Effect of Transport Properties on Supersonic Expansion around a Corner

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(Received 16 December 1965; final manuscript received 23 January 1967)

The compressible flow of a viscous, heat-conducting gas around a corner is considered; in particular, the viscous corrections in the expansion region are calculated. The solutions are written in terms of asymptotic expansions, valid in the region far, compared to a viscous length, from the corner, so that the zeroth-order solutions are the classical Prandtl–Meyer solutions. The method of inner and outer expansions is used where the inner region encloses the first Mach line emanating from the corner. It is shown that the first effect of the transport properties in the expansion region is to generate terms either of order Re^{-1} (inverse Reynolds number) or of order $\mathrm{Re}^{-1} \log \mathrm{Re}$, depending on the dependent variable considered.

1. INTRODUCTION

THE Prandtl–Meyer solution[1] for the steady supersonic flow of a compressible inviscid gas around a corner is well known, and has been used extensively. In this solution, viscous and heat-transfer effects are ignored; since there is then no physical characteristic length associated with the problem, the flow variables are independent of the radial distance from the corner, depending only on the angle of turning. Although it is clear that such assumptions are justified for a large class of problems, the extent of the approximation involved by the use of the inviscid equations is not known. More precisely, in terms of an approximate solution the zeroth-order term of which is the Prandtl–Meyer solution, the magnitude and functional form of the terms due to the inclusion of transport effects are unknown.

The general problem of the supersonic flow of a compressible viscous heat-conducting gas around a corner must include the effects of the wall before, at, and beyond the corner, unless the wall ends at the corner, in which case a mixing region exists downstream of the wall (nozzle problem). While a complete solution of the boundary layer flow around a corner has not been found, Lighthill[2] has shown that the boundary layer becomes thinner upstream of the corner. When the external flow is supersonic and the turning angle is small, the dimensionless extent of this thinning region, Δ in Fig. 1, is shown to be of order $\mathrm{Re}^{-3/8}$, where Re is the Reynolds number based on external flow conditions and a characteristic length in the flow direction, L, and Δ is dimensionless with respect to L. Also shown in Fig. 1 is a viscous region which includes the first Mach line. The expansion wave has discontinuities in the normal derivatives at the point where the uniform and Prandtl–Meyer flow solutions join, so the first effects of viscosity are felt in smoothing this discontinuity in derivatives. It is easily seen from the Prandtl–Meyer solutions that when the uniform flow upsteam of the expansion is supersonic the discontinuities appear in the first derivatives and when the upsteam flow is sonic, the discontinuities appear in the second derivatives. As a result, it is shown later that δ, the dimensionless order of the thickness of this viscous region, has a different magnitude for each of these cases. In addition, the order of the viscous corrections in this region depends on the case being considered. In the remainder of the expansion fan, viscous terms are simply corrections to the Prandtl–Meyer solutions and are less pronounced.

Upon consideration of the flow picture as sketched in Fig. 1, it seems clear that the relative importance of viscous terms due to the influence of the boundary layer as compared to those due to the viscous region at the beginning of the expansion can be characterized by a comparison of δ and Δ.[3] That is if, in the limit as $\mathrm{Re} \rightarrow \infty$, δ becomes large compared to Δ, then the viscous correction terms found from the inclusion of transport properties in the expansion wave solution are more important than those due to the thinning of the boundary layer. On the other hand, if in this limit δ is small compared to Δ, then the opposite effect occurs, and the flow picture must be changed to that where the expansion takes place around a curved wall, to first order, and the expansion wave is not really a centered wave. In the latter case, which is shown to be valid when the initial flow is supersonic, the viscous effects found from the expansion wave solution would be expected to

[1] W. G. Bickley, in *Modern Developments in Fluid Dynamics, High Speed Flow*, L. Howarth, Ed. (Clarendon Press, Oxford, England, 1953), Vol. I, pp. 164–173.
[2] M. J. Lighthill, Proc. Roy. Soc. (London) **A217**, 478 (1953).
[3] The author is indebted to Dr. Toshi Kubota, California Institute of Technology, for clarification of this point.

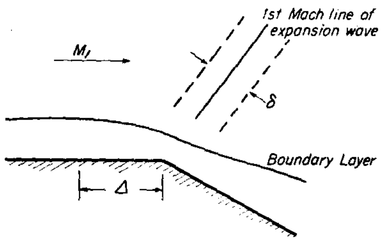

FIG. 1. Sketch of viscous regions formed as flow turns corner.

be of higher order than the boundary layer effects.

In this paper, δ is found for both the sonic and supersonic cases, and the first viscous and heat-conduction corrections to the Prandtl–Meyer solution are calculated. The problem is obviously a singular perturbation problem; the technique employed here is the so-called method of matched asymptotic expansions, developed by Kaplun, Lagerstrom, and Cole[4–6] and described in detail by Van Dyke.[7] The expansion wave is pictured as consisting of two parts; the first, hereafter referred to as the inner region, consists of the region described above, where the effects of viscosity are to smooth the discontinuities in derivatives, while the second, hereafter referred to as the outer region, contains the remainder of the expansion wave.

2. FORMULATION OF THE PROBLEM

A. General Conditions

In the following, attention is directed mainly to the expansion fan. However, it is necessary to discuss certain general aspects of a boundary-layer solution, as becomes apparent when the boundary and matching conditions are considered.

Insofar as the expansion problem is concerned, the characteristic length introduced by the inclusion of transport effect is the viscous length, $\bar{\nu}_1/\bar{q}_1$. Here, ν and q are the kinematic viscosity and velocity, respectively, the bar indicates a dimensional quantity, and the subscript one refers to quantities in the undisturbed stream. Since it is desired to find a solution outside the boundary layer, then certainly $\bar{r} \gg \bar{\nu}_1/\bar{q}_1$, where r is the radial distance form the corner. However, it is convenient to introduce the characteristic length L as an extraneous length which scales the independent variables such that $r = \bar{r}/L$ is of order unity, and

$$L/(\bar{\nu}_1/\bar{q}_1) = \mathrm{Re} \gg 1.$$

All flow variables are made dimensionless with respect to their values in the original undisturbed flow, with the exception of the total enthalpy; thus,

$$P = \bar{P}/\bar{P}_1, \qquad \rho = \bar{\rho}/\bar{\rho}_1, \qquad T = \bar{T}/\bar{T}_1,$$
$$h_t = \bar{h}_t/\bar{q}_1^2, \qquad \mathbf{q} = \bar{\mathbf{q}}/\bar{q}_1, \qquad \mathbf{X} = \bar{\mathbf{X}}/L,$$

where the symbols refer to pressure, density, temperature, total enthalpy, velocity vector, and position vector. Figure 2 is a sketch of the coordinate systems employed. Note that both a Cartesian and a polar coordinate system are indicated, with velocity components U, V and u, v, respectively. The equations relating the velocity components of each coordinate system are simply

$$u = U \sin\varphi + V \cos\varphi, \qquad v = U \cos\varphi - V \sin\varphi. \quad (1)$$

In general, it is assumed that the specific heats are constant, bulk viscosity is negligible, and the coefficients of viscosity and thermal conductivity are related to the temperature by a simple power law. Finally, the gas is assumed to follow the perfect gas law.

