Suddenly Pressurized Elastomagnetohydrodynamic Channel Flow

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The channel flow of a suddenly pressurized visco-elastic and electrically conducting fluid under the influence of a constant and transversal magnetic field is given analytic treatment. Flow oscillations resulting from the elasticity of the fluid and damping effect of the magnetic field on these oscillations are shown in terms of three parameters.

INTRODUCTION

ADVANCES in mechanics of continua now make it possible to include non-Newtonian and electromagnetic effects into viscous flow problems. The laminar flow problems containing the effect of elasticity alone have been treated by Broer[1] and by Thomas and Walters,[2,3] and the problem including the effect of a magnetic field alone by Yen and Chang.[4] It is the purpose of this paper to investigate, by including both elastic and electromagnetic effects, the channel flow of a suddenly pressurized visco-elastic and electrically conducting fluid under the influence of a magnetic field.

FORMULATION

A visco-elastic fluid is considered between parallel plates ($2L$ distance apart) extending to infinity in the directions of x_1 and x_3 axes (Fig. 1). Following assumptions are made: (i) fluid is Maxwellian and has constant properties, (ii) plates are perfect electrical conductors, (iii) a magnetic field of constant strength H_0 is transversally applied to plates, and (iv) fluid is suddenly pressurized in the direction of x_1 axis by a constant pressure gradient $-\partial p/\partial x_1$ from an initial condition of rest.

Under the assumptions above, the general formulation of the problem given in Gaussian units reduces to

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} - \frac{1}{\rho}\frac{\partial \tau_{21}}{\partial x_2} + \frac{\mu_e}{4\pi\rho} H_0 \frac{\partial H_1}{\partial x_2},$$

$$\frac{\partial H_1}{\partial t} = H_0 \frac{\partial u_1}{\partial x_2} + \eta \frac{\partial^2 H_1}{\partial x_2^2},$$

$$\tau_{21} + \frac{\mu}{G}\frac{\partial \tau_{21}}{\partial t} = -\mu \frac{\partial u_1}{\partial x_2},$$

where ρ denotes density, μ viscosity, μ_e magnetic permeability, σ electric conductivity, c speed of light, $\eta = c^2/4\pi\mu_e\sigma$ magnetic diffusivity, G elastic shear rigidity, p static pressure, and τ_{21}, u_1, and H_1, components of stress tensor, velocity and magnetic field vectors, respectively.

Introducing the kinematic viscosity ν, and the dimensionless variables $\theta = \nu t/L^2$, $y = x_2/L$, $V = u_1 L/\nu$, $H = H_1/H_0$, $P = -(\partial p/\partial x_1)L^3/\rho\nu^2$, $\tau = \tau_{21}L^2/\rho\nu^2$, $\xi = \rho\nu^2/GL^2$, a (Alfvén wave speed) $= (\mu_e H_0^2/4\pi\rho)^{\frac{1}{2}}$, $\zeta = (aL/\nu)^2$, and $\chi = \eta/\nu$, the formulation may conveniently be rearranged in the form

$$\partial V/\partial\theta = P - (\partial\tau/\partial y) + \zeta(\partial H/\partial y), \tag{1}$$

$$\partial H/\partial\theta = (\partial V/\partial y) + \chi(\partial^2 H/\partial y^2), \tag{2}$$

$$\tau + \xi(\partial\tau/\partial\theta) = -\partial V/\partial y, \tag{3}$$

subject to the initial and boundary conditions

$$V(y, 0) = H(y, 0) = \tau(y, 0) = 0, \tag{4}$$

$$V(\pm 1, \theta) = \partial H(\pm 1, \theta)/\partial y = 0. \tag{5}$$

As to be seen in the next section, the boundary conditions related to τ need not be considered here.

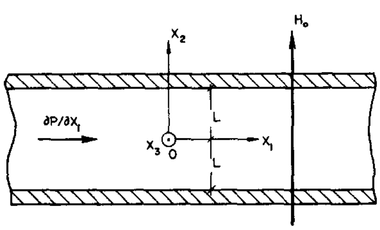

FIG. 1. Channel flow and coordinate system.

