

Thermal stability of radiating fluids: The Bénard problem

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The Bénard problem of the radiating nongray fluids is examined in terms of the Eddington approximation. The nongrayness of radiation is prescribed by the ratio and product of the Planck and Rosseland means of the absorption coefficient, $\eta = (\alpha_P/\alpha_R)^{1/2}$ and $\alpha_M = (\alpha_P\alpha_R)^{1/2}$, respectively. Effects of radiation on the classical problem are then characterized by four parameters: the Planck number, ϕ_0 (the ratio of conduction to radiation), optical thickness, $\tau = \alpha_M d$ (d being the distance between the plates) nongrayness of the fluid η and the emissivity of boundaries ϵ_0 and ϵ_1 , respectively. The radiation in general has a stabilizing effect; decreasing ϕ_0 , increasing degree of nongrayness for $\eta > 1$, changing color of boundaries from black to mirror all delay the onset of instability. The boundary color and nongrayness of gas are responsible for the extrema observed in stability curves. Accuracy of the Eddington approximation is checked with the exact solution and the convergence of the approximate solution is studied in terms of the first and second approximations. Results are given for black-black, mirror-mirror, and black-mirror boundaries.

I. INTRODUCTION

The Bénard problem of radiating fluids received considerable attention in the past. In his original paper Goody¹ investigated the problem for free boundaries by considering the thin gas and thick gas approximations. Following Goody's approach Murgai and Khosla² and Khosla and Murgai,³ respectively, included the effects of magnetic field and rotation. Spiegel⁴ reconsidered the problem for rigid boundaries and for the entire range of optical thickness but neglected the effect of conduction; his formulation employed the integral form of the radiative transfer equation but neglected the effect of radiative boundaries from the disturbance equations. In a recent paper of mathematical context, Davis⁵ investigated the validity of the exchange of stabilities for slightly nonself-adjoint problems, and following Spiegel's formulation, applied his results to the Bénard problem. Recently, Christophorides and Davis⁶ included the effect of conduction. The above studies are all based on the gray gas. Only Gille and Goody⁷ considered the spectral effect of nongrayness and compared their results with some measurements.

The foregoing literature survey reveals that the effect of radiative boundary (color, temperature jump, perturbations), weighted nongrayness of gas, and the use and accuracy of the Eddington approximation have apparently been left untreated. The present study is aimed at these points.

II. FORMULATION

As is well known, an exact treatment of radiative transfer in a fluid leads to a formulation in terms of integrodifferential equations. The solution of these equations appears to be rather involved.⁴ Approximate theories have been developed which permit a formulation involving only differential equations. One such theory expresses radiation in terms of spherical harmonics, another in terms of a moment sequence. These

theories were originally developed for astrophysical studies⁸⁻¹¹ and were later employed in neutron transport theory.¹² Together with a brief reference in a footnote of the paper by Traugott,¹³ the works of Cheng¹⁴ and of Unno and Spiegel¹⁵ treat the general three-dimensional case with application to gasdynamics and astrophysics, respectively.

In the present study we confine ourselves to the first-order spherical harmonics. Later, we shall comment on the accuracy of the first, third, and fifth-order harmonics which are usually referred to as the P_1 , P_3 , and P_5 approximations in the neutron transport theory. Since the formulation based on the first-order spherical harmonics is identical to that based on the first two moments of the radiative transfer equation, the following brief review employs the latter approach because of its simplicity.

The usual (frequency averaged) transfer equation may be replaced, in terms of the first three moments of the intensity over the solid angle Ω ,

$$j = \int_{\Omega} I \, d\Omega = c_0 u^R, \quad q_i^R = \int_{\Omega} l_i I \, d\Omega,$$

$$\pi_{ij} = \int_{\Omega} l_i l_j I \, d\Omega = c_0 \tau_{ij}^R,$$

by its first two moments,

$$\partial_i q_i^R = \alpha_P (4E_b - j), \quad (1)$$

$$\partial_i \pi_{ij} = -\alpha_R q_j^R, \quad (2)$$

to be closed by the assumption of local isotropy,

$$\pi_{ij} = \frac{1}{3} j \delta_{ij} \quad (3)$$

which is the so-called Eddington approximation. Combining (2) and (3), we have

$$\partial_i j = -3\alpha_R q_i^R. \quad (4)$$

Here, l_i denotes the unit vector in the direction of propagation, c_0 is the velocity of light, u^R is the radiant internal energy, q_i^R is the radiant flux, τ_{ij}^R is the radiant