B. Dimensionless Equations

The dimensionless equations expressing the conservation of mass, momentum, and energy are written below in vector form. In the inner region of the expansion, Cartesian coordinates are employed, whereas in the outer region, polar coordinates are more convenient. The set of governing equations is completed with the addition of the equation of state and the relation defining the total enthalpy.

$$\begin{aligned}
&\text{(a)} \quad \operatorname{div}(\rho\mathbf{q}) = 0,\\
&\text{(b)} \quad \rho\mathbf{q}\cdot\operatorname{grad}\mathbf{q} + (\sin^2\bar{\mu}_1/\gamma)\operatorname{grad}P = (1/\mathrm{Re})\mathbf{R}_q,\\
&\text{(c)} \quad \rho\mathbf{q}\cdot\operatorname{grad}h_t = (1/\mathrm{Re})R_h, \qquad (2)\\
&\text{(d)} \quad P = \rho T,\\
&\text{(e)} \quad h_t = (\sin^2\bar{\mu}_1)\beta T + \tfrac{1}{2}q^2,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{R}_q &= \operatorname{div}(\mu\,\operatorname{def}\mathbf{q}) - \operatorname{grad}(\tfrac{2}{3}\mu\operatorname{div}\mathbf{q}),\\
R_h &= \operatorname{div}\{(\beta\sin^2\bar{\mu}_1/\mathrm{Pr})\lambda\operatorname{grad}T\\
&\qquad + \mu[\operatorname{def}\mathbf{q} - \tfrac{2}{3}(\operatorname{div}\mathbf{q})\mathsf{I}]\mathbf{q}\},
\end{aligned} \qquad (3)$$

and where I is the unit tensor. In the above, μ and λ are the dimensionless coefficients of viscosity and

[4] S. Kaplun and P. A. Lagerstrom, J. Math. Mech. 6, 585 (1957).
[5] S. Kaplun, J. Math. Mech. 6, 595 (1957).
[6] P. A. Lagerstrom and J. D. Cole, J. Ratl. Mech. Anal. 4, 817 (1955).
[7] M. Van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic Press Inc., New York, 1964).

thermal conductivity, $\bar{\mu}$ is the Mach angle, $\mathrm{Pr} = c_p\mu/\lambda$ is the Prandtl number, $\beta = (\gamma - 1)^{-1}$, where $\gamma = c_p/c_v$ is the ratio of specific heats, and def $\mathbf{q}$ is the deformation tensor (e.g., Ref. 8).

C. Magnitude of the Inner Region

The inner region, defined as that region where the effects of viscosity are to smooth discontinuities in the derivatives, contains the first Mach line, and its thickness is small compared to the scale length, L. It is desired to calculate δ, the order of this thickness, both for comparison with Δ, and for use as a stretching factor making the inner dependent variable measuring the thickness, of order unity. In the present problem, it is possible to infer the proper form for this stretching factor from the equations and the zeroth-order matching condition.

First, the momentum equation [Eq. (2b)] in the x direction may be written in Cartesian form as follows:

$$\left(\frac{P}{\gamma} + \rho U^2\right)_x - U(\rho U)_x + \rho V U_y$$
$$= \frac{1}{\mathrm{Re}}\,\{[\mu(\tfrac{4}{3}U_x - \tfrac{2}{3}V_y)]_x + [\mu(U_y + V_x)]_y\}, \qquad (4)$$

where subscripts x and y indicate partial differentiation, and where the left-hand side of the equation has been slightly rearranged. It is clear that the solutions in the inner region must join the values of the flow variables in the uniform steam to their corresponding outer values found from the Prandtl–Meyer solution for $\varphi \ll 1$. Now, the terms in Eq. (4) may be ordered by considering the total change in the functions across the inner region, so that the equation is used effectively as a jump condition across the inner region. Then, any excess momentum flux or pressure forces (higher-order terms from Prandtl–Meyer solution) must be balanced by viscous forces.

The Prandtl–Meyer solutions, for any uniform stream Mach number, are as follows:

$$u_0 = (\mathfrak{M}/\Gamma)\sin[\Gamma(\varphi + \theta_1)], \quad P_0 = T_0^{\gamma\beta},$$
$$v_0 = \mathfrak{M}\cos[\Gamma(\varphi + \theta_1)], \quad \rho_0 = T_0^{\beta}, \qquad (5)$$
$$T_0 = v_0^2/\sin^2\bar{\mu}_1, \quad h_{t0} = \mathfrak{M}^2/2\Gamma^2,$$

where

$$\mathfrak{M}^2 = [1 + \Gamma^2(M_1^2 - 1)]/M_1^2, \qquad \Gamma^2 = (\gamma - 1)/(\gamma + 1),$$

and where θ_1 is the Prandtl–Meyer turning angle associated with M_1.

[8] P. A. Lagerstrom, in *Theory of Laminar Flows*, F. K. Moore, Ed. (Princeton University Press, Princeton, New Jersey, 1964), p. 24.

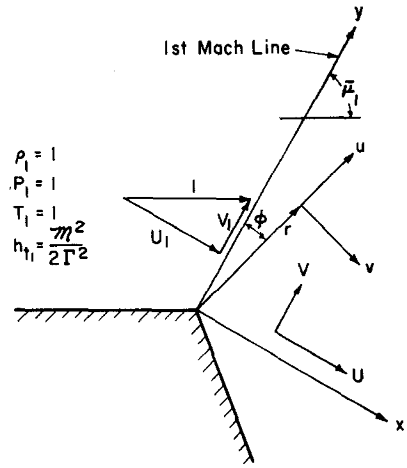

FIG. 2. Coordinate systems and initial conditions in undisturbed flow.

If u_0 and v_0 are substituted in Eqs. (1), U and V may be found in terms of φ, and the resulting equations, as well as those for P_0 and ρ_0, may be expanded for small x/y, since for $\varphi \ll 1$, $\varphi \cong x/y$. Finally, if x is ordered by a function of the Reynolds number, $\delta(\mathrm{Re})$, and y is of order unity,

$$x/y = O(\delta),$$

and across the inner region it can be shown, for example, that if $M = 1$, such that $\cos\bar{\mu}_1 \ll \delta$, then $\Delta U = O[\delta^2]$, $U_x = O(\delta)$. It should be noted that only the two limiting cases, $M_1 > 1$ such that $\cos\bar{\mu}_1 \gg \delta$, and $M_1 = 1$ so that $\cos\bar{\mu}_1 = 0$, are considered in this analysis. They are denoted by $M_1 > 1$, and $M_1 = 1$, respectively.