SOLUTION

Applying Laplace transforms to (1), (2), (3), and (5) in the usual manner [using (4), and eliminating the transform of τ between transforms of (1) and (3)], gives

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[4] J. T. Yen and C. C. Chang, Phys. Fluids **4**, 1355 (1961).

$$s\bar{V} = s^{-1}P + \zeta(d\bar{H}/dy) + (1+\xi s)^{-1}(d^2\bar{V}/dy^2), \quad (6)$$

$$s\bar{H} = (d\bar{V}/dy) + \chi(d^2\bar{H}/dy^2), \quad (7)$$

$$\bar{V}(\pm 1, s) = d\bar{H}(\pm 1, s)/dy = 0, \quad (8)$$

where s is the Laplace transform variable in time, and $\bar{V}$ and $\bar{H}$ denote the Laplace transforms of V and H, respectively.

Clearly, the boundary conditions given by (8) are satisfied by assuming solutions in the form

$$\bar{V}(y, s) = \sum_{n=0}^{\infty} \bar{A}_n(s) \cos \lambda_n y, \quad (9)$$

$$\bar{H}(y, s) = \sum_{n=0}^{\infty} \bar{B}_n(s) \sin \lambda_n y, \quad (10)$$

provided $\lambda_n = (n + \frac{1}{2})\pi$, $n = 0, 1, 2, \cdots$. Inserting (9) and (10) into (6) and (7), and using the appropriate series expansion for P, it is found that

$$\lambda_n \bar{A}_n + (s + \chi\lambda_n^2)\bar{B}_n = 0,$$

$$s[s + \lambda_n^2/(1+\xi s)]\bar{A}_n - s\zeta\lambda_n\bar{B}_n = 2P(-1)^n/\lambda_n.$$

Solving these equations for $\bar{A}_n$ and $\bar{B}_n$ gives

$$\bar{A}_n = \frac{2P(-1)^n(1+\xi s)(s+\chi\lambda_n^2)}{\lambda_n s[(1+\xi s)(s^2+\chi\lambda_n^2 s+\zeta\lambda_n^2)+\lambda_n^2(s+\chi\lambda_n^2)]}, \quad (11)$$

$$\bar{B}_n = \frac{2P(-1)^{n+1}(1+\xi s)}{s[(1+\xi s)(s^2+\chi\lambda_n^2 s+\zeta\lambda_n^2)+\lambda_n^2(s+\chi\lambda_n^2)]}. \quad (12)$$

Finally, introducing (11) and (12) into (9) and (10), respectively, expanding in partial fractions following the discussion of the roots of third-order denominators in s, and inverting the results, yields the solution of the problem,

$$V(y, \theta) = 2P \sum_{n=0}^{\infty} (-1)^n F_n(\theta; \xi, \zeta, \chi) \cos \lambda_n y, \quad (13)$$

$$H(y, \theta) = 2P \sum_{n=0}^{\infty} (-1)^{n+1} G_n(\theta; \xi, \zeta, \chi) \sin \lambda_n y. \quad (14)$$

Explicit forms of F_n and G_n are given in the Appendix. In the limit as $\chi \to \infty$, $H(y, \theta) = 0$ according to (2), (5), and (4), and the solution is reduced to

$$V(y, \theta) = \tfrac{1}{2}P(1-y^2) - 2P \sum_{n=0}^{\infty} (-1)^n F_n^*(\theta; \xi) \cos \lambda_n y. \quad (15)$$

Explicit form of F_n^* is given in the Appendix.

RESULTS

The physical significance of the parameters involved in (13) and (14) is readily found to be:

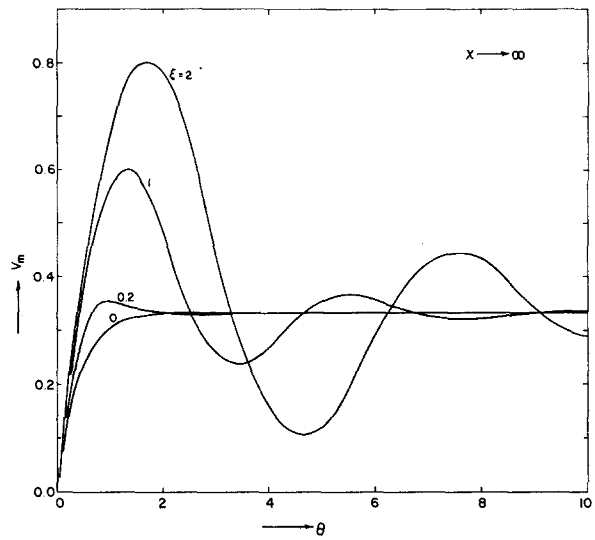

FIG. 2. Oscillations in mean velocity for limiting nonmagnetic case $\chi \to \infty$ depending on elasticity of fluid $\xi = 0$, 0.2, 1, 2.