stress, $E_b = \sigma T^4$ is the blackbody radiation, σ is the Stefan-Boltzmann constant, T is the absolute temperature, and j and π_{ij} are introduced for notational convenience. Noting that $4E_b \gg j$ as $\tau \rightarrow 0$ and (1) should give the flux for thin gas, $\partial_i q_i^R = 4\alpha_P E_b$, and that $4E_b \rightarrow j$ for $\tau \rightarrow \infty$ and (4) should yield the flux for thick gas, $q_i^R = -(4/3\alpha_R)\partial_i E_b$, we, respectively, replaced, following Traugott,¹⁶ the gray absorption coefficient involved in (1) and (4) by its Planck mean α_P and Rosseland mean α_R . These mean values bring a weighted effect of nongrayness into our discussion. Combining (1) and (4), we get the balance of radiative energy,

$$\partial_i \partial_j q_j^R - 3\alpha_P \alpha_R q_j^R = 4\alpha_P \partial_j E_b, \quad (5)$$

or, in terms of j ,

$$\partial_i \partial_j j - 3\alpha_P \alpha_R j = -12\alpha_P \alpha_R E_b \quad (6)$$

which proves convenient in the following formulation.

Assuming the fluid incompressible, viscosity, and conductivity constant, neglecting viscous dissipation and radiative contributions to momentum, modifying the thermal energy by the radiant flux, and relating the blackbody radiation to temperature, we have

$$\partial_i u_i + u_k \partial_k u_i = X_i - (1/\rho_0)\partial_i p + \nu \partial_k \partial_k u_i,$$

$$\partial_i T + u_k \partial_k T = \kappa \partial_k \partial_k T + (\alpha_P/\rho_0 c_v)(j - 4\sigma T^4), \quad (7)$$

$$\partial_k \partial_k j - 3\alpha_P \alpha_R j = -12\alpha_P \alpha_R \sigma T^4,$$

where u_i is the velocity, X_i is the body force per unit mass, p is the pressure, ρ_0 is the reference value of density, ν is the kinematic viscosity, T is the absolute temperature, κ is the thermal diffusivity, and c_v is the specific heat at constant volume.

Although the fluid dynamical boundary conditions remain identical to those of the classical Bénard problem, thermal boundary conditions require the additional radiative conditions which we consider next. The boundary conditions compatible with the Eddington approximation and for one-dimensional problems may be found in Goody.¹⁷ Also Varma *et al.*¹⁸ developed, in terms of spherical harmonics, the conditions for black boundaries and gray gas (in this connection Mark^{19,20} and Marshak²¹ conditions of neutron transport theory may be noted). Here, we extend Goody's approach to three-dimensional problems, and further include the color of boundaries and the weighted nongrayness of gas. This approach appears to be simpler conceptually and shorter algebraically than that of Varma *et al.*¹⁸

In terms of the hemispherical intensities, the first moment of the intensity may be expressed as

$$j = j^+ + j^-, \quad (8)$$

and for boundaries in the x_1, x_2 -plane (1) reduces to

$$\frac{\partial q_w^R}{\partial x_3} = \alpha_P [4E_{b_0} - (j_w^+ + j_w^-)]. \quad (9)$$

Furthermore, the assumption of hemispherical isotropy gives

$$q_w^R = \frac{1}{2}(j_w^+ - j_w^-). \quad (10)$$

The hemispherical intensities, obtained from (9) and (10), may then be written as

$$j_w^+ = 2E_{b_0} + q_w^R - \frac{1}{2}\alpha_P \partial q_w^R / \partial x_3, \quad (11)$$

$$j_w^- = 2E_{b_0} - q_w^R - \frac{1}{2}\alpha_P \partial q_w^R / \partial x_3, \quad (12)$$

where E_b denotes the blackbody emissive power, the subscript 0 and w refer to fluid and wall values on the boundary. Furthermore, from the balance among j_w^+ , j_w^- , and E_{b_w} ,

$$j_w^+ = 2\epsilon E_{b_w} + \rho j_w^-, \quad (13)$$

where ϵ and ρ are the hemispherical (diffuse) emissivity and reflectivity of the wall, respectively. Combination of (10) and (13) yields

$$j_w^+ = 2E_{b_w} - 2(\rho/\epsilon)q_w^R. \quad (14)$$

Finally, elimination of j_w^+ between (11) and (14) gives the radiative boundary condition in terms of q_w^R ,

$$4(E_{b_w} - E_{b_0}) = q_w^R/\lambda - (1/\eta\tau)\partial q_w^R/\partial z, \quad (15)$$

