theory has been carried out for a strip rather than a cylinder, and (3) the analysis was based on the model equation rather than the Boltzmann equation, the agreement between theory and experiment indicated in Figs. 5, 6, and 7 appears to be quite satisfactory.

V. CONCLUSIONS

(1) Experimental values for the drag coefficient of cross-stream cylinders in supersonic flow have been determined for a particular set of thermal conditions but for Knudsen numbers extending from continuum to free-molecule flow.

(2) The results are a smooth interpolation between inviscid values at low Knudsen number and free-molecule-flow predictions for diffuse reflection at high Knudsen number.

(3) There seems to be reasonable agreement between the experimental values and the theoretical predictions for the near-free-molecule-flow range of high Knudsen number.

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Stability of Liquid Flow down an Inclined Plane

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The stability of a liquid layer flowing down an inclined plane is investigated. A new perturbation method is used to furnish information regarding stability of surface waves for three cases: the case of small wavenumbers, of small Reynolds numbers, and of large wavenumbers. The results for small wavenumbers agree with Benjamin's result obtained by the use of power series expansion, and the results for the two other cases are new. The results for large wavenumbers, zero surface tension, and vertical plate contradict the tentative assertion of Benjamin. The three cases are then re-examined for shear-wave stability, and the results compared with those for confined plane Poiseuille flow. The comparison serves to indicate the vestiges of shear waves in the free-surface flow, and to give a sense of unity in the understanding of the stability of both flows. The case of large wavenumbers also serves as a new example of the dual role of viscosity in stability phenomena.

The topological features of the c_i curves for four cases (surface tension $= 0$ or $\neq 0$ and angle of plate inclination $=$ or $< \frac{1}{2}\pi$) are depicted. The effect of variability of surface tension is briefly assessed.

I. INTRODUCTION

THE stability of the laminar flow of a liquid layer, analyzed inexactly by Kapitza,[1] was first rigorously formulated by Yih[2] (henceforth referred to as I), who solved the Orr–Sommerfeld equation by an expansion in powers of αR. The resulting secular equation was solved by numerical computation, involving the solution of simultaneous nonlinear algebraic equations. Whereas the numerical computation produced the result that the flow down a vertical plane is unstable for Reynolds numbers larger than 1.5, thus establishing the instability of the flow at low Reynolds numbers, it was not accurate enough, and both the shape of the neutral stability curve and the values of the wave speed given in I are incorrect. In a paper based on Yih's formulation and on a variation of his method, Benjamin[3] performed a new calculation, with the important difference that his neutral-stability curves were obtained analytically, instead of numerically. His calculation established the result that free-surface flow down a vertical plane is unstable for all finite Reynolds numbers, and

[1] P. L. Kapitza, Zh. Eksperim. i Teor. Fiz. **18**, 3 (1948); **18**, 20 (1948); **19**, 105 (1949).

[2] C.-S. Yih, "Stability of Parallel Laminar Flow with a Free Surface," *Proceedings of the Second U. S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1955), pp. 623–628.

[3] T. B. Benjamin, J. Fluid Mech. **2**, 554 (1957).

gave values for the wave speed which are more in accord with experiments.[4] The present paper, on the same subject, has been written for the following reasons.

Yih's numerical computation and Benjamin's power expansion are both very laborious. It is desirable to have a simple method for the solution of problems of the same kind. A perturbation procedure based on Yih's expansion (I) provides just such a method. The agreement of the results obtained by this new method with Benjamin's should dispel the feeling in the minds of some of the people working on free-surface instability that there is a fundamental difference between Yih's expansion and Benjamin's. But quite apart from this, the perturbation procedure provides a powerful method for solving stability problems involving free surfaces or interfaces, and is itself worth presenting. It is presented in this paper.

The nature of the axis $\alpha = 0$ (α is the wavenumber) and the topology of the curves for constant c_i in the α–R plane ($\alpha c_i \sim$ rate of amplification or damping, R is the Reynolds number) have not been clarified. It is hoped that this paper will provide such a clarification.

The plane Poiseuille flow is known to be unstable only at rather high Reynolds numbers. Since free-surface flow is one-half the plane Poiseuille flow, it is rather surprising that the free surface should make it unstable at very much lower Reynolds numbers. Should not there remain some features of the stability of the free-surface flow which are similar to those of the plane Poiseuille flow? Why should the features of the stability of the plane Poiseuille flow disappear so completely when a free surface is present? Clarification of this point leads not only to the understanding of the correct choice of mathematical approximations to be made in dealing with problems of free-surface instability, but also to a better understanding of the physics of the phenomenon. This paper contains such a clarification.

