MONOTONICITY IN MATHEMATICAL PROGRAMMING

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ABSTRACT

Coordinatewise monotonicity of the objective and constraint functions with respect to the decision variables in mathematical programs may be used to identify active and inactive constraints. When applicable, this may lead to the global optimum directly or at least provide significant simplifications. Monotonicity in multidimensional spaces is studied for systems of inequalities and two necessary conditions of optimality are derived. Several examples demonstrate how to use monotonicity analysis. The ideas discussed originated from engineering design applications.

1. Introduction. Mathematical programming models of engineering design problems generally result in nonlinearities, nonconvexity, discrete variables and multiple optima. Identification of the global optimum for problems involving more than just few variables can be very difficult. Experience has shown, however, that these problems often have many inequality constraints which are active, i.e. satisfied as strict equalities at the optimum. Moreover due to modeling practices, it is not unusual to have a constraint-bound optimum, i.e. the number of active constraints to equal the number of design variables [1]. A reason for this widespread constraint activity is an observed frequent property of engineering design problems, namely to have the objective and constraint functions monotonic with respect to some or all the design variables. This observation led to the development of monotonicity analysis, a methodology for constraint activity identification which has been used to study and solve several design problems [1-9].

Although the motivation comes from engineering design, monotonicity arguments may apply to mathematical programming problems in general. Linear programs, for example, have always constraint bound solutions due to monotonicity. Nonlinear programs may be reduced significantly using monotonicity arguments and thus making subsequent numerical treatment easier and more reliable, particularly when a global optimum is sought.

The present paper provides a mathematical background for monotonicity properties that were used rather informally in the previous engineering publications [1-9]. A particular type of monotonicity is defined in multidimensional space, a theory on monotone inequalities that can be used for model manipulations is described and two necessary conditions for optimality are presented. A general application procedure with some examples is included to demonstrate how the theory can be used. The references cited above can provide further insight for applications.

2. Monotonicity in Multidimensional Spaces. The classical notion of monotonicity for a real-valued function f(x) of a real variable x lends itself to a variety of generalizations when we consider real-valued functions of several variables. In this section we examine some of these and indicate the one used in later sections.

Given a function $f: A \to R$, where A is a subset of the real line R, we say that f is monotone nondecreasing, if for all x_1, x_2 in A,

$$x_1 \leq x_2 \quad \text{implies} \quad f(x_1) \leq f(x_2) \qquad (1)$$

or equivalently

$$(x_1 - x_2)(f(x_1) - f(x_2)) \geq 0. \qquad (2)$$

We say that f is strictly nondecreasing or increasing, if f is monotone nondecreasing and in addition f is one to one. Consequently, for increasing functions conditions (1) and (2) should be modified to have strict inequalities.

To generalize the above concepts to functions of several variables, we introduce some notation: Let E^n denote the n-dimensional Euclidean space and E_+^n the positive orthant of E^n, that is $E^n_+ = \{ x=(x_1,\ldots,x_n) \in E^n, x_i \geq 0, i=1,2,\ldots,n \}$. Clearly E_+^n is a closed convex cone in E^n and gives rise to a partial ordering $\leq$ defined on E^n by $x \leq y$ if and only if, $x_i \leq y_i$, $i=1,\ldots,n$; that is, $x \leq y$ if and only if $(y-x) \in E^n_+$. In general, if C is a closed convex cone in E^n, then C induces a partial ordering $\leq_C$ defined by $x \leq y$ if and only if $(y-x) \in C$.

Given a real valued function f on E^n and a closed convex cone C in E^n, we can define f to be C-monotone if $f(x_1) \leq f(x_2)$ whenever $(x_2-x_1) \in C$. Also, f is said to be strictly C-monotone if f is C-monotone and one to one, that is, $(x_2-x_1) \in C$ and $x_1 \neq x_2$ imply $f(x_1) < f(x_2)$. It is to be observed that we did not use the phrase "nondecreasing" because the cone C might define an ordering which could lead either to "nondecreasing" or "nonincreasing". For example if $n=1$ and $C = \{x/x \leq 0\}$ then $(x_2-x_1) \in C$ implies $x_2 \leq x_1$ and if in this case $f(x_1) \leq f(x_2)$, then we are led to "nonincreasing" function concept. Another point of interest is that, given any real-valued one to one function f on E^n, we can define a partial order $\leq_f$ in E^n as follows: $x_1 \leq_f x_2$ if and only if $f(x_1) \leq (x_2)$. However, this partial order need not be generated by a cone in the above sense.

For a different notion of monotonicity we refer to Minty [12] and Rockafellar [13].

Let us now define another concept of monotonicity that we will use in the present discussion. Let X be an arbitrary (finite or infinite) subset of E^n and let $X_i =$ $\{x_i/$ there exist $x_1,\ldots,x_{i-1},x_{i+1},\ldots x_n$ with $(x_1,\ldots,x_n) \in$

X} that is, X_i is the projection of X in the i^{th} coordinate. In most applications, $X \subseteq E^n_+$. Given a real valued function f on X, a point $\bar{x} \in X$ and an integer i with $1 \leq i \leq n$, we shall say that f is increasing at $\bar{x}$ with respect to the i^{th} coordinate, if for all y,z real with $(\bar{x}_1,\ldots,\bar{x}_{i-1},y,\bar{x}_{i+1},\ldots,\bar{x}_n) \in X$ and $(\bar{x}_1,\ldots,\bar{x}_{i-1},z,\bar{x}_{i+1},\ldots,\bar{x}_n) \in X$, $y < z$ implies $f(y) < f(z)$. For convenience of notation we shall write

$$(\hat{x}_i,z) \equiv (x_1,\ldots,x_{i-1},z,x_{i+1},\ldots,x_n).$$

Thus, f is increasing with respect to the i^{th} coordinate in X means that at each point $x \in X$, if we fix all coordinates except the i^{th} one, then in the remaining i^{th} variable, f is monotone increasing. We define f to be decreasing with respect to the i^{th} coordinate if (-f) is increasing in the i^{th} coordinate. In summary,

DEFINITION 2.1. A function f is coordinatewise monotone on X with respect to (wrt) the i^{th} coordinate, if and only if f is either increasing or decreasing wrt the i^{th} coordinate.