If, in the case $M_1 = 1$, each term of Eq. (4) is ordered as indicated above, then if the largest viscous term is to balance the largest excess momentum flux terms

$$\delta = \mathrm{Re}^{-\frac{1}{3}}. \qquad (6)$$

The combination of terms shown on the left-hand side of Eq. (4) results in a change of smallest magnitude; hence it is clear that the proper value for δ is that given in Eq. (6).

If the above computations are repeated for the case $M_1 > 1$, the result is

$$\delta = \mathrm{Re}^{-\frac{1}{2}}. \qquad (7)$$

The thickness of a weak discontinuity is generally considered to be of order $\mathrm{Re}^{-\frac{1}{2}}$ (e.g., Ref. 9) as found for the $M_1 > 1$ case. The fact that $\delta = \mathrm{Re}^{-\frac{1}{3}}$ for the first case considered is due to the fact that the flow entering the discontinuity is sonic. This is evidenced by the fact that the relative orders of

[9] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Inc., London, 1959), p. 345.

ΔU, ΔV, Δx, etc., are consistent with those found in investigations of transonic similarity.[10]

Using the values for δ given above, then, the stretched inner variables are defined as follows:

$$\tilde{x} = x/\delta, \qquad \tilde{y} = y, \tag{8}$$

so that in terms of the stretched coordinates, both $\tilde{x}$ and $\tilde{y}$ are of order unity in the inner region. δ is given by Eqs. (6) or (7) depending on whether $M_1 = 1$ or $M_1 > 1$. Finally, the governing equations may be written in terms of the stretched variables, in Cartesian form

$$\begin{aligned}
&\text{(a)}\quad (\rho U)_{\tilde{x}} + \delta(\rho V)_{\tilde{y}} = 0,\\
&\text{(b)}\quad \rho U U_{\tilde{x}} + \delta \rho V U_{\tilde{y}} + (\sin^2 \bar{\mu}_1/\gamma)P_{\tilde{x}} = (\delta/\mathrm{Re})R_U,\\
&\text{(c)}\quad \rho U V_{\tilde{x}} + \delta \rho V V_{\tilde{y}} + \delta(\sin^2 \bar{\mu}_1/\gamma)P_{\tilde{y}} = (\delta/\mathrm{Re})R_V,\\
&\text{(d)}\quad \rho U (h_t)_{\tilde{x}} + \delta \rho V (h_t)_{\tilde{y}} = (\delta/\mathrm{Re})R_H,\\
&\text{(e)}\quad P = \rho T,\\
&\text{(f)}\quad h_t = (\sin^2 \bar{\mu}_1)\beta T + \tfrac{1}{2}U^2 + \tfrac{1}{2}V^2,
\end{aligned} \tag{9}$$

where for example,

$$R_U = (1/\delta)\{\mu[(4/3\delta)U_{\tilde{x}} - \tfrac{2}{3}V_{\tilde{y}}]\}_{\tilde{x}} + \{\mu[U_{\tilde{y}} + (1/\delta)V_{\tilde{x}}]\}_{\tilde{y}}.$$

R_V and R_H may be found from Eqs. (3) by expanding the vector forms in Cartesian coordinates and transforming to stretched coordinates (e.g., Ref. 11).

In order to simplify later calculations, it is necessary, at this point, to introduce an equation formed from the governing equations. Solution to Eqs. (9) are found by expanding the six unknown functions in appropriate forms for $\mathrm{Re} \gg 1$, substituting these expansions into the governing equations, and thus obtaining sets of equations for the zeroth-, first-, etc., order approximations. Now, when the first-order equations are solved, it is found that, although there are six equations, one of them is redundant. Hence, it is necessary to go to the second-order equations, rearrange them, and find, finally, a relation between first-order functions only. This equation completes the set of first-order equations, but then the third-order equations must be considered to find the missing second-order equation, etc. Such an occurrence is not uncommon; it is found in the weak-shock structure problem,[12] for example. However, it can be shown that, in this problem at least, an equation can be derived from the general conservation equations such that when the expansion procedure is used the "missing" equation for each order of approximation results. In effect, the rearranging is accomplished before the expansions are inserted into the equations. The following equation is formed by taking the Eulerian derivative of the equation of state and using the continuity, momentum, and energy equations to remove the resulting derivatives of pressure, temperature, and density. It is written in terms of the stretched inner coordinates:

$$\begin{aligned}
\frac{\delta}{\mathrm{Re}}\left(UR_U + VR_V - \frac{1}{\beta\gamma}R_H\right)
&= \frac{1}{\gamma}\left\{\rho U_{\tilde{x}}\left[\left(\frac{\gamma+1}{2}\right)U^2 + \frac{V^2}{2\beta} - \frac{h_t}{\beta}\right]\right.\\
&\quad + \delta\rho V_{\tilde{y}}\left[\left(\frac{\gamma+1}{2}\right)V^2 + \frac{U^2}{2\beta} - \frac{h_t}{\beta}\right]\\
&\quad \left. + \rho UV(\delta U_{\tilde{y}} + V_{\tilde{x}})\right\}.
\end{aligned} \tag{10}$$

D. Boundary and Matching Conditions

A sketch of the various regions of flow has been given in Fig. 1. The so-called inner region of the expansion wave is contained within the dashed line, while the outer region consists of the part of the expansion downstream of the inner region. Evidently, another "inner" region would exist about a final Mach line, bringing the flow to the final desired condition. This region is not considered in this paper, the emphasis being on the relationship between the indicated flow regions at the beginning of the expansion wave.

In the inner region, the Navier–Stokes equations are to be satisfied. In general, because the unknown functions appear in the second-order derivatives which occur in the viscous terms, two boundary or matching conditions are required. As $\tilde{x} \to \infty$ for all $\tilde{y} = O(1)$, a matching condition between the inner and outer solutions must be met. As $\tilde{x} \to -\infty$ for all $\tilde{y} = O(1)$, another matching condition must be met between the inner solutions and the incoming flow solutions which consist of the zeroth-order uniform flow plus higher-order terms introduced by the existence of the boundary layer. Since the boundary-layer solution for the flow around a corner is not known, neither the order of the higher-order terms, nor their functional dependence is known. However, a gross condition is known, in that as $\tilde{x} \to -\infty$, for $\tilde{y}$ fixed, any corrections to the uniform flow must be bounded and in fact tend toward zero. Hence, any solutions found in the inner region must meet this general condition.

In the outer region, because the solution is a perturbation from the inviscid solutions, the viscous

[10] T. von Kármán, J. Math. & Phys. **26**, 182 (1947).
[11] T. C. Adamson, Jr., University of Michigan, Institute of Science and Technology, Report Bamirac 4613-120-T (1966).
[12] M. Sichel, Phys. Fluids **6**, 653 (1963).

terms are calculated from known functions. Hence, only first-order differential equations must be solved for the unknown functions, and one boundary or matching condition suffices. As it turns out, the first-order effects of viscosity are given by terms of order $\ln \mathrm{Re}/\mathrm{Re}$ and Re^{-1} for both the $M_1 \simeq 1$ and $M_1 > 1$ cases. If, for a given dependent variable, the first-order correction is of order $\ln \mathrm{Re}/\mathrm{Re}$, then the next term, of order Re^{-1}, cannot be found without further information. On the other hand, if there is no term of order $\ln \mathrm{Re}/\mathrm{Re}$, so that the first perturbation is of order Re^{-1}, this solution can be found, so the first-order effect can be found for each variable. In the former case, which occurs for v, P, ρ, and T, the solution for terms of order Re^{-1} may be found only up to one unknown function of φ. Since terms of this order should result from a boundary-layer solution, it seems plausible that this missing information should be supplied by matching with a boundary-layer solution.

