Instability of a Liquid Film Flowing down an Inclined Plane

S. P. LIN*
Department of Engineering Mechanics, The University of Michigan, Ann Arbor, Michigan
(Received 28 April 1966; final manuscript received 5 October 1966)

The mechanism of the instability of a layer of liquid flowing down an inclined plane is studied. First, the instability of the flow with respect to Tollimen–Schlichting wave is investigated. The results obtained are then compared with the known results of the instability of the same flow with respect to surface wave formation. It is found that, for a given angle of inclination, there exists a critical wavelength of the surface wave. If the wavelengths of the free surface disturbances are shorter than this critical wavelength, the film can become unstable due to shear waves before it becomes unstable with respect to surface waves. On the other hand, if the wavelengths of the free surface disturbances are longer than this critical wavelength, then the film will always become unstable due to free surface disturbances.

I. INTRODUCTION

THE problem of instability of a layer of fluid flowing down an inclined plane was first studied by Kapitza[1] and others. The first complete boundary conditions were given by Yih.[2] The governing differential equation is the well-known Orr–Sommerfeld equation. The boundary conditions are the nonslip condition at the bottom plane and the stress conditions at the free surface. The Orr–Sommerfeld equation and the boundary conditions constitute an eigenvalue problem. Benjamin[3] and Yih[4] studied the instability of the problem with respect to surface wave formation. In this paper, the instability with respect to Tollmien–Schlichting waves is studied. Several neutral-stability curves for different angles of inclination and surface tension have been obtained. The reduction of the angle of inclination and the surface tension are all shown to be stabilizing factors. One of the curves corresponding to $\beta = \frac{1}{2}\pi$ and zero surface tension is shown to be the neutral-stability curve of the plane Poiseulle flow. This neutral curve checks with that obtained by Lin.[5]

By comparing the results obtained in this paper and those obtained by Yih and Benjamin, it is shown that there exists a critical wavelength of the surface wave. If the wavelengths of the free surface disturbances are shorter than this critical wavelength, the film can become unstable due to shear waves. On the other hand, if the wavelengths of the free surface disturbances are longer than this critical wavelength, then the film will always become unstable due to free surface disturbances.

II. FORMULATION OF THE PROBLEM

Consider a layer of liquid flowing down an inclined plane, under the action of gravity. The plane is of infinite length and the flow is assumed to be parallel to the plate, so that the velocity component parallel to the x axis does not change along this axis. This primary flow is given by (cf. Fig. 1)

$$U = (g \sin \beta/2\nu)(d^2 - Y^2), \tag{1}$$

in which g is the gravitational acceleration, β the angle of inclination of the plane, ν the kinematic viscosity, d the depth of the fluid, and Y the distance from the free surface (positive if measured upward from the free surface).

By introducing the following dimensionless quantities:

$$y = Y/d, \qquad U = U/U_m,$$

Eq. (1) can be written in a dimensionless form

$$U(y) = (1 - y^2), \tag{2}$$

U_m being the maximum velocity of the primary flow, i.e.,

$$U_m = gd \sin \beta/2\nu. \tag{3}$$

The governing differential equation of the instability problem of this flow with respect to in-

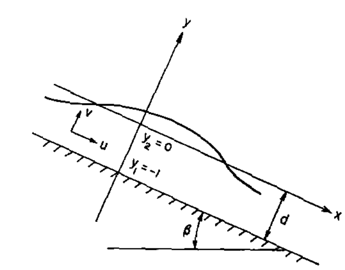

FIG. 1. Definition sketch.

* Present address: Mechanical Engineering Department, Clarkson College of Technology, Potsdam, New York.
[1] P. L. Kapitza, Zh. Eksperim. i Teor. Fiz. **18**, 3, 20 (1948); **19**, 105 (1949).
[2] C. S. Yih, in *Proceedings of the Second U. S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1955), p. 623.
[3] T. B. Benjamin, J. Fluid Mech. **2**, 554 (1957).
[4] C. S. Yih, Phys. Fluids **6**, 321 (1963).
[5] C. C. Lin, Quart. Appl. Math. **3**, 117, 218, 277 (1945).

finitesimal disturbances is the well-known Orr–Sommerfeld equation[6–8]

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(U - c)(\phi'' - \alpha^2\phi) - U''\phi], \quad (4)$$

in which ϕ is a function of y related to the velocity disturbances in the x and y directions by the following equations:

$$u' = \partial\psi/\partial y, \qquad (5)$$
$$v' = -\partial\psi/\partial x,$$

where

$$\psi = \phi(y) \exp [i\alpha(x - c\tau)]. \qquad (6)$$

In Eq. (6), c is the complex wave speed, α the wave number, and τ the time.