Note that the above definition corresponds to strict monotonicity.

If the projection X_i of X is open in E^1, if f is continuously differentiable on X_i and if f is increasing wrt the i^{th} coordinate, then the partial derivative $f_i \equiv \partial f/\partial x_i$ is nonnegative at each point $\bar{x} \in X$, since the partial derivative is defined in terms of the difference quotients, each of which is positive. Conversely, if the partial derivative $\partial f/\partial x_i$ is positive at $\bar{x} \in X$ (strictly greater than zero), then f is increasing wrt the i^{th}

coordinate at the point $\bar{x}$.

Since the monotonicity of f may be in different directions wrt different coordinates, we shall use the superscript +(-) on the i^{th} coordinate to mean <u>increasing</u> (<u>decreasing</u>). Thus, the notation $f(x_1^+, x_2^-, x_3^0, x_4)$ would mean that f is increasing wrt the 1st coordinate, decreasing wrt to the 2d, independent wrt the 3d and undetermined wrt the 4th coordinate.

Note that the concept of coordinatewise monotonicity is used also in the theory of generalized convexity of composite functions (as in Mangasarian [10] and Avriel [11]).

Two functions, one increasing and the other decreasing wrt a particular variable, are said to have <u>opposite monotonicity</u> or to be <u>monotonic in the opposite sense</u>. Two or more functions, all either increasing or decreasing, are said to have the <u>same monotonicity</u> or to be <u>monotonic in the same sense</u>.

For a function of a single variable, strict monotonicity means that the function is one to one and hence invertible. It is of interest to note that the corresponding inverse function (which obviously exists) is also monotonic and in the same sense. We may view this as an implicit function result. For example, given an (one to one) increasing function f(x), let F(x,y) denote f(x)-y so that F(x,y)=0 is the same as f(x)=y. Then for each x there is at most one y (which we shall denote by $\phi_0(x)$ such that $F(x,\phi_0(x))=0$; in fact, $\phi_0(x) = f(x)$ and clearly ϕ_0 and f are monotonic in the same sense. Also, since f is one to one, for each y in the range of f, there is at most one x (which we shall denote by $\psi_0(y)$) such that $F(\psi_0(y),y)=0$; in fact $\psi_0(y) = f^{-1}(y)$ and clearly ψ_0, f^{-1} and f are all monotonic in the same sense. It is to be noted that if we have $f(x^+)$, then $f(x)-y \equiv F(x^+,y^-)$ and in this case both $\phi_0(x)$ and $\psi_0(y)$ are increasing functions. On the other hand, if we have $f(x^-)$, then $F(x^-,y^-)$ and in this case both $\phi_0(x)$ and $\psi_0(y)$ are decreasing functions.

We shall now state and prove a theorem which shows that the above comments are true in a more general setting.

THEOREM 2.1. Let R be the set of real numbers and let X, Y, S denote (finite or infinite) subsets of R. Let $F: X\times Y \to S$ be a real-valued function (coordinatewise) monotone on X$\times$Y. Then, for each $\bar{s} \in S$, there exists monotone functions $\phi_{\bar{s}}(x)$ and $\psi_{\bar{s}}(y)$ (alternatively written as $\phi(\bar{s};x)$ and $\psi(\bar{s};y)$ such that $F(x,\phi(\bar{s};x)) = F(\psi(\bar{s};y),y) = \bar{s}$, with the understanding that these equalities hold for all x in the domain of $\phi_{\bar{s}}(.)=\{x/$ there is a y with $F(x,y)=\bar{s}\}$ and for all y in the domain of $\psi_{\bar{s}}(.)$. Furthermore, if F(x,y) is monotonic in the same (opposite) sense wrt x and y then the functions $\phi(\bar{s};x)$ and $\psi(\bar{s};y)$ are both decreasing (increasing).

PROOF. For convenience of notation and without loss of generality, let $\bar{s}=0 \in S$. For a fixed $\bar{x} \in X$, the function F(x,y) is monotonic wrt y (and one to one) and thus there is at most one value of y, denoted by $\phi_0(x)$ such that $F(\bar{x}, \phi_0(x)) = 0$. Thus $\phi_0(x)$ is a function of x with domain

$\{x \in X$ / there is a y such that $F(x,y)=0\}$. Reversing the roles of x and y we obtain a function $\psi_0(y)$ such that $F(\psi_0(y),y)=0$ with domain $\{y \in Y$ / there is an x such that $F(x,y)=0\}$

For the second part of the theorem, we assume $F(x^+,y^+)$ and take x_1, x_2 to be any two elements from the domain of ϕ_0 with $x_1 < x_2$. Let $x_1 = \phi_0(x_1)$ and $y_2 = \phi_0(y_2)$ so that $F(x_1,y_1)=0$ and $F(x_2,y_2)=0$. We wish to show that $y_1 > y_2$, i.e. ϕ_0 is decreasing wrt x. Assuming $y_1 \leq y_2$ and since F is increasing wrt y (for fixed x), we get $F(x_2,y_1) \leq F(x_2,y_2) = F(x_1,y_1)$. But since F is increasing wrt x (for fixed y) and since $x_1 < x_2$, we get $F(x_1,y_1) < F(x_2,y_1)$, a contradiction. Hence we must have $y_1 > y_2$, i.e. $x_1 < x_2$ implies $\phi_0(x_1) > \phi_0(x_2)$, or $\phi_0$is increasing wrt x. Reversing the roles of x and y we obtain that $\psi_0(y)$ is decreasing wrt y. Next let us assume $F(x^-,y^+)$. As before, let $x_1 < x_2$ and $y_1 = \phi_0(x_1)$, $y_2 = \phi_0(x_2)$. We observe again that assuming $y_1 \geq y_2$ leads to $F(x_1,y_1) \geq F(x_1,y_2)$ and since $F(x_1,y_1) = F(x_2,y_2)$ this implies $F(x_2,y_2) \geq F(x_1,y_2)$ which contradicts $F(x_2,y_2) < F(x_1,y_2)$ obtained from $x_1 < x_2$ and F decreasing wrt x. Thus, when F is monotonic in the opposite sense wrt x and y, $x_1<x_2$ implies $y_1<y_2$ and $\phi_0(x)$ is increasing wrt x. Similarly $\psi_0(y)$ is increasing wrt y. This completes the proof.

