Structure of Ališauskas–Jucys form of the 9j coefficients*

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From the Ališauakas–Jucys triple summation expression, the Wigner 9j coefficients may be visualized as boundary values of a new generalized hypergeometric function $\phi^{(3)}(\alpha_{ik};\beta_k,\gamma_m;w_k)$ in three variables. Integral representations are given both for $\phi^{(3)}$ in general and its boundary values as the 9j coefficients. The Radon structure is discussed. It is seen that $\phi^{(3)}$ and the 9j coefficients in general do not belong to the class of hypergeometric functions whose Radon transforms are products of linear forms.

I. INTRODUCTION

In a previous paper,[1] the structure of the Wigner 9j coefficients was analyzed from the Bargmann approach. The generating function was derived, the 72-element symmetry was manifest, and a sixfold summation expression for the 9j coefficient was obtained. Contrary to the situation of the lower-order coupling and recoupling coefficients where the Clebsch–Gordan coefficent may be visualized as a ${}_3F_2$ function at $x=-1$ and the Racah coefficient as a ${}_4F_3$ function at $x=1$,[2] the 9j coefficient is seen not to belong to the ${}_pF_q$ family of functions.

However, the question was unanswered as to whether the 9j coefficient may be regarded as a boundary value of a member of some other class of generalized hypergeometric functions. A particularly interesting class of generalized hypergeometric functions is the Gel'fand type[3], being the Radon transforms of products of *linear* forms. Does the 9j coefficient satisfy the Gel'fand criterion? The sixfold summation expression derived in Ref. 1, having a rather complicated Radon transform, was not suited to answer this question.

Ališauskas and Jucys[4] have derived a remarkable triple summation expression for the 9j coefficients. This triple summation expression, while lacking in the manifest symmetry of the 9j coefficients, permits a definition of a *new* generalized hypergeometric function in three variables $\phi^{(3)}(\alpha_{kl};\beta_i,\gamma_m;w_k)$ [Eq. (11) below] of which the 9j coefficient is evaluated at $w_k=1$ together with the special values of the coefficients α,β, and γ.

A ninefold integral representation is given for $\phi^{(3)}$. When restricted to the case of 9j coefficients, a sixfold integral representation is obtained. It is seen that, in general, neither $\phi^{(3)}$ nor the 9j coefficient satisfy the Gel'fand criterion.

II. ALIŠAUSKAS-JUCYS TRIPLE SUM EXPRESSION OF 9j COEFFICIENTS

The 9j coefficient in the Ališauskas–Jucys triple summation form may be written as follows:

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = K \sum_{x_1,x_2,x_3} \frac{\prod_{k=1}^{3}\prod_{l=1}^{4}(a_{kl})_{x_k}}{\prod_{k=1}^{3}(b_k)_{x_k}\prod_{\substack{m=1\\(m,n,k\\ \text{cyclic})}}^{3}(c_m)_{x_n+x_k}} \cdot \frac{1}{\prod_{k=1}^{3}x_k!} \tag{1}$$

where K is a multiplicative factor [see (6) below],

$$(a)_x \equiv \Gamma(a+x)/\Gamma(a), \tag{2}$$

a_{kl}, b_k, and c_m are certain linear combinations of the j_{pq}'s, namely,

$$(a_{kl}) \equiv \begin{pmatrix} 1+j_{21}+j_{31}-j_{11} & 1+j_{12}+j_{13}-j_{11} & j_{21}-j_{31}-j_{11} & j_{13}-j_{12}-j_{11} \\ j_{32}-j_{12}-j_{22} & 1+j_{12}+j_{32}-j_{22} & j_{23}-j_{21}-j_{22} & -1-j_{21}-j_{22}-j_{23} \\ 1+j_{31}+j_{33}-j_{32} & j_{33}-j_{13}-j_{23} & 1+j_{23}+j_{33}-j_{13} & j_{33}-j_{31}-j_{32} \end{pmatrix}, \tag{3}$$

$$(b_k) \equiv \begin{pmatrix} -2j_{11} \\ -2j_{22} \\ 2+2j_{33} \end{pmatrix}, \tag{4}$$

$$(c_m) \equiv \begin{pmatrix} j_{33}-j_{13}-j_{21}-j_{22} \\ 1+j_{33}+j_{21}-j_{11}-j_{32} \\ 1+j_{32}+j_{13}-j_{11}-j_{22} \end{pmatrix}, \tag{5}$$