The boundary conditions as formulated by Yih[2] consists of the nonslip conditions at the bottom, and the stress conditions at the free surface. They are

$$\phi'(-1) = 0, \qquad \phi(-1) = 0,$$
$$\phi''(0) + (\alpha^2 - 2/c')\phi(0) = 0,$$
$$[\alpha(2 \cot \beta + \alpha^2 SR)/c']\phi(0) + \alpha(Rc' + 3i\alpha)\phi'(0) - i\phi'''(0) = 0,$$

in which $R = U_m\, d/\nu$ is the Reynolds number, $S = T/\rho\, dU_m^2$(T is the surface tension, ρ the density), and $c' = c - 1$.

Equation (4) and the above four boundary conditions constitute an eigenvalue problem. A nontrivial solution exists if there exists a relation between R, c, S, and α. The task is to obtain solution of the form $c = c(\alpha, R)$ for given values of β and S, so that a nontrivial solution exists. In general, c is complex, i.e., $c = c_r + ic_i$, in which $c_r = c_r(R, \alpha)$, $c_i = c_i(R, \alpha)$ for given β and S. As can be seen from Eq. (6), the flow is stable if $c_i < 0$ and unstable if $c_i > 0$. If $c_i = 0$, there is a sustained oscillation. The relation $c_i(R, \alpha) = 0$ gives the neutral-stability curve in the α–R plane.

The complete solution of the Orr–Sommerfeld equation can be written as

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4,$$

where ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are four independent particular solutions of the equation.

Substituting this into the boundary conditions, one has a system of simultaneous equations in C_1, C_2, C_3 and C_4. Nontrivial solutions of this system exist if

$$\begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi'_{11} & \phi'_{21} & \phi'_{31} & \phi'_{41} \\ (\phi''_{12} + l\phi_{12}) & (\phi''_{22} + l\phi_{22}) & (\phi''_{32} + l\phi_{32}) & (\phi''_{42} + l\phi_{42}) \\ (m\phi_{12} + j\phi'_{12} - i\phi'''_{12}) & (m\phi_{22} + j\phi'_{22} - i\phi'''_{22}) & (m\phi_{32} + j\phi'_{32} - i\phi'''_{32}) & (m\phi_{42} + j\phi'_{42} - i\phi'''_{42}) \end{vmatrix} = 0, \qquad (7)$$

in which

$$\phi_{ij} = \phi_i(y_j), \qquad (i = 1, 2, 3, 4, \quad j = 1, 2),$$
$$y_1 = -1, \qquad y_2 = 0, \qquad l = \alpha^2 - 2/c',$$
$$m = (\alpha/c)(2 \cot \beta + \alpha^2 SR), \qquad j = \alpha(Rc' + 3i\alpha).$$

The four particular solutions used in this analysis are

$$\phi_1 = \sum_{n=0} a_n(y - y_c)^{n+1}, \qquad (8)$$

$$y_c = -(1 - c)^{\frac{1}{2}}, \quad a_0 = 1, \quad a_1 = 1/2y_c, \quad a_2 = \tfrac{1}{6}\alpha^2,$$
$$a_n = -\{[n(n - 1) - 2]a_{n-1}2y_c\alpha^2 a_{n-2} - \alpha^2 a_{n-3}\}/2n(n + 1)y_c, \qquad (n \geq 3),$$
$$\phi_2 = \phi_1 \ln (y - y_c) + \sum_{n=0} b_n(y - y_c)^n, \qquad (9)$$
$$b_0 = -(1 - c)^{\frac{1}{2}}, \quad b_1 = 1, \quad b_2 = (-1/2y_c) + \tfrac{1}{2}y_c\alpha^2,$$
$$b_n = [n(n - 3)b_{n-1} + 2(1 - c)^{\frac{1}{2}}\alpha^2 b_{n-2} - \alpha^2 b_{n-3} + (2n - 3)a_{n-2} - 2y_c(1 - 2n)a_{n-1}], \quad (n \geq 3),$$

