where the double prime indicates that the terms in the sum over k with $k = \alpha$ or $k = \beta$ should be omitted.

It is straightforward and easy to check that for 2×2 and 3×3 matrices conditions (28) coincide with (6) and (18), respectively. It is also straightforward but less than easy to check that (28), and also (25), are also necessary and sufficient conditions for the existence of a solution of the system (3) for 4×4 matrices. (Because it involves extremely tedious calculations, the case of 4×4 matrices is not discussed here.) We are aware though that there is no substitute for rigorous proof and that we failed to produce such a proof for $n > 4$. But even if (25) and (28) were not necessary conditions for the existence of a solution of (3), they clearly are sufficient and significantly less restrictive than the Martin–Newton condition. Because of this, and particularly in view of the fact that it has been pointed out[2] that the Martin–Newton condition is not fulfilled in a number of physical cases, it would be interesting to extend the results given in this paper to infinite matrices in general and specifically to the non-linear integral equation of Ref. 1.

---

[1] R. G. Newton, J. Math. Phys. 9, 2050 (1968) ; A. Martin, Nuovo Cimento 59A, 131 (1969).

[2] I. A. Sakmar, Lett. Nuovo Cimento 2, 256 (1969) ; H. Goldberg, Phys. Rev. D 1, 1242 (1970).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 10 OCTOBER 1971

A Convergent Expansion for the Resolvent of: $\varphi^4:_{1+1}$*

Paul Federbush

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

(Received 12 April 1971)

For the model theory, $N + \lambda \int :\varphi^4: dx$ in a box, a convergent expansion of the resolvent is exhibited. This also provides another proof of boundedness below for the model.

We consider the field theory model with Hamiltonian $N + \int : \varphi^4 : dx$ in a box of length 1. We obtain a convergent expansion for the resolvent of this model and at the same time another proof of boundedness below of the Hamiltonian. The main idea is to consider the 'subblocks' of the Hamiltonian obtained by restricting to states with particle number spectrum lying between N and $2N$. The resolvent of a subblock is shown to have very small matrix elements connecting states with a large difference in particle number. The extension of the present results to the more general model $N_\tau + \lambda \int : \varphi^{2s} : dx$ has not yet been achieved.

We begin with the basic theorem to be used.

Theorem: Let A be a positive self-adjoint operator of norm $\leqslant M$, and $|\alpha\rangle$ and $|\beta\rangle$ be two vectors of unit length. Suppose $\langle \alpha | A^k | \beta \rangle = 0, 0 \leqslant k \leqslant N$. Then, for any $\lambda > 0$, a real number,

$$\left| \left\langle \alpha \left| \frac{1}{\lambda + A} \right| \beta \right\rangle \right| \leqslant \frac{4\sqrt{M}}{\lambda\sqrt{2\lambda}} \cdot \left(\frac{1}{1 + \sqrt{2\lambda/M}} \right)^N \tag{1}$$

$$\sim \exp(-\sqrt{2\lambda/M}\, N), \tag{2}$$

where in (2) it is assumed M and N are large.

Proof: $(\lambda + A)^{-1}$ and $(\lambda + A)^{-1} - P_N(A)$ have the same matrix elements between $|\alpha\rangle$ and $|\beta\rangle$, where $P_N(x)$ is any polynomial of degree N. This implies the matrix element is smaller than the supremum of $|(\lambda + x)^{-1} - P_N(x)|$ for values of x in the spectrum of A. We make a linear change of variables moving the spectrum from $[0, M]$ to $[-1, 1]$. Now one has the function $[\lambda + (x+1)\frac{1}{2}M]^{-1}$ on the interval $[-1, 1]$. There is a basic theorem[1] in the theory of polynomial approximation stating that if f is analytic in an ellipse with foci at -1 and 1 and major and minor radii a and b, then it may be approximated on $[-1, 1]$ by a polynomial of degree N within

$$\frac{2f\,\max}{(a + b - 1)} \left(\frac{1}{a + b} \right)^N \tag{3}$$

in the uniform norm. Here f max is the supremum of the absolute value of f in the ellipse. The theorem is obtained applying this result to the ellipse with $a = 1 + \lambda/M$ and $f = [\lambda + (x+1)\frac{1}{2}M]^{-1}$.