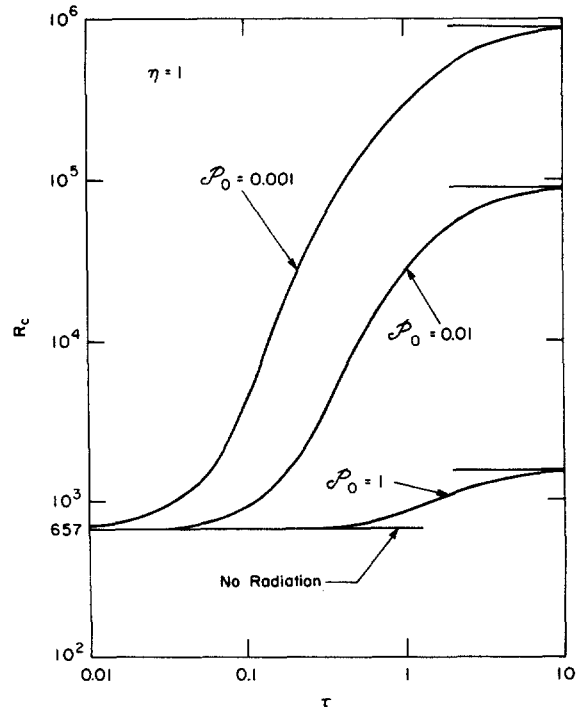


FIG. 1. The effect of Planck number on the gray gas with free boundaries.

where $1/\lambda = 4(1/\epsilon - \frac{1}{2})$, $\tau = \alpha_M d$, $\alpha_M = (\alpha_P \alpha_R)^{1/2}$, $\eta = (\alpha_P/\alpha_R)^{1/2}$, $z = x_3/d$, and d is the distance between the plates.

Since we formulated the governing equations in terms of j , the radiative boundary condition should be in terms of j_w rather than q_w^R . Accordingly, combining (8), (11), and (12) we have

$$j_w = 4E_{b_0} - (1/\eta\tau)\partial q_w^R/\partial z. \quad (16)$$

Elimination of q_w^R and $\partial q_w^R/\partial z$ among (2), (3), (9), (15), and (16) results in the appropriate form of the radiative boundary condition,

$$j_w = 4E_{b_0} + (\eta/3\lambda\tau)\partial j_w/\partial z + 4(E_{b_w} - E_{b_0}), \quad (17)$$

where the last term on the right-hand side (related to a temperature jump) exists only for the case of pure radiation (see Ref. 17, p. 54 for the boundary condition corresponding to gray gas and no temperature jump, expressed in terms of the heat flux). For mirror boundaries $q_w^R = 0$ which implies, in terms of (2) and (3),

$$\partial j_w/\partial z = 0. \quad (18)$$

Here, we conclude our formulation by stating the initial problem and the perturbation equations, respectively.

The initial steady formulation may be given most conveniently in terms of the temperature and the radiant flux as

$$\begin{aligned} k \frac{d^2 \bar{T}}{dx_3^2} - \frac{d\bar{q}_s^R}{dx_3} &= 0, \\ \frac{d^2 \bar{q}_s^R}{dx_3^2} - 3\alpha_P \alpha_R \bar{q}_s^R &= 4\alpha_P \frac{d\bar{E}_b}{dx_3}, \\ \bar{T}(0) = T_0, \quad \bar{T}(d) &= T_1, \\ \eta \bar{q}_s^R(0) - \frac{\lambda_0}{\alpha_M} \frac{d\bar{q}_s^R(0)}{dx_3} &= 0, \\ \eta \bar{q}_s^R(d) + \frac{\lambda_1}{\alpha_M} \frac{d\bar{q}_s^R(d)}{dx_3} &= 0. \end{aligned} \quad (19)$$

Since we are interested in the high-temperature level but not the large temperature differences, the last term of the radiative energy equation dE_b/dx_3 may be linearized. The solution of the foregoing problem is trivial and not elaborated here. The stability problem requires only the gradient of the initial state which is for nongray gas and gray boundaries

$$\beta/\beta_0 = M + N \sinh \varphi \tau (z - \frac{1}{2}) + K \cosh \varphi \tau (z - \frac{1}{2}), \quad (20)$$

where

$$\begin{aligned} \beta &= -\frac{d\bar{T}}{dx_3}, \quad \beta_0 = \frac{(T_0 - T_1)}{d}, \quad \varphi = (3 + 4\chi)^{1/2}, \quad \chi = \frac{\eta}{\varphi_0}, \quad \varphi_0 = \frac{\alpha_M k}{4\sigma T_0^3}, \\ H &= \frac{[1 + \lambda_0 \lambda_1 (\varphi/\eta)^2] \sinh \varphi \tau + (\varphi/\eta) (\lambda_0 + \lambda_1) \cosh \varphi \tau}{2 \sinh \varphi \tau / 2 + (\varphi/\eta) (\lambda_0 + \lambda_1) \cosh \varphi \tau / 2}, \quad M = \frac{H}{H + (8\chi/3\varphi\tau) \sinh \varphi \tau / 2}, \\ N &= \frac{(\lambda_0 - \lambda_1) (4\chi\varphi/3\eta) \sinh \varphi \tau / 2}{[2 \sinh \varphi \tau / 2 + (\varphi/\eta) (\lambda_0 + \lambda_1) \cosh \varphi \tau / 2] [H + (8\chi/3\varphi\tau) \sinh \varphi \tau / 2]}, \quad K = \frac{4\chi/3}{H + (8\chi/3\varphi\tau) \sinh \varphi \tau / 2}. \end{aligned}$$

The results for the mirror and black surfaces are readily obtainable from the foregoing equations by considering the limits $\lambda_0, \lambda_1 = 0$ and $\lambda_0, \lambda_1 = \frac{1}{2}$, respectively. For gray gas (20) reduces to that obtained by Goody¹ (see also Goody¹⁷).

