac{1}{2}\sqrt{\frac{(2S+3)(n+T-S)(n-S-T)}{2(n-1)(n+1)(S+1)}}$	$-\frac{1}{2}\sqrt{\frac{(2S+3)(n+T-S+1)(n-S-T-1)}{2(n-1)(n+1)(S+1)}}$
$S \quad T$ $S_2 = 1$	$-\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)S}{2(n+1)(n-1)(S+1)}}$ $+\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)(S+1)}{2(n+1)(n-1)S}}$	$-\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)S}{2(n+1)(n-1)(S+1)}}$ $+\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)(S+1)}{2(n+1)(n-1)S}}$
$S-1 \quad T$	$\frac{1}{2}\sqrt{\frac{(n-T+S)(n+S+T+2)(2S-1)}{2(n+1)(n+3)S}}$	$\frac{1}{2}\sqrt{\frac{(n-T+S+1)(n+S+T+1)(2S-1)}{2(n+1)(n-1)S}}$
$S \quad T+1$	$-\frac{1}{2}\sqrt{\frac{(2T+3)(n+S-T)(n-S-T)}{2(n-1)(n+1)(T+1)}}$	$-\frac{1}{2}\sqrt{\frac{(2T+3)(n+S-T+1)(n-S-T-1)}{2(n-1)(n+1)(T+1)}}$
$S \quad T$ $T_2 = 1$	$\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)T}{2(n-1)(n+1)(T+1)}}$ $-\frac{1}{2}\sqrt{\frac{(n-S-T)(n+S+T+2)(T+1)}{2(n-1)(n+1)T}}$	$-\frac{1}{2}\sqrt{\frac{(n-S+T+1)(n+S-T+1)(T+1)}{2(n-1)(n+1)T}}$ $+\frac{1}{2}\sqrt{\frac{(n+S-T+1)(n-S+T+1)T}{2(n-1)(n+1)(T+1)}}$
$S \quad T-1$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n+S+T+2)(n-S+T)}{2(n-1)(n+1)T}}$	$\frac{1}{2}\sqrt{\frac{(2T-1)(n-S+T+1)(n+S+T+1)}{2(n-1)(n+1)T}}$

CHAPTER IV

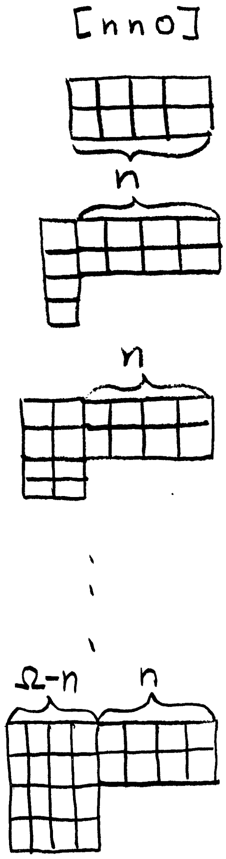
THE O(8)/O(6) PART OF THE PAIRING PROBLEM

A. INTRODUCTION

To complete the calculation of the matrix elements of the pair creation and annihilation operators, the factor dealing with the $O(8) \supset O(6)$ segment of the group chain remains to be evaluated. Unfortunately the canonical group chain $O(8) \supset O(7) \supset O(6)$ gives a basis in which the number operator is not diagonal. For the case of seniority $v = 0$ and $v = 1$, only a single quantum number is needed to specify the possible O(7) representations in the $O(8) \supset O(7) \supset O(6)$ chain. To obtain a physically meaningful basis, however, it is necessary to choose nucleon number in place of the O(7) quantum number. It is the purpose of this chapter to buildup a vector basis which is a linear combination of different O(7) representations sandwiched in between fixed O(8) and O(6) representations such that the number operator is diagonal in this bases. After the basis has been found the fundamental matrix elements of the twelve pair operators A^+ and A, which involve the 6,7,8-space, are evaluated in terms of their reduced matrix elements which are labeled by the quantum numbers of the O(8), O(6) representations and the nucleon number, N.

B. BUILDING UP OF BASIS WHICH DIAGONALIZES THE NUMBER OPERATOR—THE CASE V = 0

For $u = 0$, the O(8) representations are characterized by (Ωooo) and the possible O(6) representations by (noo). The possible values of N (nucleon number) for fixed Ω and n can be determined by counting the number of nodes in the possible Young tableau for (noo), that is SU(4) representation [nno]. These are shown below together with the eigenvalues of the operator $Q_0 = J_{78}$.



N	$Q_0 = \Omega - \frac{N}{2}$
$2n$	$\Omega - n$
$2n+4$	$\Omega - n - 2$
$2n+8$	$\Omega - n - 4$
$\vdots$	$\vdots$
$2n+4(\Omega-n)$	$-\Omega+n$

The maximum number of columns in the tableau is limited by Ω, the spatial degeneracy.

The O(7) representations are characterized by (moo) where $n \leq m \leq \Omega$. There are altogether $(\Omega-n+1)$ different values of m. On the other hand Q_0 or J_{78} have just $(\Omega-n+1)$ distinct eigenvalues, so that nucleon number can be used in place of m.

Taking the linear combination

$$\sum_{m=n}^{\Omega} C_{mn}^{(\lambda)} \left| \begin{matrix} \Omega\,0\,0\,0 \\ m\,0\,0 \\ n\,0\,0 \end{matrix} \right\rangle$$

such that it is an eigenbasis for the number operator, it must satisfy

$$J_{78} \sum_{m=n}^{\Omega} C_{mn}^{(\lambda)} \left| \begin{matrix} \Omega 000 \\ m00 \\ n00 \end{matrix} \right\rangle = \lambda_{\Omega n} \sum_{m=n}^{\Omega} C_{mn}^{(\lambda)} \left| \begin{matrix} \Omega 000 \\ m00 \\ n00 \end{matrix} \right\rangle \tag{4.B.1}$$

where

$$\begin{aligned} \lambda_{\Omega n} &= (\Omega - n) - 2(k-1) \qquad k = 1, 2, \ldots (\Omega - n + 1) \\ &= \Omega - \frac{N}{2} \end{aligned}$$

The properties of J_{78} is the canonical basis $O(8) \supset O(7) \supset O(6)$ are known. The matrix elements of $J_{i-1,i}$ are evaluated in the supplement. In particular

$$J_{78} \left| \begin{matrix} \Omega 000 \\ m00 \\ n00 \end{matrix} \right\rangle = A_{\Omega mn+} \left| \begin{matrix} \Omega 000 \\ m{+}1\,00 \\ n\,00 \end{matrix} \right\rangle + A_{\Omega mn-} \left| \begin{matrix} \Omega 000 \\ m{-}1\,00 \\ n\,00 \end{matrix} \right\rangle$$

where the coefficients are provided by (5.44) of the supplement

$$\begin{aligned} A_{\Omega mn+} &= -i \sqrt{\frac{(\Omega - m)(\Omega + m + 6)(m + n + 5)(m - n + 1)}{(2m+5)(2m+7)}} \\ A_{\Omega mn-} &= i \sqrt{\frac{(\Omega - m + 1)(\Omega + m + 5)(m + n + 4)(m - n)}{(2m+3)(2m+5)}} \end{aligned} \tag{4.B.2}$$

From these one gets the recursion relations

$$\begin{aligned} & C_{m-1\,n}^{(\lambda)} \sqrt{\frac{(m+n+4)(m-n)(\Omega+m+5)(\Omega-m+1)}{(2m+3)(2m+5)}} \\ &= C_{m+1\,n}^{(\lambda)} \sqrt{\frac{(m+n+5)(m-n+1)(\Omega+m+6)(\Omega-m)}{(2m+5)(2m+7)}} + i\lambda_{\Omega n} C_{mn}^{(\lambda)} \end{aligned} \tag{4.B.3}$$

In order to start the recursion, two boundary conditions are put in. They are

$$C^{(\lambda)}_{\Omega+1\, n} = 0$$

and (4.B.4)

$$C^{(\lambda)}_{\Omega n} = 1$$

After the successive application of the recursion, all the $C^{(\lambda)}_{mm}$ are determined. Then, define

$$A^{(\lambda)} = \sqrt{\sum_{m=n}^{\Omega} \left| C^{(\lambda)}_{mn} \right|^2}$$

$$D^{(\lambda)}_{mn} = \frac{C^{(\lambda)}_{mn}}{A^{k}}$$

and (4.B.5)

$$\varphi_{\Omega m n} = \left| \begin{matrix} \Omega & 0 & 0 \\ m & 0 & 0 \\ n & 0 & 0 \end{matrix} \right\rangle$$

The final normalized basis, with nucleon number a good quantum number, is

$$\Psi^{(\lambda)}_{\Omega n} = \sum_{m=n}^{\Omega} D^{(\lambda)}_{mn}\, \varphi_{\Omega m n}$$

(4.B.6)

Not all $D^{(\lambda)}_{m\,n}$ are needed for the latter work, in fact only two are required, they are

$$D^{(\lambda)}_{\Omega n} = \sqrt{\frac{(\Omega+n+4)!\,(\Omega-n)!\,(\Omega+2)!\,(\Omega+2)!}{\left(\frac{\Omega-n+\lambda}{2}\right)!\left(\frac{\Omega-n-\lambda}{2}\right)!\left(\frac{\Omega+n+\lambda+4}{2}\right)!\left(\frac{\Omega+n-\lambda+4}{2}\right)!\,(2\Omega+4)!}}$$

$$D^{\lambda}_{\Omega-1\, n} = i\lambda\sqrt{\frac{(\Omega+n+3)!\,(\Omega-n-1)!\,(\Omega+2)!\,(2\Omega+3)!}{\left(\frac{\Omega-n+\lambda}{2}\right)!\left(\frac{\Omega-n-\lambda}{2}\right)!\left(\frac{\Omega+n+\lambda+4}{2}\right)!\left(\frac{\Omega+n-\lambda+4}{2}\right)!\,(2\Omega+4)!}}$$

(4.B.7)

C. NUCLEON NUMBER RAISING OR LOWERING OPERATORS IN v = 0

The basis that has been built in the previous section is an eigenbasis for the number operator. Since the pair operators A^+ and A lead to an increase or decrease of two particles, it is necessary to investigate the connection of the basis with nucleon number N, (that is J_{78} eigenvalue λ), to the bases with nucleon number $N \mp 2$ (J_{78} eigenvalues $\lambda \pm 1$). From the point of view of the canonical chain $O(8) \supset O(7) \supset O(6)$ and the results of the work of Gelfand and the supplement of the thesis applied to the 6,7,8-space, the simplest operators which make this connection are the combinations $J_{67} \pm iJ_{68}$. Matrix elements of these operators are calculated in this section.

Define a sub-basis

$$\Psi^{(\lambda)}_{\Omega n p} = \sum_{m=n}^{\Omega} D^{(\lambda)}_{mn} \left| \begin{matrix} \Omega & 0 & 0 & 0 \\ m & 0 & 0 & \\ n & 0 & 0 & \\ p & 0 & & \end{matrix} \right\rangle = \sum_{m=n}^{\Omega} D^{(\lambda)}_{mn} \varphi_{\Omega m n p} \tag{4.C.1}$$

where

$$\varphi_{\Omega m n p} = \left| \begin{matrix} \Omega & 0 & 0 & 0 \\ m & 0 & 0 & \\ n & 0 & 0 & \\ p & 0 & & \end{matrix} \right\rangle$$

p describes the O(5) representation. Then,

$$J_{78} \Psi^{(\lambda)}_{\Omega n p} = \lambda \Psi^{(\lambda)}_{\Omega n p} \tag{4.C.2}$$

and

$$(J_{67} + iJ_{68}) \Psi^{(\lambda)}_{\Omega n p} = C^{+}_{\lambda \Omega n p} \Psi^{(\lambda+1)}_{\Omega n+1 p} + C^{-}_{\lambda \Omega n p} \Psi^{(\lambda+1)}_{\Omega n-1 p} \tag{4.C.3}$$

where the $C^{\pm}_{\lambda\Omega np}$ are the matrix elements of basic interest.

The operator J_{68} can be written in terms of J_{67} and J_{78} for which matrix elements are given by (5.46) (5.45) and (5.44) of the supplement.

$$J_{78}\varphi_{\Omega mnp} = A_{\Omega mn+}\varphi_{\Omega m+1\, np} + A_{\Omega mn-}\varphi_{\Omega\, m-1\, np}$$

$$J_{67}\varphi_{\Omega mnp} = B_{mnp+}\varphi_{\Omega\, m\, n+1\, p} + B_{mnp-}\varphi_{\Omega\, m\, n-1\, p} \tag{4.C.4}$$

where the coefficients A is given by (4.B.2), and

$$B_{mnp+} = -\frac{i}{2}\sqrt{\frac{(n-p+1)(n+p+4)(m-n)(m+n+5)}{(n+2)(n+3)}}$$

$$B_{mnp-} = \frac{i}{2}\sqrt{\frac{(n-p)(n+p+3)(m-n+1)(m+n+4)}{(n+1)(n+2)}}$$

(4.C.5)

With

$$J_{67} + iJ_{68} = J_{67} + J_{78}J_{67} - J_{67}J_{78} \tag{4.C.6}$$

and with (4.B.6), (4.C.5) one can compare the coefficients of $\phi_{\Omega\Omega n+1p}$ and $\phi_{\Omega\Omega n-1p}$ on the L.H.S. and R.H.S. of (4.C.3), to obtain the matrix elements of $J_{67}+iJ_{68}$.