REMARK 2.1. The main property of <u>order</u> in R that we needed in the above proof is that the set R is totally ordered under <. Thus, for $y_1 \neq y_2$, we have either $y_1 < y_2$ or $y_2 < y_1$. The above theorem is valid for any totally ordered set

R.

Theorem 2.1 can be generalized easily to functions of several variables to get the following

PROPOSITION 2.1. Let $X_i \subseteq R$, $i=1,\dots,n$ be n subsets (finite or infinite) of R and let $X=\{x/x=(x_1,\dots,x_n),\ x_i \in X_i,\ i=1,\dots,n\}$. Let $F: X \to R$ be (coordinatewise) monotone on X. Then, for each $\bar{s}$ in the range of F and for each $i=1,\dots,n$ there exists a function $\phi(i,\bar{s};x_i')$ of the variable $x_i'=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ such that $\phi(i,s;x_i')$ is (coordinatewise) monotone wrt x_i'. Furthermore, for $1 \leq j \leq n$ and $i \neq j$, if F is monotone in the same (opposite) sense wrt x_i and x_j, then $\phi(i,s;x_i')$ is decreasing (increasing) wrt x_j.

REMARK 2.2. In all the above considerations, no assumption is made of continuity, nor differentiablity and in fact the domains of the functions need not be infinite so that the above results can be stated in terms of sequences.

3. Monotone Inequalities. In this section we consider inequalities involving monotone functions. For a discussion of linear inequalities in the context of optimization we refer to Mickle and Sze [14] and Shefi [15]. Here we consider real-valued functions f(x) which are (coordinatewise) monotone on some subset X of E_+^n, the set X being finite or infinite. Following a design practice stemming from geometric programming we shall consider inequalities written in the normalized form $f(x) \leq 1$.

An inequality is said to be tight at a point $\bar{x}$ if the inequality is satisfied as an equality at x. Thus an inequality $f(x) \leq 1$ is tight at x if $f(x)=1$. If f is coordinatewise monotone, then by definition f is one to one and thus there can be at most one solution for the equation $f(x)=1$ and so there can be at most one point x X where $f(x) \leq 1$ is tight. An inequality is said to be redundant if it is not tight at any point. Thus $f(x) \leq 1$ is redundant if the equation $f(x)=1$ has no real solutions.

An inequality of the form $t \leq a$ or $t \geq a$, where a is a fixed real number, is called a simple bound on the (real) variable t. When a simple bound is written in normalized form $t^{-1}a \leq 1$ or $ta^{-1} \leq 1$ it is referred to as a simple inequality.

In the following we shall consider mainly normalized inequalities of the form $f(x) \leq 1$. A normalized inequality is called monotone wrt a coordinate x_i if the function f(x) is (coordinatewise) monotone wrt x_i. Without loss of generality, in the statements involving a monotone function f we can assume that f is monotone increasing wrt any particular x_i. In fact, if f is not increasing wrt x_i, then f must be decreasing wrt x_i, so that if we rewrite f using the reciprocal x_i^{-1} instead of x_i, we can obtain a function f' which is increasing wrt the new variable $x_i'=x_i^{-1}$.

The following theorem shows that a monotone inequality, however complicated its algebraic form is equivalent to a simple inequality.

THEOREM 3.1. Every nonredundant increasing inequality in one variable can be reduced to a simple inequality providing a simple upper bound on the variable.

PROOF. According to the above assumptions the inequality is of the form $f(x^+) \leq 1$. Since the inequality is not redundant, there exists a (single) point $\bar{x}$ such that $f(\bar{x})=1$ and $f(x) \leq f(\bar{x})$. But then $x \leq \bar{x}$, since otherwise $x > \bar{x}$ and $f(x^+)$ would imply $f(x) > f(\bar{x})=1$, a contradiction. Thus $x \leq \bar{x}$, or $x\bar{x}^{-1} \leq 1$.

REMARK 3.1. Clearly $f(x^+) \leq 1$ implies an upper bound on x, while $f(x^-) \leq 1$ implies a lower bound on x.

REMARK 3.2. THe above theorem relies on the fact that $\leq$ is a total order on the real line so that if $x \nleq x_0$ then $x > x_0$. Since coordinatewise ordering in the n-dimensional Euclidean space is only a partial order, the above theorem does not hold for inequalities with several variables. However we can use the argument above to conclude the following:

PROPOSITION 3.1. If $f(x) \leq 1$ for all x in X, the domain of f, with f increasing wrt each coordinate, and $f(\bar{x})=1$ for some $\bar{x} \in X$, then $\bar{x}$ is a maximal element in X wrt the coordinatewise ordering; that is, for each $x \in X$, if $x_i \geq \bar{x}_i$ for each i, then $x=\bar{x}$.

REMARK 3.3. By suitably modifying the definition of partial order on R^n, the above proposition on the existence of maximal element $\bar{x}$ can be proved for functions which are coordinatewise monotone, even if monotonicity may not be in the same sense wrt all the coordinates.

We shall now use the above observations on the existence of maximal elements to derive some important results for simplification of systems of inequalities.

THEOREM 3.2. Let $f_1,\ldots,f_n$ be n given increasing functions with a common domain $D \subseteq R$. Let $F_i = \{x/f_i(x) \leq 1;\ 1 \leq i \leq n\}$ and let there exist an $\bar{x}_i \in F_i$ such that $f_i(\bar{x}_i) = 1$. Let $F = \bigcap_i F_i$ be nonemply. If $\bar{x}_1,\ldots,\bar{x}_n$ are all distinct, then the following two statements are equivalent:

(a) For some i_0 with $1 \leq i_0 \leq n$, $\bar{x}_{i_0}$ is the lowest value of the numbers $x_1,\ldots,x_n$.