Following the standard procedure, the linear stability problem may be analyzed in terms of the normal modes applied to x_3 components of the velocity and vorticity, w and ζ , the temperature, θ , and the first moment of intensity, j . Considering disturbances characterized by a particular wavenumber \bar{k} , we suppose that they (symbolized by b') have the general form

$$b'(x_1, x_2, x_3, t) = B(x_3) \exp[i(k_1 x_1 + k_2 x_2) + \bar{p}t],$$

$$\bar{k} = (k_1^2 + k_2^2)^{1/2}.$$

Then, the formulation of the stability problem (deleting

the uncoupled vorticity) may be given as

$$(D^2 - a^2)(D^2 - a^2 - c)W = (\kappa a^2 R / \beta_0 a^2) \Theta, \quad (21a)$$

$$(D^2 - a^2 - Pc - 4\chi r^2) \Theta + 3\chi r^2 J = -(\beta \bar{a}^2 / \kappa) W, \quad (22a)$$

$$(D^2 - a^2 - 3r^2)J = -4r^2 \Theta, \quad (23a)$$

where $D \equiv d/dz$, $c = \bar{p} \bar{a}^2 / \nu$, $a = d(k_1^2 + k_2^2)^{1/2}$ is the non-dimensional wavenumber, $R = g\alpha_0 \beta_0 d^4 / \kappa \nu$ is the Rayleigh number, P is the Prandtl number, α_0 is the coefficient of volumetric expansion, J is dimensionalized by $12\sigma T_0^3$, and $\bar{T} \sim T_0$ is assumed in (22).

Concerning the boundary conditions, those of the classical Bénard problem,

$$\Theta = W = 0 \quad \text{for } z = 0 \text{ and } 1, \quad (24)$$

$$DW = 0 \quad \text{for a rigid surface,} \quad (25a)$$

$$D^2 W = 0 \quad \text{for a free surface,} \quad (25b)$$

must be supplemented by the radiative conditions for a mirror surface

$$DJ=0, \tag{26}$$

and for gray surfaces,

$$\begin{aligned} J - (\eta/3\lambda_0\tau)DJ &= 0 & \text{for } z=0, \\ J + (\eta/3\lambda_1\tau)DJ &= 0 & \text{for } z=1. \end{aligned} \tag{27}$$

For $\lambda_0, \lambda_1 = \frac{1}{2}$, Eqs. (27) apply to free as well as black-rigid boundaries.

III. A VARIATIONAL FORMULATION

In this section we give a variational formulation for the critical Rayleigh number employing the concept of adjoint system (of differential equations) developed by Roberts.²² Other variational forms related to different formulations of the same problem already exist in the literature.^{4-6,17}

The adjoint system of the present formulation,

$$\lambda = \frac{(\beta_0 d^2 / \kappa) \int_0^1 [D^2 W D^2 \tilde{W} + (2a^2 + c) D W D \tilde{W} + a^2 (a^2 + c) W \tilde{W}] dz}{\int_0^1 \{D \Theta D \tilde{\Theta} + (a^2 + P c + 4\chi\tau^2) \Theta \tilde{\Theta} - (3\chi/4) [D J D \tilde{J} + (a^2 + 3\tau^2) J \tilde{J}]\} dz - (3\chi/4) J_0^1}, \tag{29}$$

where $J_0^1 = (3\tau/\eta)[\lambda_1 J(1)\tilde{J}(1) + \lambda_0 J(0)\tilde{J}(0)]$. Equation (29) reduces, with vanishing radiative effects, to the second variational principle given by Chandrasekhar.²³ It may readily be shown that λ obtained from (28) or (29) is stationary to the variations in Θ and $\tilde{\Theta}$, and that the approximate solution technique employed in the next section is identical to the foregoing variational formulation.²³

Since we neglected radiative contributions to the momentum equation, the onset of instability is expected to be stationary as it is in the classical Bénard problem. In two special cases considered so far, Spiegel⁴ assumed the base temperature constant and included the entire optical thickness; Davis,⁵ on the other hand, restricted himself to small optical thicknesses but incorporated the variation of the base temperature. It may further be shown in terms of approximate profiles, and for free boundaries, that the exchange of stabilities continues to be valid when both the variable base temperature and the entire optical thickness are included. However, the development appears to be lengthy and not interesting, and is not given here. In the next section, assuming the validity of the exchange of stabilities, we consider the marginal state corresponding to the stationary instability.