$$\Phi_3 = \int_\infty^\eta d\eta \int_\infty^\eta d\eta\ \eta H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}], \qquad (10)$$

$$\Phi_4 = \int_\infty^\eta d\eta \int_\infty^\eta d\eta\ \eta H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(i\alpha_0\eta)^{\frac{3}{2}}], \qquad (11)$$

where

$$\eta = (\alpha R)^{\frac{1}{3}}(y - y_c), \qquad \alpha_0 = [U'(y_c)]^{\frac{1}{3}},$$

and $H_{\frac{1}{3}}$ denotes the Hankel function of the third order. Φ_3 and Φ_4 were first used by Lin[5] for the problem of plane Poiseuille flow. It can be easily varified that Φ_3 decreases and Φ_4 increases exponentially as η while ϕ_1 and ϕ_2 remain their order of magnitude of unity. ϕ_1 and ϕ_2 are actually the power series solutions of the inviscid equation. The proper branch of $\ln (y - y_c)$ in (9) is $-\pi < \arg (y - y_c) < 0$.

[6] C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, New York, 1955).
[7] H. B. Squire, Proc. Roy. Soc. (London) **A142**, 621 (1933).
[8] C. S. Yih, Quart. Appl. Math. **12**, 434 (1955).

Having four particular solutions ϕ_1, ϕ_2 and Φ_3, Φ_4, one is in a position to use them to solve the secular equation. Before doing so, one can simplify (7) by order of magnitude analysis. To start with, one divides the third column by Φ_{31} and the fourth column by Φ_{42}. Thus Eq. (7) is rewritten as

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 & \Phi_{41}/\Phi_{42} \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} & \Phi_{41}'/\Phi_{42} \\ (\phi_{12}'' + l\phi_{12}) & (\phi_{22}'' + l\phi_{22}) & \left(\frac{\Phi_{32}''}{\Phi_{31}} + l\frac{\Phi_{32}}{\Phi_{31}}\right) & \left(\frac{\Phi_{42}''}{\Phi_{42}} + l\right) \\ \left(\phi_{12} + \frac{j}{m}\phi_{12}' - \frac{j}{m}\phi_{12}'''\right) & \left(\phi_{22} + \frac{j}{m}\phi_{22}' - \frac{i}{m}\phi_{22}'''\right) & \left(\frac{\Phi_{32}}{\Phi_{31}} + \frac{j}{m}\frac{\Phi_{32}'}{\Phi_{31}} - \frac{\Phi_{32}'''}{m\Phi_{31}}\right) & \left(1 + \frac{j}{m}\frac{\Phi_{42}'}{\Phi_{42}} - \frac{i}{m}\frac{\Phi_{42}'''}{\Phi_{42}}\right) \end{vmatrix} = 0. \tag{12}$$

The last row in Eq. (12) has been divided through by m. Each element in the first two columns are of order unity (if $j > m$, then j should be factored out instead of m, and the same is true). If one neglects the elements of exponentially small order compared with elements of order one, (12) can be reduced to

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 & 0 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} & 0 \\ (\phi_{12}'' + l\phi_{12}) & (\phi_{22}'' + l\phi_{22}) & 0 & \left(\frac{\Phi_{42}''}{\Phi_{42}} + l\right) \\ \left(\phi_{12} + \frac{j}{m}\phi_{12}' - \frac{i}{m}\phi_{12}'''\right) & \left(\phi_{22} + \frac{j}{m}\phi_{22}' - \frac{i}{m}\phi_{22}'''\right) & 0 & \left(1 + \frac{j}{m}\frac{\Phi_{42}'}{\Phi_{42}} - \frac{i}{m}\frac{\Phi_{42}'''}{\Phi_{42}}\right) \end{vmatrix} = 0. \tag{13}$$