(b) For some i_0 with $1 \leq i_0 \leq n$, $\bar{x}_{i_0} \in F$ and for each $x \in F$, $f_{i_0}(x) \leq 1$ and $f_i(x) < 1$ for each $i \neq i_0$.

PROOF. (a) implies (b): Since each $f_i(x)$ is increasing wrt x, we have from theorem 3.1 that $\bar{x}_i$ is the maximum of F_i so that, since $F \subseteq F_i$ we have $x \leq \bar{x}_i$ for each $x \in F$ and each i. But since $\bar{x}_i$ are all distinct and $\bar{x}_{i_0}$ is the lowest of them, we obtain that $\bar{x}_{i_0}$ is the lowest upper bound of F, i.e. $x \leq \bar{x}_{i_0}$ and $x < \bar{x}_i$ for $i \neq i_0$. But then, since f_i is increasing (and one to one) we get for all $x \in F$, $f_i(x) < f_i(\bar{x}_i)$ or $f_i(x) < 1$ for $i \neq i_0$, while $f_{i_0}(x) \leq 1$. Also, $\bar{x}_{i_0} \in F$ because for each i, $\bar{x}_{i_0} < x_i$ and as such, $f_i(\bar{x}_{i_0}) \leq f_i(\bar{x}_i) = 1$.

(b) implies (a): For all $x \in F$ let $f_i(x) < 1$ for $i \neq i_0$ and $f_{i_0}(x) \leq 1$. Also, for the same index i_0, let $\bar{x}_{i_0} \in F$. If for some $i \neq i_0$ we have $\bar{x}_i < \bar{x}_{i_0}$, then $f_i(\bar{x}_i) < f_i(\bar{x}_{i_0})$ or $1 < f_i(\bar{x}_{i_0})$ which contradicts the assumption that $\bar{x}_{i_0} \in F \subseteq F_i = \{x/f_i(x) \leq 1\}$. This completes the proof.

REMARK 3.4. In design optimization it is important to identify the design requirements which are critical at the optimum. A critical requirement corresponds to an inequality constraint which is active, i.e. one whose presence defines the location of the optimal point. Usually an active inequality constraint is tight at the optimum. The above theorem 3.2 states that in a set of increasing inequalities, the active one has the smallest root for the corresponding (tight) equation. From a practical standpoint this is useful in eliminating several inactive constraints in a problem with monotone inequality constraints. One should notice the distinction between redundant and inactive: redundant inequalities are never tight, while inactive inequalities are not tight at the optimal point though they may be tight at some other points.

Inactive constraints can be ignored in locating an optimal point. This leads naturally to the concept of dominance similar to that studied by Wilde [2].

DEFINITION 3.1. An inequality $f(x) \leq 1$, $x \in R^n$ is dominant over the inequality $g(x) \leq 1$, if and only if $f(x) \leq 1$ implies $g(x) \leq 1$ for all x.

LEMMA 3.1. Let $f(x) \leq 1$ be dominant over $g(x) \leq 1$ and let f and g be both increasing wrt $x \varepsilon R$. Let $f(x_f) = 1$ and $g(x_g) = 1$ for x_f, $x_g \varepsilon R$. Then $x_f \leq x_g$.

PROOF. This follows immediately from Theorem 3.1 which shows that x_g is an upper bound for $G = \{x/g(x) \leq 1\}$ and $x_f \varepsilon G$ because $f(x) = 1$ implies $g(x) \leq 1$ by dominance.

REMARK 3.5. In an optimization problem we can omit all nondominant constraints from consideration.

We shall now return to inequalities with several variables. In Proposition 3.1 we obtained the existence of maximal elements for such inequalities. We shall now consider bounds in a different sense.

DEFINITION 3.2. An inequality $f(x) \leq 1$ is said to be partially simple wrt x_i, if and only if $f(x_1,\ldots,x_n)$ can be written as $x_i g(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$.

The following theorem shows how to simplify the constraint set using the fact that a tight multivariable inequality represents a hypersurface and thus there are infinite solutions for the corresponding equation. For brevity, we shall consider a function of two variables $f(x,y)$.

THEOREM 3.3. Let $f(x^+,y^+)$ and $f(x_0,y_0) = 1$. Then, for any (x,y) such that $f(x,y) \leq 1$, we have $x \geq x_0$ implies $y \leq y_0$ and $y \geq y_0$ implies $x \leq x_0$.

PROOF. Let (x,y) be such that $f(x,y) \leq 1$ and let $x \geq x_0$. If possible let $y > y_0$. Then, since f is increasing wrt both x and y, we have $f(x,y) \geq f(x_0,y) > f(x_0,y_0) = 1$, a contradiction. Thus, $y \leq y_0$. Similar arguments show that $y \geq y_0$ implies $x \leq x_0$.

REMARK 3.6. Using the above theorem, a simple inequality of the form $x \leq x_0$ can be combined with another monotonic inequality of the form $f(x^-,y^-) \leq 1$ to yield another simple inequality. More precisely, let $f(x,y)$ be monotone decreasing wrt x and y and let $f(x,y) \leq 1$ for all $x > 0$, $y > 0$. Let there exist some $x_0 > 0$, $y_0 > 0$ such that $f(x_0,y_0) = 1$. Then, the pair of inequalities $x \leq x_0$ and $f(x,y) \leq 1$ is equivalent to the pair $xx_0^{-1} \leq 1$ and $y_0y^{-1} \leq$

1. Similarly, if $f(x^-,y^+) \leq 1$ and $x \leq x_0$ with $f(x_0,y_0) = 1$, then $y \leq y_0$. In fact, if $y > y_0$ then $f(x,y) \geq f(x_0,y) > f(x_0,y_0) = 1$, a contradiction. These observations may help significantly the simplification of a system of nonlinear monotone inequalities [5].