Benjamin's calculation is based on the assumption that α is small. For this reason Benjamin did not consider his calculation applicable to values of α which are not small. For the case of vertical flow with zero surface tension, he gave the dashed line $\alpha = 0.43$ (approximately) as the estimated neutral-stability curve. This is incorrect, and has misled some people to obtain such a neutral-stability curve with a high-speed computer. Here Yih's method,[2] coupled with the new perturbation procedure, provides results at low Reynolds numbers for any value of α, however large. These results, which cannot be obtained by Benjamin's power series expansion, are presented in this paper. They show that the entire axis $R = 0$ is part of a neutral-stability curve for vertical film flows if surface tension is zero, and that there is no bifurcation point enabling the curve to branch out. The greater versatility of the expansion in powers of αR is thus demonstrated.

The question at large values of the wavenumber α has so far not been touched. It will be discussed in this paper. The pertinent result furnishes a new example of the dual role of viscosity, i.e., a new example of the destabilizing effect of viscosity.

The free-surface boundary condition involving shear will be formulated with variable surface tension taken into account, and the effect of this variability is briefly assessed in this paper.

[4] A. M. Binnie, J. Fluid Mech. 2, 551 (1957).

II. FORMULATION OF THE PROBLEM

For the sake of completeness, the formulation of the problem is presented. With reference to Fig. 1, the primary flow, assumed steady, is parallel to the X axis, with the velocity $\bar{u}$ varying only with Y. Since the pressure gradient in the X direction and the velocity component parallel to Y are zero, the Navier–Stokes equations are simply

$$\rho g \sin \beta + \mu \, d^2\bar{u}/dY^2 = 0, \tag{1}$$

$$d\bar{p}/dY = \rho g \cos \beta, \tag{2}$$

in which ρ is the (constant) density, g the gravitational acceleration, μ the viscosity, and $\bar{p}$ is the pressure of the primary flow. The coordinates X and Y and the angle of inclination β of the plane boundary are all defined in Fig. 1.

Equation (1) can be integrated with the boundary conditions $\bar{u} = 0$ at $Y = d$, and $du/dY = 0$ at $Y = 0$, since d is the depth of the primary flow and the mean free surface is at $Y = 0$, where the shear stress must vanish. The result is

$$\bar{u} = (g \sin \beta/2\nu)(d^2 - Y^2),$$

or

$$U(y) = \tfrac{3}{2}(1 - y^2), \tag{3}$$

in which

$$U = \bar{u}/\bar{u}_a\,,\ \bar{u}_a = \text{average velocity of the primary flow} = g\, d^2 \sin \beta/3\nu,\ y = Y/d. \tag{4}$$

The Reynolds number and Froude number will be defined to be

$$R = \bar{u}_a d/\nu, \qquad F = \bar{u}_a/(gd)^{\frac{1}{2}}. \tag{5}$$

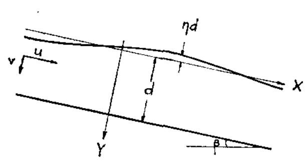
The second equation in (4) can be written as

$$3F^2 = R \sin \beta. \tag{6}$$

Squire's result[5] for the stability of three-dimensional disturbances in unidirectional flows between rigid boundaries has been extended to flows with free surfaces, interfaces, or density stratification.[6] For these flows, the primary flow is stable or unstable for a three-dimensional disturbance according as it is stable or unstable for a two-dimensional disturbance at a lower Reynolds number, a milder slope, and a reduced pressure gradient in the direction of flow. (For the flow under study, the pressure gradient in the X direction is zero. Hence no reduction is necessary.) Consequently, it is sufficient to consider two-dimensional disturbances only.

