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$SU(4) \supset SU(2) \otimes SU(2)$ Projection Techniques*

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The twofold multiplicity problem associated with the Wigner supermultiplet reduction $SU(4) \supset SU(2) \otimes SU(2)$ is resolved by spin–isospin projection techniques analogous to the angular momentum projection technique introduced by Elliott to resolve the $SU(3) \supset R(3)$ multiplicity problem. The projection quantum numbers, which furnish either an integer or half-integer characterization of the multiplicity, are assigned according to an (ST)-multiplicity formula derived from a consideration of the symmetry properties of spin–isospin degeneracy diagrams. An expression is obtained for the coefficients which relate the $SU(4) \supset SU(2) \otimes SU(2)$ projected basis states to states labeled according to the natural $U(4) \supset U(3) \supset U(2) \supset U(1)$ chain. General expressions for $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients and tensorial matrix elements are given in terms of the corresponding $U(4) \supset U(3) \supset U(2) \supset U(1)$ quantities.

1. INTRODUCTION

In 1937 Wigner[1] pioneered work that established $SU(4)$ as a group of major importance in nuclear structure studies. Its basis, the charge independence of nuclear forces, followed from an observed approximate fourfold degeneracy of nuclear energy levels. The result was the introduction of a nucleon distinguishing isospin quantum number which was combined with that of ordinary spin in the development of a spin–isospin supermultiplet theory. Group-theoretically, it corresponds to a state labeling scheme based upon the spin–isospin reduction $SU(4) \supset SU(2) \otimes SU(2)$.

In general, a complete specification of states in the supermultiplet scheme requires six labels in addition to those of the irreducible representation (IR) of $SU(4)$. The direct product $SU(2) \otimes SU(2)$ provides only four; two additional labels are needed. Techniques that can be used to resolve the multiplicity have been proposed by several authors.[2] In particular, Moshinsky and Nagel[2] have given a recipe for the construction of two operators whose eigenvalues may be used to complete the labeling. Labels obtained in this manner do not, however, exhibit any obvious symmetry properties, nor do they correspond in any way to know quantities of physical interest. In addition, the labels are not necessarily rational numbers.

A mathematically more convenient reduction is the natural or Gel'fand[3] chain $U(4) \supset U(3) \supset U(2) \supset U(1)$. In this case, the IR labels of $U(3)$, $U(2)$, and $U(1)$ provide the required six labels. Unfortunately, the reduction is unphysical. Nevertheless, since calculations are simpler within such a framework, the scheme has been used to calculate quantities of physical interest which depend only upon the IR labels of $SU(4)$. An example in point is that of the $SU(4)$ unitary recoupling coefficients (U functions) given by Hecht and Pang.[4]

The purpose of the present paper is to state and prove the existence of another solution to the $SU(4) \supset SU(2) \otimes SU(2)$ multiplicity problem, one in which the two additional labels are chosen so as to furnish an integer or half-integer characterization of the multiplicity that exhibits spin-isospin symmetry properties. The technique used is one of spin–isospin projection; it parallels closely Elliott's[5] resolution of the multiplicity problem in the $SU(3) \supset R(3)$ reduction. The simplifications associated with the $U(4) \supset U(3) \supset U(2) \supset U(1)$ reduction are incorporated into the scheme via coefficients which relate the projected $SU(4) \supset SU(2) \otimes SU(2)$ basis states to those labeled according to the $U(4) \supset U(3) \supset U(2) \supset U(1)$ chain.

To establish notation, Sec. 2 is devoted to a brief review of $SU(4)$ operator and state labeling techniques. In Sec. 3 a discussion of $SU(4)$ spin–isospin degeneracy diagrams is given, and a new rule for determining the number of occurrences of a spin–isospin pair (ST) in a given IR of $SU(4)$ is derived. In Sec. 4 the projection hypothesis is stated, and the completeness of the states so defined is proved. In Sec. 5 an expression is obtained for the coefficients which relate the projected basis states to those labeled according to the canonical $U(4) \supset U(3) \supset U(2) \supset U(1)$ reduction; general expressions for $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients and tensorial matrix elements in terms of the corresponding $U(4) \supset U(3) \supset U(2) \supset U(1)$ quantities are also given.

2. BASIC NOTATION

A. Infinitesimal Generators

The 16 infinitesimal generators of $U(4)$ are given in terms of nucleon spin-charge creation and annihilation operators by

$$A_{\alpha\beta} = \sum_s a_\alpha^{s\dagger} a_\beta^s, \tag{2.1}$$

where s denotes the full set of space quantum numbers. The $A_{\alpha\beta}$ satisfy the $U(4)$ commutation relations

$$[A_{\alpha\beta}, A_{\rho\sigma}] = \delta_{\beta\rho}A_{\alpha\sigma} - \delta_{\sigma\alpha}A_{\rho\beta}. \quad (2.2)$$

Deletion of the operator $N = \sum_\alpha A_{\alpha\alpha}$ which commutes with the $A_{\alpha\beta}$ leads to a set of 15 infinitesimal generators for the group $SU(4)$. If $\alpha = 1, 2, 3$, and 4 represent the spin–isospin quantum numbers m_s and m_t in the sense

$$|1\rangle = |+\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \quad |2\rangle = |+\tfrac{1}{2}, -\tfrac{1}{2}\rangle,$$
$$|3\rangle = |-\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \quad |4\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \quad (2.3)$$

then the $SU(4)$ generators can be expressed in terms of $SU(4) \supset SU(2) \otimes SU(2)$ tensors as[4]

$$\begin{aligned}
S_0 &= \tfrac{1}{2}(A_{11} - A_{33} + A_{22} - A_{44}),\\
T_0 &= \tfrac{1}{2}(A_{11} - A_{22} + A_{33} - A_{44}),\\
E_{00} &= \tfrac{1}{2}(A_{11} - A_{22} - A_{33} + A_{44}),\\
S_+ &= A_{13} + A_{24}, \qquad S_- = A_{31} + A_{42},\\
T_+ &= A_{12} + A_{34}, \qquad T_- = A_{21} + A_{43}, \quad (2.4)\\
E_{10} &= A_{13} - A_{24}, \qquad E_{-1\,0} = A_{31} - A_{42},\\
E_{01} &= A_{12} - A_{34}, \qquad E_{0\,-1} = A_{21} - A_{43},\\
E_{11} &= A_{14}, \qquad E_{-1\,-1} = A_{41},\\
E_{1\,-1} &= A_{23}, \qquad E_{-1\,1} = A_{32}.
\end{aligned}$$

The commutation properties of $\bar{S}$, $\bar{T}$, and $\bar{E}$ follow from the commutation properties of the $A_{\alpha\beta}$ given by Eq. (2.2).

B. Irreducible Representations

Gel'fand patterns of the type

$$|G\rangle = \left|\begin{matrix} h_{14} & h_{24} & h_{34} & h_{44} \\ & h_{13} & h_{23} & h_{33} \\ & h_{12} & h_{22} & \\ & & h_{11} & \end{matrix}\right\rangle \quad (2.5)$$

furnish a complete set of labels for the basis states of an IR of $U(4)$. The $h_{\alpha\beta}$, $1 \le \alpha \le \beta \le 4$, specify the IR's of $U(\beta)$ in the canonical chain $U(4) \supset U(3) \supset U(2) \supset U(1)$ to which the state belongs. The $h_{\alpha\beta}$ are integral and satisfy the Young tableau or betweenness conditions

$$h_{\alpha\beta} \ge h_{\alpha,\beta-1} \ge h_{\alpha+1,\beta} \ge 0. \quad (2.6)$$

Replacing each $h_{\alpha\beta}$ by $h_{\alpha\beta} - h_{44}$ leads to the corresponding basis state for $SU(4)$; it differs from the $U(4)$ state by at most an h_{44}-dependent phase factor.

Other characterizations for the IR's of $SU(4)$ include the set of three numbers $(\lambda_1\lambda_2\lambda_3)$ given by $\lambda_1 = h_{14} - h_{24}$, $\lambda_2 = h_{24} - h_{34}$, and $\lambda_3 = h_{34} - h_{44}$. $SU(4)$ conjugation properties can then be expressed as relating the $(\lambda_1\lambda_2\lambda_3)$ and $(\lambda_3\lambda_2\lambda_1)$ IR's.[6] Wigner[1] introduced the triplet of numbers $(PP'P'')$ given by $P = \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)$, $P' = \frac{1}{2}(\lambda_1 + \lambda_3)$, and $P'' = \frac{1}{2}(\lambda_1 - \lambda_3)$. They are associated with the maximum eigenvalues for the operators E_{00}, S_0, and T_0 (e.g., P = maximum eigenvalue of E_{00} contained in the IR, P' = maximum eigenvalue of S_0 for states with $E_{00} = P$, and P'' = maximum eigenvalue of T_0 for states with $E_{00} = P$ and $S_0 = P'$).[7] In what follows, simplicity of formulation will determine which labels are used. In all cases the relationships as given above apply.

The states $|G\rangle$ are eigenstates of the operators $A_{\alpha\alpha}$ with eigenvalues w_α,

$$\begin{aligned}
A_{\alpha\alpha}|G\rangle &= w_\alpha |G\rangle,\\
w_\alpha &= \textstyle\sum \text{row}\,\alpha - \sum \text{row}\,(\alpha - 1)\\
&= \sum_\beta h_{\beta\alpha} - \sum_\beta h_{\beta,\alpha-1}. \quad (2.7)
\end{aligned}$$

States of particular interest in the present development are those for which the operator $E_{00} = \frac{1}{2}(A_{11} - A_{22} - A_{33} + A_{44})$ assumes either its (a) maximum $(E_{00}^{\max} = P)$ or (b) minimum $(E_{00}^{\min} = -P)$ eigenvalue. The $h_{\alpha\beta}$ for such states are uniquely specified by K_S and K_T, the eigenvalues of $\frac{1}{2}(A_{11} + A_{22} - A_{33} - A_{44}) = S_0$ and $\frac{1}{2}(A_{11} - A_{22} + A_{33} - A_{44}) = T_0$, respectively. Explicitly,

$$|G_{E\uparrow}\{K_SK_T\}\rangle = \left|\begin{matrix} h_1 & h_2 & h_3 & h_4 \\ & h_1-p & h_3 & h_4 \\ & h_1-p & h_3-q & \\ & & h_1-p & \end{matrix}\right\rangle,$$
$$0 \le p \le \lambda_1,\ 0 \le q \le \lambda_3, \quad (2.8a)$$

$$|G_{E\downarrow}\{K_SK_T\}\rangle = \left|\begin{matrix} h_1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3-q \\ & h_1-p & h_3-q & \\ & & h_3-q & \end{matrix}\right\rangle,$$
$$0 \le p \le \lambda_1,\ 0 \le q \le \lambda_3, \quad (2.8b)$$

where

$$\begin{aligned}
K_S + K_T &= h_1 - h_2 - 2p = \lambda_1 - 2p,\\
K_S - K_T &= h_2 - h_3 - 2q = \lambda_3 - 2q, \quad (2.9a)\\
K_S + K_T &= h_2 - h_3 - 2q = \lambda_3 - 2q,\\
K_S - K_T &= h_1 - h_2 - 2p = \lambda_1 - 2p \quad (2.9b)
\end{aligned}$$

for $|G_{E\uparrow}\rangle$ and $|G_{E\downarrow}\rangle$, respectively.[8] The solid curves in Fig. 2 of Sec. 4 illustrate the result schematically.