It should be emphasized that the main point of this calculation is to find the first effects of viscosity and thermal conduction generated in the expansion region. These terms can be found, and matching and boundary conditions met. In doing this, it is not necessary that every possible term due to effects other than those being studied (e.g., boundary-layer effects) be carried along in the general expansion. For example, if the first-order effects due to the boundary layer are of different order than those generated in the expansion, there is no interaction between the terms, and one should be able to find all constants in the expansion solution, exactly. If, on the other hand, first-order terms from each effect are of the same order, then it should be that undetermined constants exist in the expansion solutions, so that matching with the boundary-layer solution may be accomplished. Of course, the above argument holds only for lower-order terms since it is possible for different multiples of various-order terms to result in terms of the same higher order. In any event, insofar as the general expansions for the flow variables are concerned, some terms are generated by the boundary layer and some by the viscous expansion region. If they are not the same order, then for a complete solution in each viscous region, terms of order of those unique to the expansion process have to be added to the boundary-layer solution, whereas terms like those unique to the boundary layer have to be added to the expansion region solution. This must be done essentially in the same spirit in which $\ln \mathrm{Re}$ terms are added to expansions as a result of matching difficulties in other inner and outer expansion problems. Again, as long as the orders are different, no change in original first-order solutions results from the addition of terms; when orders are the same, of course, no addition need be made.

3. SOLUTIONS

In the following, detailed calculations are given only for the algebraically simpler case, $M_1 = 1$, but results for both $M_1 = 1$ and $M_1 > 1$ are given. Details for the latter calculation may be found in Ref. 11.

The outer solution is desired in the form which indicates the first-order effects of transport properties for the condition $\mathrm{Re} \gg 1$. Thus, we wish to write the flow variables in the form of asymptotic expansions valid in the limit as $\mathrm{Re} \to \infty$. For example, the expansion for the radial flow variable, u, is of the form

$$u \sim \sum_n \epsilon_n(\mathrm{Re}) u_n(r, \varphi), \tag{11}$$

where $\epsilon_0 = 1$, $u_0 = u^{(0)}(\varphi) =$ Prandtl–Meyer solution, and

$$\lim_{\mathrm{Re}\to\infty} (\epsilon_{n+1}/\epsilon_n) = 0.$$

The superscript notation for the outer flow variables (e.g., $u^{(0)}$, $u^{(1)}$) indicates that they are functions of φ alone.

In view of the fact that the zeroth-order outer solution is already known, the next step is to find the zeroth- and first-order inner expansions, where the inner expansions are defined in the same general form as above. For example, the x component of the velocity is expanded as follows:

$$U \sim \sum_n \tilde{\epsilon}_n(\mathrm{Re}) \tilde{U}_n(\tilde{x}, \tilde{y}),$$

where $\tilde{\epsilon}_0 = 1$, $\tilde{U}_0 = \sin \bar{\mu}_1$, and $\tilde{\epsilon}_{n+1} = o(\tilde{\epsilon}_n)$. Note that in the inner expansion, the zeroth-order solution corresponds to the zeroth order oncoming flow.

In order to find $\tilde{\epsilon}_1$, one must examine the zeroth-order outer solution in light of the matching condition which must be met. For example, for $M_1 = 1$, if the Prandtl–Meyer solutions for U and V are expanded for $\varphi \ll 1$ and written in terms of the inner variables, $\tilde{x}$ and $\tilde{y}$, then

$$\begin{aligned} U &\sim 1 + \tfrac{1}{2}(1 - \Gamma^2)(1/\mathrm{Re}^{\frac{2}{3}})(\tilde{x}/\tilde{y})^2 + \cdots, \\ V &\sim -\tfrac{1}{3}(1 - \Gamma^2)(1/\mathrm{Re})(\tilde{x}/\tilde{y})^3 + \cdots, \end{aligned} \tag{12}$$

and it can easily be shown that the remaining dependent variables have the same form as U. Equations (12) are the solutions which must be matched, term by term, by the inner solutions. Their form suggests that $\tilde{\epsilon}_1 = \mathrm{Re}^{-\frac{2}{3}}$ and the corresponding small parameter in the V expansion is Re^{-1}. Then, the expansions are

$$\begin{aligned} U &\sim 1 + (1/\mathrm{Re}^{\frac{2}{3}})\tilde{U}_1 + \cdots \\ &= 1 + (1/\mathrm{Re}^{\frac{2}{3}})(\tilde{U}^{(1)}/\tilde{y}^{\frac{2}{3}}) + \cdots , \\ V &\sim (1/\mathrm{Re})\tilde{V}_1 + \cdots \\ &= (1/\mathrm{Re})(V^{(1)}/\tilde{y}) + \cdots , \end{aligned} \tag{13}$$

where the principle of eliminability[13] has been employed to infer the $\tilde{y}$ dependence of the above terms. That is, since L is an extraneous parameter, Re and $\tilde{y}$ must appear in a form such that L may be eliminated from each term. By the same token, then, the various order functions must be independent of L. Hence, for example,

$$\tilde{U}^{(n)} = \tilde{U}^{(n)}(t),$$

where $t = t(\tilde{x}, \tilde{y})$ is a similarity variable independent of L. The superscript notation, then, indicates a function of t alone.

Since t is independent of L and is a function of $\tilde{x}$ and $\tilde{y}$, it must be of the form $\tilde{x}\tilde{y}^{-m}$. Now,

$$\tilde{x}\tilde{y}^{-m} = \mathrm{Re}^{\frac{1}{3}}\, xy^{-m} \propto L^{\frac{1}{3}-1+m}.$$

Hence $m = \frac{2}{3}$ and $t \propto \tilde{x}\tilde{y}^{-\frac{2}{3}}$. It is shown that with the above constructions, the governing partial differential equations are reduced to ordinary differential equations, so t is a similarity variable and the above arguments are consistent.