With the origin of the Cartesian coordinates (shown in Fig. 1) at the free surface, and with u

FIG. 1. Definition sketch.



and v denoting the velocity components in the directions of X and Y, respectively, the Navier–Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial X} + g \sin\beta + \nu\Delta u,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial Y} + g \cos\beta + \nu\Delta v,$$

in which t is the time, p the pressure, and Δ the Laplacian operator. The equation of continuity is

$$\partial u/\partial X + \partial v/\partial Y = 0.$$

By the substitutions

$$(u_1, v_1) = (u, v)/\bar{u}_a, \qquad (x, y) = (X, Y)/d,$$

$$p_1 = p/\rho\bar{u}_a^2, \qquad \tau = t\bar{u}_a/d,$$

the equations of motion and of continuity can be written in the following dimensionless forms:

$$\frac{\partial u_1}{\partial \tau} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} = \frac{\partial p_1}{\partial x} + \frac{\sin\beta}{F^2} + \frac{1}{R}\Delta u_1, \tag{7}$$

$$\frac{\partial v_1}{\partial \tau} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{\cos\beta}{F^2} + \frac{1}{R}\Delta v_1, \tag{8}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0. \tag{9}$$

Let

$$u_1 = U + u', \qquad v = v', \qquad p = P + p', \tag{10}$$

in which U and P are the (dimensionless) velocity and pressure of the primary flow, and the accented quantities are the velocity and the pressure perturbations. Substitution of (10) into (5), (6), and (7) yields

$$u'_\tau + Uu'_x + U_y v' = -p'_x + (1/R)\Delta u', \tag{11}$$

$$v'_\tau + Uv'_x = -p'_y + (1/R)\Delta v', \tag{12}$$

$$u'_x + v'_y = 0, \tag{13}$$

if terms quadratic in the perturbation quantities are neglected. In obtaining (11) and (12), the fact that U and P satisfy (5) and (6) has been utilized. The subscripts in (11), (12), and (13) denote partial differentiation.

Equation (13) allows the use of a (dimensionless) stream function ψ, in terms of which u' and v' can be expressed as follows:

$$u' = \psi_y, \qquad v' = -\psi_x.$$

Equations (11) and (12) can then be written as

$$\psi_{y\tau} + U\psi_{xy} - U_y\psi_x = -p'_x + (1/R)\Delta\psi_y, \tag{14}$$

$$\psi_{x\tau} + U\psi_{xx} = -p'_y + (1/R)\Delta\psi_x. \tag{15}$$

The boundary conditions at the bottom ($y = 1$) are

$$\text{(i)} \quad u' = \psi_y = 0, \qquad \text{(ii)} \quad v' = -\psi_x = 0.$$

At the free surface the shear stress must vanish and the normal stress must just balance the normal stress induced by surface tension. Hence the boundary conditions at the free surface are

$$\text{(iii)} \quad \frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y} = 0,$$

$$\text{(iv)} \quad \left(-p_1 + \frac{2}{R}\frac{\partial v_1}{\partial y}\right)\rho\bar{u}_a^2 + T\frac{\partial^2(\eta d)}{\partial X^2} = 0,$$

or

$$-p_1 + \frac{2}{R}\frac{\partial v_1}{\partial y} + S\frac{\partial^2\eta}{\partial x^2} = 0, \qquad S = \frac{T}{\rho d\bar{u}_a^2},$$

in which ηd is the displacement of the free surface from its mean position. The free-surface conditions must be applied at $y = \eta$, not at $y = 0$, because the gradients of the shear stress and of the pressure

[5] H. B. Squire, Proc. Roy. Soc. (London) **A142**, 621 (1933).
[6] C.-S. Yih, Quart. Appl. Math. **12**, 434 (1955).

of the primary flow at $y = 0$ are not zero. Hence (iii) and (iv) can be rewritten in the forms

$$\text{(iii)}\quad (d^2U/dy^2)\eta + \psi_{yy} - \psi_{xx} = 0,$$

$$\text{(iv)}\quad -P - P_y\eta - p' - (2/R)\psi_{xy} + S\eta_{xx} = 0,$$

which, apart from the terms containing η, are now to be applied at $y = 0$. Since $P(0) = 0$, and $P_y(0) = \cos\beta/F^2$, (iv) can be written further as

$$\text{(iv)}\quad (\cos\beta/F^2)\eta + p' + (2/R)\psi_{xy} - S\eta_{xx} = 0.$$

The (dimensionless) displacement η is related to ψ by the kinematic condition at the free surface

$$-\psi_x = \eta_\tau + U\eta_x. \tag{16}$$

As usual, a sinusoidal disturbance is assumed. If the disturbance is in the form of a "cloud" vanishing at $x = \pm\infty$, it can be expressed in terms of the elemental sinusoidal disturbances by means of a Fourier integral. Assume

$$\psi = \phi(y) \exp[i\alpha(x - c\tau)], \quad p' = f(y) \exp[i\alpha(x - c\tau)], \tag{17}$$