Note that for (a) $(\lambda_1\lambda_3)$-(odd, even) K_S and K_T are half-integral with K_S differing from K_T by twice an integer, for (b) $(\lambda_1\lambda_3)$-(odd, odd) K_S and K_T are integral with K_S differing from K_T by twice an integer plus one, for (c) $(\lambda_1\lambda_3)$-(even, odd) K_S and K_T are half-integral with K_S differing from K_T by twice an integer plus one, and for (d) $(\lambda_1\lambda_3)$-(even, even) K_S and K_T are integral with K_S differing from K_T by twice an integer. That is, the odd–even characteristics of λ_1 and λ_3 furnish a complete characterization of distinct symmetry types for the $\{K_S K_T\}$-values associated with the $|G_E\rangle$.

3. SPIN-ISOSPIN MULTIPLICITIES

Racah[9] has given a relatively simple algebraic formula for determining the multiplicity $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ of (ST)-values in an IR $(\lambda_1\lambda_2\lambda_3)$ of $SU(4)$. Some simplifications in his result follow from the investigations of Kretzschmar[10] and Perelomov and Popov.[11] In each case the expressions given are based upon the Littlewood rules[12] which allow $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ to be related to a sum over terms of the type $N_{(ST)}(\lambda_1'\lambda_2'\lambda_3')$, where the IR's $(\lambda_1'\lambda_2'\lambda_3')$ have particularly simple multiplicity structures. In this section an expression for $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ is given which involves a sum over terms of the type $N_{(S'T')}(\lambda_1 0 \lambda_3)$ where the $(S'T')$-values are related to the (ST)-values in a very simple way. Since Racah's expression for $N_{ST}(\lambda_1 0 \lambda_3)$ is quite transparent, the result is particularly convenient for a study of the origin of (ST)-multiplicities and leads quite naturally to a rule for the projection numbers of Sec. 4.

A. Degeneracy Diagrams

A spin–isospin degeneracy diagram for the IR $(\lambda_1\lambda_2\lambda_3)$ of $SU(4)$ is a regular lattice of points (ST) each of which is labeled by the numerical value of $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$, the multiplicity of the pair (ST) in the IR $(\lambda_1\lambda_2\lambda_3)$. Figure 4 of Sec. 4 gives examples. The spin–isospin symmetry property $N_{(ST)}(\lambda_1\lambda_2\lambda_3) = N_{(TS)}(\lambda_1\lambda_2\lambda_3)$ corresponds to reflection symmetry in the $S = T$ plane. The conjugation properties of $SU(4)$ imply that $N_{(ST)}(\lambda_1\lambda_2\lambda_3) = N_{(ST)}(\lambda_3\lambda_2\lambda_1)$. A systematic study of $SU(4)$ spin–isospin degeneracy diagrams can therefore be limited to a consideration of those IR's of $SU(4)$ for which $\lambda_1 \geq \lambda_3$ and within such IR's those (ST)-values for which $S \leq T$.

Figure 1 illustrates features common to all $SU(4)$ spin–isospin degeneracy diagrams. The heavy solid curve $EP(\lambda_1\lambda_2\lambda_3)$ is, in the terminology of Perelomov and Popov,[11] the enveloping polygon for the spin–isospin degeneracy diagram associated with the $(\lambda_1\lambda_2\lambda_3)$ IR of $SU(4)$. It circumscribes all (ST)-values

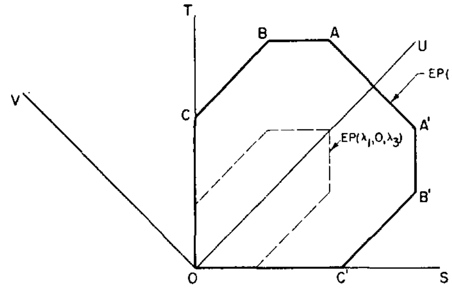

FIG. 1. General features of an $SU(4)$ spin–isospin degeneracy diagram. The heavy solid curve $EP(\lambda_1\lambda_2\lambda_3)$ is the enveloping polygon for the spin–isospin degeneracy diagram associated with the $(\lambda_1\lambda_2\lambda_3)$ IR of $SU(4)$. The (ST)- and $[UV]$-coordinates of the boundary points are given by

$A:(P',P),[Q,+\lambda_2]$; $B:(P'',P),[Q',+Q'']$; $C:(0,Q''),[Q'',+Q'']$,

$A':(P,P'),[Q,-\lambda_2]$; $B':(P,P''),[Q',-Q'']$; $C':(Q'',0),[Q'',-Q'']$,

where $(PP'P'')$ are the Wigner supermultiplet quantum numbers

$$P = \tfrac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3), \quad P' = \tfrac{1}{2}(\lambda_1 + \lambda_3), \quad P'' = \tfrac{1}{2}(\lambda_1 - \lambda_3)$$

and the $(QQ'Q'')$ triplet of numbers is given by

$$Q = \lambda_1 + \lambda_2 + \lambda_3, \quad Q' = \lambda_1 + \lambda_2, \quad Q'' = \lambda_2 + \lambda_3.$$

The dashed curve $EP(\lambda_1 0 \lambda_3)$ is the corresponding result for $\lambda_2 = 0$.

for which $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ is nonzero. The boundary points for the polygon are as given in the figure. The axes $U = T + S$ and $V = T - S$ have been included as a simplifying feature for the discussion that is to follow. The dashed curve $EP(\lambda_1 0 \lambda_3)$ is the corresponding result for $\lambda_2 = 0$. As shown, the figure corresponds to $\lambda_1 + \lambda_3$ even and hence integral (ST)-values. For $\lambda_1 + \lambda_3$ odd and hence half-integral (ST)-values, the schematics are identical, the only difference being that the lines OC and OC' are shifted one-half unit from the coordinate axes.

As can be seen from Fig. 1, $EP(\lambda_1\lambda_2\lambda_3)$ and $EP(\lambda_1 0 \lambda_3)$ are simply related; for $S \leq T$, $EP(\lambda_1\lambda_2\lambda_3)$ corresponds to $EP(\lambda_1 0 \lambda_3)$ shifted λ_2 units along the T axis, and, for $S > T$, $EP(\lambda_1\lambda_2\lambda_3)$ corresponds to $EP(\lambda_1 0 \lambda_3)$ shifted λ_2 units along the S axis. More precisely, $EP(\lambda_1\lambda_2\lambda_3)$ is the envelope of all isosceles right triangles built by λ_2 regular lattice displacements[13] upon the (ST)-values of $EP(\lambda_1 0 \lambda_3)$. Therefore, $EP(\lambda_1 0 \lambda_3)$ is a characteristic structure common to all IR $(\lambda_1\lambda_2\lambda_3)$ (λ_1 and λ_3 fixed; λ_2 arbitrary) of $SU(4)$. Furthermore, note that for $\lambda_2 = 0$ the boundary points B and B' coincide with the boundary points $\{P'', P'\}$ and $\{P', P''\}$ of Fig. 2a (Sec. 4). Therefore, like rule (2.9) for the $\{K_S K_T\}$-values associated with $|G_E\rangle$, a classification scheme based on the odd–even characteristics of the fundamental lengths $U_A - U_C = \lambda_1$ and $V_B - V_A = \lambda_3$ furnishes a complete characterization of distinct $EP(\lambda_1 0 \lambda_3)$ and hence $EP(\lambda_1\lambda_2\lambda_3)$ symmetry types.

The results for $EP(\lambda_1\lambda_2\lambda_3)$ suggest that $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ may be simply related to $N_{(S'T')}(\lambda_1 0\lambda_3)$ and, furthermore, that the classification scheme (a) $(\lambda_1\lambda_3)$-(odd, even), (b) $(\lambda_1\lambda_3)$-(odd, odd), (c) $(\lambda_1\lambda_3)$-(even, odd), and (d) $(\lambda_1\lambda_3)$-(even, even) may furnish a complete characterization of distinct $N_{(ST)}(\lambda_1 0\lambda_3)$ and hence $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ symmetry types. To test the hypothesis, a quantitative study of the numerology of related degeneracy diagrams was made (e.g., see Fig. 4 in Sec. 4). In terms of $N_{[UV]}(\lambda_1\lambda_2\lambda_3) \equiv N_{(ST)}(\lambda_1\lambda_2\lambda_3)$, $U = T + S$, and $V = T - S$, the result of the investigation, with $V \geq 0$, is that

$$N_{[UV]}(\lambda_1\lambda_2\lambda_3) = N_{[UV]}(\lambda_1, \lambda_2 - 1, \lambda_3) + N_{[U'V']}(\lambda_1 0\lambda_3) + \delta_{[UV]}(\lambda_1\lambda_2\lambda_3),$$
$$U' = U - \lambda_2,$$
$$V' = \text{map}\,[V - \lambda_2, |\text{mod}\,(V - \lambda_2, 2)|], \quad (3.1)$$

where $\delta_{[UV]}(\lambda_1\lambda_2\lambda_3) = 0$ for cases (a), (b), and (c) and, for case (d),

$$\begin{aligned}\delta_{[UV]}(\lambda_1\lambda_2\lambda_3) &= 1, \quad \lambda_2 > U \geq V,\ U - \lambda_2 \text{ even},\\ &= -1, \quad \lambda_2 > U \geq V,\ U - \lambda_2 \text{ odd},\\ &= 0, \quad \text{otherwise}.\end{aligned} \quad (3.2)$$

The formula is recursive and therefore may be iterated to yield

$$N_{[UV]}(\lambda_1\lambda_2\lambda_3) = \sum_m N_{[U'V']}(\lambda_1 0\lambda_3),$$
$$U' = U - m, \quad (3.3)$$
$$V' = \max\,[V - m, |\text{mod}\,(V - m, 2)|],$$
$$0 \leq m \leq \lambda_2, \quad m \neq U \quad \text{if} \quad U - \lambda_2 \text{ odd},$$

which is applicable to all four cases (a)–(d). In terms of $N_{(ST)}(\lambda_1 0\lambda_3)$, Eq. (3.3) has the form

$S > T$:

$$\begin{aligned}N_{(ST)}(\lambda_1\lambda_2\lambda_3) &= N_{(ST)}(\lambda_1 0\lambda_3) + N_{(S-1,T)}(\lambda_1 0\lambda_3) + \cdots\\ &\quad + N_{(TT)}(\lambda_1 0\lambda_3) + N_{(T,T-1)}(\lambda_1 0\lambda_3) + N_{(T-1,T-1)}(\lambda_1 0\lambda_3) + \cdots\\ &\quad + N_{(S'T')}(\lambda_1 0\lambda_3),\end{aligned} \quad (3.4a)$$
$$S' + T' = S + T - \lambda_2,$$

$S \leq T$:

$$\begin{aligned}N_{(ST)}(\lambda_1\lambda_2\lambda_3) &= N_{(ST)}(\lambda_1 0\lambda_3) + N_{(S,T-1)}(\lambda_1 0\lambda_3) + \cdots\\ &\quad + N_{(SS)}(\lambda_1 0\lambda_3) + N_{(S,S-1)}(\lambda_1 0\lambda_3) + N_{(S-1,S-1)}(\lambda_1 0\lambda_3) + \cdots\\ &\quad + N_{(S'T')}(\lambda_1 0\lambda_3),\end{aligned} \quad (3.4b)$$
$$S' + T' = S + T - \lambda_2,$$

where $N_{(00)}(\lambda_1 0\lambda_3)$ is *not* included if $S + T - \lambda_2$ is odd. The next section is devoted to an analytic proof of this result.