If Eqs. (13), and expansions similar to U for the remaining dependent variables, are substituted into Eqs. (9) and (10), a set of equations for the first-order functions results. Three of these equations may be integrated by inspection, and if the boundary condition that these first-order terms tend to zero as $\tilde{x} \to -\infty$ is invoked, the resulting equations are

$$\begin{aligned} &\text{(a)}\quad \tilde{\rho}_1 = \tilde{P}_1/\gamma = \beta\tilde{T}_1 = -\tilde{U}_1, \\ &\text{(b)}\quad \tilde{h}_{t1} = 0, \\ &\text{(c)}\quad (\tilde{V}_1)_{\tilde{x}} - (\tilde{U}_1)_{\tilde{y}} = 0, \\ &\text{(d)}\quad \tfrac{4}{3}[\tilde{\mu}_0(\tilde{U}_1)_{\tilde{x}}]_{\tilde{x}} - (1/\mathrm{Pr})[\tilde{\lambda}_0(\tilde{T}_1)_{\tilde{x}}]_{\tilde{x}} \\ &\qquad = [(\gamma + 1)\tilde{U}_1 - (\tilde{h}_{t1}/\beta)](\tilde{U}_1)_{\tilde{x}} - (\tilde{V}_1)_{\tilde{y}}. \end{aligned} \tag{14}$$

In view of the assumed power law dependence, $\tilde{\lambda}_0 = \tilde{\mu}_0 = 1$.

Equation (14c) is an irrotationality condition. Hence a velocity potential function, Φ_1, may be defined such that

$$\begin{aligned} \tilde{U}_1 &= b\tfrac{1}{2}(1 - \Gamma^2)(\Phi_1)_{\tilde{x}}, \\ \tilde{V}_1 &= b\tfrac{1}{2}(1 - \Gamma^2)(\Phi_1)_{\tilde{y}}, \\ b &= \tfrac{4}{3}\gamma\{1 - (1/\gamma\beta)[1 - (3/4\,\mathrm{Pr})]\}, \end{aligned} \tag{15}$$

and Eq. (14c) is satisfied identically. If Eqs. (15) and (14a) are substituted into Eq. (14d), an equation for Φ_1 results. This equation may be simplified by the transformation

$$X = \tilde{x}b^{-\frac{1}{3}}, \qquad Y = \tilde{y},$$

to the following:

$$(\Phi_1)_{XXX} = -(\Phi_1)_{YY} + (\Phi_1)_X(\Phi_1)_{XX}. \tag{16}$$

Equation (16) is the so-called viscous transonic equation. It has been studied recently by Sichel[14] in connection with his work on shock-wave structure and nozzle flow, and by Szaniawski[15] with reference, again, to nozzle flow.

If a new dimensionless potential function, $F_1(t)$, is defined as follows:

$$\Phi_1(X, Y) = F_1(t), \qquad t = X/Y^{\frac{2}{3}}, \tag{17}$$

and if the substitution

$$g_1(t) = F_1'(t)$$

is made, Eq. (16) may be written in terms of g_1 as follows:

$$g_1'' - g_1 g_1' + \tfrac{4}{9}t^2 g_1' + \tfrac{10}{9}t g_1 = 0. \tag{18}$$

This is the equation studied by Sichel.[14] However, he was interested in solutions which are bounded as $t \to \pm\infty$. In the present problem, we are interested in solutions which are bounded, and in fact tend to zero as $t \to -\infty$. As $t \to \infty$, the solutions must match with the outer solution. If $\tilde{U}_1$ and $\tilde{V}_1$ are written in terms of g_1, and the resulting equations are compared with Eqs. (12), it can be seen that as $t \to \infty$, the matching condition implies that

$$g_1/y^{\frac{2}{3}} \propto X^2/Y^2 = t^2/Y^{\frac{2}{3}}.$$

Hence, the first term in the asymptotic expansion for g_1 must be t^2. It can be seen from Eq. (18) that $g_1 = t^2$ is an exact solution to the inviscid equation, i.e., Eq. (18) with $g_1'' = 0$. Standard techniques may be used to show that the asymptotic form of the solution which approaches t^2 as $t \to \infty$ is

$$g_1(t) \sim t^2 + \frac{6}{t} - \frac{36}{t^4} + \cdots + A_0\left(\frac{1}{t^{8/5}} + \cdots\right) + \frac{B_0}{t^{8/5}} \cdot \int \exp\left[\frac{5}{27}t^3\right] dt + \cdots , \tag{19}$$

where A_0 and B_0 are undetermined constants. Thus, the solution for the present problem would be that for which $B_0 = 0$.

As $t \to -\infty$, under the condition that $g_1 \to 0$,

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[15] A. Szaniawski, Arch. Mech. Stosowanej. **16**, 643 (1964).

it can be shown[14] that the asymptotic solution is

$$g_1 \sim \bar{A}_0\left[(-t)^{-5/2} - \frac{105}{16}(-t)^{-11/2} + \cdots\right] + \bar{B}_0(-t)^{1/2} \exp\left[\frac{4}{27}(-t)^3\right] + \cdots, \qquad (20)$$

where $\bar{A}_0$ and $\bar{B}_0$ are undetermined constants. Thus, $\bar{B}_0 = 0$ satisfies the condition that tg_1 (from $\tilde{V}_1$) tends to zero as $t \to -\infty$.

Finally, the question arises as to whether the solution which is asymptotic to Eq. (19) as $t \to \infty$ is the same as that which is asymptotic to Eq. (20) as $t \to -\infty$. Figure 3 shows the results of numerical computations made on an IBM 7090 to resolve this point. Solutions were started at $t = -10$, running forward, and at $t = +10$, running backward. At $t = -10$, Eq. (20) was used to calculate g_1 and g_1' for various values of $\bar{A}_0$, while at $t = 10$, Eq. (19) was used and the same procedure followed. In Fig. 3, four solutions are presented, two started at

$$t = -10[g_1(-10) = 0.38639 \times 10^{-2} \text{ and } 0.38640 \times 10^{-2}]$$

and two starting at $t = +10$ [$g_1(10) = 100.499$ and 100.500]. The arrows on the curves indicate the direction of integration of Eq. (18). It can be seen that there is a common region of solution, roughly between $t = -2$ and $t = +3$. Outside this region, the solutions diverge from the desired solution, one in a positive direction and one in a negative direction, with only a very small change in initial conditions. This is a result of the fact that it is impossible, in the machine solution, to set B_0 and $\bar{B}_0$ equal to zero. A slight deviation from the exact solution, as $|t|$ increases, allows the exponential part of the solutions to overcome the other terms, and divergence results. However, that fact that there is a common region of solution, where all four solutions are very nearly identical, regardless of the direction of integration, indicates that a solution which joins the given asymptotic forms, does exist.