REMARK 3.7. If a nonredundant inequality is written in two different equivalent ways, such as $f(x) \leq 1$ and $g(x) \leq 1$ with f and g both monotone (and dominant over each other), then f and g must have the same monotonicity. In fact, if x_f and x_g are such that $f(x_f) = 1$, $g(x_g) = 1$ and $f(x^+)$, then $f(x) \leq 1$ is equivalent to $x \leq x_f$ while if g is decreasing wrt x, then $g(x) \leq 1$ is equivalent to $x \geq x_g$. Since $x \leq x_f$ and $x \geq x_g$ cannot be equivalent, we conclude that g must be increasing, if f is increasing. Similarly for f,g decreasing.

REMARK 3.8. Monotonicity of normalized inequalities is affected by the algebraic manipulations that convert the original inequality to a normalized one. Given two nonredundant equivalent inequalities $f(x) \leq 1$ and $g(x) \leq 1$, the fact that f is monotone does not imply that g has to be monotone. However, if both are monotone, then by remark 3.7 they must be monotonic in the same sense. This observation has direct implication on the modeling of constraints that arise from physical considerations in design optimization problems. Careless algebraic manipulations may lead to models with lack of monotonicity or even with nonsimply connected feasible domains, while the original physics of the problem indicate otherwise.

The next theorem is useful when an active inequality constraint that cannot be solved explicitly for a particular variable is to be used for eliminating this variable from the problem while retaining existing monotonicities.

THEOREM 3.4. Every monotone nonredundant inequality can be transformed into a partially simple one wrt each monotonic variable with the original monotonicities preserved.

PROOF. For convenience of notation, let us consider an inequality with only two variables such as $f(x^+,y^+) \leq 1$. We need to show that $f(x^+,y^+) \leq 1$ is equivalent to $xg(y^+) \leq 1$. We assume as usual that the domain of f is $\{(x,y)/x>0, y>0\}$. Since $f(x,y) \leq 1$ is assumed to be nonredundant, there exist $x_0 > 0$, $y_0 > 0$ such that $f(x_0,y_0) = 1$. Then by Theorem 2.1, there exists a function $\psi(y^-)$ such that $x_0(\psi(y_0))^{-1} = 1$. Since $x_0 > 0$, obviously $\psi(y_0) > 0$. Let us now consider $x > 0$, $y > 0$ and the product $x(\psi(y))^{-1}$. Since f is increasing wrt x and y and $f(x_0,y_0) = 1$ while $f(x,y) \leq 1$ for all $x > 0$, $y > 0$, we must have $x \leq x_0$ and $y \leq y_0$. Since $\psi(y^-)$, we get $\psi(y) \geq \psi(y_0) > 0$ and $(\psi(y))^{-1} \leq (\psi(y_0))^{-1}$, so that $x(\psi(y))^{-1} \leq x_0(\psi(y_0))^{-1} = 1$. It suffices to take $g(y) = (\psi(y))^{-1}$ to have $g(y^+)$ and complete the proof.

4. Necessary Conditions at Optimality.

In the present section we shall obtain two necessary conditions for mathematical programs that possess certain monotonicity properties. Principles exploiting systematically monotone properties of functions in the present context were first developed by Wilde [2,3]. The major thrust here is that we do not assume continuity or differentiability of the constraint functions and thus the results are valid for descrete problems as well, where the independent variables may take descrete values, a situation very common in engineering design.

Let us pose the following nonlinear monotone programming problem in normalized form

<u>Problem P1</u>

<u>minimize</u> $f_0(x)$, $x \varepsilon X \subseteq R_+^n$

<u>subject to</u> $f_i(x) \leq 1$, $i = 1,\ldots,m$

where $f_0, f_1, \ldots, f_m$ are all coordinatewise monotone and where R_+^n denotes the <u>strict</u> positive orthant of the n-dimensional Euclidean space, so that for $x \varepsilon X$, we have $x = (x_1,\ldots,x_n)$ and $x_i > 0$ for $1 \leq i \leq n$. In the theorems below we assume that Problem P1 is feasible and it has an optimal solution.

THEOREM 4.1. Let all the constraints in Problem P1 be nonredundant. If in Problem P1 the objective function f_0 is monotone wrt x_i, then there exists at least one active constraint with opposite monotonicity to f_0 wrt x_i.

PROOF. For convenience of notation, let i = 1 and assume that f_0 is decreasing wrt x_i. Then the value of f_0 can be indefinitely decreased by increasing values of x_1, unless x_1 is bounded above and this upper bound can be found only if there is an index j with f_j increasing wrt x_1, so that $f_j(x) \leq 1$ would lead to an upper bound on x_1 of the form x_1

$\leq x_1^j$ (see Remark 3.1). If on the other hand f_0 is increasing wrt x_1, then there is some j, $1 \leq j \leq m$ such that f_j is decreasing wrt x_1. Otherwise all f_j are increasing wrt x_1 and thus $f_j(x) \leq 1$ leads to an upper bound on x_1, for each j, so that we obtain zero to be the only lower bound for x_1 and f_0 can be lowered indefinitely by getting x_1 closer and closer to zero. Since no vector with a zero component can be a valid optimal solution, we get a contradiction to the existence of optimal solution. We conclude that there is at least one f_j which is decreasing wrt x_1.

REMARK 4.1. The above theorem can be applied to any program that has an objective monotonic wrt one or more variables in order to test whether a bounded (optimal) solution exists. The presence of a nonmonotonic constraint would generally satisfy boundedness.

COROLLARY 4.1. If the objective f_0 is monotone wrt x_i, then the dominant inequality of the set of constraints with opposite monotonicities is active.

PROOF. The dominant constraint provides the glb or lub on x_i (depending on whether the constraint is decreasing or increasing).

Now we pose the following Mayer type problem in normalized form

Problem P2

minimize $\quad f_0(x) \quad , \quad x \in X \subseteq R_+^n$

subject to $\quad f_i(x,u) \leq 1, \quad i = 1,\ldots,p$

where $x = (x_1,\ldots,x_n)$; $x_i > 0$ for $1 \leq i \leq n$ and u =

$(u_1,\ldots,u_m)$, $u_j > 0$ for $1 \le j \le m$.