B. Proof of the Multiplicity Formula

Racah[9] has shown that

$$\begin{aligned}N_{[UV]}(\lambda_1\lambda_2\lambda_3) &= \omega_{[UV]}(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)\\ &\quad - \omega_{[UV]}(\lambda_1 + \lambda_2 + \lambda_3 + 1, \lambda_2 - 1)\\ &\quad - \omega_{[UV]}(\lambda_1 - 1, \lambda_3 - 1),\end{aligned} \quad (3.5)$$

where $\omega_{[UV]}(xy)$ vanishes unless

$$x + y \geq \max\,(U + V, U - V),$$
$$x + y \equiv U + V \equiv U - V\,(\text{mod}\,2),$$

and that, if these conditions are satisfied and $x \geq y$,

$$\begin{aligned}\omega_{[UV]}(xy) = \omega_{[UV]}(yx) &= \varphi(y + 2 - |V|)\\ &\quad - \varphi(y + 1 - U)\\ &\quad + \varphi(U - x + 1)\\ &\quad - \tfrac{1}{2}\varphi(U - |V| - x + y + 1).\end{aligned} \quad (3.6)$$

The function $\varphi(z)$ is given by

$$\begin{aligned}\varphi(z) &= [z^2/4], \quad z \geq 0,\\ &= 0, \quad z < 0,\end{aligned} \quad (3.7)$$

where the boldface brackets indicate the greatest integer contained in the argument.

Define

$$\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) = N_{[UV]}(\lambda_1\lambda_2\lambda_3) - N_{[UV]}(\lambda_1, \lambda_2 - 1, \lambda_3), \quad (3.8a)$$
$$\Delta\omega_{[UV]}(xy) = \omega_{[UV]}(xy) - \omega_{[UV]}(x - 1, y - 1), \quad (3.8b)$$
$$\Delta\varphi(z) = \varphi(z) - \varphi(z - 1). \quad (3.8c)$$

Then, to prove Eq. (3.1), it is sufficient to demonstrate the equivalence of

$$\begin{aligned}\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) &= \Delta\omega_{[UV]}(\lambda_1+\lambda_2, \lambda_2+\lambda_3) \\ &\quad - \Delta\omega_{[UV]}(\lambda_1+\lambda_2+\lambda_3+1, \lambda_2-1) \\ &= \Delta\varphi(\lambda_2+\lambda_3+2-V) - \Delta\varphi(\lambda_2+\lambda_3+1-U) \\ &\quad + \Delta\varphi(U-\lambda_1-\lambda_2-1) - \Delta\varphi(\lambda_2+1-V) \\ &\quad + \Delta\varphi(\lambda_2-U) - \Delta\varphi(U-\lambda_1-\lambda_2-\lambda_3) \quad (3.9)\end{aligned}$$

and

$$\begin{aligned}N_{[U'V']}(\lambda_1 0\lambda_3) &= \Delta\omega_{[U'V']}(\lambda_1\lambda_3) \\ &= \Delta\varphi(\lambda_3+2-V') - \Delta\varphi(\lambda_3+1-U') \\ &\quad + \Delta\varphi(U'-\lambda_1-1). \quad (3.10)\end{aligned}$$

For $(\lambda_1\lambda_3)$-(even, even) and $\lambda_2 > U \geq V$, the factor $\delta_{[UV]}(\lambda_1\lambda_2\lambda_3)$ must, of course, be added to

$$N_{[U'V']}(\lambda_1 0\lambda_3).$$

Consider the following special cases:

Case 1: $U \geq V \geq \lambda_2$.
Case 2: $U \geq \lambda_2 \geq V$:
(a) $V - \lambda_2 = -2n$,
(b) $V - \lambda_2 = -2n - 1$.
Case 3: $\lambda_2 > U \geq V$:
(a) $U = V + 2n + 1$:
(1) $(\lambda_1\lambda_3)$-(odd, even),
(2) $(\lambda_1\lambda_3)$-(even, odd);
(b) $U = V + 2n$:
(1) $(\lambda_1\lambda_3)$-(odd, odd),
(2) $(\lambda_1\lambda_3)$-(even, even).

For case 1 the result is trivial since $U' = U - \lambda_2$, $V' = V - \lambda_2$ makes $\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3)$ and $N_{[U'V']}(\lambda_1 0\lambda_3)$ identical functions in φ. In both (a) and (b) of case 2 an application of the result $\Delta\varphi(m+2n) = \Delta\varphi(m) + n$, m, n integer, leads to the desired conclusion. Case 3 is somewhat more complicated because $U' = U - \lambda_2 < 0$ implies that $N_{[U'V']}(\lambda_1 0\lambda_3) = 0$. In this case it is therefore necessary to demonstrate the equivalence of $\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3)$ and $\delta_{[UV]}(\lambda_1\lambda_2\lambda_3)$. The substitution $\lambda_2 - U = 2m + \delta$ and $\lambda_2 + 1 - V = 2n + \nu$, m, n integer and μ, ν being 0 or 1, simplifies $\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3)$ to

$$\begin{aligned}\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) &= \Delta\varphi(\lambda_3+1+\nu) \\ &\quad - \Delta\varphi(\lambda_3+1+\mu). \quad (3.11)\end{aligned}$$

For 3(a) $\mu = \nu$ so that $\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) = 0$. For 3(b) $\mu \neq \nu$, but the substitution $\lambda_3 + 1 = 2k + \kappa$, k integer and κ being 0 or 1, leads to

$$\begin{aligned}\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) &= \Delta\varphi(\nu) - \Delta\varphi(\mu) \\ &= 0 \quad (3.12)\end{aligned}$$

for (b1) and

$$\begin{aligned}\Delta N_{[UV]}(\lambda_1\lambda_2\lambda_3) &= \Delta\varphi(\nu+1) - \Delta\varphi(\mu+1) \\ &= \begin{cases} 1, & U - \lambda_2 \text{ even}, \quad \mu = 0, \nu = 1 \\ -1, & U - \lambda_2 \text{ odd}, \quad \mu = 1, \nu = 0 \end{cases}, \quad (3.13)\end{aligned}$$

for (b2), which is the desired result.

4. SPIN-ISOSPIN PROJECTION

The additional quantum numbers that are required to resolve the twofold multiplicity associated with the reduction $SU(4) \supset SU(2) \otimes SU(2)$ may be chosen in a variety of ways. The solution proposed by Moshinsky and Nagel[2] is not necessarily the most convenient because of the algebraic diffculties inherent with the corresponding eigenvalue problem. In this section the existence of another solution to the multiplicity problem is stated and proved. It is based upon spin–isospin projection techniques in which the $\{K_S K_T\}$-pairs associated with the states $|G_E\rangle$ furnish the required labels.

A. Projection Hypothesis

A projection operator for a state of total angular momentum J with projection M may be expressed in Hill–Wheeler integral form[14] as

$$P^J_{MK} = (2J+1)\int d\Omega D^{J*}_{MK}(\Omega) R_J(\Omega), \quad (4.1)$$

where $D^J_{MK}(\Omega)$ is an $R(3)$ rotation matrix and $R_J(\Omega)$ is an $R(3)$ rotation operator,

$$\begin{aligned}R_J(\Omega) &= e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}, \\ [J_1, J_2] &= iJ_3 \quad \text{(cyclic)}. \quad (4.2)\end{aligned}$$

The integration is over Euler angles $(\alpha\beta\gamma)$. From this definition it follows that

$$P^{J'}_{M'K'} P^J_{MK} = \delta_{J'J}\delta_{K'M} P^J_{M'K}, \quad (4.3)$$

$$P^{J\dagger}_{MK} = P^J_{KM}, \quad (4.4)$$

where $P^{J\dagger}_{MK}$ indicates the Hermitian conjugate of P^J_{MK}. Cases of interest in the present analysis are those for which J is either the spin S or the isospin T of Eq. (2.4).

Since eigenstates of the total spin and isospin operators may be obtained from a state $|G\rangle$ by simply applying the projection operators $P^S_{M_SK_S}$ and $P^T_{M_TK_T}$, we define

$$|GK_SSM_SK_TTM_T\rangle \equiv P^S_{M_SK_S}P^T_{M_TK_T}|G\rangle. \quad (4.5)$$

The complete G symbol has been retained in the projected ket as a reminder of the Gel'fand state from which it was derived; only the IR labels $h_{\alpha 4}$, however, remain valid state labels. In many cases the $|GK_SSM_SK_TTM_T\rangle$ will turn out to be identically zero. It remains to specify the $|G\rangle$ and pairs $\{K_SK_T\}$ with their corresponding (ST)-values for which projected states span the IR space.

The Projection Hypothesis

The projected states

$$|G_EK_SSM_SK_TTM_T\rangle \equiv P^S_{M_SK_S}P^T_{M_TK_T}|G_E\rangle, \quad (4.6)$$

with $|G_E\rangle$ the Gel'fand states for which the operator E_{00} assumes its maximum $(\lambda_1 \geq \lambda_3)$ or minimum $(\lambda_1 < \lambda_3)$ eigenvalue, span the $(\lambda_1\lambda_2\lambda_3)$ IR space of $SU(4)$ if with each integer $(\lambda_1 + \lambda_3$ even) or half-integer $(\lambda_1 + \lambda_3$ odd) pair $\{K_SK_T\}$ satisfying

$$\begin{aligned} &K_S + K_T = \max(\lambda_1\lambda_3) - 2p,\\ &K_S - K_T = \min(\lambda_1\lambda_3) - 2q,\\ &\quad 0 \leq p \leq [\max(\lambda_1\lambda_3)/2],\\ &\quad \kappa \leq q \leq \min(\lambda_1\lambda_3),\\ &\qquad \kappa = 0, \quad K_S + K_T \neq 0,\\ &\qquad \kappa = [\min(\lambda_1\lambda_3)/2], \quad K_S + K_T = 0, \end{aligned} \quad (4.7)$$

is associated the (ST)-values

$$\begin{aligned} \sigma > \tau: \quad & (ST) = (\sigma + \mu, \tau + \nu),\\ & 0 \leq \mu \leq \lambda_2,\\ & 0 \leq \nu < \sigma - \tau + \lambda_2 - \mu, \end{aligned} \quad (4.8a)$$

$$\begin{aligned} \sigma \leq \tau \neq 0: \quad & (ST) = (\sigma + \mu, \tau + \nu),\\ & 0 \leq \nu \leq \lambda_2,\\ & 0 \leq \mu \leq \tau - \sigma + \lambda_2 - \nu, \end{aligned} \quad (4.8b)$$

$$\begin{aligned} \sigma = \tau = 0: \quad & (ST) = (\lambda_2 - 2\mu - \nu, \nu),\\ & 0 \leq \mu \leq [\lambda_2/2],\\ & 0 \leq \nu \leq \lambda_2 - 2, \end{aligned} \quad (4.8c)$$

where $\sigma = |K_S|$ and $\tau = |K_T|$. The projections M_S and M_T assume the usual values $-S \leq M_S \leq S$ and $-T \leq M_T \leq T$.