With g_1 known, $\tilde{U}_1$ and $\tilde{V}_1$ are known, and $\tilde{P}_1$, $\tilde{\rho}_1$, $\tilde{T}_1$, and $\tilde{h}_{t1}$ may be calculated from Eqs. (14). Hence, all first-order inner terms are known and the next step is to find the first-order outer solutions. In order to find ϵ_1, and the corresponding gauge functions for the remaining variables, it is necessary to write the known inner solutions for large t, in terms of the outer variables. Using the velocity components again as examples, for large t, one finds that $\epsilon_1 = \mathrm{Re}^{-1}$. However, when expansions with this form for ϵ_1 are substituted in the governing

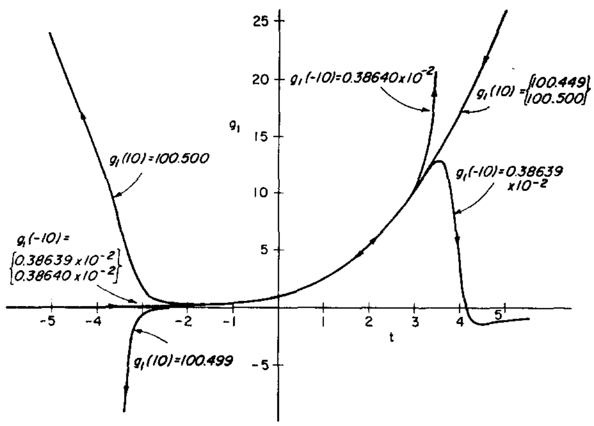

FIG. 3. Numerical solutions to Eq. (18), started with asymptotic solutions given in Eqs. (19) and (20). Arrowheads indicate direction of integration.

equations [Eqs. (2) and (3) written in polar coordinates], it is found that solutions exist only when the form of u_1, for example, is

$$u_1 = (\ln r/r)f_1(\varphi) + (1/r)f_2(\varphi).$$

Again, employing the principle of eliminability, this means that a $\mathrm{Re}^{-1} \ln \mathrm{Re}$ term must be added, so that the proper form of the expansion is

$$\begin{aligned} u &\sim u_0 + (\ln \mathrm{Re}/\mathrm{Re})u_1(r, \varphi) + (1/\mathrm{Re})u_2(r, \varphi) + \cdots \\ &= u_0 + (\ln \mathrm{Re}/\mathrm{Re})(u^{(1)}/r) + (1/\mathrm{Re})[(\ln r/r)u^{(1)} + (u^{(2)}/r)] + \cdots. \end{aligned} \qquad (21)$$

Similar expansions hold for the remaining outer dependent variables.

When expansions of the form shown in Eq. (21) are substituted in the governing equations, the resulting zeroth-order equations are those satisfied by the Prandtl–Meyer solutions. The first-order equations may be integrated to give the following results (details of the calculation may be found in Ref. 11):

$$\begin{aligned} &\text{(a)} \quad \rho^{(1)} = -(\rho^{(0)}v^{(1)}/v^{(0)}) + (\eta^{(1)}/v^{(0)}), \\ &\qquad \eta^{(1)} = \text{const}, \\ &\text{(b)} \quad u^{(1)} = -\eta^{(1)}u^{(0)}/\eta^{(0)}, \\ &\text{(c)} \quad P^{(1)} = -\gamma[\eta^{(0)}v^{(1)} + \eta^{(1)}v^{(0)}], \\ &\text{(d)} \quad h_t^{(1)} = H_1/\eta^{(0)}, \qquad H_1 = \text{const}, \\ &\text{(e)} \quad \eta^{(1)} = -\Gamma^2 H_1, \\ &\text{(f)} \quad T^{(1)} = -(v^{(0)}v^{(1)}/\beta) + (H_1/\beta\eta^{(0)}) + (\eta^{(1)}u^{(0)^2}/\beta\eta^{(0)}), \end{aligned} \qquad (22)$$

where $\eta = \rho v$ so that $\eta^{(0)} = \rho^{(0)}v^{(0)}$, $\eta^{(1)} = \rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)}$, etc.

Thus, again, a redundancy exists in that Eq. (22e) gives only a relationship between two constants which already appear in other equations. As a result, only $u^{(1)}$ and $h_t^{(1)}$ are known, with $P^{(1)}$, $\rho^{(1)}$, and $T^{(1)}$ being given in terms of $v^{(1)}$. To find $v^{(1)}$, it is necessary to go to the equations involving terms of order Re^{-1}. It is not possible to remove this redundancy, in general, as was done for the inner equations. In an effort to investigate the extent of this redundancy, several higher-order sets of equations were derived; beyond the terms of order Re^{-1}, no other redundant equations were found at least up to terms of order Re^{-2}.

In order to complete the solution, then, two more solutions are necessary. First, the inner terms corresponding to the outer terms of order $\mathrm{Re}^{-1} \ln \mathrm{Re}$ must be found and matched. Finally, the necessary outer terms of order Re^{-1} must be found and matched.

The gauge functions for the inner expansions may be inferred as follows. From the form of the governing differential equations, it is clear that, in the expansion for U, for example, there must be terms of order $\mathrm{Re}^{-4/3}$, the product of terms of order $\mathrm{Re}^{-2/3}$. Furthermore, since a term involving ln Re has been found in the outer expansion, one must occur in the inner expansion, and is presumably of the order $\mathrm{Re}^{-4/3} \ln \mathrm{Re}$. Similar arguments may be made for V, so that the final expansions for U and V are

$$\begin{aligned} U &\sim 1 + \frac{1}{\mathrm{Re}^{2/3}}\frac{\tilde{U}^{(1)}}{\tilde{y}^{2/3}} + \frac{\ln \mathrm{Re}}{\mathrm{Re}^{4/3}}\frac{\tilde{U}^{(2)}}{\tilde{y}^{4/3}} + \frac{1}{\mathrm{Re}^{4/3}}\left(\frac{\ln \tilde{y}}{\tilde{y}^{4/3}}\tilde{U}^{(2)} + \frac{\tilde{U}^{(3)}}{\tilde{y}^{4/3}}\right) + \cdots, \\ V &\sim \frac{1}{\mathrm{Re}}\frac{\tilde{V}^{(1)}}{\tilde{y}} + \frac{\ln \mathrm{Re}}{\mathrm{Re}^{5/3}}\frac{\tilde{V}^{(2)}}{\tilde{y}^{5/3}} + \frac{1}{\mathrm{Re}^{5/3}}\left(\frac{\ln \tilde{y}}{\tilde{y}^{5/3}}\tilde{V}^{(2)} + \frac{\tilde{V}^{(3)}}{\tilde{y}^{5/3}}\right) + \cdots, \end{aligned} \tag{23}$$

where the remaining dependent variables are expanded in the same form as U. While it may be argued that there might be other terms intermediate to those found by the above considerations, this question can be answered by demonstrating that matching does occur between inner and outer solutions with the above forms.

The second-order inner equations are found by substituting expansions of the form given in Eqs. (23) into Eqs. (9). Just as in the first-order solutions, several of the equations are easily integrated, and in fact, the first three of Eqs. (14) are reproduced with subscript 2 replacing subscript 1. Again, a velocity potential function may be defined. After going through the same type of calculation as performed for the first-order inner solutions, it can be shown that a dimensionless potential function, $F_2(t)$, exists, such that

$$\tilde{U}_2 = b^{-1/3}Y^{-4/3}F_2', \qquad \tilde{V}_2 = -\tfrac{2}{3}(F_2 + tF_2')Y^{-5/3}. \tag{24}$$

Further, the solution, which exhibits the highest-order allowable terms (i.e., nonexponential) for $t \gg 1$, is

$$F_2(t) = A_1 g_1(t), \tag{25}$$

where A_1 is a constant. From Eq. (20), it is apparent that the boundary conditions on $\tilde{U}_2$ and $\tilde{V}_2$ are satisfied. With F_2 known, asymptotic forms for all second-order inner terms may be found.