$$K \equiv (-1)^{a_{23}+a_{34}} K_1K_2/K_3, \tag{6}$$

$$K_1 \equiv \frac{\nabla(j_{21}j_{22}j_{23})\,\nabla(j_{31}j_{32}j_{33})\,\nabla(j_{13}j_{23}j_{33})}{\nabla(j_{11}j_{21}j_{31})\,\nabla(j_{12}j_{22}j_{32})\,\nabla(j_{11}j_{12}j_{13})}, \tag{7}$$

$$\nabla(abc) \equiv [\Gamma(2+a+b+c)\Gamma(1+a+b-c) \times \Gamma(1+c+a-b)/\Gamma(1+b+c-a)]^{\frac{1}{2}},$$

$$K_2 \equiv \prod_{k=1}^{3}\Gamma(a_{kk})\Gamma(a_{12})\Gamma(a_{31})\Gamma(1-b_1)\Gamma(1-b_2)\Gamma(1-c_1), \tag{8}$$

$$K_3 \equiv \Gamma(1-a_{13})\Gamma(1-a_{21})\Gamma(1-a_{23})\Gamma(1-a_{32}) \times \prod_{k=1}^{3}\Gamma(1-a_{k4})\,\Gamma(b_3)\Gamma(c_2)\Gamma(c_3). \tag{9}$$

The apparent lack of symmetry among the entries in a_{kl}, b_k, and c_m in (3)–(5) is perhaps mitigated by the summation simplicity of Eq. (1).

III. 9j COEFFICIENT AS BOUNDARY VALUE OF A NEW HYPERGEOMETRIC FUNCTION $\phi^{(3)}(\alpha_{kl};\beta_k,\gamma_m;w_k)$

Equation (1) immediately suggests that the 9j coefficients may be regarded as boundary values of a function in three variables at $w_k=1$, $k=1,2,3$, namely,

$$\{9j\} = K\phi^{(3)}(a_{kl};b_k,c_m;w_k=1) \tag{10}$$

with the a's, b's, and c's given by (3)–(5). The $\phi^{(3)}$ function is defined as follows:

$$\phi^{(3)}(\alpha_{kl};\beta_k,\gamma_m;w_k) \equiv \sum_{x_1,x_2,x_3} \frac{\prod_{k=1}^{3}\prod_{l=1}^{4}(\alpha_{kl})_{x_k}}{\prod_{k=1}^{3}(\beta_k)_{x_k}\prod_{\substack{m=1\\(m,n,k\\ \text{cyclic})}}^{3}(\gamma_m)_{x_n+x_k}} \prod_{k=1}^{3}\frac{w_k^{x_k}}{x_k!} \tag{11}$$

$\phi^{(3)}$ does not seem to be a known function. In the next section, we examine its integral representation.

IV. INTEGRAL REPRESENTATION FOR $\phi^{(3)}$

Using the identity

$$(\gamma_m)_{x_n+x_k} = (\gamma_m + x_k)_{x_n}(\gamma_m)_{x_k}, \tag{12}$$

we see that the triple sum in Eq. (11) may be viewed as a folded produce of three ${}_4F_3$ functions, namely

$$\phi^{(3)} = \sum_{x_1}\frac{\prod_{l=1}^{4}(\alpha_{1l})_{x_1}}{(\beta_1)_{x_1}(\gamma_2)_{x_1}(\gamma_3)_{x_1}}\cdot\frac{w_1^{x_1}}{x_1!} \times \sum_{x_2}\frac{\prod_{l=1}^{4}(\alpha_{2l})_{x_2}}{(\beta_2)_{x_2}(\gamma_3+x_1)_{x_2}(\gamma_1)_{x_2}}\cdot\frac{w_2^{x_2}}{x_2!} \times \sum_{x_3}\frac{\prod_{l=1}^{4}(\alpha_{3l})_{x_3}}{(\beta_3)_{x_3}(\gamma_2+x_1)_{x_3}(\gamma_1+x_2)_{x_3}}\cdot\frac{w_3^{x_3}}{x_3!}. \tag{13}$$