The proof of the hypothesis will be made in two steps. First, the value of $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$ predicted by the rule will be shown to be precisely that derived in Sec. 3. And, secondly, the assumption that there exists a function belonging to the IR space but orthogonal to the projected states will be shown to lead to a contradiction. Before proceeding, however, we first consider in more detail the structure of the rule as given by Eqs. (4.7) and (4.8).

Since the Gel'fand states $|G_E\rangle$ are eigenstates of S_0 and T_0, the $\{K_SK_T\}$-pairs of Eq. (4.7) are necessarily a subset of the allowed $\{K_SK_T\}$-pairs given by Eq. (2.9). The choice made (see Fig. 2) is not, however, unique; other possibilities exist. For example, simply replacing each $\{K_SK_T\}$-pair of Eq. (4.7) by $\{-K_S, -K_T\}$ (inversion in the $\{K_SK_T\}$-plane) provides an equally acceptable set of projection numbers. It is also true that any partial inversion in the $\{K_SK_T\}$-plane provides an acceptable set of projection numbers. The essential feature of any such choice is that only one of the pairs, $\{K_SK_T\}$ or its inversion $\{-K_S, -K_T\}$, be included. Inclusion of both pairs leads to states which are not linearly independent. The choice made by Eq. (4.7) is therefore one of convention; its simplifying feature is that it maximizes the number of $\{K_SK_T\}$-pairs contained within $EP(\lambda_10\lambda_3)$.

In some applications it is convenient to know the rule corresponding to Eq. (4.7) for projection from $|G_{E\uparrow}\rangle$ if $\lambda_1 < \lambda_3$ and from $|G_{E\downarrow}\rangle$ if $\lambda_1 \geq \lambda_3$. It can be obtained from Eq. (4.7) by simply interchanging the max–min specifications. It follows that the rules for determining the $\{K_SK_T\}$-pairs for projection from $|G_{E\uparrow}\rangle$ and $|G_{E\downarrow}\rangle$ without regard to the relationship of λ_1 and λ_3 are given by the following:

projection from $|G_{E\uparrow}\rangle$:

$$\begin{aligned} &K_S + K_T = \lambda_1 - 2p,\\ &K_S - K_T = \lambda_3 - 2q,\\ &\quad 0 \leq p \leq [\lambda_1/2],\\ &\quad \kappa \leq q \leq \lambda_3,\\ &\qquad \kappa = 0, K_S + K_T \neq 0,\\ &\qquad \kappa = [\lambda_3/2], K_S + K_T = 0; \end{aligned} \quad (4.9a)$$

projection from $|G_{E\downarrow}\rangle$:

$$\begin{aligned} &K_S + K_T = \lambda_3 - 2q,\\ &K_S - K_T = \lambda_1 - 2p,\\ &\quad 0 \leq q \leq [\lambda_3/2],\\ &\quad \kappa \leq p \leq \lambda_1,\\ &\qquad \kappa = 0, K_S + K_T \neq 0,\\ &\qquad \kappa = [\lambda_1/2], K_S + K_T = 0. \end{aligned} \quad (4.9b)$$

Figure 2 illustrates the result schematically. The dashed curves ($K_S + K_T = 0$ not allowed) and the broken curves ($K_S + K_T = 0$ allowed) divide the $\{K_SK_T\}$-pairs of Eq. (2.9) into two sets equivalent under

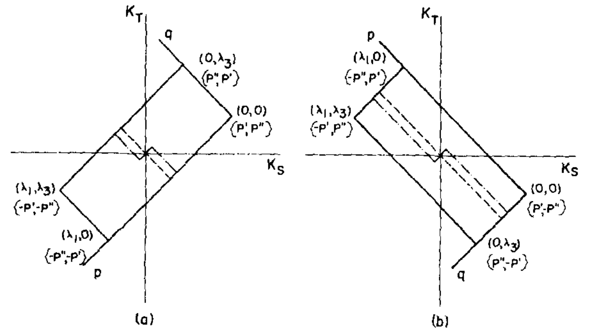

FIG. 2. The envelope of $\{K_SK_T\}$-pairs associated with $|G_E\rangle$. (a) $|G_E\rangle = |G_E\uparrow\rangle$, (b) $|G_E\rangle = |G_E\downarrow\rangle$. The boundaries are denoted by their (pq)- and $\{K_SK_T\}$-values. The dashed curves ($K_S + K_T = 0$ not allowed) and the broken curves ($K_S + K_T = 0$ allowed) divide the $\{K_SK_T\}$-pairs into two sets equivalent under inversion; the pairs for which $K_S + K_T \geq 0$ are by convention the projection numbers of Eq. (4.9).

inversion; the pairs for which $K_S + K_T \geq 0$ are by convention the projection numbers of Eq. (4.9). In any case the spectrum of (ST)-values given by Eq. (4.8) depends only upon σ and τ and is therefore independent of the $\{K_SK_T\}$-rule chosen as long as all $\{K_SK_T\}$-pairs belonging to the Gel'fand state $|G_E\rangle$ under consideration, but not equivalent under inversion, are included in the rule specification.

Figure 3 illustrates Eq. (4.8) by giving the spectrum of (ST)-values associated with a given $\{K_SK_T\}$-pair for the cases $\sigma < \tau$, $\sigma = \tau = \sigma'$, and $\sigma = \tau = 0$. The schematics of the figure are such that the (ST)-values labeled by the same symbol are those derived from the same $\{K_SK_T\}$-pair. In the examples shown, $\lambda_2 = 4$. For $\sigma < \tau$, both $\{K_SK_T\} = \{\sigma\tau\}$ and $\{K_SK_T\} = \{\tau\sigma\}$ have been given. In the case $\sigma < \tau$, note that except for $(ST) = (\tau + \lambda_2 - \nu, \tau + \nu)$, $0 \leq \nu \leq \lambda_2$, for each $(ST)_{\{K_SK_T\}}$ (labeled by +) there exists the transpose set $(TS)_{\{K_TK_S\}}$ (labeled by ○). The asymmetry can be removed for λ_2 odd by relating $(ST) = (\tau + \lambda_2 - \nu, \tau + \nu)$, $0 \leq \nu \leq [\lambda_2/2]$, to $\{\sigma\tau\}$ and $(ST) = (\tau + \lambda_2 - \nu, \tau + \nu)$, $[\lambda_2/2] + 1 \leq \nu \leq \lambda_2$, to $\{\tau\sigma\}$. For λ_2 even, however, the asymmetry associated with $(ST) = (\tau + \frac{1}{2}\lambda_2, \tau + \frac{1}{2}\lambda_2)$ cannot be removed. The choice made by Eqs. (4.8) is therefore again one of convention. Its simplifying feature is manifest in the form of Eqs. (4.8a) and (4.8b). For $\sigma = \tau = \sigma'$, an asymmetry only exists if $\{K_SK_T\} = \{-\sigma', \sigma'\}$. It is related to the fact that the transpose of $(ST)_{\{K_S, -K_S\}}$ is not allowed because $\{-K_S, K_S\}$ is related to $\{K_S, -K_S\}$ by inversion. The singularity of the point $\{K_SK_T\} = \{00\}$ is manifest in the form of Eq. (4.8c).

The eight degeneracy diagrams of Fig. 4 illustrate in complete detail the result of associating (ST)-values as prescribed by Eqs. (4.8) with the $\{K_SK_T\}$-pairs defined by Eqs. (4.7). The examples shown correspond to symmetry types (a) $(\lambda_1\lambda_3)$-(odd, even), (b) $(\lambda_1\lambda_3)$-(odd, odd), (c) $(\lambda_1\lambda_3)$-(even, odd), (d) $(\lambda_1\lambda_3)$-(even, even) for two cases, λ_2 zero and λ_2 such that the degeneracy of $S = T = P'$ is a maximum. On each degeneracy diagram the $\{K_SK_T\}$-lattice corresponding to Eqs. (4.7) is given in outline form. Note that for symmetry types (a) and (b) the $\{K_SK_T\}$-lattices are rectangular ($K_S + K_T = 0$ not allowed). The corresponding degeneracy diagrams reflect a maximum degree of regularity. For symmetry types (c) and (d) the $\{K_SK_T\}$-lattices are not rectangular ($K_S + K_T = 0$ allowed). Nevertheless, since symmetry type (c) is equivalent to symmetry type (a) under conjugation ($\lambda_1\lambda_3$ interchange), degeneracy diagrams of type (c) also possess a maximum degree of regularity. For symmetry type (d), however, the singularity of the point $\{K_SK_T\} = \{00\}$ is an inherent feature which propagates an irregularity into the multiplicities of the (ST)-values associated with $(ST) = (00)$ by $\lambda \leq \lambda_2$ regular lattice displacements.

B. Completeness of the Projected States

First of all, consider the multiplicity $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3)$ of (ST)-values predicted by Eqs. (4.8). As can be seen from Fig. 3, the basic structure of the rule is one of triangulation. That is, the (ST)-values associated with each $\{K_SK_T\}$-pair for $\lambda_2 \geq 0$ are simply those

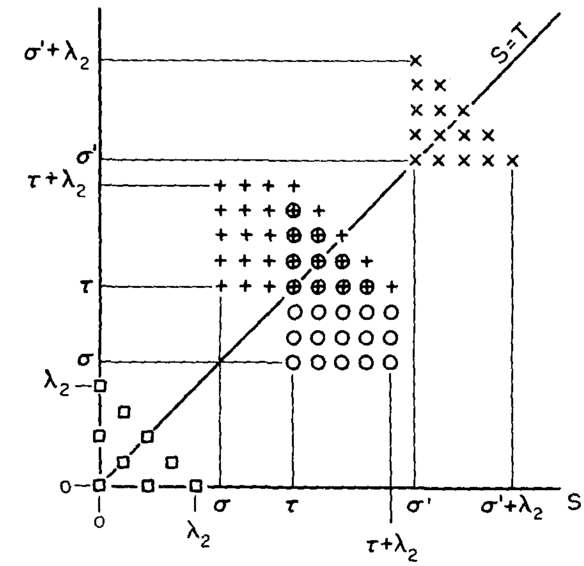

FIG. 3. Spectrum of (ST)-values associated with the projection numbers $\{K_SK_T\}$:

$$\{K_SK_T\} = \{\sigma\tau\}: +, \quad \{K_SK_T\} = \{\tau\sigma\}: \bigcirc,$$
$$\{K_SK_T\} = \{\sigma'\sigma'\}: \times, \quad \{K_SK_T\} = \{0, 0\}: \square.$$

In the examples shown, $\lambda_2 = 4$.