Expansion of the above determinant with respect to its last column gives

$$\left(1 + \frac{j}{m}\frac{\Phi_{42}'}{\Phi_{42}} - \frac{i}{m}\frac{\Phi_{42}'''}{\Phi_{42}}\right)\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ (\phi_{12}'' + l\phi_{12}) & (\phi_{22}'' + l\phi_{22}) & 0 \end{vmatrix}$$
$$- \left(\frac{\Phi_{42}''}{\Phi_{42}} + l\right)\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \left(\phi_{12} + \frac{j}{m}\phi_{12}' - \frac{i}{m}\phi_{12}'''\right) & \left(\phi_{22} + \frac{j}{m}\phi_{22}' - \frac{j}{m}\phi_{22}'''\right) & 0 \end{vmatrix} = 0. \tag{14}$$

The first term of the above equation is of the order (αR) whereas the second term is of order $(\alpha R)^{\frac{2}{3}}$. Hence, for αR so large that $(\alpha R)^{-\frac{1}{3}}$ can be neglected compared with unity, one can approximate (14) by

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \left(\phi_{12} + \frac{j}{m}\phi_{12}' - \frac{i}{m}\phi_{12}'''\right) & \left(\phi_{22} + \frac{j}{m}\phi_{22}' - \frac{i}{m}\phi_{22}'''\right) & 0 \end{vmatrix} = 0, \tag{15}$$

since

$$\Phi_{42}''/\Phi_{42} + l \neq 0.$$

For the case of infinite surface tension and any β, or $\beta = 0$ and any surface tension, one has $m = \infty$. Thus (15) can be further reduced to

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \phi_{12} & \phi_{22} & 0 \end{vmatrix} = 0.$$

This is the secular equation for the stability of plane Poiseuille flow with respect to symmetric disturbances (ϕ odd in y).[5]

By use of the relations, $\phi_{12}''' = [\alpha^2 - 2(1 - c)^{-1}]\phi_{12}'$ and $\phi_{22}''' = [\alpha^2 - 2(1 - c)^{-1}]\phi_{22}'$ one can, for the case of $T = 0$ and $\beta = \frac{1}{2}\pi$, reduce (15) to

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \phi_{12}' & \phi_{22}' & 0 \end{vmatrix} = 0,$$

which is the secular equation of the stability problem of the plane Poiseuille flow with respect to antisymmetric disturbances. Thus, the neutral-stability curves, for any finite surface tension and any β of the present problem, can be expected to lie between the two neutral stability curves of plane Poiseuille flow. One of which is for the symmetric disturbances, the other for the antisymmetric disturbances.

III. METHODS OF COMPUTATION AND THE RESULTS

In terms of ϕ_1, ϕ_2 and Φ_3 the secular equation (15) can be written as

$$(\Phi_{31}/\Phi'_{31}) - (X + iY)/(U + iV) = 0, \quad (16)$$

where

$$\begin{aligned} X &= \phi_{11}(m\phi_{22} + fc'\phi'_{22}) \\ &\quad - (\phi_{11} \ln(1-a) + \psi_{31})(m\phi_{12} + fc'\phi'_{12}) \\ &\quad + \pi\phi_{11}(\phi'''_{12} - 3\alpha^2\phi'_{12}), \\ Y &= \phi_{11}(3\alpha^2\phi'_{22} - \phi'''_{22}) + \pi\phi_{11}(m\phi_{12} + fc'\phi'_{12}) \\ &\quad + [\phi_{11} l_n(1-a) + \psi_{31}](\phi'''_{12} - 3\alpha^2\phi'_{12}), \\ U &= \phi'_{11}(m\phi_{22} + fc'\phi'_{22}) \\ &\quad - \{\phi'_{11} \ln(1-a) + [\phi_{11}/(a-1)] + \psi_{31}\} \\ &\quad \cdot(m\phi_{12} + fc'\phi'_{12}) + \pi\phi'_{11}(\phi'''_{12} - 3\alpha^2\phi'_{12}), \\ V &= \phi'_{11}(3\alpha^2\phi_{22} - \phi'''_{22}) + \pi\phi'_{11}(m\phi_{12} + fc'\phi'_{12}) \\ &\quad - [\phi'_{11} \ln(1-a) + \phi_{11}(a-1) + \psi'_{31}] \\ &\quad \cdot(3\alpha^2\phi'_{12} - \phi'''_{12}), \end{aligned} \quad (17)$$