Referring to the expansion for U given in Eq. (23), it appears that there is a possibility that the term $\tilde{U}^{(3)}$ could contribute to those terms which match with the outer terms of order Re^{-1}. However, a simple calculation shows that the only possible contribution must be of lower order than the term given by the first-order solution. Hence it is not necessary to compute the third-order term except in the case of h_t. Since $\tilde{h}_{t1}$ and $\tilde{h}_{t2}$ are identically zero, the only contribution is that given by $\tilde{h}_{t3}$, so it must be calculated. The equation for $\tilde{h}_{t3}$ involves only known first-order terms and is easily integrated. Thus,

$$\tilde{h}_{t3} = b^{-1/3}[\tfrac{4}{3} - (1/\mathrm{Pr})](\tilde{U}^{(1)})' Y^{-4/3}. \tag{26}$$

In order to facilitate matching, the most important inner solutions are summarized below. The remaining functions may be constructed easily from them. The asymptotic forms for $t \gg 1$ have been employed, with the resulting expressions then being written in terms of the outer variables. Equation (1), for $\varphi \ll 1$, has been used to form the outer velocity components

$$\begin{aligned} &\text{(a)} \quad u \sim \varphi - \frac{\Gamma^2\varphi^3}{3!} + \frac{1}{\mathrm{Re}}\frac{b(1-\Gamma^2)}{r} + \cdots, \\ &\text{(b)} \quad v \sim 1 - \frac{\Gamma^2\varphi^2}{2!} + \frac{\ln \mathrm{Re}}{\mathrm{Re}}\frac{2A_1}{b^{2/3}}\frac{\varphi}{r} + \frac{1}{\mathrm{Re}}\left(\frac{2A_1}{b^{2/3}}\frac{\varphi \ln r}{r} + \frac{3b(1-\Gamma^2)}{r\varphi}\right) + \cdots, \\ &\text{(c)} \quad h_t \sim \frac{1}{2\Gamma^2} + \frac{1}{\mathrm{Re}}\left(\frac{4}{3} - \frac{1}{\mathrm{Pr}}\right)(1-\Gamma^2)\frac{\varphi}{r} + \cdots, \\ &\qquad \varphi \ll 1, r \quad \text{fixed}, \qquad M_1 = 1. \end{aligned} \tag{27}$$

The outer solutions must match the above expansions term by term. It should be noted that no $\mathrm{Re}^{-1} \ln \mathrm{Re}$ term has been written for u, since this term was zero at least up to order φ^2; this is sufficient information for matching.

Before proceeding with the final outer calculations, a simplification of the outer expansions may be achieved by matching the two known first-order outer solutions for $u^{(1)}$ and $h_t^{(1)}$ with the corresponding terms in Eqs. (27). For matching to occur,

$$u^{(1)} = h_t^{(1)} = 0. \tag{28}$$

The equations for the outer terms of order Re^{-1}, derived by substituting the outer expansions into the governing equations, involve both first- and second-order functions. They may be integrated[11] to the following forms:

$$\begin{aligned}
&\text{(a)}\quad v^{(1)} = (N'/\rho^{(0)}u^{(0)}) + (\Gamma^2 N/\eta^{(0)}),\\
&\text{(b)}\quad u^{(2)} = (\gamma G + \tfrac{3}{4}bf^{(0)} - N)/\rho^{(0)}u^{(0)},\\
&\text{(c)}\quad h_t^{(2)} = \{[\tfrac{4}{3} - (1/\mathrm{Pr})](1 - \Gamma^2)\mu^{(0)}u^{(0)}/\rho^{(0)}\}\\
&\qquad\qquad + (C_0/\eta^{(0)}),\\
&\text{(d)}\quad (P^{(2)}/\gamma) + \eta^{(0)}v^{(2)} = G - \rho^{(0)}u^{(0)}u^{(2)} + f^{(0)},\\
&\text{(e)}\quad \rho^{(0)}v^{(2)} + \rho^{(2)}v^{(0)} = N/v^{(0)},\\
&\text{(f)}\quad \rho^{(0)}T^{(2)} + \rho^{(2)}T^{(0)} = P^{(2)},
\end{aligned} \tag{29}$$

where

$$f^{(0)} = \tfrac{4}{3}\mu^{(0)}[(v^{(0)})' + u^{(0)}],$$

$$\begin{aligned}
N = {} & (\rho^{(0)}u^{(0)}v^{(0)})^{\frac{1}{2}}\\
& \cdot\left(C_2 + \int^{\varphi} (\rho^{(0)}u^{(0)}v^{(0)})^{-\frac{1}{2}}\left\{\tfrac{1}{2}\gamma\left[\left(G + \frac{3bf^{(0)}}{4\gamma}\right)'\right.\right.\right.\\
& \left.- \left(\frac{v^{(0)}}{u^{(0)}} - \frac{(1-\Gamma^2)u^{(0)}}{v^{(0)}}\right)\left(G + \frac{3bf^{(0)}}{4\gamma}\right)\right]\\
& \left.\left.- \frac{u^{(0)}G}{2v^{(0)}}\right\} d\varphi\right),
\end{aligned} \tag{30}$$

$$\begin{aligned}
G &= -\tfrac{3}{4}b(1 - \Gamma^2)v^{(0)} \int^{\varphi} \left(\frac{u^{(0)}f^{(0)}}{v^{(0)2}}\right) d\varphi + C_1 v^{(0)}\\
&= v^{(0)}[C_1 + \beta b(1 - \Gamma^2)(u^{(0)}v^{(0)} - \varphi)],
\end{aligned}$$

for a linear viscosity–temperature relationship. Again, the prime denotes differentiation with respect to φ, and C_0, C_1, and C_2 are constants of integration.

It is seen that due to the redundancy, $v^{(1)}$, $u^{(2)}$, and $h_t^{(2)}$ may be found as functions of φ, but $P^{(2)}$, $\rho^{(2)}$, and $T^{(2)}$ are known only in terms of $v^{(2)}$. Thus, although $u^{(1)}$ and $h_t^{(1)}$ are zero, $u^{(2)}$ and $h_t^{(2)}$ may be calculated, so that for each dependent variable, the first correction term may be found, with no additional information necessary. In order to complete the solution for the second-order terms, an additional function of φ (e.g., $v^{(2)}$) is necessary. It appears most likely that this unknown function must be found from matching with a boundary-layer solution; since a term of order Re^{-1} will certainly occur in the boundary-layer solution, such a matching is possible.