in which α is the wavenumber defined by $2\pi d/\lambda$, λ being the wavelength, and $c = c_r + ic_i$, c_r being the wave velocity and αc_i being the rate of amplification or damping. The kinematic condition at the free surface becomes

$$\eta = [\phi(0)/c'] \exp[i\alpha(x - c\tau)], \quad c' = c - \tfrac{3}{2}, \tag{18}$$

and the equations of motion become

$$-i\alpha c\phi' + i\alpha U\phi' - i\alpha U'\phi = -i\alpha f + (1/R)(\phi''' - \alpha^2\phi'), \tag{19}$$

$$\alpha^2 c\phi - \alpha^2 U\phi = f' + (1/R)(i\alpha\phi'' - i\alpha^3\phi), \tag{20}$$

in which the primes now denote differentiation with respect to y. If f is eliminated from (19) and (20) by cross differentiation, the well-known Orr–Sommerfeld equation results:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(U - c)(\phi'' - \alpha^2\phi) - U''\phi]. \tag{21}$$

The boundary conditions (i), (ii), and (iii) now assume the forms

$$\text{(i)}\quad \phi'(1) = 0, \tag{22}$$

$$\text{(ii)}\quad \phi(1) = 0, \tag{23}$$

$$\text{(iii)}\quad \chi(0) \equiv \phi''(0) + (\alpha^2 - 3/c')\phi(0) = 0. \tag{24}$$

The p' in boundary condition (iv) can be evaluated from the second equation in (17), with f given by (19). The final form of the boundary condition (iv) is

$$\text{(iv)}\quad \theta(0) \equiv [\alpha(3\cot\beta + \alpha^2 SR)/c']\phi(0) + \alpha(Rc' + 3\alpha i)\phi'(0) - i\phi'''(0) = 0. \tag{25}$$

Equation (21) and the boundary conditions constitute an eigenvalue problem. For a nontrivial solution, a relationship

$$c = c(R, F, \alpha)$$

must exist between R, F, α, and c. This relationship is complex, and can be resolved into the relationships

$$c_r = c_r(R, F, \alpha), \qquad c_i = c_i(R, F, \alpha).$$

Since R, F, and the angle β are related by (6), on putting $c_i = 0$ the equation

$$c_i(R, F, \alpha) = 0 \tag{26}$$

defines a relationship between R and α for a given value of β, and the graph depiciting this relationship is the neutral-stability curve.

III. SOLUTION FOR LONG WAVES

For wavelengths that are long compared with the depth d, the wavenumber α is small, and a solution of the differential system defining stability by successive perturbation is possible. For the first approximation, α can be simply set equal to zero, and the differential equation becomes

$$\phi'''' = 0, \tag{27}$$

with the boundary conditions

$$\text{(i)}\quad \phi(1) = 0, \qquad \text{(ii)}\quad \phi'(1) = 0,$$

$$\text{(iii)}\quad \phi''(0) - (3/c')\phi(0) = 0, \qquad \text{(iv)}\quad \phi'''(0) = 0.$$

The differential system is indeed an extremely simple one. Integration of (27) produces

$$\phi = A + By + Cy^2 + Dy^3, \tag{28}$$

and (iv) demands $D = 0$. Boundary conditions (i) and (ii) then give

$$A + B + C = 0, \quad \text{and} \quad B + 2C = 0,$$

or

$$C = A, \quad \text{and} \quad B = -2A.$$

Then (iii) demands

$$2C - (3/c')A = 0.$$

Hence

$$c' = \tfrac{3}{2}, \quad \text{or} \quad c = 3. \tag{29}$$

Since the differential system is linear and homogeneous, the eigenfunction is determined only up to a multiplicative constant. We shall take $A = 1$. The eigenfunction is then simply

$$\phi(y) = 1 - 2y + y^2 = (1 - y)^2. \qquad (30)$$

For the sake of clarity, this expression will be denoted by $\phi_0(y)$ in subsequent approximations.

Equation (29) gives a value of c which is $\frac{3}{2}$ times that of Benjamin. This is because the reference velocity is taken to be the average velocity of the primary flow in this paper (and in I), whereas Benjamin uses the surface velocity of the primary flow as the reference velocity. The present result is therefore in agreement with Benjamin's.