obtained by the usual steps, is

$$(D^2 - a^2)(D^2 - a^2 - c)\tilde{W} = (\kappa a^2 R / \beta_0 d^2)(\beta / \beta_0)\tilde{\Theta}, \tag{21b}$$

$$(D^2 - a^2 - P c - 4\chi\tau^2)\tilde{\Theta} + 3\chi\tau^2\tilde{J} = -(\beta_0 d^2 / \kappa)\tilde{W}, \tag{22b}$$

$$(D^2 - a^2 - 3\tau^2)\tilde{J} = -4\tau^2\tilde{\Theta}, \tag{23b}$$

with boundary conditions given by (24)-(27). Here, the superscript \sim denotes the adjoint problem. Defining

$$L \equiv (D^2 - a^2)(D^2 - a^2 - c), \quad \lambda \equiv \kappa a^2 R / \beta_0 d^2,$$

the characteristic value λ may be obtained from (21a) or (21b) as

$$\begin{aligned} \lambda &= \int_0^1 \tilde{W} L W dz / \int_0^1 \tilde{W} \Theta dz, \\ \lambda &= \int_0^1 W L \tilde{W} dz / \int_0^1 (\beta / \beta_0) W \tilde{\Theta} dz, \end{aligned} \tag{28}$$

respectively. After elementary manipulations based on several integration by parts, and on the appropriate use of (22a), (23a) or (22b), (23b), (28) may alternatively be stated as

IV. MARGINAL STATE

Equations (21), (22), and (23) with $c=0$ govern the marginal state subject to the boundary conditions (24)-(27). For symmetric (two black, two mirror, or two identical gray surfaces) the origin of the coordinates may be shifted to the middle plane. However, we also have cases with asymmetric boundaries. This suggests that the origin be retained at the lower plate and

$$\Theta = \sum_n A_n \Theta_n = \sum_n A_n \sin n\pi z \tag{30}$$

be considered with $n=1, 3, \dots$ for symmetric problems ($\lambda_0 = \lambda_1$), and with $n=1, 2, \dots$ for asymmetric problems ($\lambda_0 \neq \lambda_1$). Clearly, with respect to the middle plane, the first series is made of only even functions, whereas the second series is composed of both even and odd functions. When W and J are expressed as

$$W = (\kappa a^2 R / \beta_0 d^2) \sum_n A_n W_n, \quad J = \sum_n A_n J_n, \tag{31}$$

W_n and J_n satisfy

$$(D^2 - a^2)^2 W_n = \Theta_n = \sin n\pi z, \tag{32}$$

$$(D^2 - a^2 - 3\tau^2) J_n = -4\tau^2 \Theta_n = -4\tau^2 \sin n\pi z, \tag{33}$$

subject to boundary conditions (24)–(27). Having determined W_n and J_n from (32) and (33), we insert the expansions (30) and (31) in (22) to obtain

$$\sum_n A_n (n^2 \pi^2 + a^2 + 4\chi r^2) \sin n\pi z - 3\chi r^2 \sum_m A_m J_m = (\beta/\beta_0) a^2 R \sum_m A_m J_m \quad (34)$$

Multiplying (34) by $\sin n\pi z$ and integrating over the range of z we get an infinite set of linear homogeneous equations for A_m which in turn leads to the secular equation

$$|(1/R)[\frac{1}{2}(n^2 \pi^2 + a^2 + 4\chi r^2)\delta_{mn} - (m/n)_1] - (m/n)_2| = 0, \quad (35)$$

where

$$\left(\frac{m}{n}\right)_1 = 3\chi r^2 \int_0^1 J_m \sin n\pi z \, dz,$$

$$\left(\frac{m}{n}\right)_2 = a^2 \int_0^1 \frac{\beta}{\beta_0} W_m \sin n\pi z \, dz.$$

The critical value of R is obtained by satisfying the determinant (35) for an arbitrary wavenumber a and then obtaining its minimum with respect to a .