THEOREM 4.2. Let there be some j, $1 \le j \le m$, such that $f_i(x,u)$ is monotone wrt u_j for all i. Let us assume that Problem P2 has a solution (x,u). Then there exists an optimal solution (x^*,u^*) of Problem P2 such that the following holds: Either all the constraints $f_i(\bar{x},\bar{u}) \le 1$ are inactive, or there is a pair of constraints with opposite monotonicities, both of which are active at (x^*,u^*).

PROOF. (i) Let us assume that $f_i(u_j^+)$ for all i. (The argument is the same for $f_i(u_j^-)$ for all i). Suppose $(\bar{x},\bar{u})$ is an optimal solution for which $f_{i_0} \le 1$ is active, so that $f_{i_0}(\bar{x},\bar{u}) = 1$, for some i_0. Then, we choose u^* such that $0 < u_j^* < \bar{u}_j$ and $u_i^* = \bar{u}_i$ for $i \ne j$. Since $f_i(u_j^+)$ for all i, we get $f_i(\bar{x},u^*) < f_i(\bar{x},\bar{u}) \le 1$ for all i and the value of the objective $f_0(x)$, being independent of u, is same for $(\bar{x},\bar{u})$ and $(\bar{x},u^*)$. Thus, all constraints are inactive for $(\bar{x},u^*)$, which is also optimal.

(ii) By renumbering if necessary, let us assume that $f_1,\ldots,f_k$ are decreasing wrt u_j and $f_{k+1},\ldots,f_p$ are increasing wrt u_j for $1 \le k \le p$. Let $f_i(\bar{x},\bar{u}) \le 1$ be equivalent to $\bar{u}_j \ge \bar{\phi}_{ij}$ for $1 \le i \le k$ and to $\bar{u}_j \le \bar{\phi}_{ij}$ for $k+1 \le i \le p$. In fact, $\bar{\phi}_{ij} = \phi_i(\bar{x},u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_m)$. Let us also assume without loss of generality that $\bar{\phi}_{1j} \ge \bar{\phi}_{ij}$ for $1 \le i \le k$ and $\bar{\phi}_{pj} \le \bar{\phi}_{ij}$ for $k+1 \le i \le p$. Then the above inequalities imply

$$\bar{\phi}_{1j} \le \bar{u}_j \le \bar{\phi}_{pj} .$$

Let us now consider two cases. Case (a): $\bar{\phi}_{1j} \le \bar{\phi}_{pj}$. Then there is some u_j^* such that $\overline{\phi}_{1j} < u_j^* < \overline{\phi}_{pj}$. We choose u^* so that $u_i^* = \overline{u}_i$ for $i \neq j$. Then $(\overline{x},u^*)$ is optimal and at this point all f_i are inactive. Case (b): $\overline{\phi}_{1j} = \overline{\phi}_{pj}$. In this case the constraints $f_1 \leq 1$ and $f_p \leq 1$ are both active at $(\overline{x},\overline{u})$ and they have opposite monotonicities wrt u_j.

REMARK 4.2. In theorem 4.2 no assumption of differentiability was made. If the functions f_i, $0 \leq i \leq p$,are differentiable and Kuhn-Tucker conditions are applicable, then we can say that for every optimal solution (x^*,u^*) the conclusion of the theorem holds. Related results can be found in Wilde [2,3] for single-term continuous, differentiable functions and in Papalambros [7] for posynomials.

REMARK 4.3. In view of theorem 4.2 we can simplify the process of locating optimal solutions--if any one solution would be acceptable. For example, in the case where there exists a control variable (that is a variable wrt which the objective is independent) such that all constraints involving this variable are of the same monotonicity wrt it, all these constraints can be ignored.

5. Suggested Procedure. There are certain steps suggested by the discussion in the previous sections which can be taken in the study of programs in general. They are particularly important for problems arising from physical applications where the adequacy and completeness of the optimization model are not known with certainty. Here we

give some suggestions that have been found very useful in practical applications.

1. Since a given inequality may be equivalent to several normalized forms, we choose, when possible, that form where the corresponding function is monotone. Thus, for example, if $f_1(x) \leq 1$ and $f_2(x) \leq 1$ are equivalent (dominant over each other) and if $f_1(x)$ is monotone wrt more coordinates than $f_2(x)$, then we prefer the representation $f_1(x) \leq 1$.

2. We look for simple inequalities of the form $x_i a_i \leq 1$ and use these to simplify, when possible, other monotone inequalities, in a manner described in section 3. Thus, for example, if $xa \leq 1$ and $f_1(x^-, y^-) \leq 1$, then we can replace $f_1(x,y) \leq 1$ by a simple inequality of the form $yb \leq 1$.

3. Negative variables should be replaced by new variables with opposite sign. In a problem where variables may take both positive and negative values we may transform them into strictly positive ones by a change of datum. When this is not possible, we can consider several subproblems in each of which, the variables all keep the same sign. The globally optimal solution can be then found by comparing the solutions of the subproblems. This case decomposition can be applied also to problems with nonmonotonic functions by examining subproblems defined over regions of the feasible space, where monotonicities may be determined [6].

6. Examples. In this section we apply the ideas of monotonicity analysis to some simple examples of the type used to test numerical algorithms. Applications to

engineering design problems can be found in the references given in the introduction.