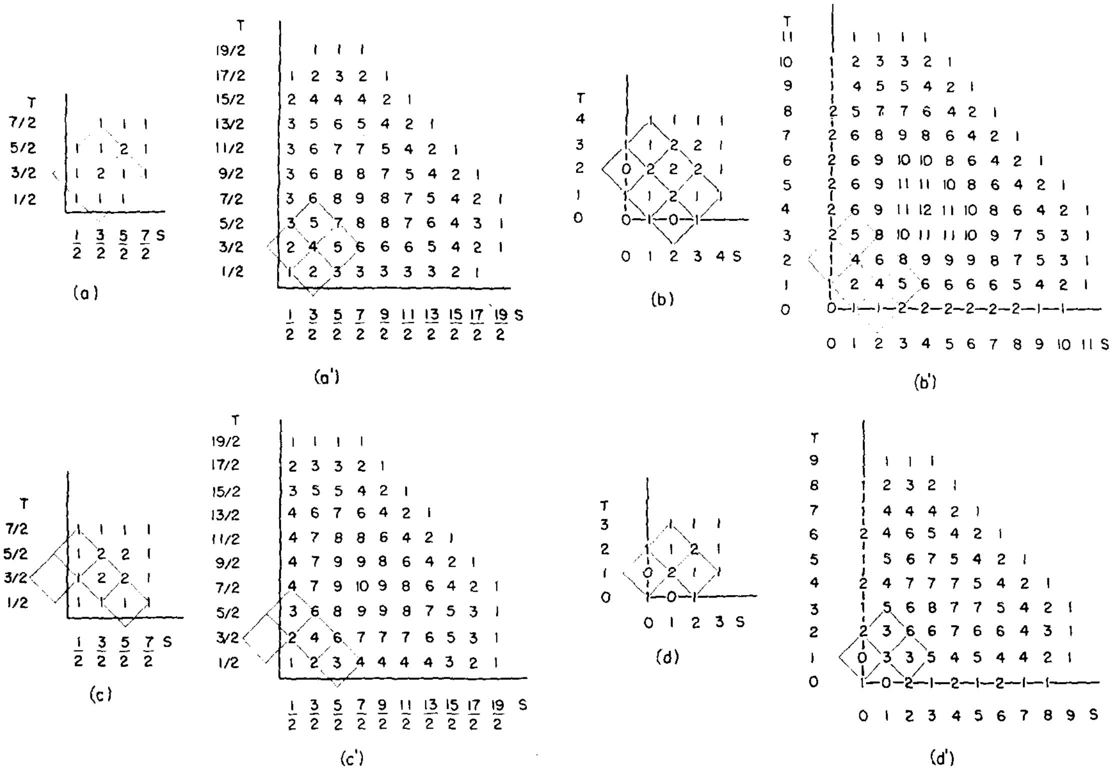

FIG. 4. Spin–isospin degeneracy diagrams for the $(\lambda_1\lambda_2\lambda_3)$ IR of $SU(4)$. (a) $N_{(ST)}(5, 0, 2)$, (b) $N_{(ST)}(5, 0, 3)$, (c) $N_{(ST)}(4, 0, 3)$, (d) $N_{(ST)} \times (4, 0, 2)$, (a′) $N_{(ST)}(5, 6, 2)$, (b′) $N_{(ST)}(5, 7, 3)$, (c′) $N_{(ST)}(4, 6, 3)$, (d′) $N_{(ST)}(4, 6, 2)$. The $\{K_SK_T\}$-lattices given by Eq. (4.7) are included in outline form. The value of λ_2 in (a′), (b′), (c′), and (d′) corresponds to a maximum value for the degeneracy of $S = T = P'$.

(ST)-values contained within the envelope of isosceles right triangles built by λ_2 regular lattice displacements from the (ST)-values associated with $\{K_SK_T\}$ for $\lambda_2 = 0$. The one exception, $\{K_SK_T\} = \{00\}$, admits only the subset of these (ST)-values for which $S + T$ differs from λ_2 by twice an integer ($U - \lambda_2$ even). It therefore follows that the $\{K_SK_T\}$-pairs that contribute to $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3)$ are the $\{K_SK_T\}$-pairs that contribute to the $N^R_{(S'T')}(\lambda_10\lambda_3)$ related to $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3)$ in the same way as the $N_{(S'T')}(\lambda_10\lambda_3)$ are related to $N_{(ST)}(\lambda_1\lambda_2\lambda_3)$. That is, $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3)$ satisfies Eqs. (3.4). It remains to prove that

$$N^R_{(ST)}(\lambda_10\lambda_3) = N_{(ST)}(\lambda_10\lambda_3).$$

Consider Eqs. (4.8) for the special case $\lambda_2 = 0$:

$$\sigma > \tau\colon\ (ST) = (\sigma, \tau + \nu), \quad 0 \le \nu < \sigma - \tau; \tag{4.10a}$$

$$\sigma \le \tau\colon\ (ST) = (\sigma + \mu, \tau), \quad 0 \le \mu \le \tau - \sigma. \tag{4.10b}$$

Then $N^R_{(ST)}(\lambda_10\lambda_3)$ is equal to the number of $\{K_SK_T\}$-pairs given by Eqs. (4.7) for which (ST) is contained in the set given by Eqs. (4.10):

$$S > T\colon\quad N^R_{(ST)}(\lambda_10\lambda_3) = \text{number of } \{K_SK_T\}\text{-pairs for which } \sigma = S,\ \ \tau \le T; \tag{4.11a}$$

$$S \le T\colon\quad N^R_{(ST)}(\lambda_10\lambda_3) = \text{number of } \{K_SK_T\}\text{-pairs for which } \sigma \le S,\ \ \tau = T. \tag{4.11b}$$

The algebraic formulation is straightforward; it leads directly to the result that $N^R_{(ST)}(\lambda_10\lambda_3) = N_{(ST)}(\lambda_10\lambda_3)$ and hence $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3) = N_{(ST)}(\lambda_1\lambda_2\lambda_3)$. On the degeneracy diagrams of Fig. 4 the $\{K_SK_T\}$-lattices corresponding to Eqs. (4.7) have been included. By using Eqs. (4.11) the result can be verified for each of the four cases (a) $(\lambda_1\lambda_3)$-(odd, even), (b) $(\lambda_1\lambda_3)$-(odd, odd), (c) $(\lambda_1\lambda_3)$-(even, odd), and (d) $(\lambda_1\lambda_3)$-(even, even).

To complete the proof of the projection hypothesis, an adaptation of the method first given by Elliott[5] for the $SU(3) \supset R(3)$ reduction and subsequently used by Williams and Pursey[15] in considering the $R(5) \supset R(3)$ reduction problem will be used. It proceeds by *reductio ad absurdum*. That is, the consequence of assuming that the projected states do not span the IR space is shown to be a contradiction. Explicitly, suppose there exists a function $|\varphi(S'M'_ST'M'_T)\rangle$ belonging to the IR but orthogonal to all the $|G_EK_SSM_SK_TTM_T\rangle$,

$$\langle\varphi(S'M'_ST'M'_T) \mid G_EK_SSM_SK_TTM_T\rangle = 0. \quad (4.12)$$

Since $N^R_{(ST)}(\lambda_1\lambda_2\lambda_3) = N_{(ST)}(\lambda_1\lambda_2\lambda_3)$, the only nontrivial implications of such an assumption are those which follow for $S' = S$, $M'_S = M_S$, $T' = T$, and $M'_T = M_T$, namely,

$$\begin{aligned}\langle\varphi(SM_STM_T) \mid G_E\{K_SK_T\}\rangle &= \langle P^S_{M_SM_S}P^T_{M_TM_T}\varphi(SM_STM_T) \mid G_E\{K_SK_T\}\rangle \\ &= \langle\varphi(SM_STM_T| \, P^{S\dagger}_{M_SM_S}P^{T\dagger}_{M_TM_T} \, |G_E\{K_SK_T\}\rangle \\ &= \langle\varphi(SM_STM_T)| \, P^S_{M_SM_S}P^T_{M_TM_T} \, |G_E\{K_SK_T\}\rangle \\ &= \delta_{M_SK_S}\delta_{M_TK_T} \\ &\quad \times \langle\varphi(SM_STM_T) \mid G_EK_SSM_SK_TTM_T\rangle = 0. \end{aligned} \quad (4.13)$$

As is shown below, Eq. (4.13) implies that

$$\langle\varphi(SM_STM_T)| \, \mathcal{O} \, |G_E\{K_SK_T\}\rangle = 0, \quad (4.14)$$

where $\mathcal{O}$ is an arbitrary element of $SU(4)$. But, by definition of irreducibility, functions of the type $\mathcal{O} \, |G_E\{K_SK_T\}\rangle$ span the IR space. Hence a contradiction exists; the hypothesis that there exists a function $|\varphi(SM_STM_T)\rangle$ belonging to the IR which is orthogonal to all the $|G_EK_SSM_SK_TTM_T\rangle$ is false. It follows that the $|G_EK_SSM_SK_TTM_T\rangle$ span the IR space.

The argument given above hinges upon a proof that Eq. (4.13) implies Eq. (4.14). For this, note that the operator $\mathcal{O}$ being an element of $SU(4)$ implies that it can be expressed as a power series in the generators of the group. Furthermore, note that the commutation properties of the generators imply that the order of the generators within each term of such an expansion can be chosen in any desired manner. Then we define

$$\begin{aligned}\xi^1_\pm &= \tfrac{1}{2}(S_\pm + E_{\pm1\,0}), \\ \xi^2_\pm &= \tfrac{1}{2}(S_\pm - E_{\pm1\,0}), \\ \eta^1_\pm &= \tfrac{1}{2}(T_\pm + E_{0\,\pm1}), \\ \eta^2_\pm &= \tfrac{1}{2}(T_\pm - E_{0\,\pm1}),\end{aligned} \quad (4.15)$$

and consider the case of projection from $|G_{E\uparrow}\rangle$. It is convenient to divide the generators into the two sets

$$\begin{aligned}\text{A:} \quad E_{00} &= \tfrac{1}{2}(A_{11} - A_{22} - A_{33} + A_{44}), \\ S_0 &= \tfrac{1}{2}(A_{11} + A_{22} - A_{33} - A_{44}), \\ T_0 &= \tfrac{1}{2}(A_{11} - A_{22} + A_{33} - A_{44}), \\ E_{11} &= A_{14}, \quad E_{-1\,-1} = A_{41}, \end{aligned} \quad (4.16a)$$

$$\begin{aligned} E_{1\,-1} &= A_{23}, \quad E_{-1\,1} = A_{32}, \\ \xi^1_+ &= A_{13}, \quad \xi^2_- = A_{42}, \\ \eta^1_+ &= A_{12}, \quad \eta^2_- = A_{43}, \\ \text{B:} \quad S_+ &= A_{13} + A_{24}, \quad S_- = A_{31} + A_{42}, \\ T_+ &= A_{12} + A_{34}, \quad T_- = A_{21} + A_{43}. \end{aligned} \quad (4.16b)$$