Note, that for this purpose, only the coefficients $D^{(\lambda)}_{\Omega n}$ and $D^{(\lambda)}_{\Omega-1n}$ are needed in the expansion of $\Psi^{\lambda}_{\Omega np}$ in terms of $\phi_{\Omega mnp}$. With (4.B.6) one therefore gets

$$c^{+}_{\lambda\Omega np} = -\frac{i}{2}\sqrt{\frac{(n-p+1)(n+p+4)(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{(n+2)(n+3)}}$$

$$c^{-}_{\lambda\Omega np} = \frac{i}{2}\sqrt{\frac{(n-p)(n+p+3)(\Omega-n+\lambda+2)(\Omega+n-\lambda+4)}{(n+2)(n+1)}}$$

(4.C.7)

Similarly

$$(J_{67}-iJ_{68})\Psi^{(\lambda)}_{\Omega np} = b^{+}_{\lambda\Omega np}\Psi^{(\lambda-1)}_{\Omega(n+1)p} + b^{-}_{\lambda\Omega np}\Psi^{(\lambda-1)}_{\Omega(n-1)p} \tag{4.C.8}$$

where

$$b^{+}_{\lambda\Omega np} = -c^{-}_{(\lambda-1)\Omega(n+1)p}$$

$$b^{-}_{\lambda\Omega np} = -c^{+}_{(\lambda-1)\Omega(n-1)p} \tag{4.C.9}$$

D. O(8)/O(6) REDUCED WIGNER COEFFICIENTS IN A BASIS IN WHICH N IS DIAGONAL

From a knowledge of the matrix elements of the operators $J_{67}\pm iJ_{68}$ in the basis on which N is diagonal, the matrix elements of all pair operators A and A^{+} can be obtained through a complete tensor classification of all operators and an application of the Wigner Eckart theorem. (The relation between operators A and A^{+} and the J_{ij} has been given in (1.B.9).)

D.1 Tensor Properties of J_{67} and J_{68}

Since J_{67}, J_{68} are infinitesimal operators for O(8) then O(8) tensor character is that of the regular representation (1100). They are both the sixth components of vectors J_{i7}, J_{i8} with $i = 1,\ldots,6$. Their O(6) tensor character is thus that of the 6-dimensional vector representation (100). (The equivalent SU(4) representation [110] is that of the antisymmetrically coupled pair.)

In the isospin space

$$T_0 = J_{56}$$
$$T_\pm = J_{45} \pm i J_{46}$$

If R is a tensor of rank k with respect to the isospin space, it satisfies

$$[T_0, R^k_q] = q R^k_q$$
$$[T_+, R^k_q] = \sqrt{(k-q)(k+q+1)}\; R^k_{q+1}$$
$$[T_-, R^k_q] = \sqrt{(k+q)(k-q+1)}\; R^k_{q-1}$$

(4.D.1)

With the relations of (4.D.1), then

$$R^{1\,\prime}_0 = J_{47} \qquad R^{1\,\prime\prime}_0 = J_{48}$$
$$R^{1\,\prime}_1 = \frac{-J_{67} + iJ_{57}}{\sqrt{2}} \qquad R^{1\,\prime\prime}_1 = \frac{-J_{68} + iJ_{58}}{\sqrt{2}}$$
$$R^{1\,\prime}_{-1} = \frac{J_{67} + iJ_{57}}{\sqrt{2}} \qquad R^{1\,\prime\prime}_{-1} = \frac{J_{68} + iJ_{58}}{\sqrt{2}}$$

(4.D.2)

Re-expressing the J_{67}, and J_{68} in the tensor form with respect to isospin space

$$J_{67} = \frac{R^{1\prime}_{-1} - R^{1\prime}_{1}}{\sqrt{2}}$$

$$J_{68} = \frac{R^{1\prime\prime}_{-1} - R^{1\prime\prime}_{1}}{\sqrt{2}} \qquad (4.D.3)$$

Since J_{67} and J_{68} do not interact with 1,2,3-space, they are scalars with respect to spin space. Therefore they can be written in full eight-dimensional tensor notation as the following tensor operators

$$J_{67} = \frac{1}{\sqrt{2}} \left[T'^{\,(1100)}_{\substack{(100) \\ S=M_S=0 \\ T=1\ M_T=-1}} - T'^{\,(1100)}_{\substack{(100) \\ S=M_S=0 \\ T=M_T=1}} \right]$$

$$J_{68} = \frac{1}{\sqrt{2}} \left[T''^{\,(1100)}_{\substack{(100) \\ S=M_S=0 \\ T=1\ M_T=-1}} - T''^{\,(1100)}_{\substack{(100) \\ S=M_S=0 \\ T=M_T=1}} \right] \qquad (4.D.4)$$

D.2 Reduced Wigner Coefficient O(8)/O(6)

To calculate the reduced O(8)/O(6) Wigner coefficients, any simple choice of state vector within O(6) can be made. In the number basis established in (4.B.6), one can take a vector with $S = m_S = n$ and $T = m_T = 0$,

$$\Phi^{\lambda}_{\Omega n (nn)(00)} = \sum_{m=n}^{\Omega} D^{(\lambda)}_{mn} \left| \begin{matrix} \Omega & 0 & 0 & 0 \\ m & 0 & 0 & \\ n & 0 & 0 & \\ n & 0 & & \\ n & 0 & & \\ n & & & \\ n & & & \end{matrix} \right\rangle \qquad (4.D.5)$$

and a vector with $S = m_S = n-1$ and $T = 1$

$$\Phi^{\lambda}_{\Omega n\,(n-1\,n-1)\,(11)} = \frac{-1}{\sqrt{2}} \sum_{m=n}^{\Omega} D^{(\lambda)}_{mn} \left\{ \left| \begin{matrix} \Omega & 0 & 0 & 0 \\ m & 0 & 0 & \\ n & 0 & 0 & \\ n & 0 & & \\ n-1 & 0 & & \\ n-1 & & & \\ n-1 & & & \end{matrix} \right\rangle + i \left| \begin{matrix} \Omega & 0 & 0 & 0 \\ m & 0 & 0 & \\ n & 0 & 0 & \\ n-1 & 0 & & \\ n-1 & 0 & & \\ n-1 & & & \\ n-1 & & & \end{matrix} \right\rangle \right\}$$

(4.D.6)

The latter is obtained by applying the O_{-11} operator of Chapter II, expressed in terms of $J_{\alpha\beta}$, on the base vector $\left| \begin{matrix} n & 0 & 0 \\ n & 0 & \\ n & 0 & \\ n & & \\ n & & \end{matrix} \right\rangle$, which is identified with $\left| \begin{matrix} [n\,n\,0] \\ \{n\,0\} \end{matrix} \right\rangle$. With the above knowledge, one can find the Wigner coefficients by

$$\left\langle \Phi^{(\lambda+1)}_{\Omega n+1\,(nn)(11)} \middle| J_{67} + iJ_{68} \middle| \Phi^{(\lambda)}_{\Omega n\,(nn)(00)} \right\rangle$$

$$= \frac{i}{\sqrt{2}} \left\langle \Psi^{(\lambda+1)}_{\Omega n+1\, n} \middle| J_{67} + iJ_{68} \middle| \Psi^{(\lambda)}_{\Omega nn} \right\rangle = \frac{1}{\sqrt{2}}\, i\, C^{+}_{\lambda\Omega nn}$$

$$= \frac{1}{2} \sqrt{\frac{(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{(n+3)}} \qquad (4.D.7)$$

Where operators in the 6,7,8-space do not have any effect on S. On the other hand by the use of the Wigner-Eckart theorem, the matrix element can be split into

$$\left\langle \Phi^{(\lambda+1)}_{\Omega n+1\,(nn)(11)} \middle| J_{67} + iJ_{68} \middle| \Phi^{(\lambda)}_{\Omega n\,(nn)(00)} \right\rangle$$

$$= F(\Omega; \lambda, \lambda+1; n, n+1) \left\langle \begin{matrix} [n\,n\,0] \\ n\,0 \end{matrix} \;\begin{matrix} [1\,1\,0] \\ 0\,1 \end{matrix} \middle\| \begin{matrix} [n+1\;n+1\;0] \\ n\;1 \end{matrix} \right\rangle$$

$$\times \left\{ \frac{\langle 0\,0\,1\,{-1} | 1\,1 \rangle - \langle 0\,0\,1\,1 | 1\,1 \rangle}{\sqrt{2}} \right\} \qquad (4.D.8)$$

Where the physical content of the O(8)/O(6) part of the chain is given by the factor F. Specifically, F is the product of the O(8) reduced matrix element with an O(8)/O(6) reduced Wigner coefficient in the number basis. Equating (4.D.8) with (4.D.7), and substituting the values for the SU(4) and ordinary isospin Wigner coefficients, one obtains

$$F(\Omega; \lambda, \lambda+1; n, n+1) = -\sqrt{\frac{(n+1)(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{2(n+3)}}$$

The other F factors are obtained by similar techniques. The results are tabulated as follows:

TABLE 4.1

F FACTORS FOR EVEN PARTICLE NUMBER

$$F(\Omega; \lambda, \lambda+1; n\ n+1) = -\sqrt{\frac{(n+1)(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{2(n+3)}}$$

$$F(\Omega; \lambda, \lambda+1; n\ n-1) = -\sqrt{\frac{(n+3)(\Omega+n-\lambda+4)(\Omega-n+\lambda+2)}{2(n+1)}}$$

$$F(\Omega; \lambda+1\ \lambda; n+1\ n) = -\sqrt{\frac{(n+4)(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{2(n+2)}}$$

$$F(\Omega\ \lambda+1\ \lambda; n-1\ n) = -\sqrt{\frac{(\Omega-n-\lambda+2)(\Omega+n-\lambda+4)\,n}{2(n+2)}}$$

With these F factors and the SU(4) Wigner coefficients of Tables 3.12 and 3.13, the matrix element of any pair operator A^+ or A can be evaluated. The full tensor classification of A^+ and A is given in Table 2.1. The matrix element of a tensor operator $T^{\substack{(1100) \\ (100)^{\lambda}}}_{(SM_S)(TM_T)}$ for the v = 0 states is given by

$$\langle \Omega, \lambda'', n''; (S''M_S'')(T''M_T'') \mid T^{(1100)\lambda}_{(100)\,(SM_S)(TM_T)} \mid \Omega \lambda' n' (S'M_S')(T'M_T') \rangle$$

$$= F(\Omega; \lambda', \lambda''; n', n'') \left\langle \begin{matrix} [n'n'0] \\ S'T' \end{matrix} \; \begin{matrix} [110] \\ ST \end{matrix} \middle\| \begin{matrix} [n''n''0] \\ S''T'' \end{matrix} \right\rangle$$

$$\cdot \langle S'M_S' \, SM_S \mid S''M_S'' \rangle \langle T'M_T' \, TM_T \mid T''M_T'' \rangle$$

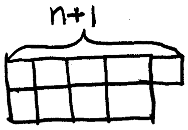
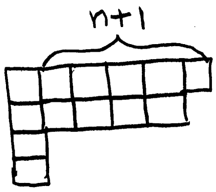
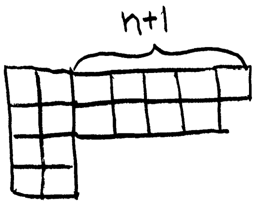
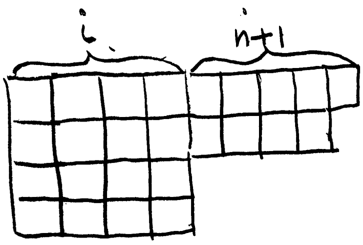
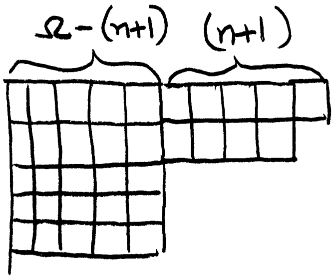
(4.D.9)

E. The F Functions for v = 1

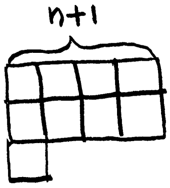
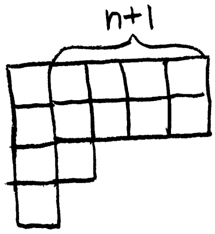
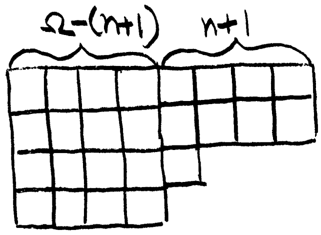
The technique used to calculate the F functions for the v = 1 states is similar to that for the v = 0 states. The following section tries to make a few remarks on the intermediate steps and the results, stressing these steps which are different from the v = 0 cases.

For v = 1, the O(8) irreducible representation is characterized by $(\Omega - \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$, and the possible O(6) representations by $(n + \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$ or $(n + \frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2})$. The possible value of nucleon number, N, or eigenvalue of $Q_0 = J_{78}$ can again be read from the possible Young Tableaux. They are

(1) For $(n + \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$

	N	$Q_0 = \Omega - \frac{N}{2}$
	$(2n+1)$	$\Omega - n - \frac{1}{2}$
	$(2n+1)+4$	$\Omega - n - \frac{1}{2} - 2$
	$(2n+1)+8$	$\Omega - n - \frac{1}{2} - 4$
	$(2n+1)+4i$	$\Omega - n - \frac{1}{2} - 2i$
	$(2n+1)+4(\Omega-n-1)$	$-(\Omega-n-\frac{1}{2})+1$

However,

(2) For $(n+\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2})$	N	$Q_0 = \Omega - \frac{N}{2}$
	$(2n+3)$	$\Omega - n - \frac{3}{2}$
	$(2n+3)+4$	$\Omega - n - \frac{3}{2} - 2$
⋮		⋮
	$(2n+3)+4[\Omega-(n+1)]$	$-\Omega + n + \frac{1}{2}$

The two sets of eigenvalues λ of Q_0 for $(\Omega - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $(n + \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\Omega - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $(n + \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ are opposite in sign but equal in magnitude.