ξ (viscous shear stress/elastic shear rigidity)/Reynolds number, ζ (magnetic pressure/viscous shear stress) Reynolds number, χ (magnetic diffusivity/Momentum diffusivity). A complete study of (13) and (14) in terms of the variables y, θ, and the parameters ξ, ζ, χ is somewhat lengthy. The y dependence in (13), however, may be eliminated by considering the mean velocity

$$V_m(\theta) = \frac{1}{2P}\int_{-1}^{1} V(y, \theta)\, dy,$$

which is a measure of the flow rate. Oscillations in the mean velocity for the limiting nonconductive case $\chi \to \infty$ are shown in Fig. 2, and those for the conductive case $\chi = 20$ under the influence of the transversal magnetic field $\zeta = 10$ in Fig. 3, depending on the elasticity of the fluid $\xi = 0$, 0.2,

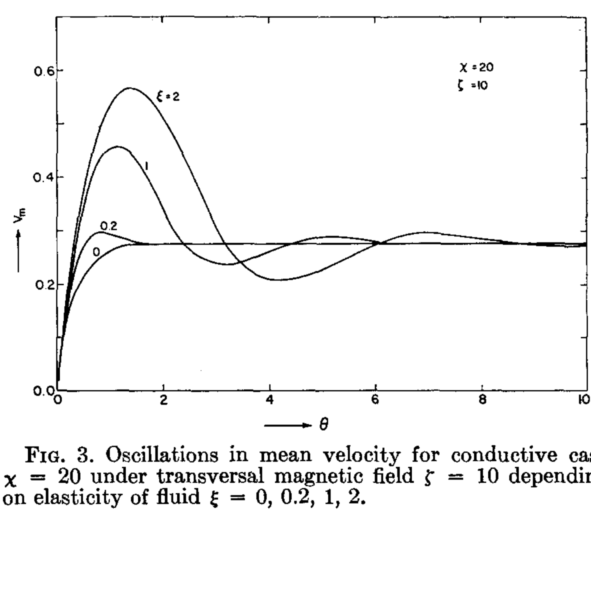

FIG. 3. Oscillations in mean velocity for conductive case $\chi = 20$ under transversal magnetic field $\zeta = 10$ depending on elasticity of fluid $\xi = 0$, 0.2, 1, 2.

1, and 2. These results are valid provided the current induced in the direction of x_3 axis flows freely in a circuit closed appropriately in the same direction. Otherwise the effect of a polarized electric field must be taken into account. In the present case, the x_2 component of the momentum gives rise to a magnetic pressure equivalent to a hydrostatic pressure. Oscillations in the induced electric current and magnetic fields, and the magnetic pressure are not given here because of space considerations.

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APPENDIX

Explicit Values of F_n and G_n

(i) $\xi \neq 0$

$$F_n = \left\{\frac{\chi\lambda_n^2}{(g_0+g_1)[(g_0-\frac{1}{2}g_1)^2+(\frac{1}{2}g_2)^2]} - 4f_5\frac{\exp[-(g_0+g_1)\theta]}{(g_0+g_1)(9g_1^2+g_2^2)} + \left(\frac{2}{f_0}\right)(f_3^2+f_4^2)^{\frac{1}{2}}\exp[(\tfrac{1}{2}g_1-g_0)\theta]\sin(\tfrac{1}{2}g_2\theta-\psi)\right\}(\xi\lambda_n)^{-1},$$