When a generator of the set A operates on $|G_{E\uparrow}\rangle$, the result is either another intrinsic state of the same type $(E_{00}, S_0, T_0, E_{11}, E_{-1\,-1}, E_{1\,-1}, E_{-1\,1})$ or zero $(\xi^1_+, \xi^2_-, \eta^1_+, \eta^2_-)$. Generators of the set B do not reproduce intrinsic states but are operators which act only in the direct product space $SU(2) \otimes SU(2)$. Express $\mathcal{O}$ in the form

$$\mathcal{O} = \sum_\alpha C_\alpha \pi_{B_\alpha}\pi_{A_\alpha}, \quad (4.17)$$

where the C_α are constants and π_{A_α} and π_{B_α} are products of generators of the type A and B, respectively. Then consider

$$\begin{aligned}\langle\varphi(SM_STM_T)| \, \mathcal{O} \, |G_E\{K_SK_T\}\rangle = \sum_\alpha C_\alpha \langle\varphi(SM_STM_T)| \, \pi_{B_\alpha}\pi_{A_\alpha} \, |G_E\{K_SK_T\}\rangle.\end{aligned} \quad (4.18)$$

Each factor π_{A_α} acting to the right changes at most K_S and K_T, and the π_{B_α} factors acting to the left change at most M_S and M_T. Therefore,

$$\begin{aligned}\langle\varphi(SM_STM_T)| \, \mathcal{O} \, |G_E\{K_SK_T\}\rangle &= \sum_\alpha C_\alpha \langle\varphi(SM_STM_T)| \, \pi_{B_\alpha}\pi_{A_\alpha} \, |G_E\{K_SK_T\}\rangle \\ &= \sum_{\alpha M'_SM'_TK'_SK'_T} C'_\alpha \langle\varphi(SM'_STM'_T \mid G_E\{K'_SK'_T\}\rangle = 0.\end{aligned} \quad (4.19)$$

The equivalent proof for the case of projection from $|G_{E\downarrow}\rangle$ follows by merely replacing the ξ^1_+, ξ^2_-, η^1_+, η^2_- operators of set A by the operators ξ^1_-, ξ^2_+, η^1_-, η^2_+.

5. TRANSFORMATION BRACKETS

Although the projection numbers $\{K_SK_T\}$ furnish an integral or half-integral solution exhibiting spin–isospin symmetry properties for the $SU(4) \supset SU(2) \otimes SU(2)$ multiplicity problem, the projected states are not normalized nor are they necessarily orthogonal on the K_S and K_T labels. The difficulties associated with the nonorthonormality of the projected states can be resolved, however, if an expression for the coefficients (transformation brackets) which relate the projected states to the orthonormal Gel'fand basis vectors is known. This section is devoted to deriving such an expression. The method used is similar to that developed in Ref. 16, where the analogous problem in the $SU(3) \supset R(3)$ reduction was considered; it is based on the results of Moshinsky and Chacón[17] for the matrix elements of the permutations (1, 2), (2, 3), and (3, 4) between the $U(4)$ basis states $|G\rangle$.

A. The Expression

Since the Gel'fand basis vectors $|G\rangle$ for a given IR of $U(4)$ form an orthonormal set which spans the representation space, an arbitrary projected state $|GK_SSM_SK_TTM_T\rangle$ belonging to the IR may be expanded in terms of the $|G\rangle$ as

$$|GK_SSM_SK_TTM_T\rangle = \sum_{G'} \langle G' \mid GK_SSM_SK_TTM_T\rangle \, |G'\rangle, \quad (5.1)$$

where it is to be understood that $h'_{\alpha 4} = h_{\alpha 4}$. The $\langle G' \mid GK_SSM_SK_TTM_T\rangle$ in Eq. (5.1) are the transformation brackets which relate the $U(4) \supset SU(2) \otimes SU(2)$ scheme of Sec. 4 to the Gel'fand $U(4) \supset U(3) \supset U(2) \supset U(1)$ scheme. By definition of the projected states, we have

$$\begin{aligned}\langle G' \mid GK_SSM_SK_TTM_T\rangle &= \langle G'| P^S_{M_SK_S} P^T_{M_TK_T} |G\rangle \\ &= (2S+1)\int d\Omega_S D^{S*}_{M_SK_S}(\Omega)(2T+1) \\ &\quad \times \int d\Omega_T D^{T*}_{M_TK_T}(\Omega_T)\langle G'|\, R_S(\Omega_S)R_T(\Omega_T)\, |G\rangle.\end{aligned} \quad (5.2)$$

Therefore, an expression for the

$$\langle G' \mid GK_SSM_SK_TTM_T\rangle$$

can be obtained if the matrix elements

$$\langle G'|\, R_S(\Omega_S)R_T(\Omega_T)\, |G\rangle$$

are known. Note that the inverse of the transformation matrix defined by Eq. (5.1) is only guaranteed to exist if the $|GK_SSM_SK_TTM_T\rangle$ are restricted to the projected basis vectors $|G_EK_SSM_SK_TTM_T\rangle$ defined in Sec. 4 by the projection hypothesis. An expression for the $\langle G' \mid G_EK_SSM_SK_TTM_T\rangle$ follows as a special case of the general result for $\langle G' \mid GK_SSM_SK_TTM_T\rangle$.

For notational convenience let

$$|G\rangle = \left| \begin{matrix} h_1 & h_2 & h_3 & h_4 \\ x & y & z & \\ p & q & & \\ r & & & \end{matrix} \right\rangle . \quad (5.3)$$

The infinitesimal generators of $SU(2)$ corresponding to $U(2)$ in the chain $U(4) \supset U(3) \supset U(2) \supset U(1)$ are given by

$$J_+ = A_{12}, \quad J_- = A_{21}, \quad J_0 = \tfrac{1}{2}(A_{11} - A_{22}), \quad (5.4)$$

where

$$J_\pm = J_1 \pm iJ_2. \quad (5.5)$$

Then, for

$$\mathcal{R}(\Omega) = e^{-i\alpha J_0} e^{-i\beta J_2} e^{-i\gamma J_0}, \quad (5.6)$$

it follows that

$$\begin{aligned}\langle G'|\, \mathcal{R}(\Omega)\, |G\rangle &= \delta_{x'x}\delta_{y'y}\delta_{z'z}\delta_{p'p}\delta_{q'q} D^j_{m'm}(\Omega), \\ j = \tfrac{1}{2}(p-q), &\quad m = r - \tfrac{1}{2}(p+q), \\ m' &= r' - \tfrac{1}{2}(p+q).\end{aligned} \quad (5.7)$$

To relate $R_S(\Omega)$ and $R_T(\Omega)$ to operators of the type $\mathcal{R}(\Omega)$, the permutation operators (1, 2), (2, 3), and (3, 4) can be used. For example, consider $R_S(\Omega)$. Let

$$\begin{aligned}S_0 &= S_0^1 + S_0^2, \\ S_0^1 &= \tfrac{1}{2}(A_{11} - A_{33}) = (2,3)J_0(2,3), \\ S_0^2 &= \tfrac{1}{2}(A_{22} - A_{44}) \\ &= (1,2)(3,4)(2,3)J_0(2,3)(3,4)(1,2), \\ [S_0^1&, S_0^2] = 0,\end{aligned} \quad (5.8a)$$

$$\begin{aligned}S_2 &= S_2^1 + S_2^2, \\ S_2^1 &= (2i)^{-1}(A_{13} - A_{31}) = (2,3)J_2(2,3), \\ S_2^2 &= (2i)^{-1}(A_{24} - A_{42}) \\ &= (1,2)(3,4)(2,3)J_2(2,3)(3,4)(1,2), \\ [S_2^1&, S_2^2] = [S_0^1, S_2^2] = [S_0^2, S_2^1] = 0.\end{aligned} \quad (5.8b)$$

Then

$$\begin{aligned}R_S(\Omega) &= e^{-i\alpha S_0} e^{-i\beta S_2} e^{-i\gamma S_0} \\ &= e^{-i\alpha S_0^1} e^{-i\alpha S_0^2} e^{-i\beta S_2^1} e^{-i\beta S_2^2} e^{-i\gamma S_0^1} e^{-i\gamma S_0^2} \\ &= e^{-i\alpha S_0^1} e^{-i\beta S_2^1} e^{-i\gamma S_0^1} e^{-i\alpha S_0^2} e^{-i\beta S_2^2} e^{-i\gamma S_0^2} \\ &= (2,3)\mathcal{R}(\Omega)(2,3)(1,2)(3,4)(2,3) \\ &\quad \times \mathcal{R}(\Omega)(2,3)(3,4)(1,2).\end{aligned} \quad (5.9)$$

In a similar fashion it can be shown that

$$\begin{aligned}R_T(\Omega) &= \mathcal{R}(\Omega)(2,3)(1,2)(3,4)(2,3) \\ &\quad \times \mathcal{R}(\Omega)(2,3)(3,4)(1,2)(2,3).\end{aligned} \quad (5.10)$$

From Eqs. (5.9) and (5.10) it follows that

$$\begin{aligned}R_S(\Omega_S)R_T(\Omega_T) &= R_T(\Omega_T)R_S(\Omega_S) \\ &= \mathcal{R}(\Omega_T)(2,3)(1,2)(3,4)(2,3) \\ &\quad \times \mathcal{R}(\Omega_T)(2,3)(1,2)(3,4) \\ &\quad \times \mathcal{R}(\Omega_S)(2,3)(1,2)(3,4)(2,3) \\ &\quad \times \mathcal{R}(\Omega_S)(2,3)(1,2)(3,4).\end{aligned} \quad (5.11)$$

Define

$$\begin{aligned}M_{G'G}(\Omega) &= \langle G'|\, \mathcal{R}(\Omega)(2,3)(1,2)(3,4)(2,3) \\ &\quad \times \mathcal{R}(\Omega)(2,3)(1,2)(3,4)\, |G\rangle\end{aligned} \quad (5.12)$$

so that

$$\langle G'|\, R_S(\Omega_S)R_T(\Omega_T)\, |G\rangle = \sum_{G''} M_{G'G''}(\Omega_T) M_{G''G}(\Omega_S). \quad (5.13)$$

Let

$$\mathcal{M}_{G'G}(KJM) = (2J+1)\int d\Omega D^{J*}_{MK}(\Omega) M_{G'G}(\Omega). \quad (5.14)$$

The transformation brackets of Eq. (5.2) are then given by

$$\langle G' \mid GK_S SM_S K_T TM_T\rangle = \sum_{G''} \mathcal{M}_{G'G''}(K_T TM_T)\mathcal{M}_{G''G}(K_S SM_S). \quad (5.15)$$