Define

$$\varphi_{\Omega-\frac{1}{2}\, m+\frac{1}{2}\, n+\frac{1}{2}} = \left| \begin{array}{llll} \Omega-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ m+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ n+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array} \right/$$

and (4.E.1)

$$\varphi'_{\Omega-\frac{1}{2}\, m+\frac{1}{2}\, n+\frac{1}{2}} = \left| \begin{array}{llll} \Omega-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ m+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ n+\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \end{array} \right/$$

and the basis in which J_{78} is diagonal

$$\Psi^{(\lambda)}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}} = \sum_{m=n}^{\Omega-1} D^{(\lambda)}_{m+\frac{1}{2}\, n+\frac{1}{2}}\, \varphi_{\Omega-\frac{1}{2},\, m+\frac{1}{2},\, n+\frac{1}{2}}$$

and (4.E.2)

$$\Psi^{(\lambda')}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}} = \sum_{m=n}^{\Omega-1} D^{(\lambda')'}_{m+\frac{1}{2}\, n+\frac{1}{2}}\, \varphi'_{\Omega-\frac{1}{2},\, m+\frac{1}{2},\, n+\frac{1}{2}}$$

The coefficients satisfy the following recursion equations, respectively.

$$-i\, D^{(\lambda)}_{m-\frac{1}{2}\, n+\frac{1}{2}} \sqrt{\frac{(m+n+5)(m-n)(\Omega+m+5)(\Omega-m)}{(2m+5)}}$$

$$+i\, D^{(\lambda)}_{m+\frac{3}{2}\, n+\frac{1}{2}} \sqrt{\frac{(m+n+6)(m-n+1)(\Omega+m+6)(\Omega-m-1)}{(2m+7)}}$$

$$= \left[\lambda - \frac{(2n+5)(2\Omega+5)}{2(2m+5)(2m+7)}\right] D^{(\lambda)}_{m+\frac{1}{2}\, n+\frac{1}{2}}$$

$$\lambda = (\Omega-n-\tfrac{1}{2}) - (2k-1) \qquad k=1,2,\ldots(\Omega-n) \qquad (4.E.3)$$

and

$$-i\, D^{(\lambda')'}_{m-\frac{1}{2}\, n+\frac{1}{2}} \sqrt{\frac{(m+n+5)(m-n)(\Omega+m+5)(\Omega-m)}{(2m+5)}}$$

$$+i\, D^{(\lambda')'}_{m+\frac{3}{2}\, n+\frac{1}{2}} \sqrt{\frac{(m+n+6)(m-n+1)(\Omega+m+6)(\Omega-m-1)}{(2m+7)}}$$

$$= \left[\lambda' + \frac{(2n+5)(2\Omega+5)}{2(2m+5)(2m+7)}\right] D^{(\lambda')'}_{m+\frac{1}{2}\, n+\frac{1}{2}}$$

$$\lambda' = -(\Omega-n-\tfrac{1}{2}) + 2(k-1) \qquad k=1,2,\ \ldots\ (\Omega-n)$$

Since $\lambda' = -\lambda$, the two types of coefficients are related

$$D^{(-\lambda)}_{m+\frac{1}{2}\, n+\frac{1}{2}} = \left(D^{\lambda}_{m+\frac{1}{2}\, n+\frac{1}{2}}\right)^*$$

Through successive application of the recursion, one obtains

$$D^{(\lambda)}_{\Omega-\frac{1}{2} n} = \sqrt{\frac{(\Omega+2)!\,(\Omega+1)!\,(\Omega+n+4)!\,(\Omega-n-1)!}{\left(\frac{\Omega-n-\lambda-\frac{1}{2}}{2}\right)!\left(\frac{\Omega+n+\lambda+\frac{9}{2}}{2}\right)!\left(\frac{\Omega+n-\lambda+\frac{7}{2}}{2}\right)!\left(\frac{\Omega-n+\lambda-\frac{3}{2}}{2}\right)!(2\Omega+3)!}} \qquad (4.E.4)$$

And the other D needed, can be obtained through (4.E.3)

$$D^{(\lambda)}_{\Omega-\frac{3}{2} n} = \frac{i(2\Omega+3)}{\sqrt{(\Omega+n+4)(\Omega-n-1)(2\Omega+4)}}\left(\lambda - \frac{(2n+5)}{2(2\Omega+3)}\right) D^{(\lambda)}_{\Omega-\frac{1}{2} n} \qquad (4.E.5)$$

Similar to (3.C.3)

$$\begin{aligned}
(J_{67}+iJ_{68})\,\Psi^{(\lambda)}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}} &= C^{+}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi^{(\lambda+1)}_{\Omega-\frac{1}{2}\, n+\frac{3}{2}\, p+\frac{1}{2}} \\
&+ C^{-}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi^{(\lambda+1)}_{\Omega-\frac{1}{2}\, n-\frac{1}{2}\, p+\frac{1}{2}} \\
&+ C^{0}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi^{(\lambda+1)'}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}} \\
(J_{67}+iJ_{68})\,\Psi'^{(\lambda)}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}} &= \tilde{C}^{+}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi'^{(\lambda+1)}_{\Omega-\frac{1}{2}\, n+\frac{3}{2}\, p+\frac{1}{2}} \\
&+ \tilde{C}^{-}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi'^{(\lambda+1)}_{\Omega-\frac{1}{2}\, n-\frac{1}{2}\, p+\frac{1}{2}} \\
&+ \tilde{C}^{0}_{\lambda\,\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}\,\Psi^{(\lambda+1)}_{\Omega-\frac{1}{2}\, n+\frac{1}{2}\, p+\frac{1}{2}}
\end{aligned} \qquad (4.E.6)$$

where

$$\Psi^{(\lambda)}_{\Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \sum_{m=n}^{\Omega-1} D^{(\lambda)}_{m+\frac{1}{2}\; n+\frac{1}{2}} \left| \begin{matrix} \Omega-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ m+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ n+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ p+\frac{1}{2} & \frac{1}{2} & & \end{matrix} \right\rangle$$

and

$$\tilde{\Psi}^{(\lambda)'}_{\Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \sum_{m=n}^{\Omega-1} D^{(-\lambda)*}_{m+\frac{1}{2}\; n+\frac{1}{2}} \left| \begin{matrix} \Omega-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ m+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\ n+\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \\ p+\frac{1}{2} & \frac{1}{2} & & \end{matrix} \right\rangle$$

By comparing coefficients, one obtains

$$C^{+}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{-i}{4}\frac{1}{(n+3)}\sqrt{(2\Omega-2n-2\lambda-1)(2\Omega+2n+2\lambda+13)(n+p+5)(n-p+1)}$$

$$C^{-}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{i}{4}\frac{1}{(n+2)}\sqrt{(2\Omega+2n-2\lambda+7)(2\Omega-2n+2\lambda+1)(n+p+4)(n-p)}$$

$$C^{0}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{i}{4}\frac{p+2}{(n+2)(n+3)}\sqrt{(2\Omega-2n+2\lambda+1)(2\Omega-2n-2\lambda-1)}$$

$$\tilde{C}^{+}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{-i}{4}\frac{1}{(n+3)}\sqrt{(2\Omega-2\lambda-2n-3)(2\Omega+2n+2\lambda+11)(n+p+5)(n-p+1)}$$

$$\tilde{C}^{-}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{i}{4}\frac{1}{(n+2)}\sqrt{(2\Omega-2n+2\lambda+3)(2\Omega+2n-2\lambda+9)(n+p+4)(n-p)}$$

$$\tilde{C}^{0}_{\lambda\; \Omega-\frac{1}{2}\; n+\frac{1}{2}\; p+\frac{1}{2}} = \frac{-i}{4}\frac{(p+2)}{(n+2)(n+3)}\sqrt{(2\Omega+2n+2\lambda+11)(2\Omega-2n-2\lambda+9)} \tag{4.E.7}$$

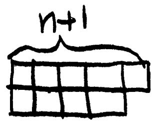
With these coefficients and the application of the Wigner-Eckart theorem the F functions are found by exactly the same techniques as that used in (4.D.5)-(4.D.8). The results are tabulated as follows:

TABLE 4.2

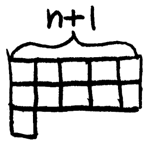
F FACTORS FOR ODD PARTICLE NUMBERS

$F(\Omega; \lambda-1\ \lambda; n-1\ n)$	$\frac{1}{4}\sqrt{\frac{2n(2\Omega-2n-2\lambda+3)(2\Omega+2n+2\lambda+9)}{(n+2)}}$
$F(\Omega; \lambda+1\ \lambda; n+1\ n)$	$\frac{1}{4}\sqrt{\frac{2(n+5)(2\Omega-2n-2\lambda-1)(2\Omega+2n+2\lambda+13)}{(n+3)}}$
$F(\Omega; \lambda-1\ \lambda; n+1\ n)$	$\frac{1}{4}\sqrt{\frac{2(n+5)(2\Omega+2n-2\lambda+11)(2\Omega-2n+2\lambda-3)}{(n+3)}}$
$F(\Omega; \lambda+1\ \lambda; n-1\ n)$	$\frac{1}{4}\sqrt{\frac{2n(2\Omega+2n-2\lambda+7)(2\Omega-2n+2\lambda+1)}{(n+2)}}$
$F(\Omega; \lambda+1\ \lambda; \tilde{n}\ n)$	$\frac{i}{2}\sqrt{\frac{(2\Omega-2n+2\lambda+1)(2\Omega-2n-2\lambda-1)}{(n+2)(n+3)}}$
$F(\Omega; \lambda-1\ \lambda; n\ \tilde{n})$	$\frac{-i}{2}\sqrt{\frac{(2\Omega-2n+2\lambda-1)(2\Omega-2n-2\lambda+1)}{(n+2)(n+3)}}$

where

n stands for 

$(n+\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2})$

$\tilde{n}$ stands for 

$(n+\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2})$

With these F factors and the SU(4) Wigner coefficients of Table 3.14-Table 3.19, one can get the matrix element of a tensor $T^{(1100)\lambda}_{\substack{(100)\\(SM_S)(TM_T)}}$ similar to (4.D.9).

CHAPTER V

SOME APPLICATIONS OF THE QUASISPIN METHOD

A. INTRODUCTION

With the preparation of the last four chapters, one is ready for some applications of the technique. The building up of the wave functions and the classification of the infinitesimal operators has been tailored for the pairing Hamiltonian, and this Hamiltonian is now investigated more thoroughly in section B. As mentioned before only sneiority $v = 0$ and $v = 1$ wave functions are treated in the present work. The $v = 0$ case is chosen to demonstrate the present technique in full detail. The pairing interaction in a single level is discussed in section C, and the two-level case is taken up in Section D. The method is exactly the same for the corresponding $v = 1$ cases, and is also easily extended to the case of several single particle levels. In Section D with the aid of the computer the effect of the pairing interaction is studied by working out the case of 4, 6, 8 and 10 particles for a specific two level problem which may have some relevance for actual nuclei.

B. THE PAIRING HAMILTONIAN

The general Hamiltonian can be written in the form

$$H = H_{s.p.} + H_{2\text{-body}}$$

where the single particle Hamiltonian is given by

$$H_{s.p.} = \sum_i \varepsilon_i N_i \qquad \text{with} \qquad N_i = \sum_{m_i m_s m_t} a^{\dagger}_{i m_i m_s m_t} a_{i m_i m_s m_t}$$

The indicies i characterize the single particle levels, and ε_i are the single particle energies. The spatial degeneray of the levels is given by $\Omega_i = \sum_{m=1} 1$ for example $\Omega_i = 2\ell_i+1$ if i denotes ℓ_i.

The general 2-body interaction is replaced by a pairing Hamiltonian in this work.

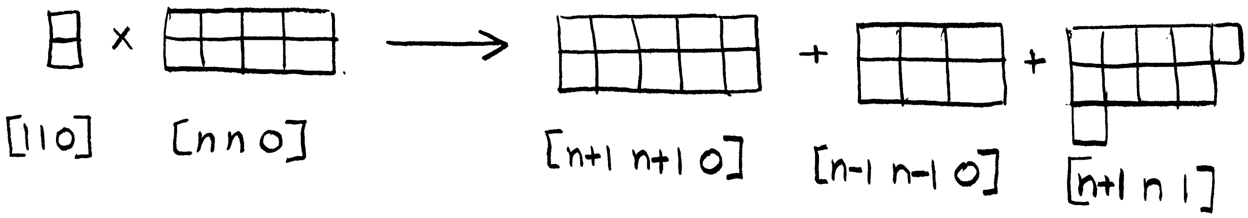
The pairing Hamiltonian in the ΩST scheme for the case of several single particle energy levels can be written in the general form as follows

$$\begin{aligned} H_{pairing} = & \sum_{ij} g^{ij}_{31} \sum_{M_S} {}^{31}A^{\dagger}_i(M_S 0)\, {}^{31}A_j(M_S 0) \\ & + \sum_{ij} g^{ij}_{13} \sum_{M_T} {}^{13}A^{\dagger}_i(0 M_T)\, {}^{13}A_j(0 M_T) \end{aligned} \tag{5.B.1}$$

The $g^{ij}_{(2SH)(2T+1)}$ give the strength of the interaction between a pair of particles in the ith single particle energy level and a piar of particles in the jth single particle energy level with S = 0(T = 1) or S = 1 (T = 0). Since the A^+ and A are the priamry block in the Hamiltonian, their properties are studied in detail here.