EXAMPLE 1 (F. J. Gould [18] in Lootsma [16]). The objective function is nonconvex and the constraint region is a narrow, half-moon shaped valley of the type described in Fletcher and McCann [17]. The normalized problem is

$\underline{\text{minimize}}\ f(x_1,x_2) = (x_1-10)^3 + (x_2-20)^3$

$\underline{\text{subject to}}$

$$R_1(x_1^-) = 13x_1^{-1} \leq 1$$

$$R_2(x_1,x_2) = 100[(x_1-5)^2 + (x_2-5)^2]^{-1} \leq 1$$

$$R_3(x_1,x_2) = (1/82.81)[(x_1-6)^2 + (x_2-5)^2] \leq 1$$

with $x_2 \geq 0$

To determine a smaller feasible domain we observe that

$$(x_1-6)^2 + (x_2-5)^2 = (x_1-5)^2 + (x_2-5)^2 + (11-2x_1)$$

and from constraint R_2 and R_3 we see that

$$100 \leq (x_1-5)^2 + (x_2-5)^2 \leq 82.81 - (11-2x_1)$$

or

$$x_1 \geq 14.095 \qquad (3)$$

Since any feasible solution must satisfy (3), constraint R_1 is redundant. From constraint R_3 we have

$$(x_2-5)^2 \leq 82.81 - |x_1-6|^2 \leq 82.81 - |14.095-6|^2$$

since the right-hand side is strictly decreasing wrt x_1 (this being true because of (3)). Therefore

$$0.8429 \leq x_2 \leq 9.1569 \qquad (4)$$

To determine the monotonicities we observe that a difficulty arises because the square terms in parentheses in the problem statement can be either increasing or decreasing depending on the sign of the quantity in parentheses. But since $x_1 \geq 14.095$ all terms containing x_1 are positive. This is not true for x_2 so we circumvent this difficulty by decomposing the problem into Case A with $0.8429 \leq x_2 \leq 5$ and Case B with $5 \leq x_2 \leq 9.1569$.

Case A. We rewrite the problem as follows:

minimize $f(x_1^+, x_2^+) = |x_1-10|^3-|20-x_2|^3$

subject to

$$R_2(x_1^-,x_2^+) = 100[|x_1-5|^2 + |5-x_2|^2]^{-1} \leq 1$$

$$R_3(x_1^+,x_2^-) = (1/82.81)[|x_1-6|^2 + |5-x_2|^2] \leq 1$$

and allowable range

$$14.095 \leq x_1 \quad \text{and} \quad 0.8429 \leq x_2 \leq 5 \qquad (5)$$

The objective is increasing wrt both x_1 and x_2. Constraint R_2 being the only one with opposite monotonicity wrt x_1 must be active. Then $R_2(x_1^-,x_2^+) = 1$ implies $x_1 = \phi_2(x_2^+)$ and thus elimination of x_1 from the problem gives an objective function increasing wrt x_2. Then R_3 must be active (whether it is monotonic or not) if the problem has a solution. Thus the optimum is given by the simultaneous activity of R_2 and R_3 and is

$$f^* = -6961.8,\ x_1^* = 14.095,\ x_2^* = 0.8429 \qquad (6)$$

We note that the allowable range (5) is equivalent to the two constraints R_2, R_3 and for $f(x_1^+,x_2^+)$ the minimum is at the lower bounds for both x_1 and x_2.

Case B. The problem is the same as in case A but with $5 \leq x_2 \leq 9.1569$. Again R_2 must be active, but R_3 cannot be simultaneously active since solution (6) resulting from $R_2=R_3=1$ gives $x_2<5$. We observe that now $R_2(x_1^-,x_2^-)=1$ implies $x_2= \phi_2(x_1^-)$ so the objective may become nonmonotonic. This, however, is irrelevant since the greatest lower bound on any solution in case B is

$$f_{l.b.} = (14.095-10)^3 + (5-20)^3 = -3306.3 \qquad (7)$$

Clearly, since $f_{l.b.} \geq f^*$ no better solution than (6) can be obtained in case B and the global optimum is (6).

EXAMPLE 2. (Collatz and Wetterling, cited by Eckhardt [19] in Lootsma [16]). The problem in normalized form is as follows:

<u>minimize</u> $f(x_1^-, x_2^-) = -2x_1(24-x_1) - x_2(40-x_2)$

<u>subject to</u>

$$R_1(x_1^+, x_2^+) = 0.125\ x_1 + 0.125\ x_2 \leq 1$$

$$R_2(x_1^+, x_2^0) = 0.1667\ x_1 \leq 1$$

$$R_3(x_1^+, x_2^+) = 0.0556\ x_1 + 0.1667\ x_2 \leq 1$$

and $x_1 \geq 0$, $x_2 \geq 0$.

We note that since f is decreasing wrt both x_1 and x_2, the optimum would have $x_1 > 0$, $x_2 > 0$ unless this is the only feasible solution, which is clearly not true. Next we observe that, because of the monotonicity of f wrt x_2, at least one of the constraints R_1 and R_3 must be active. To obtain a dominance condition we observe that

$$R_3 = (0.125\ x_1 + 0.125\ x_2) + (0.0417\ x_2 - 0.0694\ x_1)$$

$$= R_1 + (0.0417\ x_2 - 0.0694\ x_1) \equiv R_1 + d \qquad (8)$$

so that if $d \leq 0$, R_1 is dominant and if $d \geq 0$, R_3 is dominant. This leads to two cases:

<u>Case A</u>. The condition $d \leq 0$ holds and it corresponds to an added constraint R_3', while R_3 is redundant. Since R_1 is active, it can be used to eliminate x_2 and the resulting problem is

minimize $f = 3x_1^2-24x_1-256$

subject to

$R_2 = x_1/6 \leq 1$

$R_3' = 3x_1^{-1} \leq 1$

which has an interior minimum with R_2, R_3' inactive, i.e.

$$f^* = -304, \; x_1^* = 4, \; x_2^* = 4 \tag{9}$$

Case B. The condition $d \geq 0$ holds corresponding to an added constraint R_1'

$$R_1': \; 1.667 \, x_1 x_2^{-1} \leq 1 \tag{10}$$

while R_1 is redundant and R_3 is active. Simultaneous activity of R_3 and R_2 leads to a violation of R_1', while simultaneous activity of R_3 and R_1' leads to $x_1 = 3$ and $x_2 = 5$, giving $f=-301$. Thus the global optimum is given by (9). The minimum reported in the original reference above is $f=-292$, $x_1=6$, $x_2=2$. This is in fact a local (boundary) minimum for case A above.