An expression for the matrix $\mathcal{M}_{G'G}(KJM)$ can be obtained by using the completeness of the orthonormal set of states $|G\rangle$ and Eq. (5.7) to put $M_{G'G}(\Omega)$ into the form

$$\begin{aligned} M_{G'G}(\Omega) &= \sum_{G_\alpha} \langle G'|\, \mathfrak{R}(\Omega)\, |G_1\rangle\langle G_1|\, (2,3)\, |G_2\rangle\langle G_2|\, (1,2)\, |G_3\rangle \\ &\quad \times \langle G_3|\, (3,4)\, |G_4\rangle\langle G_4|\, (2,3)\, |G_5\rangle\langle G_5|\, \mathfrak{R}(\Omega)\, |G_6\rangle \\ &\quad \times \langle G_6|\, (2,3)\, |G_7\rangle\langle G_7|\, (1,2)\, |G_8\rangle\langle G_8|\, (3,4)\, |G\rangle \\ &= \sum_{\substack{G_\alpha(\alpha\neq 1,5)\\ K'M''}} D^{J'}_{M'K'}(\Omega) D^{J''}_{M''K''}(\Omega)\langle G_1|\, (2,3)\, |G_2\rangle \\ &\quad \times \langle G_2|\, (1,2)\, |G_3\rangle\langle G_3|\, (3,4)\, |G_4\rangle\langle G_4|\, (2,3)\, |G_5\rangle \\ &\quad \times \langle G_6|\, (2,3)\, |G_7\rangle\langle G_7|\, (1,2)\, |G_8\rangle\langle G_8|\, (3,4)\, |G\rangle, \end{aligned}$$

$$\begin{aligned} J' &= \tfrac{1}{2}(p'-q'), \quad M' = r' - \tfrac{1}{2}(p'+q'), \\ K' &= r_1 - \tfrac{1}{2}(p'+q'), \\ J'' &= \tfrac{1}{2}(p_6-q_6), \quad M'' = r_5 - \tfrac{1}{2}(p_6+q_6), \\ K'' &= r_6 - \tfrac{1}{2}(p_6+q_6), \end{aligned} \quad (5.16)$$

where, except for r_1 (determined by K') and r_5 (determined by M''), the elements of G_1 and G_5 are equal to the corresponding elements in the G' and G_6, respectively. Then, by using the well-known result expressing the integral of three rotation matrices in terms of a product of two $SU(2)$ Wigner (Clebsch–Gordan) coefficients, it follows that

$$\begin{aligned} \mathcal{M}_{G'G}(KJM) &= \sum_{G_\alpha(\alpha\neq 1,5)} \langle J'M'; J''M'' \mid JM\rangle\langle J'K'; J''K'' \mid JK\rangle \\ &\quad \times \langle G_1|\, (2,3)\, |G_2\rangle\langle G_2|\, (1,2)\, |G_3\rangle\langle G_3|\, (3,4)\, |G_4\rangle \\ &\quad \times \langle G_4|\, (2,3)\, |G_5\rangle\langle G_6|\, (2,3)\, |G_7\rangle\langle G_7|\, (1,2)\, |G_8\rangle \\ &\quad \times \langle G_8|\, (3,4)\, |G\rangle. \end{aligned} \quad (5.17)$$

The permutation matrices $\langle G'|\, (n-1,n)\, |G\rangle$, $n = 2, 3, 4$, required for an evaluation of Eq. (5.17), have been given by Moshinsky and Chacón[17]; they are equivalent to special unitary recoupling coefficients for the groups $U(1)$, $U(2)$, and $U(3)$, respectively. Note that $(n-1, n)$ operating on $|G\rangle$ changes only the $h_{\alpha\beta}$ for which $\beta = n-1$ and these in such a manner that the result is zero unless $w'_{n-1} = w_n$. The apparent $6 \times 6 = 36$-fold sum over the G_α in Eq. (5.17) is therefore in actual fact at worst a sixfold sum. The result as given by Eq. (5.17) may, however, be the most convenient for the purposes of machine coding since the summations over G_2, G_3, G_4 and G_7, G_8 are matrix multiplications involving the permutation matrices. The remaining summation over G_6 then involves simply the product of two Clebsch–Gordan coefficients and one element from each of the matrix products.

It is to be noted that the transformation brackets are equivalent to normalization and overlap integrals of the projected states. This may be seen by considering

$$\begin{aligned} \langle G'K'_S SM_S K'_T TM_T \mid GK_S SM_S K_T TM_T\rangle &= \langle G'|\, P^{S\dagger}_{M_S K'_S} P^{T\dagger}_{M_T K'_T} |GK_S SM_S K_T TM_T\rangle \\ &= \langle G'|\, P^{S}_{K'_S M_S} P^{T}_{K'_T M_T}\, |GK_S SM_S K_T TM_T\rangle \\ &= \langle G' \mid GK_S SK'_S K_T TK'_T\rangle. \end{aligned} \quad (5.18)$$

B. The Application

In general, the transformation brackets[18]

$$A(G' \mid G_E K_S SM_S K_T TM_T)$$

relate the set of nonorthogonal basis vectors $|G_E K_S SM_S K_T TM_T\rangle$ to the set of orthonormal basis vectors $|G'\rangle$ and are therefore the elements of a nonorthogonal matrix A. The inverse expansion of the $|G\rangle$ in terms of the $|G_E K'_S S' M'_S K'_T T' M'_T\rangle$ exists, and the coefficients $B(G_E K'_S S' M'_S K'_T T' M'_T \mid G)$ can be obtained by inverting the appropriate A matrix. An equivalent but perhaps somewhat simpler evaluation of these coefficients can be obtained by considering directly the expansion

$$|G\rangle = \sum_{\substack{K'_S S' M'_S\\ K'_T T' M'_T}} B(G_E K'_S S' M'_S K'_T T' M'_T \mid G) \times |G_E K'_S S' M'_S K'_T T' M'_T\rangle. \quad (5.19)$$

Then

$$\begin{aligned} |GK_S SM_S K_T TM_T\rangle &= P^S_{M_S K_S} P^T_{M_T K_T}\, |G\rangle \\ &= \sum_{\substack{K'_S S' M'_S\\ K'_T T' M'_T}} B(G_E K'_S S' M'_S K'_T T' M'_T \mid G) \\ &\quad \times P^S_{M_S K_S} P^T_{M_T K_T}\, |G_E K'_S S' M'_S K'_T T' M'_T\rangle \\ &= \sum_{\substack{K'_S S' M'_S\\ K'_T T' M'_T}} B(G_E K'_S S' M'_S K'_T T' M'_T \mid G) \\ &\quad \times \delta_{S'S}\delta_{M'_S K_S}\delta_{T'T}\delta_{M'_T K_T}\, |G_E K'_S SM_S K'_T TM_T\rangle \\ &= \sum_{K'_S K'_T} B(G_E K'_S SK_S K'_T TK_T \mid G) \\ &\quad \times |G_E K'_S SM_S K'_T TM_T\rangle. \end{aligned} \quad (5.20)$$

That is, the $B(G_E K'_S S' M'_S K'_T T' M'_T \mid G)$ are not only the coefficients in the expansion of the $|G\rangle$ in terms of the $|G_E K'_S S' M'_S K'_T T' M'_T\rangle$, but they are also the coefficients in the expansion of $|GK_SSM_SK_TTM_T\rangle$ in terms of the $|G_EK'_SSM_SK'_TTM_T\rangle$. Using this result, we can determine a unique solution for the

$$B(G_EK'_SS'M'_SK'_TT'M'_T \mid G)$$

from the set of simultaneous equations

$$A(G' \mid GK_SSM_SK_TTM_T) = \sum_{K'_SK'_T} B(G_EK'_SSK_SK'_TTK_T \mid G) \times A(G' \mid G_EK'_SSM_SK'_TTM_T). \quad (5.21)$$

In those cases for which the $\{K_SK_T\}$-labels are redundant, it follows that the $B(G_EK'_SS'M'_SK'_TT'M'_T \mid G)$ are simply given as the ratio of two transformation brackets. Since B is the inverse of A, Eq. (5.21) also shows that

$$\sum_{SM_STM_T} A(G' \mid GM_SSM_SM_TTM_T) = \delta_{G'G} \quad (5.22)$$

and

$$\sum_G A(G \mid GK_SSM_SK_TTM_T) = \delta_{M_SK_S}\delta_{M_TK_T} \sum_{K'_SK'_T} = \delta_{M_SK_S}\delta_{M_TK_T}N_{(ST)}(\lambda_1\lambda_2\lambda_3). \quad (5.23)$$

In a fashion similar to that demonstrated in detail in Ref. 16 for the $SU(3) \supset R(3)$ case, quantities of physical interest which depend upon the $SU(4) \supset SU(2) \otimes SU(2)$ labels can be expressed in terms of the corresponding quantities labeled according to the canonical $U(4) \supset U(3) \supset U(2) \supset U(1)$ scheme by means of the A's and B's. For example, for the $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients defined by

$$|\rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3}\rangle = \sum_{\substack{K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1} \\ K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2}}} C_1(G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1};\, G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2} \mid \rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3}) \times |G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1}\rangle\, |G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2}\rangle, \quad (5.24a)$$

$$|G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1}\rangle\, |G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2}\rangle = \sum_{\rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3}} C_2(\rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3} \mid G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1};\, G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2}) \times |\rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3}\rangle, \quad (5.24b)$$

it can be shown that

$$C_1(G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1};\, G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2} \mid \rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3}) = \langle S_1M_{S_1};\, S_2M_{S_2} \mid S_3M_{S_3}\rangle\langle T_1M_{T_1};\, T_2M_{T_2} \mid T_3M_{T_3}\rangle \sum_{\substack{G'_1M'_{S_1}M'_{T_1} \\ G'_2M'_{S_2}M'_{T_2}}} \langle S_1M'_{S_1};\, S_2M'_{S_2} \mid S_3K_{S_3}\rangle \times \langle T_1M'_{T_1};\, T_2M'_{T_2} \mid T_3K_{T_3}\rangle B(G_{1E}K_{S_1}S_1M'_{S_1}K_{T_1}T_1M'_{T_1} \mid G'_1) \times B(G_{2E}K_{S_2}S_2M'_{S_2}K_{T_2}T_2M'_{T_2} \mid G'_2)\langle G'_1;\, G'_2 \mid \rho G_{3E}\rangle, \quad (5.25a)$$

$$C_2(\rho G_{3E}K_{S_3}S_3M_{S_3}K_{T_3}T_3M_{T_3} \mid G_{1E}K_{S_1}S_1M_{S_1}K_{T_1}T_1M_{T_1};\, G_{2E}K_{S_2}S_2M_{S_2}K_{T_2}T_2M_{T_2}) = \langle S_1M_{S_1};\, S_2M_{S_2} \mid S_3M_{S_3}\rangle\, \langle T_1M_{T_1};\, T_2M_{T_2} \mid T_3M_{T_3}\rangle \frac{(2S_1+1)(2T_1+1)}{(2S_3+1)(2T_3+1)} \times \sum_{\substack{G'_3M'_{S_2}M'_{T_2} \\ G'_3M'_{S_3}M'_{T_3}}} \langle S_1K_{S_1};\, S_2M'_{S_2} \mid S_3M'_{S_3}\rangle\langle T_1K_{T_1};\, T_2M'_{T_2} \mid T_3M'_{T_3}\rangle B(G_{3E}K_{S_3}S_3M'_{S_3}K_{T_3}T_3M'_{T_3} \mid G'_3) \times \langle \rho G'_3 \mid G_{1E};\, G'_2\rangle A(G'_2 \mid G_{2E}K_{S_2}S_2M'_{S_2}K_{T_2}T_2M'_{T_2}), \quad (5.25b)$$

where ρ is a label that distinguishes multiple occurrences of a given IR of G_3 in the reduction of the direct product $G_1 \otimes G_2$. In Eqs. (5.25), $\langle G_1;\, G_2 \mid \rho G_3\rangle$ and $\langle \rho G_3 \mid G_1;\, G_2\rangle$ are $U(4) \supset U(3) \supset U(2) \supset U(1)$ Wigner coefficients, and the $\langle J_1M_1; J_2M_2 \mid J_3M_3\rangle$ are ordinary $SU(2)$ Wigner coefficients.