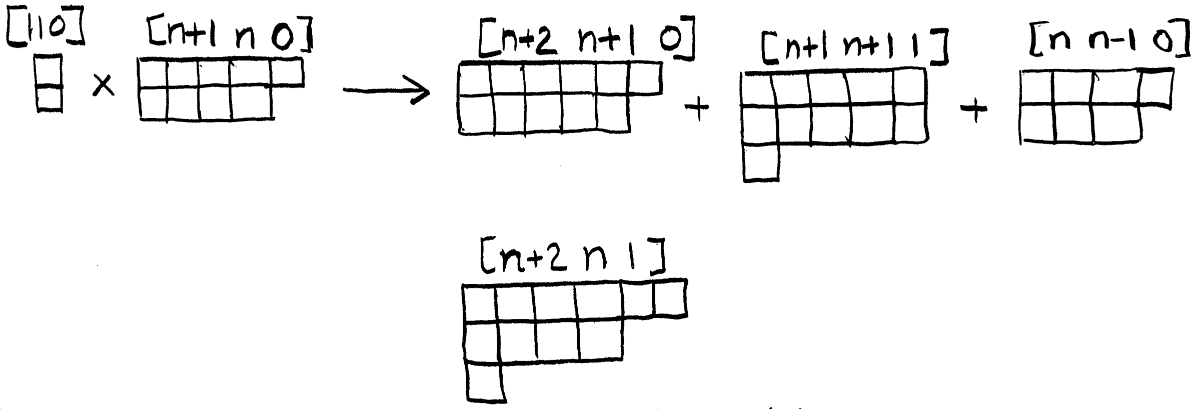
B.1. Properties of A^+ and A

The $A^+(A)$ create (destroy) a pair of nucleons which give zero spatial angular momentum quantum number, and therefore they decrease (increase) λ by 1, and do <u>not</u> change the seniority of any state. It is because of this conservation of seniority that $A^+(A)$ when acting on a state of SU(4) character [nno] can only give states of character [n+1, n+1, 0] and [n-1, n-1, 0], although the general SU(4) product of [nno] with [110] (the representation of A^+)



The operators $A^+(A)$ do not give wave functions of [n+1 n 1] in SU(4) when they act on $v = 0$ wave functions.

Similarly, although the general SU(4) product has the form



$A^+(A)$ do not give wave functions of [n+2 n 1] in SU(4) when they act on $v = 1$ wave functions. As for S and T space, A^+ behaves like the corresponding angular momentum in the relevant SU(2) space. Therefore

$$^{(2S+1)\,(2T+1)}A^{\dagger} = T^{(1100)}_{\substack{(100)^{(-1)} \\ (SM_S)(TM_T)}}$$

(5.B.2)

If v = 0, a state can be written as

$$\left| \Omega, \lambda \begin{matrix} [n_1 n_1 0] \\ (S_1 M_{S_1})(T_1 M_{T_1}) \end{matrix} \right\rangle$$

(5.B.3)

Where Ω stadns for the spatial quantum number, and $\lambda = \Omega - \frac{N}{2}$, then

$$A^{\dagger}(M_{S_2} M_{T_2}) \left| \Omega, \lambda \begin{matrix} [n_1 n_1 0] \\ (S_1 M_{S_1})(T_1 M_{T_1}) \end{matrix} \right\rangle = \sum_{\substack{n, S, M_S \\ T, M_T}} A^{\Omega\ \lambda-1\ n\ S M_S T M_T}_{\Omega\ \lambda\ n_1\ S_1 M_{S_1} T_1 M_{T_1}} \left| \Omega, \lambda-1 \begin{matrix} [n n 0] \\ (S M_S)(T M_T) \end{matrix} \right\rangle$$

(5.B.4)

where

$$A^{\Omega\ \lambda-1\ n\ S M_S T M_T}_{\Omega\ \lambda\ n_1\ S_1 M_{S_1} T_1 M_{T_1}} = F(\Omega; \lambda\, \lambda-1; n_1\, n) \left\langle \begin{matrix} [n_1 n_1 0] \\ S_1 T_1 \end{matrix} \begin{matrix} [110] \\ S_2 T_2 \end{matrix} \right\| \left. \begin{matrix} [n n 0] \\ S T \end{matrix} \right\rangle$$

$$\langle S_1 M_{S_1} S_2 M_{S_2} | S M_S \rangle \langle T_1 M_{T_1} T_2 M_{T_2} | T M_T \rangle$$

The F are listed in Table 4.1 and the SU(4) Wigner coefficients are listed in Table 3.12 and 3.13 with $n_1 = n \pm 1$.

and

$$A(M_{S_2} M_{T_2}) \left| \Omega \lambda \begin{matrix} [n_1 n_1 0] \\ (S_1 M_{S_1})(T_1 M_{T_1}) \end{matrix} \right\rangle = \sum_{\substack{n S M_S \\ T M_T}} B_{\Omega \lambda\ n_1\ S_1 M_{S_1} T_1 M_{T_1}}^{\Omega \lambda+1\ n\ S M_S\ T M_T} \times \left| \Omega \lambda+1 \begin{matrix} [n n 0] \\ (S M_S)(T M_T) \end{matrix} \right\rangle \qquad (5.B.6)$$

Since all the coefficients are real, then

$$A_{\Omega \lambda\ n_1\ S_1\ M_{S_1} T_1 M_{T_1}}^{\Omega \lambda-1\ n\ S\ M_S T\ M_T} = B_{\Omega \lambda-1\ n\ S\ M_S\ T M_T}^{\Omega \lambda\ n_1\ S_1\ M_{S_1}\ T_1 M_{T_1}} \qquad (5.B.7)$$

Completely analogous relations hold for the v = 1 case.

B.2. Symmetry Properties of the Hamiltonian

The pairing Hamiltonian in (5.B.1) has the most generalform, with the least possible additional symmetry, when the various coefficients g are all different. It is important to see what types of energy degeneracy will occur under different additional symmetry restrictions in the Hamiltonian. If one defines

$$\mathcal{A}_{ij}^{13} = \sum_{M_S} {}^{13}A_i^{+}(M_S 0)\, {}^{13}A_j(M_S 0) \qquad (5.B.8)$$

and

$$\mathcal{A}_{ij}^{31} = \sum_{M_T} {}^{31}A_i^{+}(0 M_T)\, {}^{31}A_j(0 M_T)$$

The total number of particles for the whole system are conserved. And since A^{13}_{i} and A^{31}_{ij} are scalars in spin and isospin, the pairing Hamiltonian

$$H_{pairing} = \sum_{ij} g^{13}_{ij} A^{13}_{ij} + g^{31}_{ij} A^{31}_{ij} \tag{5.B.9}$$

does not split S or T into "magnetic" components. Also, the total S and T are good quantum numbers. However, the total Wigner supermultiplet numbers are not good quantum numbers. The Hamiltonian (5.B.1) is off-diagonal in these. If the pairing Hamiltonian is further degenerated, and $g^{13}_{ij} = g^{31}_{ij}$, then the total Wigner super multiplet is a good quantum number and is not split into its sub-levels of different S and T.

C. THE SINGLE LEVEL CASE

In order to demonstrate the general technique, the single level case is worked out here as the first simple example. For a single level, the pairing Hamiltonian is reduced to

$$H_{pairing} = g^{13} \sum_{M_T'} A^{\dagger}(0M_T')A(0M_T') + g^{31} \sum_{M_S'} A^{\dagger}(M_S'0)A(M_S'0) \tag{5.C.1}$$

With v = 0, the wave function is defined by

$$\left| \begin{matrix} \Omega \lambda n \\ (SM_S)(TM_T) \end{matrix} \right\rangle = \left| \Omega \lambda \begin{matrix} [n\, n\, 0] \\ (SM_S)(TM_T) \end{matrix} \right\rangle \tag{5.C.2}$$

and the computation can be outlined as follows.

C1. Evaluation Of The Matrix Element

$$\sum_{M_s'} \left\langle {\Omega \lambda \bar{n} \atop (SM_s)(TM_T)} \right| {}^{13}A^{+}(M_s'0)\, {}^{13}A(M_s'0) \left| {\Omega \lambda n \atop (SM_s)(TM_T)} \right\rangle$$

It is necessary to sum over the intermediate states

$$\sum_{S''M_s'M_s''n''\lambda''} \left\langle {\Omega \lambda \bar{n} \atop (SM_s)(TM_T)} \right| {}^{13}A^{+}(M_s'0) \left| {\Omega \lambda'' n'' \atop (S''M_{s''})(TM_T)} \right\rangle \times$$
$$\left\langle {\Omega \lambda'' n'' \atop (S''M_{s''})(TM_T)} \right| {}^{13}A(M_s'0) \left| {\Omega \lambda n \atop (SM_s)(TM_T)} \right\rangle \qquad (5.C.3)$$

It is best to carry out the sum over the M_s' M_s'' first, then that over S'', and then n''.

(a) Summing over M_s' and M_s''

Since A behaves like $(-)^{1-M_s'}|\, SM_s >$ in S space the M_s-dependent part of the above sum gives.

$$\sum_{M_s' M_s''} (-)^{1-M_s'} \langle SM_s\, 1 -M_s' \,|\, S''M_s'' \rangle \langle S''M_s''\, 1 M_s' \,|\, SM_s \rangle$$

$$= (-)^{S+1-S''} \sqrt{\frac{2S''+1}{2S+1}} \sum_{M_s' M_s''} \langle S''M_s''\, 1 M_s' \,|\, SM_s \rangle^2$$

$$S'' = S \pm 1$$

so that

$$(-)^{S+1-S''} = 1$$

and the semmation over M'_s and M''_s gives the results $\sqrt{\frac{2S''+1}{2S+1}}$ via the orthonormality of the Wigner coefficients.

(b) Summing over S"

$$\chi_{n''} = \sum_{S''} \left\langle \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n''\,n''\,0] \\ S''T\end{matrix} \right\rangle \left\langle \begin{matrix}[n''\,n''\,0] \\ S''T\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \right\rangle \sqrt{\frac{2S''+1}{2S+1}}$$

$$= \left\langle \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n''\,n''\,0] \\ S+1\ T\end{matrix} \right\rangle \left\langle \begin{matrix}[n''\,n''\,0] \\ S+1\,T\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \right\rangle \sqrt{\frac{2S+3}{2S+1}}$$

$$+ \left\langle \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n''\,n''\,0] \\ S-1\ T\end{matrix} \right\rangle \left\langle \begin{matrix}[n''\,n''\,0] \\ S-1\,T\end{matrix} \begin{matrix}[1\,1\,0] \\ 10\end{matrix} \middle\| \begin{matrix}[n\,n\,0] \\ ST\end{matrix} \right\rangle \sqrt{\frac{2S-1}{2S+1}}$$

By using Table 3.12 and 3.13, the above summation is equal to

(1) For n" = n+1

$$\chi_{n+1} = \sqrt{\frac{1}{4(n+1)(n+2)(n+3)(n+4)(2S+1)^2}} \Big[(S+1)(n+S+T+4)(n+S-T+3) + S(n-S+T+3)(n-S-T+2)\Big] \tag{5.C.4}$$

(2) For n" = n-1

$$\chi_{n-1} = \sqrt{\frac{1}{4n(n+1)(n+2)(n+3)(2S+1)^2}} \Big[(S+1)(n-S+T+1)(n-S-T) + S(n+S-T+1)(n+S+T+2)\Big] \tag{5.C.5}$$

(c) Summing over n" and λ": The final matrix element is given by

$$F(\Omega;\lambda,\lambda+1;n\,n+1)F(\Omega;\lambda+1\,\lambda;n+1\,\bar{n})\chi_{n+1} + F(\Omega;\lambda\,\lambda+1;n\,n-1)F(\Omega,\lambda+1,\lambda;n-1\,\bar{n})\chi_{n-1} \tag{5.C.6}$$

By using Table 4.1, one immediately gets the result. With a similar claculation, or by interchanging S and T, one also obtains the matrix elements of ${}^{13}A^{+13}A$.

C.2. $\left\langle \begin{matrix}\Omega & \lambda & \bar{n}\\ (SM_S) & (TM_T) & \end{matrix} \middle| H_{pairing} \middle| \begin{matrix}\Omega & \lambda & n\\ (SM_S) & (TM_T) & \end{matrix} \right\rangle$

From the discussion of Section B.2, the pairing Hamiltonian given by (5.C.1) connects only states of the same (SM_S) (TM_T), and from (5.C.6) $\bar{n}$ can be n, n-2 and n+2

(a) Pairing Hamiltonian connection $n \to \bar{n} = n$.

$$H_{nn} = g_{13}\left\{\begin{array}{l} \dfrac{(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{4(n+2)(n+3)}\left[\dfrac{(S+1)(n+S+T+4)(n+S-T+3)+S(n-S+T+3)(n-S-T+2)}{(2S+1)}\right] \\ \dfrac{(\Omega-n+\lambda+2)(\Omega+n-\lambda+4)}{4(n+1)(n+2)}\left[\dfrac{(S+1)(n-S+T+1)(n-S-T)+S(n+S-T+1)(n+S+T+2)}{(2S+1)}\right] \end{array}\right\}$$

$$+ g_{31}\left\{\begin{array}{l} \dfrac{(\Omega+n+\lambda+6)(\Omega-n-\lambda)}{4(n+2)(n+3)}\left[\dfrac{(T+1)(n+S+T+4)(n-S+T+3)+T(n-T+S+3)(n-S-T+2)}{(2T+1)}\right] \\ \dfrac{(\Omega-n+\lambda+2)(\Omega+n-\lambda+4)}{4(n+1)(n+2)}\left[\dfrac{(T+1)(n-T+S+1)(n-S-T)+T(n+T-S+1)(n+S+T+2)}{(2T+1)}\right] \end{array}\right\}$$

(b) Pairing Hamiltonian connection $n \to \bar{n} = n+2$

$$H_{n+2,n} = (g_{31}-g_{13})\frac{1}{4(n+3)}\sqrt{\frac{(\Omega+n+\lambda+6)(\Omega+n-\lambda+6)(\Omega-n+\lambda)(\Omega-n-\lambda)\cdot(n+S+T+4)(n+S-T+3)(n-S+T+3)(n-S-T+2)}{(n+2)(n+3)}}$$

(c) Pairing Hamiltonian connection $n \to \bar{n} = n-2$.

$$H_{n-2,n} = (g_{31}-g_{13})\frac{1}{4(n+1)}\sqrt{\frac{(\Omega+n+\lambda+4)(\Omega+n-\lambda+4)(\Omega-n+\lambda+2)(\Omega-n-\lambda+2)(n+S+T+2)(n+S-T+1)(n-S+T+1)(n-S-T)}{(n+1)(n)}}$$

C.3. General Discussion of the Pairing Effect for the Single Level Case

Single level pairing effects can be applied to those cases where the energy gaps between different single particle levels are very large compared with the pairing strength; that is, the interaction between different shells is very small and the contribution of the pairing energies is due to the interaction within each level.