Equation (13) has an immediate integral representation by iterating the well-known representation for the ${}_4F_3$ function. The result is

$$\phi^{(3)} = \frac{\prod_{i=1}^{3}\Gamma(\beta_i)\Gamma(\gamma_i)}{\prod_{i,k=1}^{3}\Gamma(\alpha_{ik})\prod_{i\geq k=1}^{3}\Gamma(\beta_{ik})}\int_0^1\cdots\int\prod_{i,k=1}^{3}dt_{ik} \times t_{ik}^{\alpha_{ik}-1}(1-t_{ik})^{\beta_{ik}-1}\prod_{k=1}^{3}(1-w_k\tau_k)^{-\alpha_{k4}}, \tag{14}$$

where

$$(\beta_{ik}) \equiv \begin{pmatrix} \beta_1-\alpha_{11} & \gamma_3-\alpha_{21}-\alpha_{12} & \gamma_2-\alpha_{31}-\alpha_{13}\\ \gamma_3-\alpha_{21} & \beta_2-\alpha_{22} & \gamma_1-\alpha_{32}-\alpha_{23}\\ \gamma_2-\alpha_{31} & \gamma_1-\alpha_{32} & \beta_3-\alpha_{33}\end{pmatrix}, \tag{15}$$

$$\tau_k \equiv \prod_{3\geq l>k}(1-t_{lk})\prod_{m=1}^{3}t_{km}. \tag{16}$$

V. INTEGRAL REPRESENTATION FOR THE 9j COEFFICIENTS

When the boundary values are taken according to Eq. (10), the matrix (β_{ik}) of (15) may become triangular on account of a set of unexpected identities which come about by a judicious arrangement of the elements a_{kl} as done in (3):

$$c_m - a_{ik} - a_{ki} = 0, \quad i,k,m \text{ cyclic}. \tag{17}$$

The net effect of this is to reduce from a general ninefold integral of (14) for $\phi^{(3)}$ to a sixfold integral representation for the 9j coefficient. Thus

$$\{9j\} = KK'\int_0^1\cdots\int\prod_{i\geq k=1}^{3}dt_{ik} \times t_{ik}^{a_{ik}-1}(1-t_{ik})^{b_{ik}-1}\prod_{k=1}^{3}(1-\hat{\tau}_k)^{-a_{k4}}, \tag{18}$$

where

$$K' = K_4/K_5, \tag{19}$$

$$K_4 = \prod_{i=1}^{3}\Gamma(b_i)\Gamma(c_i),$$

$$K_5 = \prod_{i\geq k=1}^{3}\Gamma(a_{ik})\Gamma(b_{ik}),$$

$$(b_{ik}) \equiv \begin{pmatrix} b_1-a_{11} & 0 & 0\\ c_3-a_{21} & b_2-a_{22} & 0\\ c_2-a_{31} & c_1-a_{32} & b_3-a_{33}\end{pmatrix}, \tag{20}$$

$$\hat{\tau}_k \equiv \prod_{3\geq l>k}(1-t_{lk})\prod_{l\leq k}t_{kl}, \quad k=1,2,3. \tag{21}$$

VI. RADON STRUCTURE

From the integral representation (14), we see that the folded (multi-loop-like) products of integration variables appearing in (16) in general would not render the integrand of (14) to be products of linear forms even after appropriate change of variables. This is true even for the boundary values (18) as far as the nondegenerate cases are concerned. By degenerate cases we mean when any one (or more) of the sixteen parameters a_{ik}, b_{ik} $(i \geq k = 1,2,3)$ and a_{k4} vanishes. When that happens, the multiloop structure is broken, and we are back in the more familiar situation of satisfying the Gel'fand criterion.[5] In this regard, we recall an analogous situation in the Radon structure of the multiperipheral versus multiloop (nonplanar) Veneziano functions.[6]

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[1] A. C. T. Wu, J. Math. Phys. **13**, 84 (1972).

[2] See, e.g., G. Racah, Phys. Rev. **62**, 438 (1942); M. E. Rose, *Multipole Fields* (Wiley, New York, 1955), Appendix B.

[3] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. V.

[4] S. J. Ališauskas and A. P. Jucys, J. Math. Phys. **12**, 594 (1971), esp. Appendix; Note Erratum, J. Math. Phys. **13**, 575 (1972).

[5] A. C. T. Wu, J. Math. Phys. **12**, 437 (1971).

[6] A. C. T. Wu, J. Math. Phys. **12**, 2035 (1971), especially the last statement in Sec. 1.