EXAMPLE 3 (Op. cit. [16,19]). The normalized problem is

minimize $f(x_1^-) = -x_1$

subject to

$R_1(x_1^+, x_2^-) = x_2^{-1} \exp(x_1) \leq 1$

$R_2(x_2^-, x_3^-) = x_3^{-1} \exp(x_2) \leq 1$

$R_3(x_3^+) = 0.1 \, x_3 \leq 1$

and $x_1, x_2, x_3 \geq 0$

Again we observe that only strictly positive values of the variables need be considered. Successive application of the two theorems in section 4 shows that all constraints must be active and the optimum is given by

$$x_3^* = 10$$
$$x_2^* = \ln x_3^* = \ln 10 = 2.303 \qquad (11)$$
$$x_1^* = \ln x_2^* = \ln 2.303 = 0.834$$

The same problem is given also with the objective (min) $f = 0.2\, x_3 - 0.8\, x_1$ and the same constraints. This is more difficult. Constraints R_1 and R_3 are again active and the objective becomes $f = 0.2(x_3 - 4 \ln(\ln x_3))$ which is nonmonotonic with an interior minimum. In fact, for $x_3=10$ we get $f = 6.663$ while for $x_3=2$, $f=3.466$, so R_3 is definitely inactive. The optimal value of x_3 can be found by an one-dimensional search. Here monotonicity cannot solve the entire problem but reduces it to a simpler one.

EXAMPLE 4 (Pierre [20]). Now we examine a problem where the variables are unrestricted in sign. In such cases we may divide the entire space into regions, apply the theorems in each region and at the end compare the different extrema obtained in each region. It may occur, as in the present example, that in a region, extremum may not exist; in such a case, that particular region does not contribute in locating the global optimum.

The problem is to find the maximum of f on a sphere, subject to remaining on one side of a plane that passes through the sphere:

<u>maximize</u> $f = x_2$

<u>subject to</u>

$$R_1: \quad x_1^2 + x_2^2 + x_3^+ = 1$$
$$R_2: \quad 2x_2 - x_1 \leq 1$$

where x_1, x_2, x_3 are coordinates with the origin coinciding with the center of the sphere. We note that in dealing with equalities, such as R_1 above, there are two specific considerations: (i) whether the constraint can be automaticaly satisfied by a judicious choice of the value of a free (control) variable, which is not constrained in any other way; (ii) whether the equality can be replaced by an inequality, with the understanding that the (new) inequality will have to be active at the optimum, the direction of the inequality being determined by monotonicity rules.

To illustrate the above point in the present example we observe that the only constraint on x_3 is R_1 and thus for given x_1, x_2 we can always satisfy R_1 by choosing $x_3 = \pm (1-x_1^2-x_2^2)^{1/2}$ (clearly it is always $x_1^2+x_2^2 \leq 1$). Thus the problem can be simplified by deleting R_1 and replacing it by R_1' as follows:

<u>maximize</u> $f = x_2$

<u>subject to</u>

R_1': $x_1^2 + x_2^2 \leq 1$

R_2: $2x_2 - x_1 \leq 1$

In this new problem at least one constraint must be active because the objective is monotonic wrt x_2. Moreover, since the objective is independent of x_1 there must be two active constraints with opposite monotonicities wrt x_1. Thus, both R_1' and R_2 are active. Simultaneous solution of $x_1^2 + x_2^2 = 1$ and $2x_2 - x_1 = 1$ gives $(\bar{x}_1, \bar{x}_2) = (3/5, 4/5)$ or $(-1, 0)$. Obviously, $(3/5, 4/5)$ yields a larger value for the objective and the optimal solution is then

$(x_1,x_2,x_3) = (3/5,4/5,0)$. Note that we obtained the value of x_3 as $(1-x_1^2-x_2^2)^{1/2} = 0$.

The monotonicity rules have been applied above without regard to the fact that x_1, x_2, x_3 are unrestricted in sign (while the theorems assumed nonnegativity of the variables). However, if we consider the above problem divided into several cases, then in each case--except one--we would observe that there is no extremum. In fact, let us consider first three cases: (i) $x_3 = 0$, (ii) $x_3 > 0$, (iii) $x_3<0$. In case (i) with $x_3=0$ we have $x_1^2 + x_2^2 = 1$. Thus, if $x_1 \geq 0$, then $x_2 \leq (1+x_1)/2$ implies that for a fixed x_1 the maximum of the objective f is $x_2 = (1+x_1)/2$ so that $x_1^2 + x_2^2 = 1$ gives $x_1^2 + [(1+x_1)/2]^2 = 1$ or $5x_1^2 + 2x_1 - 3 = 0$ or, $x_1 = 3/5$ or -1.
Since $x_1 \geq 0$, the optimal solution in the region $x_1 \geq 0$, $x_3 = 0$ is (3/5, 4/5, 0). In the region $x_1 < 0$, $x_3 = 0$, since $x_2 \leq (1+x_1)/2$ implies $x_2 < 1/2$, then x_2 can be made arbitrarily close to 1/2 and the supremum, namely 1/2, is never attained. Thus in case (i) the only optimal solution is (3/5, 4/5, 0). Next we consider case (ii) with $x_3 > 0$. We can apply the theorems as stated. Since the objective is independent of x_3, we can always satisfy it; for example, by choosing $x_3 = \pm (1-x_1^2-x_2^2)^{1/2}$. Thus, the problem simplifies to maximizing x_2 subject to $2x_2 - x_1 \leq 1$ and the inactive $x_1^2 + x_2^2 < 1$. We examine the subcase where $x_1 \geq 0$, $x_2 \geq 0$. Applying monotonicity arguments we see that since f is independent of x_1 and the second constraint is inactive, so should be the first constraint,

i.e. $2x_2 - x_1 < 1$. But a monotone objective with no active constraints cannot have an optimum. So no extremum exists in the subcase $x_3 > 0$, $x_1 \geq 0$, $x_2 \geq 0$. All other cases with $x_1 < 0$, $x_2 < 0$, $x_3 < 0$ can be handled in entirely the same manner yielding no optimal solution. We conclude that the global optimum is the one found in case (i).

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