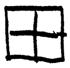
In order to see the general picture of the pairing Hamiltonian, some examples are given here. The possible Wigner supermultiplets and their (ST) values for some of the smaller nucleon numbers are shown in the following table

$N=2 \qquad \lambda=\Omega-1$

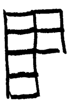
 $n=1$ ST (1 0) (0 1)

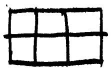
$N=4 \qquad \lambda=\Omega-2$

 $n=0$ ST (0 0)

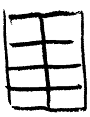
 $n=2$ ST (2 0) (1 1) (0 2) (0 0)

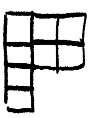
$N=6 \qquad \lambda=\Omega-3$

 $n=1$ ST (1 0) (0 1)

 $n=3$ ST (3 0) (2 1) (1 2) (1 0) (0 3) (0 1)

$N=8 \qquad \lambda=\Omega-4$

 $n=0$ ST (0 0)

 $n=2$ ST (2 0) (1 1) (0 2) (0 0)

$n=4$  (ST) (4 0) (3 1) (2 0) (2 2) (1 1) (1 3) (0 0) (0 2) (0 4)

For N = 2 the Hamiltonian matrix is 1 x 1 for each (ST).

For N = 4 the Hamiltonian matrix is 1 x 1 for (20), (11), and (02);

but it is 2 x 2 for (00) since the Hamiltonian connects n = 2 with n = 0 and each contains a state (ST) = (00).

From (5.C.6), one gets the Hamiltonian matrix for (00).

$$H_{\text{pairing}} = 4(g_{13}+g_{31})^2 \begin{pmatrix} \Omega+2 & \Delta\sqrt{\frac{(\Omega+2)(\Omega-1)}{3}} \\ \Delta\sqrt{\frac{(\Omega+2)(\Omega-1)}{3}} & \Omega-1 \end{pmatrix}$$

where

$$\Delta = \frac{2(g_{31}-g_{13})}{(g_{13}+g_{31})}$$

and

$$E_{(00)}^{N=4} = (g_{13}+g_{31})\left[(2\Omega+1) \pm \sqrt{(2\Omega+1)^2 - 4(\Omega+2)(\Omega-1)\left(1-\frac{\Delta^2}{3}\right)}\right]$$

For the case $\Delta \ll 1$, that is $g_{13} \approx g_{31}$

$$E_{00}^{N=4} \approx 2(g_{13}+g_{31})\left[1+\frac{\Delta^2(\Omega-1)}{9}\right](\Omega+2)$$

$$E_{00}^{N=4} \approx 2(g_{13}+g_{31})\left[1+\frac{\Delta^2(\Omega+2)}{9}\right](\Omega-1)$$

$$E_{(20)}^{N=4} = 2(g_{13}+g_{31})(\Omega-1)(1-\Delta)$$

$$E_{(02)}^{N=4} = 2(g_{13}+g_{31})(\Omega-1)(1+\Delta)$$

$$E_{11} = 2(g_{13}+g_{31})(\Omega-1)$$

(5.C.10)

This shows that pairing with different singlet and triplet strength does lead to a splitting of the Wigner supermultiplet. If one puts

$$E^{4}_{\boxplus} = 2(g_{13}+g_{31})(\Omega-1)$$

$$E^{4}_{\boxminus} = 2(g_{13}+g_{31})(\Omega+2)$$

Then

$$E^{N=4}_{(00)} \approx E_{\boxminus}\left[1+\frac{\Delta^2(\Omega-1)}{9}\right]$$

$$E^{N=4}_{(00)} \approx E_{\boxplus}\left[1-\frac{\Delta^2(\Omega+2)}{9}\right]$$

$$E^{N=4}_{(20)} = E_{\boxplus}(1-\Delta)$$

$$E^{N=4}_{(11)} = E_{\boxplus}$$

$$E^{N=4}_{(02)} = E_{\boxplus}(1+\Delta)$$

The splitting can be shown in Figure 5.1.

In the figure it is assumed that $g_{13} > g_{31}$; if $g_{31} > g_{13}$ the (20) and (02) level should be interchanged.

A similar effect can be shown for N = 6, where the S = 1 T = 0 and S = 0 T = 1 cases lead to 2 x 2 matrices. For N = 8, S = 0 T = 0 the matrix becomes 3 x 3; but the effect is the same.

When $g_{13} = g_{31}$ the Hamiltonian has higher symmetry and the levels become more digenerated. The Wigner supermultiplet becomes a good quantum number, and the matrix is in diagonal form since all the off diagonal elements in (5.C.8) and (5.C.9) have a common factor of $(g_{31}-g_{13})$. The diagonal elements, after

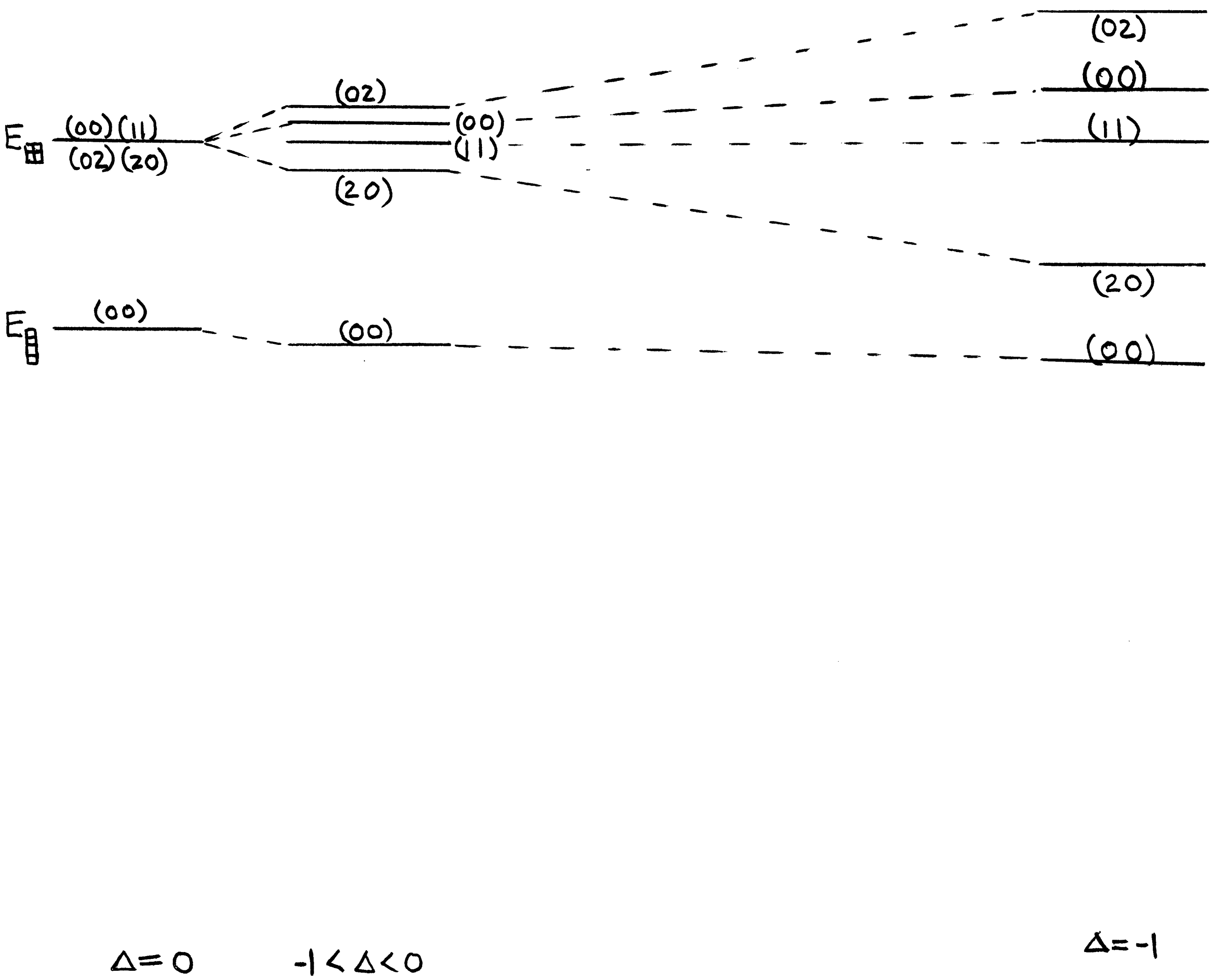


Fig. 5.1. Splitting of pairing energy due to difference in strength of g_{31} and g_{13} for N = 4 S = 0 T = 0.

taking $g_{13} = g_{31}$, contract to

$$E_{\Omega n \lambda} = \Omega^2 + 6\Omega - n^2 - 4n - \lambda^2 - 6\lambda \tag{5.C.11}$$

This can be related to the Casimir invariant of O(8) and O(6) from (4.2.6) of the supplement, and in final form can be expressed as

$$E_{\Omega n\lambda} = \mathbb{C}(8) - \mathbb{C}(6) - \lambda^2 - 6\lambda \tag{5.C.12}$$

which agrees with results derived by Flowers and Szpikowski.

D. FORMULATION OF THE MANY LEVEL PROBLEM

To study the competition between the pairing interaction and the single particle excitations, the case of a system with several single particle levels connected by a pairing interaction must be studied. The case of a system with two single particle levels will be studied in detail. Both pairing effects between levels and within levels must be taken into consideration.

D.1. The Two-Level Case

There are many ways of coupling the two wave functions for N_1 nucleons in level 1 and N_2 nucleons in level 2. One could couple the two O(8) represantations to a resultant O(8) representation specified by overall saniority and reduced supermultiplet quantum numbers. This might be useful in the extreme strong pairing limit. Without coupling O(8), one could also couple the two O(6) representations so that the state vectors are specified by overall Wigner supermultiplet quantum numbers besides those for levels 1 and 2. This would be a useful coupling scheme if the Hamiltonian has enough symmetry to make the total supermultiplet good quantum numbers; but in this scheme the matrix elements would lead to expressions involving SU(4) Racah coefficients which have not yet been calculated. Since the pairing Hamiltonian is diagonal in the total S and T, the most efficient coupling scheme is one in which S_1 and S_2 are coupled to total S and T_1 and T_2 to total T. By such a coupling the size of the Hamiltonian matrix is greatly reduced. Also, the SU(2) Racah coefficients which appear in the general expressions for the matrix elements in this coupling scheme are so well known that no additional mathematical complications are introduced as would be the case in the coupling of the higher group representations. Of course, the simplest formulation of the matrix elements could be made in the completely uncoupled scheme; but this would lead to unnecessarily large matices.

The coupling scheme chosen for the present work is therefore the one in which spins and isospins for the two levels are coupled, and the wave function is written as

$$\left| \begin{array}{cc} \Omega_1 & \Omega_2 \\ \lambda_1 & \lambda_2 \\ n_1 & n_2 \\ \underbrace{(S_1T_1) \; (S_2T_2)}_{S\,T\,M_S M_T} & \end{array} \right\rangle \tag{5.D.1}$$

The pairing Hamiltonian

$$H_{pairing} = \sum_{i,j=1}^{2} \left[g^{ij}_{31} \sum_{M_S} {}^{31}A_i^{\dagger}(M_S 0)\, {}^{31}A_j(M_S 0) + g^{ij}_{13} \sum_{M_T} {}^{13}A_i^{\dagger}(0 M_T)\, {}^{13}A_j(0 M_T) \right] \tag{5.D.2}$$

can be separated into four parts

$$(a)\quad H^{11} = g_{31}^{11} \sum_{M_{S_0}} {}^{31}A_1^{\dagger}(M_{S_0} 0)\, {}^{31}A_1(M_{S_0} 0) + g_{13}^{11} \sum_{M_{T_0}} {}^{13}A_1^{\dagger}(0 M_T)\, {}^{13}A_1(0 M_T)$$

$$(b)\quad H^{22} = g_{31}^{22} \sum_{M_{S_0}} {}^{31}A_2^{\dagger}(M_{S_0} 0)\, {}^{31}A_2(M_{S_0} 0) + g_{13}^{22} \sum_{M_{T_0}} {}^{13}A_2^{\dagger}(0 M_T)\, {}^{13}A_2(0 M_T)$$

$$(c)\quad H^{12} = g_{31}^{12} \sum_{M_{S_0}} {}^{31}A_1^{\dagger}(M_{S_0} 0)\, {}^{31}A_2(M_{S_0} 0) + g_{13}^{12} \sum_{M_{T_0}} {}^{13}A_1^{\dagger}(0 M_T)\, {}^{13}A_2(0 M_T)$$

$$(d)\quad H^{21} = g_{31}^{21} \sum_{M_{S_0}} {}^{31}A_2^{\dagger}(M_{S_0} 0)\, {}^{31}A_1(M_{S_0} 0) + g_{13}^{21} \sum_{M_{T_0}} {}^{13}A_2^{\dagger}(0 M_T)\, {}^{13}A_1(0 M_T) \qquad (5.D.3)$$

The matrix elements for part a and b have already been calculated in (5.C.7), (5.C.8), and (5.C.9). For example

$$H^{11}_{n_1' n_1} = \left\langle \underbrace{\begin{matrix} \Omega_1 & \Omega_2 \\ \lambda_1 & \lambda_2 \\ n_1' & n_2 \\ S_1 T_1 & S_2 T_2 \end{matrix}}_{S T M_S M_T} \right| H^{11} \left| \underbrace{\begin{matrix} \Omega_1 & \Omega_2 \\ \lambda_1 & \lambda_2 \\ n_1 & n_2 \\ S_1 T_1 & S_2 T_2 \end{matrix}}_{S T M_S M_T} \right\rangle \qquad (5.D.4)$$

The quantum numbers $(\Omega, \lambda, n, S, T)$ are replaced by $(\Omega_1, \lambda_1, n_1, S_1, T_1)$ in (5.C.7), (5.C.8), and (5.C.9), and similarly for $H_{n_2' n_2}$.