Similarly, consider the $SU(4) \supset SU(2) \otimes SU(2)$ tensors defined by

$$T(GK_SSM_SK_TTM_T) = (2S+1)\int d\Omega_S D^{S*}_{M_SK_S}(\Omega_S)(2T+1) \times \int d\Omega_T D^{T*}_{M_TK_T}(\Omega_T) \times R_S(\Omega_S)R_T(\Omega_T)T(G)R_T^{-1}(\Omega_T)R_S^{-1}(\Omega_S), \quad (5.26)$$

where $T(G)$ is the corresponding $U(4) \supset U(3) \supset U(2) \supset U(1)$ tensor defined by

$$[A_{\alpha\beta}, T(G)] = \sum_{G'} \langle G'| A_{\alpha\beta} |G\rangle\, T(G'). \tag{5.27}$$

The $\{K_S K_T\}$-quantum-numbers resolve the $SU(4) \supset SU(2) \otimes SU(2)$ tensorial multiplicity in precisely the same manner as described in Sec. 3 for the $SU(4) \supset SU(2) \otimes SU(2)$ basis states. It can then be shown that

$$\begin{aligned}\langle \rho G_{3E} K_{S_3} S_3 M_{S_3} K_{T_3} T_3 M_{T_3}| \, T(G_{1E} K_{S_1} S_1 M_{S_1} K_{T_1} T_1 M_{T_1}) \, |G_{2E} K_{S_2} S_2 M_{S_2} K_{T_2} T_2 M_{T_2}\rangle \\ = \langle G_3 \| \, T(G_1) \, \| G_2 \rangle_\rho \sum_{K'_{S_3} K'_{T_3}} C_2(\rho G_{3E} K'_{S_3} S_3 M_{S_3} K'_{T_3} T_3 M_{T_3} \mid G_{1E} K_{S_1} S_1 M_{S_1} K_{T_1} T_1 M_{T_1}; G_{2E} K_{S_2} S_2 M_{S_2} K_{T_2} T_2 M_{T_2}) \\ \times A(G_{3E} \mid \rho G_{3E} K'_{S_3} S_3 K_{S_3} K'_{T_3} T_3 K_{T_3}), \end{aligned} \tag{5.28}$$

where $\langle G_3 \| \, T(G_1) \, \| G_2 \rangle_\rho$ is the reduced matrix element of $T(G_1)$ corresponding to the state $|_\rho G_3\rangle$.

The particularly elegant feature of all such relationships is that a knowledge of the A's and B's allows completely general expressions for $SU(4) \supset SU(2) \otimes SU(2)$ quantities to be expressed in terms of a subset of the corresponding $U(4) \supset U(3) \supset U(2) \supset U(1)$ quantities [e.g., all $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients are determined in terms of $U(4) \supset U(3) \supset U(2) \supset U(1)$ Wigner coefficients for which one set of labels corresponds to the operator E_{00} having either its maximum or minimum eigenvalue]. Furthermore, the problems associated with phase conventions and multiplicity relate simply and directly to the corresponding problems in the canonical scheme.

6. DISCUSSION

The fact that a many-nucleon wavefunction can be decomposed into a product of its space and its spin–isospin parts allows the techniques developed in this paper to be applied quite independently of any special spatial considerations. A case of particular interest, however, is that dealing with shell-model calculations up to and through the first half of the $2s$-$1d$ shell. For such nuclei the most promising theoretical tool for the spatial part of the wavefunction is the Elliott $SU(3) \supset R(3)$ classification. For this reason the techniques developed in Ref. 16 together with those of the present paper furnish expressions which can be used to simplify as well as extend such theoretical investigations.

The simplifications are, of course, in calculational technique in that the $SU(3) \supset R(3)$ and $SU(4) \supset SU(2) \otimes SU(2)$ transformation brackets reduce the difficulties inherent in the physically significant labeling schemes, but not present in the corresponding canonical labeling schemes, to forms which can be machine coded. Nevertheless, the solution furnished by the transformation brackets to the problems associated with the nonorthonormality of the projected states is indirect and not necessarily the most convenient for purposes of machine-coding matrix element calculations. The difficulty is that the $SU(3) \supset R(3)$ coupling coefficients of Ref. 16 and the $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients of the present paper are not Wigner coefficients. That is, the coupling coefficients do not represent the scalar product of orthonormalized coupled and uncoupled basis states.

By orthonormalizing separately within each L and (ST)-multiplet according to a symmetric recipe (e.g., see Ref. 19), the transformations which orthonormalize the $SU(3) \supset R(3)$ and $SU(4) \supset SU(2) \otimes SU(2)$ basis states can be given in simple algebraic form as the ratio of normalization and overlap integrals. Since such integrals are equivalent to transformation brackets, the problems associated with the nonorthonormality of the projected states can be resolved. And, in particular, they can be resolved in a form convenient for machine coding while still maintaining all the simplifications associated with the projective processes. In fact, the $SU(3) \supset R(3)$ and $SU(4) \supset SU(2) \otimes SU(2)$ orthonormalizing transformations can be incorporated directly into programs which calculate the transformation brackets. The result is then $SU(3) \supset R(3)$ and $SU(4) \supset SU(2) \otimes SU(2)$ transformation brackets which relate physically significant orthonormal basis states to the corresponding canonical basis states. Within such a framework the $SU(3) \supset R(3)$ coupling coefficients of Ref. 13 and the $SU(4) \supset SU(2) \otimes SU(2)$ coupling coefficients of the present paper become Wigner coefficients, and hence standard algebraic techniques introduced by Racah[20] can be applied to simplify matrix element calculations.

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[1] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

[2] M. Moshinsky and J. G. Nagel, Phys. Letters **5**, 173 (1963); R. T. Sharp and C. S. Lam, J. Math. Phys. **10**, 2033 (1969); M. Brunet and M. Resnikoff, J. Math. Phys. **11**, 1471, 1474 (1970).

[3] I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk SSSR **71**, 825 (1950).

[4] K. T. Hecht and S. C. Pang, J. Math. Phys. **10**, 1571 (1969).

[5] J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128, 562 (1958).

[6] $SU(4)$ conjugation properties are defined and discussed more fully by Brunet and Resnikoff, Ref. 2.

[7] The supermultiplet quantum numbers $(PP'P'')$ specify the IR of $O(6)$ which is locally isomorphic to $SU(4)$ according to the standard Gel'fand labeling scheme.

[8] For notational convenience, states of the type (2.8) will be denoted simply by $|G_E\rangle$, an arrow being added to the E when a distinction between types (a) and (b) is required. The $\{K_SK_T\}$-label will only be included when it is necessary to distinguish between $|G_E\rangle$ states of different spin–isospin projections.

[9] G. Racah, Rev. Mod. Phys. **21**, 494 (1949).

[10] M. Kretzschmar, Z. Physik **157**, 558 (1960).

[11] A. M. Perelomov and V. S. Popov, Yad. Fiz. **2**, 738 (1965) [Sov. J. Nucl. Phys. **2**, 528 (1966)].

[12] D. E. Littlewood, *The Theory of Group Characters* (Oxford U.P., London, 1940).

[13] A regular lattice displacement is one which increases either S or T by one unit.

[14] The Löwdin–Shapiro form for the projection operator may also prove to be useful in some applications; for example, see Yu. F. Smirnov, "Projection Operators for Semisimple Lie Groups," *Proceedings on the International Conference on Clustering Phenomena in Nuclei, Bochum, Germany, 1969* (International Atomic Energy Agency, Vienna, 1969).

[15] S. A. Williams and D. L. Pursey, J. Math. Phys. **9**, 1230 (1968).

[16] J. P. Draayer and S. A. Williams, Nucl. Phys. **A129**, 647 (1969).

[17] M. Moshinsky and E. Chacón, *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Bloch, S. G. Cohen, A. de-Shalit, S. Sambursky, and I. Talmi (North-Holland, Amsterdam, 1968).

[18] To avoid possible confusion, the notational change

$$A(G' \mid GK_SSM_SK_TTM_T) \equiv \langle G' \mid GK_SSM_SK_TTM_T\rangle,$$

expressing the transformation brackets as the elements of a matrix A, with rows labeled by the $h'_{\alpha\beta}$ and columns labeled by the $\{K_SSM_SK_TTM_T\}$-values, will be adopted in this section.

[19] J. D. Vergados, Nucl. Phys. **A111**, 681 (1968).

[20] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic, New York, 1959).

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Theorem of Uniqueness and Local Stability for Liouville–Einstein Equations*

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We prove, by use of energy inequalities, a theorem of uniqueness and local (i.e., for finite time) stability for the solution of Cauchy problem relative to the integro-differential system of Einstein and Liouville. A global theorem of geometrical uniqueness follows from a general method, previously given. We will prove elsewhere an existence theorem.

INTRODUCTION

The aim of this paper is to prove a uniqueness theorem for the solution of the Cauchy problem for the coupled Liouville–Einstein equations, i.e., for a collisionless relativistic gas under its own gravitational field. Such a gas provides a model reasonably appropriate for physical systems like systems of galaxies or some systems of stars (which are then the "particles" of the gas) and certain plasmas or radiations (in this last case the particles have a zero rest mass).

With the uniqueness theorem we prove a local stability theorem; i.e., we prove that the solution (metric and distribution function) depends continuously on the initial data: such a theorem, which states that a small initial perturbation gives rise to a small perturbation during some finite time, seems the first necessary step to be assured of before any more elaborate research on stability.

The plan of this paper is the following:

In Sec. I, I give a brief review of the fundamental concepts of relativistic kinetic theory, and I recall the equations governing the motion of a self-gravitating collisionless, relativistic gas: the coupled Einstein and Liouville equations. I also recall, or establish, a few general properties of these equations which will be used in the following (i.e., local equivalence of Einstein equations in harmonic coordinates and tensorial Einstein equations, and use of bounded parameters for the momenta in the Liouville equation).

In Sec. II, I establish some inequalities satisfied by the difference of two solutions of the Cauchy problem