$$G_n = \left\{\frac{1}{(g_0+g_1)[(g_0-\frac{1}{2}g_1)^2+(\frac{1}{2}g_2)^2]} - 4[1-\xi(g_0+g_1)]\frac{\exp[-(g_0+g_1)\theta]}{(g_0+g_1)(9g_1^2+g_2^2)} + \left(\frac{2}{f_0}\right)(f_1^2+f_2^2)^{\frac{1}{2}}\exp[(\tfrac{1}{2}g_1-g_0)\theta]\sin(\tfrac{1}{2}g_2\theta-\phi)\right\}(\xi)^{-1},$$

where

$$\psi = \tan^{-1}\left(\frac{f_3}{f_4}\right), \qquad \phi = \tan^{-1}\left(\frac{f_1}{f_2}\right), \qquad f_0 = g_2^2[g_2^2(\tfrac{1}{2}g_0-g_1)^2+\tfrac{1}{16}(3g_1^2-6g_0g_1-g_2^2)^2],$$

$$f_1 = g_2^2\{(\tfrac{1}{2}g_0-g_1)-\tfrac{3}{8}\xi[(g_1-2g_0)^2+g_2^2]\},$$

$$f_2 = g_2[\tfrac{1}{2}\xi g_2^2(\tfrac{1}{2}g_0-g_1)-\tfrac{1}{4}(1+\tfrac{1}{2}\xi g_1-\xi g_0)(3g_1^2-6g_0g_1-g_2^2)],$$

$$f_3 = g_2^2\{\chi\lambda_n^2(\tfrac{1}{2}g_0-g_1)-\tfrac{1}{2}\chi\lambda_n^2\xi[(g_0-\tfrac{1}{2}g_1)^2+(\tfrac{1}{2}g_2)^2] + \tfrac{1}{4}\xi[(g_0-\tfrac{1}{2}g_1)(2g_0^2+g_0g_1-g_1^2)+\tfrac{1}{2}g_2^2(g_0+g_1)]-\tfrac{1}{2}[(g_0-\tfrac{1}{2}g_1)^2+(\tfrac{1}{2}g_2)^2]\},$$

$$f_4 = \tfrac{1}{2}g_2\{\lambda_n^2[\tfrac{1}{2}\chi\xi(g_0-\tfrac{1}{2}g_1)(3g_1^2-6g_0g_1-g_2^2)+\chi\xi g_2^2(\tfrac{1}{2}g_0-g_1) - \tfrac{1}{2}\chi(3g_1^2-6g_0g_1-g_2^2)]-\tfrac{1}{2}\xi(3g_1^2-6g_0g_1-g_2^2)[(g_0-\tfrac{1}{2}g_1)^2-(\tfrac{1}{2}g_2)^2] + \tfrac{1}{2}(g_0-\tfrac{1}{2}g_1)(3g_1^2-6g_0g_1-g_2^2)+g_2^2(\tfrac{1}{2}g_0-g_1)[1-2\xi(g_0-\tfrac{1}{2}g_1)]\},$$

$$f_5 = \chi\lambda_n^2-(1+\chi\xi\lambda_n^2)(g_0+g_1)+\xi(g_0+g_1)^2, \qquad g_0 = \tfrac{1}{3}(1/\xi+\chi\lambda_n^2),$$

$$g_1 = -\tfrac{1}{3}[(h_0+\lambda_nh_1^{\frac{1}{2}})^{\frac{1}{3}}+(h_0-\lambda_nh_1^{\frac{1}{2}})^{\frac{1}{3}}], \qquad g_2 = [(h_0+\lambda_nh_1^{\frac{1}{2}})^{\frac{1}{3}}-(h_0-\lambda_nh_1^{\frac{1}{2}})^{\frac{1}{3}}]3^{-\frac{1}{2}},$$

$$h_0 = -\tfrac{1}{2}[2\chi^3\lambda_n^6+3\chi(6-\chi-3\xi\zeta)\lambda_n^4/\xi+3(6\xi\zeta-\chi-3)\lambda_n^2/\xi^2+2/\xi^3],$$

$$h_1 = 27\left\{\frac{\chi^4\lambda_n^8}{\xi}-\chi^2[(\chi-\xi\zeta+4)^2+28\xi\zeta-24]\frac{\lambda_n^6}{4\xi^2} + [[\chi^2+2(1+\xi\zeta)^2](1-2\chi+\xi\zeta)+3\chi^3+27\chi\xi\zeta]\frac{\lambda_n^4}{2\xi^3}-[(\chi+4\xi\zeta-1)^2-24\xi^2\zeta^2+28\xi\zeta]\frac{\lambda_n^2}{4\xi^4}+\frac{\zeta}{\xi^4}\right\}.$$