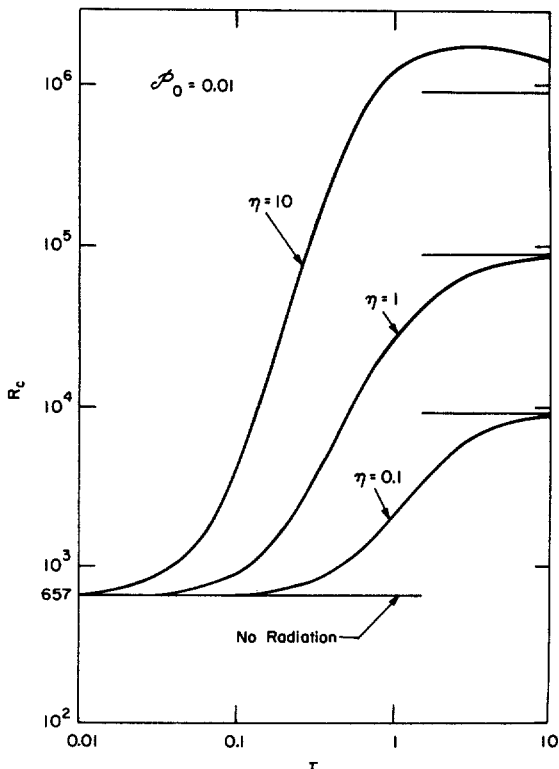


FIG. 2. The effect of nongrayness on the gas with free boundaries.

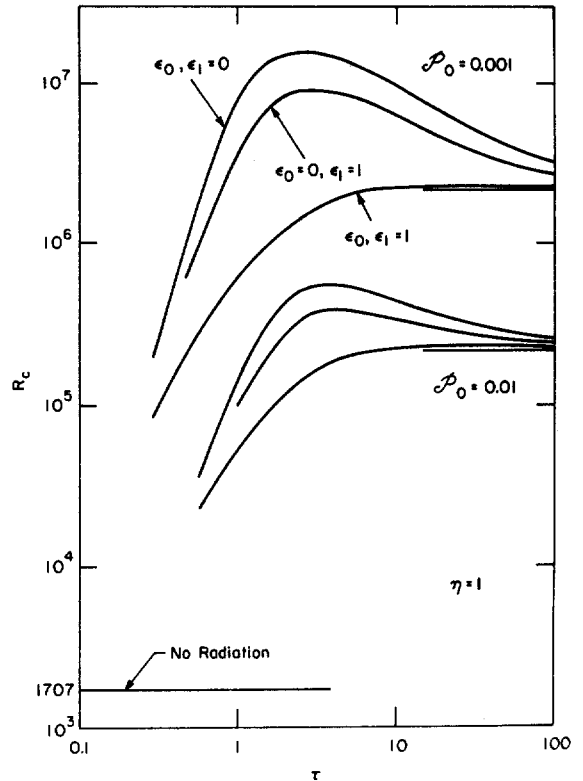


FIG. 3. The effect of nongrayness on the gas with black or mirror boundaries.

The effect of optical thickness, Planck number, nongrayness and surface color, are summarized in Figs. 1, 2, 3, and 4 for free and rigid boundaries. Calculations have been carried out in terms of the first and second approximations, the latter being based on the values of wavenumbers to the first approximation. For the computational convenience, the parameter $\varphi\tau/2$ has been taken equal to or larger than 6 in the case of rigid boundaries. This approximation allows the replacement of hyperbolic functions with exponentials, however, prevents the completion of Figs. 3 and 4 for small values of optical thickness.

V. PURE RADIATION

Although the pure radiation is a special case of the combined diffusion and radiation, it is not available from the limit of the latter for $\mathcal{P}_0 \rightarrow 0$. The discontinuity in boundary temperatures is the reason for this inconvenience. So the case of pure radiation, especially its initial steady state which requires a separate treatment, is briefly considered (see Ref. 24 for an alternative development).

The balance of steady, one-dimensional energy in the absence of diffusion is

$$d\bar{q}_s^R/dx_s = 0. \quad (36)$$

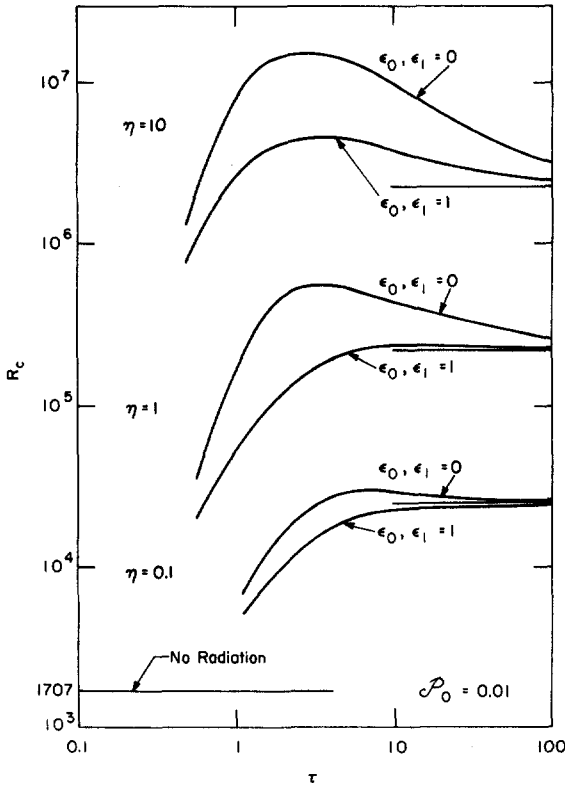


FIG. 4. The effect of Planck number on the gray gas with black, mirror, or black-mirror boundaries.