For parts c and d the matrix elements are more complicated. The following paragraph demonstrates the technique by taking one element of c

Similar to the technique used in Section C, first the SM_S, TM_T part of the matrix element is avaluated, then the n ST part, and then the $\Omega\ \lambda$ n part.

(i) The STM_SM_T part of the matrix element. For the spin triplet part of the pairing interaction chosen, the sums over the SM_S-dependent factors can be carried out, and the result expressed in terms of 9-j or 6-j symbols in the spin quantum numbers:

$$\sum_{\substack{M_{S_1} M_{S_2} \\ M_{S_0} M'_{S_1} M'_{S_2}}} \langle S_1 M_{S_1} S_2 M_{S_2} | S M_S \rangle \langle S'_1 M'_{S_1} S'_2 M'_{S_2} | S M_S \rangle \langle S M_{S_1} S_0 - M_{S_0} | S'_1 M'_{S_1} \rangle$$
$$\langle S_2 M_{S_2} S_0 M_{S_0} | S'_2 M'_{S_2} \rangle \langle S_0 - M_{S_0} S_0 M_{S_0} | 00 \rangle$$
$$\langle S M_S 0 0 | S M_S \rangle \sqrt{(2S_0+1)}$$
$$= \sqrt{(2S_0+1)(2S'_1+1)(2S'_2+1)(2S+1)} \begin{Bmatrix} S_1 & S_0 & S'_1 \\ S_2 & S_0 & S'_2 \\ S & 0 & S \end{Bmatrix}$$
$$= (-)^{S'_2+S_1+S_0+S} \sqrt{(2S'_1+1)(2S'_2+1)} \begin{Bmatrix} S_2 & S'_2 & S_0 \\ S'_1 & S_1 & S \end{Bmatrix}$$
$$= (-)^{S'_2+S_1+S+1} \sqrt{(2S'_1+1)(2S'_2+1)} \begin{Bmatrix} S_2 & S'_2 & 1 \\ S'_1 & S_1 & S \end{Bmatrix} \qquad (5.D.5)$$

It is convenient to abbreviate these resultant expression in (5.D.3) by means of the symbol $S(S_1S'_1;\ S_2S'_2;\ S)$. The needed S functions are tabulated as follows

TABLE 5.1. $S(S_1S_1';S_2S_2';\ S)$

S_1'	S_2'	$S\,(S_1S_1';\,S_2S_2';\,S)$
S_1+1	S_2+1	$\sqrt{\dfrac{(S_1+S_2+S+2)(S_1+S_2+S+3)(S_1+S_2-S+1)(S_1+S_2-S+2)}{(2S_1+1)\,(2S_2+1)\,(2S_1+2)(2S_2+2)}}$
S_1+1	S_2-1	$\sqrt{\dfrac{(S-S_1+S_2-1)\,(S-S_1+S_2)(S+S_1-S_2+1)\,(S+S_1-S_2+2)}{(2S_1+1)\,(2S_2+1)\,(2S_1+2)\,(2S_2)}}$
S_1-1	S_2+1	$\sqrt{\dfrac{(S+S_1-S_2-1)\,(S+S_1-S_2)\,(S+S_2-S_1+1)(S+S_2-S_1+2)}{(2S_1+1)\,(2S_2+1)\,(2S_2+2)\,2S_1}}$
S_1-1	S_2-1	$\sqrt{\dfrac{(S+S_1+S_2)\,(S+S_1+S_2+1)\,(S_1+S_2-S-1)\,(S_1+S_2-S)}{(2S_1)(2S_2)(2S_1+1)\,(2S_2+1)}}$

Similarly a T function is defined which is related to the S function as follows

$$T(T_1T_1';T_2T_2';T) = -S\,(T_1T_1';T_2T_2';T) \qquad (5.D.5)$$

The (-1) in (5.D.5) is ~~due to~~ the phase factor of A which can be obtained from Table 3.4. This T function is neede for the isospin triplet part of the pairing interaction.

The analogu of (5.D.3) with S = 0 has the simple value 1, so that

$$S(S_1S_1;\,S_2S_2;\,S) = T(T_1T_1;\,T_2T_2;\,T) = 1 \qquad (5.D.6)$$

(ii) The n ST part of the matrix element. The nST part can be obtained directly from Tables 3.11 and 3.12. A short hand notation is introduced.

$$P(n\,n';\,SS';\,TT') = \left\langle \begin{matrix}[n\,n\,0] & [110] \\ ST & S_0T_0\end{matrix} \right\| \left. \begin{matrix}[n'\,n'\,0] \\ S'\,T'\end{matrix} \right\rangle \tag{5.D.7}$$

(iii) The Ω λ n part of the matrix element. This can be obtained directly from Table 4.1. With the above discussion

$$\Bigg\langle \underbrace{\begin{matrix}\Omega_1 & \Omega_2 \\ \lambda_1+1 & \lambda_2-1 \\ n_1' & n_2' \\ S_1'T_1 & S_2'T_2\end{matrix}}_{STM_SM_T} \Bigg| \sum_{M_{S_0}} {}^{31}A_2^{\dagger}(M_{S_0}0)\,{}^{31}A_1(M_{S_0}0) \Bigg| \underbrace{\begin{matrix}\Omega_1 & \Omega_2 \\ \lambda_1 & \lambda_2 \\ n_1 & n_2 \\ S_1T_1 & S_2T_2\end{matrix}}_{STM_SM_T} \Bigg\rangle$$

$$= F(\Omega_1;\,\lambda_1\,\lambda_1+1;\,n_1n_1')\,F(\Omega_2;\,\lambda_2\,\lambda_2-1;\,n_2n_2')\,S(S_1S_1';\,S_2S_2'\,S)$$
$$T(T_1T_1;\,T_2T_2;\,T)\,P(n_1n_1';\,S_1S_1';\,T_1T_1)\,P(n_2n_2';\,S_2S_2';\,T_2T_2)$$

All other matrix elements can be obtained similarly. If one or both of the wave functions for levels 1 or 2 have v = 1, the expression has the same form if the F and P functions are replaced by the analogous functions valid for the v = 1 case.

D.2. The Many Level Cases

With the above two simple examples, 1 that of a single level with different pairing spin singlet and triplet strenths, and 2 that of two levels with the same pairing strengths, the technique has been illustrated. The general many level cases are the same in theory, except that the wave functions involved will lend to more and more matrix elements connections between states and therefore to a larger matrix diagonalization problem.

The choice of coupling scheme is again not unique. The simplest is not to couple at all, that is to use the scheme

$$\left| \begin{matrix} \Omega_1 & \Omega_2 & \Omega_3 & \cdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots \\ n_1 & n_2 & n_3 & \cdots \\ S_1 T_1 & S_2 T_2 & S_3 T_3 & \cdots \\ M_{S_1} M_{T_1} & M_{S_2} M_{T_2} & M_{S_3} M_{T_3} & \cdots \end{matrix} \right\rangle \tag{5.D.8}$$

By doing so the formulation of the problem is as trivial as for the single level case, but at the cost of solving a bigger matrix. However, if one explores the symmetry of the problem, and takes the best coupling scheme, one could solve a set of smaller matrices. Since for all types of pairing strengths, the total S and total T is still a good quantum number, one can therefore couple the S_i, T_i of the many levels to a resultant total S and T, and then through the suitable decoupling scheme derive the matrix elements. The following example of a pairing matrix element for the three level case is an attempt to demonstrate one of the decoupling techniques.

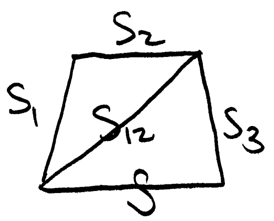
Suppose the coupling scheme for the three level case is chosen in the following form

$$\left|\begin{array}{ccc} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ n_1 & n_2 & n_3 \\ \multicolumn{3}{l}{\underbrace{\underbrace{S_1T_1 \;\; S_2T_2}_{S_{12}T_{12}} \;\; S_3T_3}_{STM_SM_T}} \end{array}\right\rangle$$

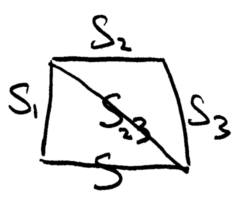
The matrix element of $A_2^{+}A_1$ has been evaluated in the two level case. The decoupling from the $|(S_{12}T_{12}S_3T_3)STM\ M>$ scheme to the $|S_{12}T_{12}M_{S12}M_{T12}S_2\ T_3M_{S3}M_{T3}>$ scheme via ordinary Clebsch-Gordan coefficients leads with the orthogonality relation, to the result that the matrix element in the above scheme is equal to

$$\left\langle\begin{array}{cc} \Omega_1 & \Omega_2 \\ \lambda_1' & \lambda_2' \\ n_1' & n_2' \\ \multicolumn{2}{c}{\underbrace{S_1'T_1' \;\; S_2'T_2'}_{\substack{S_{12}'T_{12}' \\ M_{S_{12}}M_{T_{12}}}}} \end{array}\right| A_2^{+}A_1 \left|\begin{array}{cc} \Omega_1 & \Omega_2 \\ \lambda_1 & \lambda_2 \\ n_1 & n_2 \\ \multicolumn{2}{c}{\underbrace{S_1T_1 \;\; S_2T_2}_{\substack{S_{12}T_{12} \\ M_{S_{12}}M_{T_{12}}}}} \end{array}\right\rangle \qquad (5.D.10)$$

The matrix element of $A_2^{+}A_3$ is more complicated and can be evaluated by a recoupling transformation of the type illustrated by the diagrammatic equation



$$= \sum_{S_{23}}$$



$$U(S_1S_2SS_3;\,S_{12}S_{23}) \qquad (5.D.11)$$

and a similar relation for the isospin part of the space (In the above the U-coefficient is the unitary form of the Racah coefficient). The matrix element then becomes

$$\left\langle \begin{array}{ccc} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2' & \lambda_3' \\ n_1 & n_2' & n_3' \\ \multicolumn{3}{c}{\underbrace{\underbrace{S_1T_1 \quad S_2'T_2'}_{S_{12}'T_{12}'} \quad S_3T_3}_{\substack{ST \\ M_SM_T}}} \end{array} \right| \sum_{M_{S_0}} {}^{31}A_2^{\dagger}(M_{S_0}0)\,{}^{31}A_3(M_{S_0}0) \left| \begin{array}{ccc} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ n_1 & n_2 & n_3 \\ \multicolumn{3}{c}{\underbrace{\underbrace{S_1T_1 \quad S_2T_2}_{S_{12}T_{12}} \quad S_3T_3}_{\substack{ST \\ M_SM_T}}} \end{array} \right\rangle$$

$$= \sum_{\substack{S_{23}\,S_{23}' \\ T_{23}\,T_{23}'}} U(S_1S_2SS_3;S_{12}S_{23})\,U(T_1T_2TT_3;T_{12}T_{23})\,U(S_1S_2'SS_3';S_{12}'S_{23}')\,U(T_1T_2'TT_3';T_{12}T_{23}')$$

$$\left\langle \begin{array}{ccc} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2' & \lambda_3' \\ n_1 & n_2' & n_3' \\ \multicolumn{3}{c}{\underbrace{S_1T_1 \quad \underbrace{S_2'T_2' \quad S_3'T_3'}_{S_{23}'T_{23}'}}_{ST\,M_SM_T}} \end{array} \right| \sum_{M_{S_0}} {}^{31}A_2^{\dagger}(M_{S_0}0)\,{}^{31}A_3(M_{S_0}0) \left| \begin{array}{ccc} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ n_1 & n_2 & n_3 \\ \multicolumn{3}{c}{\underbrace{S_1T_1 \quad \underbrace{S_2T_2 \quad S_3T_3}_{S_{23}T_{23}}}_{ST\,M_SM_T}} \end{array} \right\rangle \tag{5.D.12}$$

so that the problem is reduced to one of the two level type again

E. COMPUTATIONS AND RESULTS

As a specific example, a detailed computation has been carried out for a two-level case to make it possible to study the effect of the pairing interaction in the pressure of single particle excitations. The specific example chosen is one with $\Omega_1 = 1$ and $\Omega_2 = 5$; that is one with an $\ell = 0$ level and a nearby $\ell = 2$ level such as those found in the 2s1d shell of real nuclei. Fo simplicity all pairing strengths are put equal to g. The energy reference point is chosen as the single particle level 1, so that ε_1 can be put equal to zero. The full Hamiltonian can then be written as

$$H = \varepsilon N_2 - g \sum_{i,j=1}^{2} \left[\sum_{M_S} {}^{31}A_i^{\dagger}(M_S 0)\, {}^{31}A_j(M_S 0) + \sum_{M_T} {}^{13}A_i^{\dagger}(0 M_T)\, {}^{13}A_j(0 M_T) \right] \tag{5.E.1}$$

The energies for a fixed total number of particles $N = N_1 + N_2$, and fixed total S and T, can then be found by diagonalizing the matrix for the above Hamiltonian in a basis which, for seniorities $v_1 = v_2 = 0$, can be characterized by the seven numbers $N_1 n\, S_1 T_1 n_2 S_2 T_2$. (N is not listed simply because $N_2 = N - N_1$). For example, for N = 4, S = T = 0 the five states of the basis are

N_1	n_1	S_1	T_1	n_2	S_2	T_2
4	0	0	0	0	0	0
2	1	1	0	1	1	0
2	1	0	1	1	0	1
0	0	0	0	2	0	0
0	0	0	0	0	0	0

The Hamiltonian for each N,S,T is then diagonalized to give the energies and wave functions for specific values of the parameter ε/g. Since the total Wigner-supermultiplet degenerary is not removed by the Hamiltonian (5.E.1), with one common value for g, states with different ST values belonging to the same total Wigner supermultiplet will have to be degenerate. This can serve as a check on the numerical calculations which have carried out for all ST.