We now come to the Hamiltonian

$$H = N + \int_0^1 :\varphi^4: dx = N + V. \tag{4}$$

We define P_i as the projection operator onto states with numbers of particles lying in the range

$$2^i \leqslant N < 2^{i+2}, \quad i = -1, 0, \cdots, \tag{5}$$

and P_e and P_d as the projection operator onto states with numbers of particles in the ranges

$$\bigcup_{i \text{ even}} (2^i - 4 \leqslant N \leqslant 2^i + 4) \tag{6}$$

and

$$\bigcup_{i \text{ odd}} (2^i - 4 \leqslant N \leqslant 2^i + 4), \tag{7}$$

respectively. We define

$$H_i = P_i H P_i, \tag{8}$$

$$H_e = \sum_{i \text{ even}} H_i, \tag{9}$$

$$H_d = \sum_{i \text{ odd}} H_i, \tag{10}$$

$$H = H_e + L_e = H_d + L_d. \tag{11}$$

We note

$$P_e L_e P_e = L_e P_e = P_e L_e = L_e, \tag{12}$$

$$P_d L_d P_d = L_d P_d = P_d L_d = L_d. \tag{13}$$

The expansion of the resolvent we are after is the following:

$$\begin{aligned}\frac{1}{E+H} &= \frac{1}{E+H_e} - \frac{1}{E+H_d} L_e \frac{1}{E+H_e} + \frac{1}{E+H_e} L_d \frac{1}{E+H_d} L_e \frac{1}{E+H_e} - \cdots \\ &= \frac{1}{E+H_e} - \frac{1}{E+H_d} P_e L_e P_e \frac{1}{E+H_e} + \frac{1}{E+H_e} P_d L_d P_d \frac{1}{E+H_d} P_e L_e P_e \frac{1}{E+H_e} \cdots .\end{aligned} \tag{14}$$

This expansion converges for E large enough, as we will show; for E large enough $(E + H_e)^{-1}$ and $(E + H_d)^{-1}$ exist and are bounded, $|L_d P_d (E + H_d)^{-1} P_e| < \frac{1}{2}$ and $|L_e P_e (E + H_e)^{-1} P_d| < \frac{1}{2}$, and $(E + H_d)^{-1} L_e (E + H_e)^{-1}$ and $L_d (E + H_d)^{-1} L_e (E + H_e)^{-1}$ are bounded.

The following two estimates easily yield the required relations above.

Estimate 1:

$$H_i \geqslant 2^{i-1} P_i, \quad i \quad \text{large} \tag{15}$$

Estimate 2:

$$\left| P_e \frac{P_i}{E + H_i} P_d \right| \leqslant c_1 \exp(-c_2 2^{i/2}) \quad \text{for some} \quad c_1, c_2 > 0. \tag{16}$$

Proof of estimate 1: We write V as the sum $V_k + R_k$, where as usual V_k contains those terms in the expansion of V all of whose momenta are less than or equal k in absolute value

$$H_i = P_i N P_i + P_i V_{k_i} P_i + P_i R_{k_i} P_i. \tag{17}$$

We first note

$$P_i N P_i \geqslant 2^i P_i. \tag{18}$$

As in Ref. 2, one has

$$V_k \geqslant -c(\ln k)^2. \tag{19}$$

Picking

$$k_i = \exp[(1/\sqrt{c}) 2^{(i-1)/2}], \tag{20}$$

we get

$$P_i V_{k_i} P_i \geqslant -2^{i-1} P_i. \tag{21}$$

From

$$|P_i R_{k_i} P_i| \leqslant d(2^{i+2})^2 1/\sqrt{k_i}, \tag{22}$$

a standard N_τ estimate, see Ref. 3, we quickly get

$$|P_i R_{k_i} P_i| \leqslant 2^{i-1}, \quad i \quad \text{large}. \tag{23}$$

And thus, using (23), (21), and (18), we obtain estimate 1 from (17).

Proof of Estimate 2: Clearly

$$|E + H_i| \leqslant e 2^{2i} \tag{24}$$

for some e (by an N_τ estimate again), and

$$E + H_i \geqslant 2^{i-1} \tag{25}$$

for E large enough. We apply the theorem with $|\alpha\rangle = P_i P_d |a\rangle$, $|\beta\rangle = P_i P_e |b\rangle$ ($|a\rangle$ and $|b\rangle$ normalized vectors, $\lambda = 2^{i-1}$, $M = e2^{2i}$, $A = E + H_i - 2^{i-1}$, and $N < [(2^{i+1} - 4) - (2^i + 4)]/4$. This approach is easily generalized to $N_\tau + \int :\varphi^4: dx$.

The subject of obtaining convergent expansions for the resolvents of other field theory models seems interesting, as is the question of whether this is a way of obtaining lower bound estimates for other models.

---

* This work was supported in part by NSF Grant GP-17523.

[1] A. F. Timan, *Theory of Approximation of Functions of a Real Variable* (MacMillan, New York, 1963), p. 281.

[2] E. Nelson, "A Quartic Interaction in Two Dimensions," in *Mathematical Theory of Elementary Particles*, edited by R. Goodman and I. Segal (MIT Press, Cambridge, Mass., 1966).

[3] J. Glimm, Commun. Math. Phys. 5, 343 (1967).