and

$$\Phi_{31}/\Phi'_{31} = (a-1)F(z), \quad (18)$$

in which

$$a = (1-c)^{\frac{1}{2}},$$

$$F(z) = \frac{-\int_{\infty}^{-z} d\zeta \int_{\infty}^{\zeta} \zeta^{\frac{1}{2}}\, d\zeta\, H_{\frac{1}{3}}^{(1)}[\frac{2}{3}(i\zeta)^{\frac{3}{2}}]}{Z[\int_{\infty}^{-z} \zeta^{\frac{1}{2}}\, d\zeta\, H_{\frac{1}{3}}^{(1)}[\frac{2}{3}(i\zeta)^{\frac{3}{2}}]}, \quad (19)$$

$$\zeta = (U'_c)^{\frac{1}{3}}\eta,$$

$$Z = -\zeta_1 = (\alpha R U'_c)^{\frac{1}{3}}(y_c - y_1), \quad (20)$$

$$f = \alpha R.$$

Substitution of (17) and (18) into (16) yields

$$\begin{aligned} (a-1)F_r(z) - u(\alpha, c, z, \beta, s) &= 0, \\ (a-1)F_i(z) - v(\alpha, c, z, \beta, s) &= 0, \end{aligned} \quad (21)$$

where

$$u = (XU + YV)/(U^2 + V^2),$$
$$v = (YU - XV)/(U^2 + V^2),$$

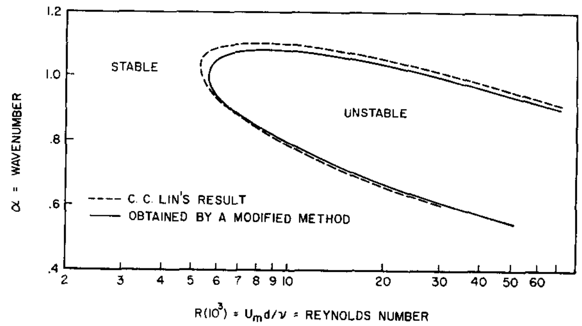

FIG. 2. Neutral stability curve for the case $S = 0.0$ and $\beta = \frac{1}{2}\pi$, i.e., neutral stability curve for plane Poiseuille flow.

$F_r(z)$ and $F_i(z)$ representing the real and the imaginary part of $F(z)$. For given values of β and S, (21) is a system of simultaneous equations in three unknowns α, c, and Z. Thus, for various values of c, one can solve (21) for α and Z. Then the corresponding R can be obtained from (20), i.e.

$$\alpha R = -[z/(a-1)]^3(1/2a).$$

A plot of the relation between α and R, thus obtained is a neutral-stability curve. In solving (21), Newton's method is used. With the above-described process, the lengthy computation was carried out by use of an electronic computer at The University of Michigan. The results are plotted in Figs. 2, 3, and 4. Only the numerical results for $\beta = 1°$, $SR = 0.0$ are given in this paper in Table I to indicate the general pattern of the variation of the wave speed with respect to wavelength.

IV. DISCUSSION ON THE RESULTS AND THE METHOD OF COMPUTATION

The neutral curve of the plane Poiseuille flow with antisymmetric disturbances is obtained as a special case of the present problem with $\beta = \frac{1}{2}\pi$ and $T = 0$. This curve checks very closely with Lin's result as shown in Fig. 2. In Fig. 3 are shown neutral-stability

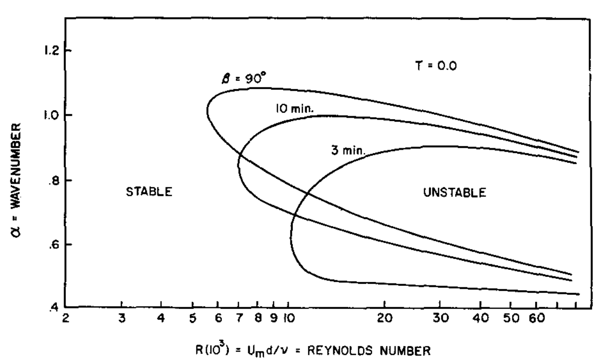

FIG. 3. The reduction of angle of inclination is a stabilizing factor.