Calculations have been carried out in particular for N = 4,6,8, and 10. The results are shown in Figures 5.2-5.5 where the energies are plotted for different values of the parameter ε/g. For $\varepsilon/g < 1$ the energies are plotted in units of g versus ε/g on the left side; on the other hand, for $\varepsilon/g > 1$, the energies are plotted in units of ε versus g/ε or the right side of the figures.

On the extreme right, that is with $g/\varepsilon = 0$, the extreme limit of weak pairing is reached; and the energies are degenerated to the single particle energies, which depend only on the number of particles in the second level. When the pairing strenght is "turned on", the levels are split. Since all $g^{ij}_{\alpha\beta}$ are assumed to have one common value in this calculation, the total Wigner supermultiplet is a good quantum number for <u>all</u> values of ε/g. Levels with different total S and T belonging to the same Wigner supermultiplet are degenerate, and only the supermultiplet lables are indicated. When the pairing strength is turned on to such a great magnitude that ε/g approaches zero, all single particle effect disappear and the extreme pairing limit of a single level with $\Omega = \Omega_1+\Omega_2 = 6$ is reached.

In this extreme pairing limit, the energies are completely specified by the overall O(8) representation labels and the total Wigner supermultiplet quantum numbers, and in terms of these are given by equation (5.C.12) with $\Omega = 6$. In order to understand this extreme pairing limit, the coupling of the simple O(8) representations corresponding to v = 0 states for levels with $\Omega_1 = 1$ and $\Omega_2 = 5$ must be worked out. The result gives the coupling rule

TABLE 5.2

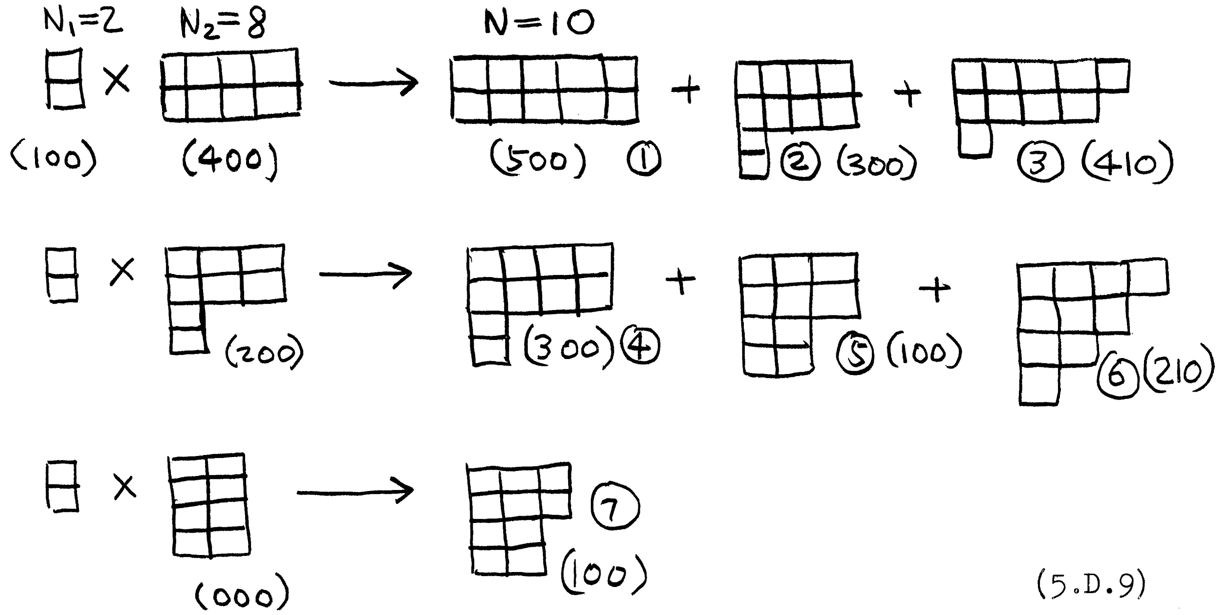
BRANCHING RULE OF (5100) → N, O(6)

N	O(6) Representations
0 (24)	(000)
2 (22)	(100)
4 (20)	(200) (110) (000)
6 (18)	(300) (210) (100)
8 (16)	(400) (310) $(200)^2$ (110) (000)
10 (14)	(500) (410) $(300)^2$ (210) $(100)^2$
12	(510) $(400)^2$ (310) $(200)^2$ (110) $(000)^2$

Since Ω is equal to 6, N_{max} is equal to 24 particles, but when N > 12, from the particle and hole relationship the O(6) representations for $N_{max}-N$ are seen to be the same as those for N.

In the extreme pairing limit, $\mathcal{E}/g = 0$, therefore, the energies are completely specified by the overall O(8) representations (6000), (5100) or (4000) and the total Wigner supermultiplets (or O(6) representations) contained in these. Certain of the levels show a two-fold Wigner supermultiplet degeneracy. These include all the two-fold cases for the O(8) representation (5100) listed in Table 5.2 as well as some additional accidental degeneracies such as that between (6000) [220], and (5100) [000] for N =4 (Fig. 5.2), which can be seen from the strong pairing energy formula (5.C.12).

In the intermediate coupling case, the total Wigner supermultiplet is a "good quantum number", and it is interesting to see how the various possible total Wigner supermultiplet, for a given total N can be obtained from the coupling of the level 1 supermultiplet $(n_1 00)$ with the level 2 supermultiplet $(n_2 00)$. This is illustrated by the following examples



(5.D.9)

From the overall results illustrated by Figs. 5.2-5.5, some general conclusions may be drawn.

1. The pairing effect is predominant for all but the very smallest values of $g/\mathcal{E}$. Except for a small splitting of the levels which are degenerate in the extreme strong pairing limit, the spectra in all cases are qualitatively very similar to the strong pairing limit down to value of $g/\mathcal{E} \approx .2$. That is, with

the exception of the extreme weak pairing limit, the pairing effects win out over the effects arising from single particle excitations.

2. The pairing interaction is most effective in those Wigner supermultiplet states which are built from the largest possible number of columns of four. These columns of four have the general symmetry and $S = T = 0$ character of an α-cluster. The pairing interaction thus seems to favor states built from and clusters in that it tends to make such states more stable. For example, for $N = 10$, the states with $N_2 = 10$ in the weak pairing limit are split such that the supermultiplet [3322] = [110] is depressed much more than [4411] = [330] which in turn lies below [5500], and the supermultiplet [3322] = [110], is depressed to an energy level lower than some of the energy levels which grow out of $N = 8$ after the pairing strength is increased only to $g/\varepsilon = .1$ For the levels which grow out of the weak pairing limit $N_2 = 8$, a similar rule can be seen to hold, so that the energy levels corresponding is states in the first row of (5.D.8) lie higher than those for the 2nd row, while those of the 3rd row built from states with two columns of four in level 2 are depressed to the lowest energies. Within the first row the ordering of energies, highest to lowest, is 3 > 1 > 2 ; that is the state with one column of four is lowest. Similarly, the energy ordering is 6 > 4 > 5 ; and the overall ordering of those levels which grow out of the weak pairing limit $N = 8$ is given by 3 > 1 > 2 > 6 > 4 > 5 > 7 in the notation of (5.D.8).

Although the numerical calculations presented here have not been extensive, the calculations which have been carried out do seem to indicate two general results. The pairing interaction is very effective compared with the particle excitation energy, and pairing effects were out in the competition with single particle excitations. Secondly, the pairing interaction tends to make more stable those states built from the largest possible number of α-like groupings of four particles, a "fouring" effect, which is not forward and could not be understood in terms of a chare-indpendent pairing interaction in the j-j coupling scheme.

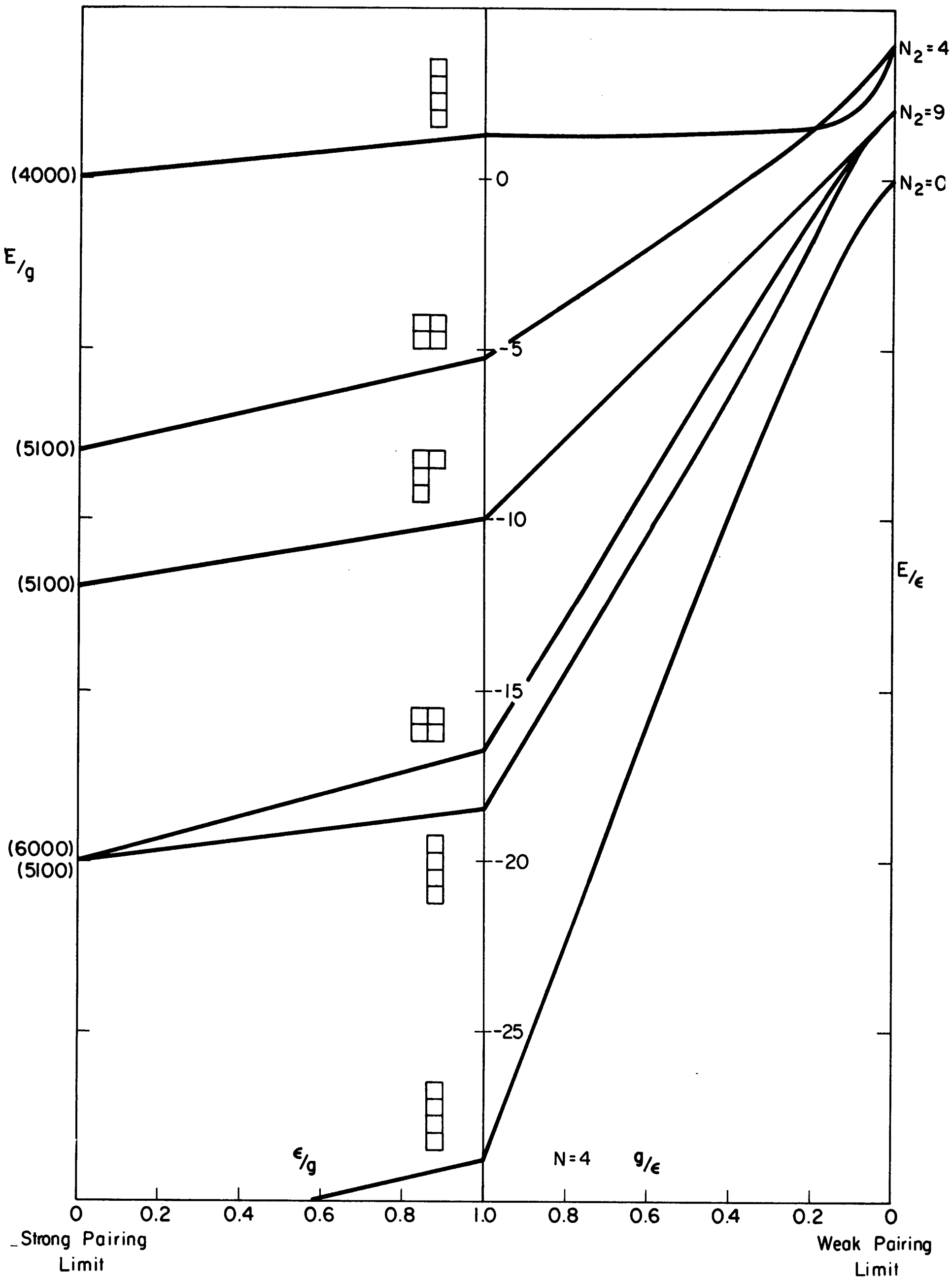


Fig. 5.2. N = 4; Pairing energy spectrum for four nucleons distributed over a spectrum of two single-particle levels of $l = 0$ (s or $\Omega_1 = 1$) and $l = 2$ (d or $\Omega_2 = 5$)—like character. Individual level seniorities are zero, $v = 0$, for the two levels. The energy of the first (s-like) level is considered as zero, while the seond level (d-like) is taken at energy $\mathcal{E}$ above the first level. The curves show the energies (in units of g) as a function of $\mathcal{E}/g$ on the left side to the limit of $g \to \infty$, in which all the nucleons are distributed in a degenerated s-d level ($\Omega = 6$). O(8) quantum number are used to denote the states in that extreme limit. The curves also show the energies (in units of $\mathcal{E}$) as a function of $g/\mathcal{E}$ on the right side to the limit of $g \to 0$, in which the N nucleons are not interacting. The SU(4) Young Tableaux which is "a good quantum number" for all pairing strengths is shown for each level. For the scalar representation U(4) is shown.