In view of Eq. (36), the radiant flux

$$\bar{q}_s^R = - (4/3\alpha_R) d\bar{E}_b/dx_s, \quad (37)$$

which is valid only for $\tau \rightarrow \infty$ when the diffusion is appreciable, now applies for all values of τ . Thus, the problem is reduced to a trivial diffusion problem in terms of $\bar{E}_b = \sigma \bar{T}^4$.

The temperature jump on boundaries, noting (36), may be written from (15) as

$$\begin{aligned} \bar{E}_{b_{w0}} - \bar{E}_{b_{00}} &= \bar{q}_s^R / 4\lambda_0, \\ \bar{E}_{b_{w1}} - \bar{E}_{b_{01}} &= -\bar{q}_s^R / 4\lambda_1, \end{aligned} \quad (38)$$

TABLE I. Critical Rayleigh numbers for gray gas with free boundaries ($\mathcal{P}_0=0.01$).

τ	a	R_c		Change in R_c %
		First approx.	Second approx.	
0.01	2.22	659.95		
0.1	2.34	908.72	908.7	0.
0.5	2.8	9 024.1	8 950.3	0.8
1.	2.67	26 635.	26 363.	1.03
3.	2.25	66 442.	65 197.	1.92
5.	2.22	79 275.	77 873.	1.8
7.	2.22	84 356.	83 281.	1.29

where $\bar{E}_{b_{w0}}$ and $\bar{E}_{b_{w1}}$ denote the wall values and $\bar{E}_{b_{00}}$ and $\bar{E}_{b_{01}}$ the fluid values on the lower and upper boundaries, respectively. Integrating (37), the constant radiant flux may be expressed in terms of the wall values of the fluid temperature,

$$\bar{q}_s^R = (4\eta/3\tau) (\bar{E}_{b_{00}} - \bar{E}_{b_{01}}), \quad (39)$$

or, by combining (38) and (39), in terms of the wall temperatures,

$$\bar{q}_s^R = \frac{\bar{E}_{b_{w0}} - E_{b_{w1}}}{1/\epsilon_0 + 1/\epsilon_1 - 1 + 3\tau/4\eta}. \quad (40)$$

Finally, the desired temperature gradient

$$\beta/\beta_0 = [\bar{T}(0) - \bar{T}(d)] / (T_0 - T_1),$$

where $\bar{T}(0)$ and $\bar{T}(d)$ denote the fluid wall temperatures, after linearizing \bar{E}_b by expanding into a Taylor series about T_0 , and employing (39) and (40), may be obtained as

$$\beta/\beta_0 = [1 + (1/\epsilon_0 + 1/\epsilon_1 - 1)(4\eta/3\tau)]^{-1}. \quad (41)$$

TABLE II. Critical Rayleigh numbers for gray gas with black boundaries ($\mathcal{P}_0=0.01$).

τ	a	R_c		Change in R_c %
		First approx.	Second approx.	
0.6	3.93	23 153.	23 575.	1.8
1.	4.16	53 401.	53 915.	0.95
3.	3.43	164 900.	164 880.	~0
5.	3.2	205 150.	205 180.	~0
10.	3.11	228 420.		
100.	3.11	231 570.		

The stability criterion for the present case can no longer be given in terms of the Rayleigh number which involves the thermal conductivity. However,

$$C = \mathcal{P}_0 R = g\alpha_0 \beta_0 d^4 / (4\sigma T_0^3 \nu / \alpha_M \rho_0 c_v)$$

becomes a suitable modified Rayleigh number. Neglecting the diffusion terms in (22) the secular determinant of the problem may be written as

$$\|(1/C)[2\eta\tau^2 \delta_{mn} - (m/n)_1] - (m/n)_2\| = 0, \quad (42)$$

where $(m/n)_1$ and $(m/n)_2$ were defined in connection with (35). It may be worth mentioning that the initial temperature gradient for the present case, although not equal to that for the case of pure diffusion, is constant. The corresponding stability formulation is self-adjoint, and the validity of the exchange of stabilities may be shown by the usual procedure.

VI. DISCUSSION

First, we examine the convergence of the approximate technique employed for the solution of the problem. Three typical computer outputs corresponding to free, black, and mirror boundaries are given in Tables I, II, and III. The critical Rayleigh numbers obtained in terms of the first and second approximations differ 3.6% at the most. In general, asymmetric cases appear to need higher order approximations for the same degree of convergence.