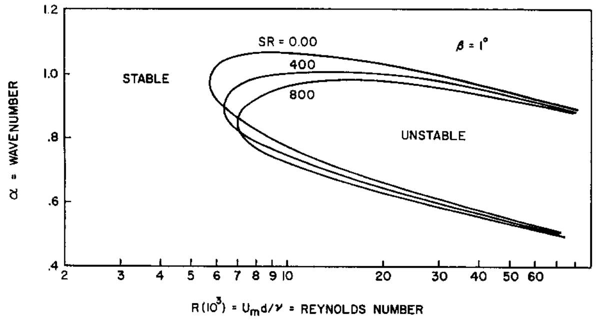

FIG. 4. Surface tension is a stabilizing factor.

curves for a layer of liquid of zero surface tension flowing down an inclined plane of different steepness. It is seen from this Fig. 3 that the reduction of the angle of inclination is a stabilizing factor. Figure 3 shows that the critical wavelength is increased as one decreases the angle of inclination. In order to see the effect of the surface tension, three neutral stability curves for given β and different values of $SR = 2T/(gd \sin \beta)$ are plotted in Fig. 4. These curves show that surface tension is a stabilizing factor. Numerical computation shows that for the case of $T = \infty$ or $\beta = 0$ no finite eigenvalues can be obtained from Eq. (16) for $0 < \alpha < 0.3$. Therefore a limiting curve of $T = \infty$ or $\beta = 0$ of this problem (which is also the neutral curve of plane Poiseuille flow with symmetric disturbances) does not exist in the finite α–R plane. For a given value of α, Grohne's[9] computation for the problem of plane Poiseuille flow with symmetric disturbances also indicates that R will approach infinity when $c_i = 0$. For various values of β and T, several neutral curves have been obtained. These neutral curves, however, are neutral curves with respect to the disturbances of a mode in which αR is much larger than unity. The instability of the same problem with

TABLE I. Eigenvalues for $SR = 0.0$, $\beta = 1°$.

Z	c	α	R
2.400196	0.070600	0.371858	4.152194×10^5
2.500516	0.129500	0.539594	5.164135×10^4
2.591080	0.172500	0.656387	1.976407×10^4
2.799280	0.236700	0.849765	7327.351990
2.995686	0.261900	0.966308	5791.709595
3.212561	0.264000	1.039895	6476.701111
3.368391	0.255000	1.062553	8126.813538
3.552588	0.237000	1.060653	1.195320×10^4
3.800499	0.204400	1.014058	2.406454×10^4
3.992671	0.175000	0.946671	4.799068×10^4
4.191120	0.143000	0.852935	1.138520×10^5
4.400189	0.110000	0.736120	3.382576×10^5
4.602193	0.081500	0.618260	1.141233×10^6

[9] D. Grohne, Z. Angew. Math. Mech. **34**, 344 (1954).

TABLE II. Critical Reynolds numbers.

β	3′	10′	1°	90°
R_s	1433	430	72	0
R_h	10 200	6900	6400	5500

respect to the disturbance of another mode in which αR is much less than unity has been studied by Yih and Benjamin. In their results