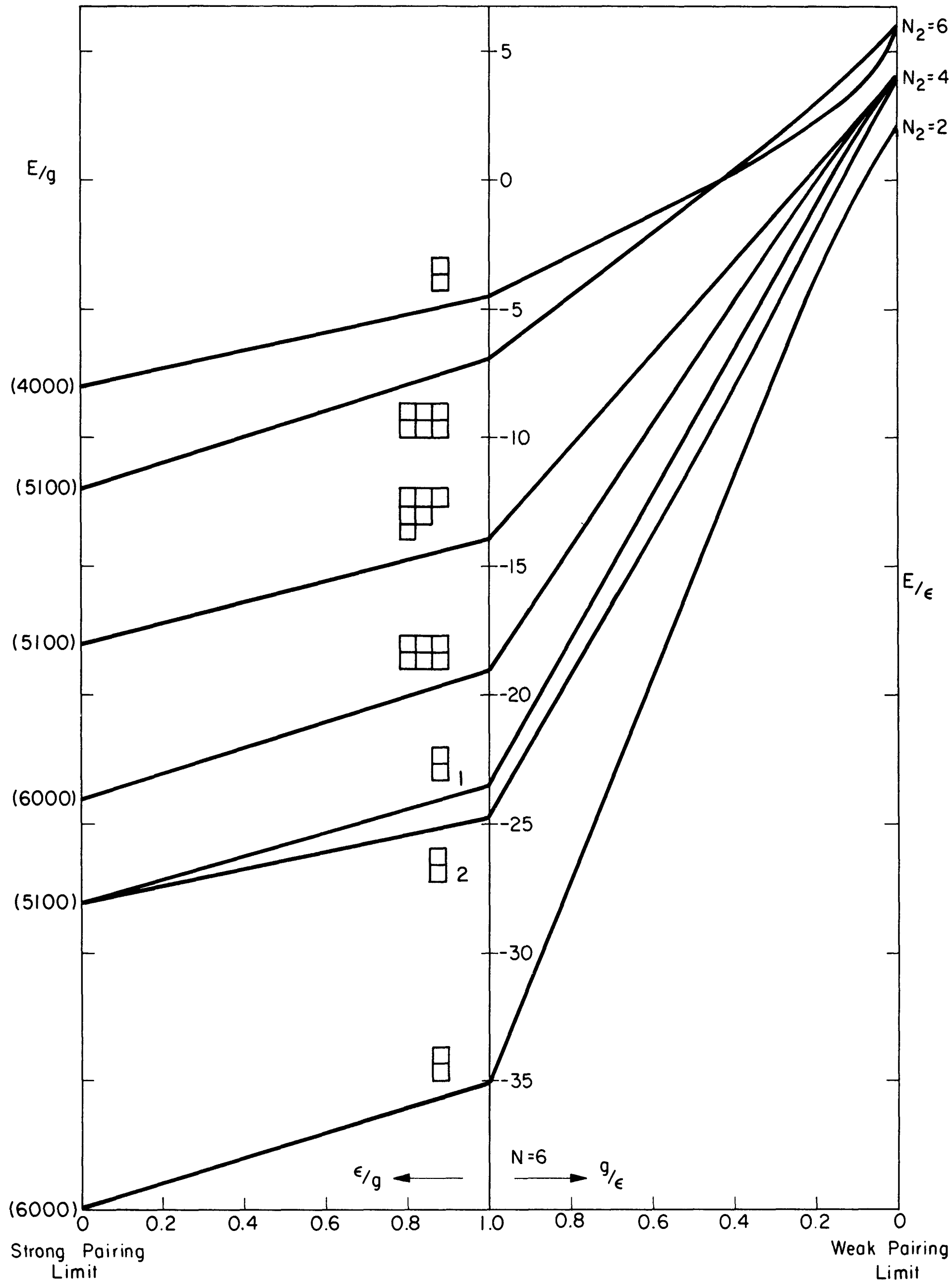


Fig. 5.3. N = 6: Pairing energy spectrum for N = 6. $\Omega_1 = 1$, $\Omega_2 = 5$, $v_1 = 0$. The notation is the same as that for Fig. 5.2.

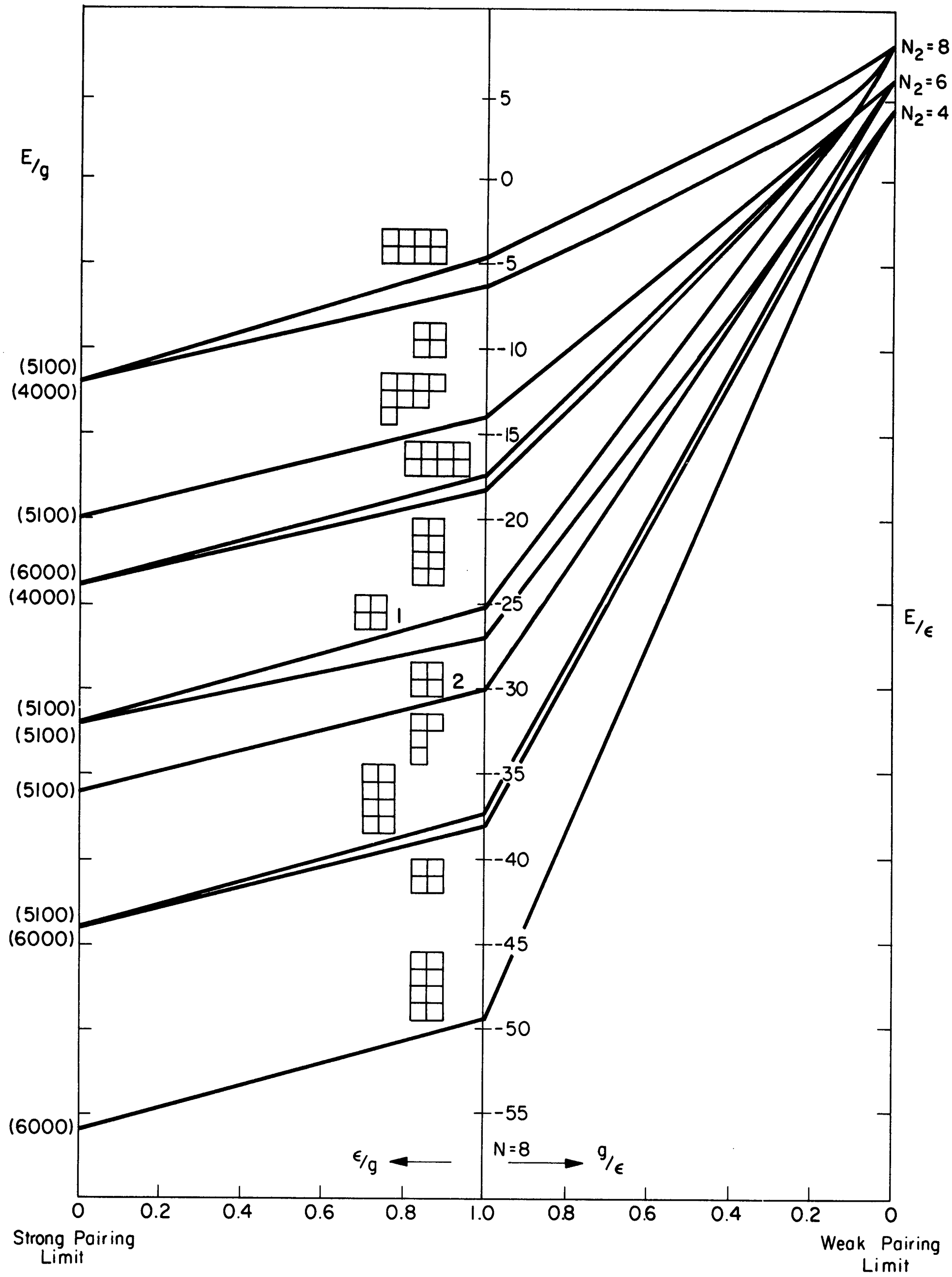


Fig. 5.4. N = 8: Pairing energy spectrum for N = 8. $\Omega_1 = 1$, $\Omega_2 = 5$, $v_1 = 0$. The notation is the same as that for Fig. 5.2.

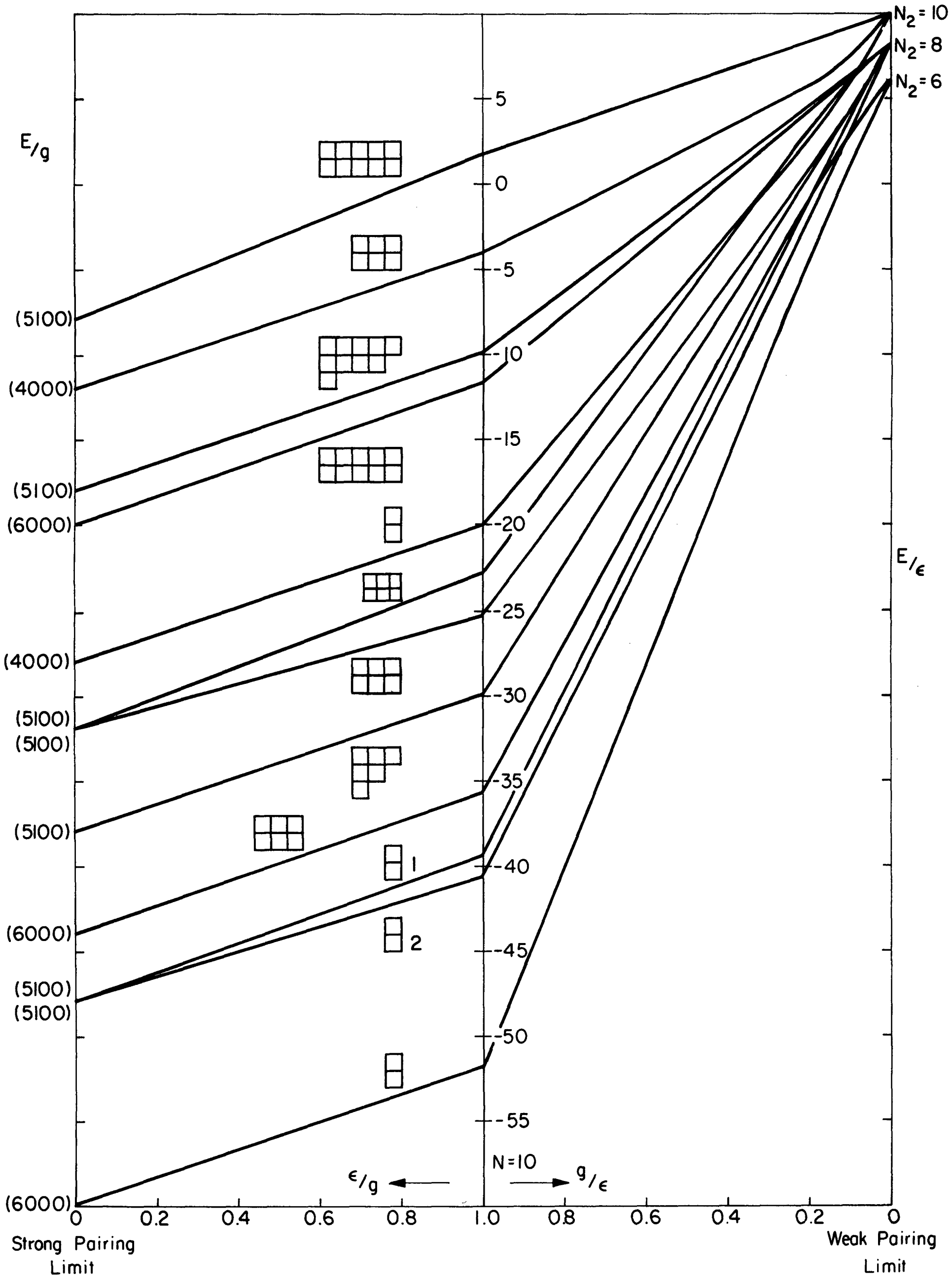


Fig. 5.5. N = 10: Pairing energy spectrum for N = 10. $\Omega_1 = 1$, $\Omega_2 = 5$, $v_1 = 0$. The notation is the same as that for Fig. 5.2.

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SUPPLEMENT

Refer to: Lowering and Raising Operators for the Orthogonal Group in the Chain $O(n) \supset O(n-1) \supset \ldots$, and Their Graphs. J. Math. Phys. $\underline{8}$, 1 33 (1967).

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The present work formulates the exact solution of the pairing problem in the Ω(spatial)-ST scheme in terms of the matrix elements of the pair creation and annihilation operators coupled to zero spatial angular momentum quantum number. This makes it possible to study the pairing interaction with different strengths for the S=0 (T=1) and S=1 (T=0) pairs, as well as for mixed configuration of several single particle levels. The mathematical formulation of the problem has involved the study of an orthogonal group in eight dimensions, the so-called quasispin group. The representations are broken down according to $O(8) \supset O(6) \sim SU(4) \supset SU(2) \times SU(2)$, where SU(4) is the usual Wigner supermultiplet.

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