In order to evaluate the constants, C_n, the matching conditions may be employed. If Eqs. (29a), (29b), (29c) are written for $\varphi \ll 1$, and matched with the corresponding terms of Eq. (27) the result is that

$$C_0 = C_2 = 0. \tag{31}$$

Likewise, if $v^{(1)}$ is multiplied by $(r\,\mathrm{Re})^{-1} \ln \mathrm{Re}$ and matched, then

$$C_1 = 0, \qquad A_1 = -\tfrac{1}{5}b^{5/3}\Gamma^2(1 - \Gamma^2)(6 - \gamma\beta). \tag{32}$$

Although matching has been demonstrated only for the velocity components and total enthalpy, there is no difficulty in demonstrating that matching does occur for the remaining variables, with no new constants arising. Hence, the inner expansions are of the form shown in Eq. (23), and the outer expansions are of the form shown in Eq. (21).

Although the above solutions have been carried out for $M_1 = 1$, the supersonic case, $M_1 > 1$, is similar in every respect. The results are summarized below; details of the calculation may be found in Ref. 11.

The inner expansions are of the form

$$U \sim \sin \bar{\mu}_1 + \frac{1}{\mathrm{Re}^{1/2}} \frac{\tilde{U}^{(1)}}{\tilde{y}^{1/2}} + \frac{\ln \mathrm{Re}}{\mathrm{Re}} \frac{\tilde{U}^{(2)}}{\tilde{y}} + \frac{1}{\mathrm{Re}}\left(\frac{\ln \tilde{y}}{\tilde{y}}\,\tilde{U}^{(2)} + \frac{\tilde{U}^{(3)}}{\tilde{y}}\right) + \cdots, \tag{33}$$

but the outer expansions are the same form as those found for $M_1 = 1$. When the matching procedure is carried out, it is found that there are only two equations for four constants. The extra unknown constants, in this case, arise from the inner expansions from terms of order $\mathrm{Re}^{-\frac{1}{2}}$ and $\mathrm{Re}^{-1} \ln \mathrm{Re}$. Hence, it appears that terms of this order will appear in the boundary-layer solution, and the constants found from matching. It must be borne in mind that the inner expansions are written in terms of stretched coordinates; in physical variables, the orders change, terms of order $\mathrm{Re}^{-\frac{1}{2}}$ in the inner velocity expansions becoming terms of order Re^{-1}

in physical variables, for example. However, no implausible orders are encountered.

4. DISCUSSION OF RESULTS

The outer solutions may be written for the velocity, pressure, temperature, density, and total enthalpy by substituting the solutions found above [e.g., Eqs. (22) and (29)] into the outer expansions [Eqs. (21)]. One of the more interesting aspects of the solution is that the first-order corrections generated in the expansion region are proportional to $\mathrm{Re}^{-1} \ln \mathrm{Re}$ for some variables (v, P, ρ, T) and to Re^{-1} for the remainder (u, h_t). Hence, $\mathrm{Re}^{-1} \ln \mathrm{Re}$ is an estimate of the error involved when viscous terms are ignored.

The change in entropy due to the transport effects may be calculated, since it involves only known functions. Because the entropy of the zeroth-order inviscid expansion is a constant, the change in entropy can be written with respect to $S^{(0)}$. Thus,

$$\Delta S = S - S^{(0)} = \ln\left(\frac{T}{T^{(0)}}\right) - \left(\frac{\gamma - 1}{\gamma}\right) \ln\left(\frac{P}{P^{(0)}}\right)$$

$$\sim \frac{1}{\mathrm{Re}\, r}\left(\frac{h_t^{(2)} - u^{(0)}u^{(2)}}{T^{(0)}} - \frac{\eta^{(2)}}{\eta^{(0)}}\right) + \cdots, \qquad (34)$$

where in the second equation, asymptotic expansions have been employed and simple substitutions made in order to include only known terms. The fact that there is no entropy change of order $\mathrm{Re}^{-1} \ln \mathrm{Re}$ is consistent with the fact that in the inner expansion, up to terms of this order, the flow was irrotational.

The value of δ was found to be of order $\mathrm{Re}^{-\frac{1}{2}}$ for $M_1 > 1$, as expected, and of order $\mathrm{Re}^{-1/3}$ for $M_1 \simeq 1$. Since $\Delta = O(\mathrm{Re}^{-3/8})$ when $M_1 > 1$, it is clear that in this case the thinning of the boundary layer will have stronger effects on the flow than the viscous effects in the expansion region. In the general solutions, therefore, one would expect the thinning of the boundary layer to contribute first-order effects, with the terms calculated above being higher order. As mentioned previously, this does not change the form of the solution of these terms, only their position in the expansions of the complete solution.

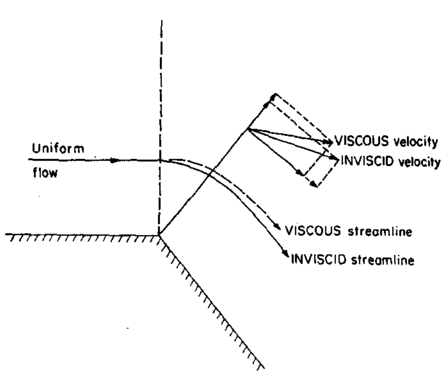

FIG. 4. Sketch of variations from Prandtl–Meyer flow due to effects of viscosity and thermal conductivity.

When $M_1 \simeq 1$, the relative orders of the terms due to boundary layer and expansion region cannot be decided, because Δ is not known. If it remains of order $\mathrm{Re}^{-3/8}$ as $M_1 \to 1$, then it appears that the expansion region effects are more important than boundary-layer effects. However, there is no reason to expect Δ to remain independent of M_1 as sonic conditions are approached.

A physical picture of the changes in the expansion flow caused by the inclusion of viscosity and thermal conduction can be gained by a consideration of the signs of the various corrections. These are indicated for the velocities and total enthalpy for $\varphi \ll 1$, for example, by the expansions given in Eq. (27). Since $A_1 < 0$, it is seen that the radial velocity component, u, is increased, while the angular velocity component v, is decreased. The magnitude of the velocity is decreased, as expected. In Fig. 4, a sketch of the differences in velocity and streamline pattern between the viscous and inviscid flows is shown. As indicated, the result of the given changes in velocity is that at a given turning angle, φ, and radius, r, the viscous, heat-conducting flow has turned less than the corresponding inviscid flow.

The total enthalpy corrections can be positive or negative, depending on whether the Prandtl number is less than or greater than three fourths. Since the uniform flow and the expansive flow at $r \to \infty$ are at the same total enthalpy, the changes in total enthalpy must be balanced by an energy exchange with the boundary layer.

Examination of the first correction terms in the expansions for the pressure, density, and temperature, indicates that they all increase over their zeroth-order (Prandtl–Meyer) solutions. This means that the effect of the transport properties is to decrease the pressure drop at a given point, for example, so that for a given pressure drop, the flow must turn through a greater angle.

ACKNOWLEDGMENTS

The author is deeply indebted to his colleague, Professor A. F. Messiter, for contributions made to this work through many discussions.

This work was supported by Bamirac, a contract funded through Project Defender and sponsored by the Advanced Research Projects Agency of the Department of Defense, under Contract SD-91.