Next, we study the accuracy of the Eddington approximation (or P_1 approximation) by comparing it with Spiegel's exact solution.⁴ Since Spiegel neglects the effect of radiation on the base temperature and that of radiative boundaries on disturbance equations, we re-evaluate our results by neglecting the same effects. Figure 5 indicates that the P_1 approximation and the exact solution differ as much as 33% around $\tau \sim \pi$. Here, it may be interesting, in view of the availability of the P_3 and P_5 approximations from the neutron transport theory (adjusted to our problem by letting $\psi_0 \rightarrow j$, $\alpha\psi_0 \rightarrow 4E_b$ and $l = 1/\alpha_M$), to consider the improvements to be brought to our results by these approximations. The transfer equations corresponding, respectively, to the P_3 and P_5 approximations are

$$-\frac{3}{8}\nabla^4 j/\alpha_M^4 + \frac{1}{2}\nabla^2(j-4E_b)/\alpha_M^2 + \nabla^2 j/3\alpha_M^2 = j-4E_b, \tag{43}$$

$$\frac{5}{8}\nabla^6 j/\alpha_M^6 - \frac{1}{8}\nabla^4(j-4E_b)/\alpha_M^4 - \frac{1}{8}\nabla^4 j/\alpha_M^4 + \frac{3}{8}\nabla^2(j-4E_b)/\alpha_M^2 + \nabla^2 j/3\alpha_M^2 = j-4E_b, \tag{44}$$

(see Ref. 12, p. 161). The formulations based on the P_3 and P_5 approximations may readily be obtained by replacing (6) successively by (43) and (44). The corresponding solutions, except for the reasonably increased and tedious algebra, follow the steps of the P_1 approximation, and are not elaborated here.

The critical Rayleigh numbers for these approxima-

TABLE III. Critical Rayleigh numbers for gray gas with mirror boundaries ($\phi_0 = 0.01$).

τ	a	R_c		Change in R_c %
		First approx.	Second approx.	
0.6	3.29	38 077.	37 341.	1.98
1.	3.39	158 680.	153 020.	3.6
3.	3.05	556 340.	544 340.	2.2
5.	3.01	546 640.	536 090.	1.96
10.	3.03	448 950.		
100.	3.1	257 800.		

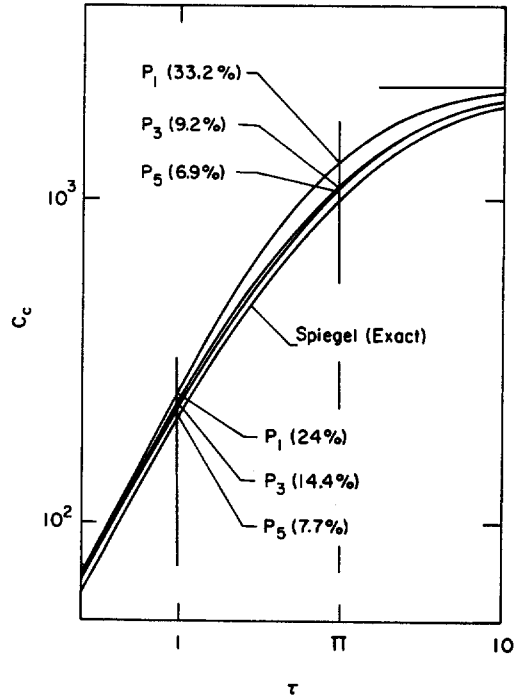


FIG. 5. Comparison of the P_1 , P_3 , and P_5 approximations with Spiegel.

tions given in Fig. 5 show a rather improved accuracy over the P_1 approximation, as expected.

The color change of boundaries from black to mirror and the increasing nongrayness of gas for $\eta > 1$ both flatten the initial temperature distribution and delay the onset of instability. These effects are responsible for the extrema of stability curves, a significant outcome of the present study, observed at intermediate optical thicknesses (about $\tau \sim \pi$).

Concerning the nongrayness of the fluids, we have not made an elaborate study of the approximate values of η for different fluids. It is known, however, that the Planck mean depends only on the temperature level, while the Rosseland mean depends on both on temperature and pressure. As we are interested in high-temperature levels but in neither large temperature differences, nor large pressure variations, the assumption of a constant η appears to be reasonable. It is also well known that the contribution to the Planck and Rosseland means comes from independent spectral regions; the former is dominated by the strong line centers, and the latter by the weak continua between bands. So depending on the temperature level and the particular gas, α_P may be one or two order of magnitude greater than α_R ; for only very low densities α_R may exceed α_P (Sampson,²⁶ p. 101). Thus, the possible range for η appears to be 0.1-10.

For the radiation parameter ϕ_0 , the numerical values

used in the literature are somewhat small and almost correspond to the case of pure radiation. This suggests an increase on the upper value to be employed for this parameter. Hence, $\varphi_0=1$ and 0.1 are considered in order to adequately demonstrate the effect of φ_0 which is shown in Figs. 1 and 4.

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