(ii) $\xi = 0$

$$F_n = \frac{\chi}{\lambda_n(\zeta+\chi\lambda_n^2)}+4\zeta^{\frac{1}{2}}\exp[-\tfrac{1}{2}(1+\chi)\lambda_n^2\theta]\frac{\sin(\frac{1}{2}g_3\theta-\psi_0)}{g_3[g_3^2+(1+\chi^2)\lambda_n^4]^{\frac{1}{2}}},$$

$$G_n = \frac{1}{\lambda_n^2(\zeta+\chi\lambda_n^2)}-4\exp[-\tfrac{1}{2}(1+\chi)\lambda_n^2\theta]\frac{\sin(\frac{1}{2}g_3\theta-\phi_0)}{g_3[g_3^2+(1+\chi^2)\lambda_n^4]^{\frac{1}{2}}},$$

where

$$g_3 = \lambda_n[4\zeta - (1-\chi)^2\lambda_n^2]^{\frac{1}{2}}, \qquad \psi_0 = \tan^{-1}\left\{\frac{2\chi g_3\lambda_n^2}{[(1-\chi^2)\lambda_n^4 + g_3^2]}\right\}, \qquad \phi_0 = \tan^{-1}\left[-\frac{g_3}{(1+\chi)\lambda_n^2}\right].$$

Explicit Value of F_n^*

(i) $\xi \neq 0$

$$F_n^* = \exp(-\theta/2\xi)\sin(g_4\theta - \psi^*)/g_4\lambda_n,$$

where

$$g_4 = (\lambda_n^2/\xi - 1/4\xi^2)^{\frac{1}{2}}, \qquad \psi^* = \tan^{-1}[-g_4/\xi(1/4\xi^2 - g_4^2)].$$

(ii) $\xi = 0$

$$F_n^* = \exp(-\lambda_n^2\theta)/\lambda_n^3.$$

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Continuum Feedback Control of Instabilities on an Infinite Fluid Interface

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Field-coupled Rayleigh–Taylor instabilities on a fluid interface can be suppressed by means of feedback. Deflections of the interface are detected to provide a signal which is amplified and fed back to a structure coupled to the interface through the fields. A theoretical study is given of the conditions for stability of an infinite interface coupled to an active structure through a perpendicular electric field and a tangential magnetic field. In both cases the interface is assumed to be perfectly conducting. A traveling wavetrain analysis is used to show the regimes of stability as they depend on the Taylor wavelength, electric or magnetic pressure, feedback gain, and technique of sampling interface deflections. Emphasis is given to the effect of feedback derived from detecting surface deflections averaged over a sampling area and feeding back to that same area.

I. INTRODUCTION

ONE can judge from the literature that instability poses a prime limitation on the engineering of fluid-dynamic and plasma-dynamic systems. Stability is also of great concern in dealing with discrete systems, where the limitations are often obviated by the use of active devices and feedback techniques. It must have occurred to many that feedback can also provide the solution to continuum instability problems. By contrast to discrete systems, in a continuum, an infinite number of degrees of freedom must be controlled. Even if complete information about all of the particles in a fluid were available, the means to influence each particle in just the right way is difficult to find. For this reason, the usefulness of feedback in controlling fluid- and plasma-dynamic instabilities is largely determined by the feasibility: (i) of experimentally determining the important "flow" variable as a function of space and time without adversely effecting the flow, (ii) of obtaining a sufficient flow sampling, since only a finite amount of information about the flow is available in spite of the infinite number of degrees of freedom to be controlled, (iii) and of influencing the flow in the desired fashion by altering a flow variable as a function of space and time.

The field-coupled Rayleigh–Taylor instabilities described here are chosen for study because electric or magnetic surface forces can be used to influence the fluid. Because the instabilities involve a surface, the fluid motions must be detected and influenced in only two dimensions

The first of two situations to be considered is shown in Fig. 1, where the interface (surface tension T) between a highly conducting liquid (density ρ^u) and an insulating liquid or gas (density ρ^l) is