$$\tfrac{5}{4}\alpha R - \tfrac{1}{3}\alpha(3 \cot \beta + \alpha^2 sR) = 0$$

gives the neutral curve, and $R = \frac{1}{4}(5 \cot \beta)$ is given as a critical Reynolds number for very long waves. The critical Reynolds number (denoted by R_h obtained in this paper for $T = 0$ are listed in Table II for comparison. Table II shows that $R_s < R_h$ for $\beta > 3'$. Although the neutral curves for $\beta < 3'$ have not been obtained, the general trend of the neutral curves for different β and the computation for the solution of Eq. (16) (which is the secular equation corresponding to the case of $\beta = 0$) indicate that $\alpha \to 0$ as $\beta \to 0$. Therefore $R_s < R_h$ also for $\beta < 3'$, since it is *a priori* known that $(\alpha R)_s < (\alpha R)_h$ for all β, and $\alpha \to 0$ for both modes when $\beta \to 0$. Thus, it is shown that $R_s < R_h$ for all β and $T = 0$. The general features of the neutral curves for the soft and hard modes are shown in Fig. 5. The two curves never intersect with each other, since in their range of validity $(\alpha R)_s < (\alpha R)_h$ along the neutral-stability curves. The above-mentioned two different modes represent two different types of disturbances. The mode in which αR is much larger than unity corresponds to hard waves or Tollmien–Schlichting waves which damp or grow rapidly. The other mode in which αR is less than unity corresponds to soft waves or surface waves which damp or grow less rapidly than the former mode. This can be seen in the following way: Given c_i, $(\alpha R)_s < (\alpha R)_h$ for all allowable values of α. Thus, $\alpha_s < \alpha_h$, for given c_i

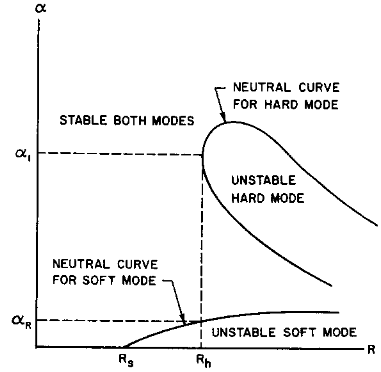

FIG. 5. General features of neutral-stability curve.

and R; consequently, $(\alpha c_i)_s < (\alpha c_i)_h$. That is to say the rate of growth or decay of soft waves is smaller than that of hard waves for given c_i. Thus, it is probable that although the flow is more stable with respect to shear waves than with surface waves of very long wavelength, the shear waves may eventually take over and the disturbances of short wavelength will spread over the entire flow. The α_R as shown in Fig. 5 is a critical wavenumber which can be obtained by intersecting $R = R_h$ and the neutral curve of the soft mode. It is clear from Fig. 5 that there exists a critical wavelength (corresponding to α_R) of the surface wave. If the wavelength of the free surface disturbances are shorter than this critical wavelength, the film can become unstable due to shear waves of shorter wavelength (corresponding to α_1).

The difficulty of the computation in this problem arises from the following two facts: First of all, the imaginary part of the secular equation is not a function of c alone; therefore, a fairly simple computational scheme used by Lin in solving the problem of plane Poiseuille flow cannot be applied. Another fact is that the eigenvalues as well as flow parameters appear in the boundary conditions. As a result the secular equation becomes considerably more complicated. The assumption made in the analysis is that the flow will not become turbulent right after the instability due to surface waves of infinitely long wavelength occurs, and the linear theory of instability still applies.

V. CONCLUSION

A layer of liquid film flowing down an inclined plane will become unstable with respect to surface wave of infinitely long wavelength at a critical Reynolds number $R = \frac{1}{4}(5 \cot \beta)$. As one increases the Reynolds number of the flow beyond this critical value, the film will also become unstable with respect to shear waves at critical Reynolds numbers which are obtained numerically by the method given in this paper. If the wavelength of the free surface wave is finite, then there exists a critical wavelength for given surface tension and angle of inclination. The film will become unstable due to shear waves if the wavelength of the surface disturbances is less than this critical length. On the other hand, if the surface wavelength exceeds this critical value, the film will always become unstable due to surface wave formation. Since the shear waves grow more rapidly than the surface waves for given c_i, the surface waves will probably be overshadowed eventually by the Tollmien–Schlichting waves as the flow becomes turbulent. It is found that both surface tension and the reduction of the angle of inclination are stabilizing factors. As a special case of this problem, the plane Poiseuille flow is shown to be always stable with respect to symmetric disturbances.

ACKNOWLEDGMENTS

This work is taken from the writer's doctoral dissertation. The writer wishes to express his appreciation to Professor Chia-Shun Yih who suggested the problem and also directed the study of the problem.

During the pursuance of this work, the writer received some financial support from the National Science Foundation and the Army Research Office (Durham).