The opinion exists that there is a difference in the results obtained, on the one hand, by putting α or R equal to zero in the differential system and, on the other, by allowing α and R to vanish in the expression for c laboriously obtained from the original, unsimplified differential system. Now the differential equation retains its order when either α or R vanishes. The eigenfunction is an entire function of α and R, as can be seen by solving (21) by a power series and considering its convergence for finite α and R, and as the boundary conditions are not singular in α or R if they are finite. Hence the aforementioned opinion has no foundation whatever. The case of infinite Reynolds number, which has occupied the attention of researchers for almost exactly three-quarters of a century, is quite another matter, for if R is set equal to infinity to start with, the order of the Orr–Sommerfeld equation is reduced by two. Hence the stability or instability of flows at large Reynolds numbers cannot be settled by considering an inviscid fluid. The opinion mentioned above results perhaps from a lack of understanding of the difference between the classical case of large Ryenolds numbers and the case under study in this paper.

Equation (28) shows that the axis $\alpha = 0$ is a part of the neutral-stability curve, whatever the angle β, the surface tension, and, indeed, the Reynolds number. It is a valid approximation for small values of α so long as αR is small, even though R may be large. The next step is to see how the eigenvalue c will be modified as α departs from zero. We note that terms of first power in α are associated with R, and that if ϕ in

$$i\alpha R[(U - c)(\phi'' - \alpha^2\phi) - U''\phi]$$

contained the first power in α that expression would contain second powers in α. Since our first approximation for ϕ is entirely independent of α, the above expression will not contain the second power in α if we use in it the first approximation for ϕ we have just obtained, as we shall. Hence it is inconsistent to include the term $-2\alpha^2\phi''$, let alone the term $\alpha^4\phi$, in the second approximation. Terms containing α^2 are also to be neglected in the boundary conditions (24) and (25). However, the term with the factor $\alpha^3 SR$ in (25) need not be neglected, because S appears here and nowhere else, and the inclusion of it will not lead to an error in the term containing S. With these considerations in mind, and with ϕ given by (30) denoted by ϕ_0, the differential equation for the second approximation is

$$\phi'''' = i\alpha R[(U - c_0)\phi_0'' - U''\phi_0], \qquad (31)$$

and the boundary conditions are

$$\text{(i)}\quad \phi(1) = 0, \qquad \text{(ii)}\quad \phi'(1) = 0,$$

$$\text{(iii)}\quad \phi''(0) - (3/c')\phi(0) = 0,$$

and

$$\text{(iv)}\quad [\alpha(3\cot\beta + \alpha^2 SR)/c_0']\phi_0(0) + \alpha R c_0'\phi_0'(0) - i\phi'''(0) = 0.$$

Note that in (31) and in (iv) above, $c_0'(=\frac{3}{2}$, given by the first approximation) and ϕ_0 are used for c and ϕ, because the use of c and ϕ would only introduce terms of a higher order in α.

Now

$$(U - c_0)\phi_0'' - U''\phi_0 = -3(y^2 + 1) + 3(1 - y)^2 = -6y.$$

Hence the solution of (31) is

$$\phi = \phi_0 + \phi_1,$$
$$\phi_1 = -(i\alpha R y^5/20) + \Delta A + \Delta B y + \Delta C y^2 + \Delta D y^3. \qquad (32)$$

In the second of Eqs. (32), the term involving α is the particular solution, and the rest constitutes the complementary solution, necessitated by the inclusion of the first-power terms in α in the differential system.

In view of the fact that

$$\phi_0''(0) - (3/c_0')\phi_0(0) = 0,$$

the boundary condition (iii) can be written as

$$\phi_1''(0) - (3/c_0')\phi_1(0) + [3\Delta c'/(c_0')^2]\phi_0(0) = 0,$$

or

$$\phi_1''(0) - 2\phi_1(0) + \tfrac{4}{3}\Delta c' = 0.$$

Substitution of (32) into the boundary conditions yield, after rearrangement and some simple divisions,

$$\Delta A + \Delta B + \Delta C + \Delta D = i\alpha R/20, \qquad (33)$$

$$\Delta B + 2\Delta C + 3\Delta D = \tfrac{1}{4} i\alpha R, \qquad (34)$$

$$\Delta C - \Delta A = -\tfrac{2}{3}\Delta c', \qquad (35)$$

$$\Delta D = -\tfrac{1}{9} i\alpha(3 \cot \beta + \alpha^2 SR) + \tfrac{1}{2} i\alpha R. \qquad (36)$$

If (33), (35), and twice of (36) are added together, and (34) is subtracted from the result, all the Δ quantities disappear, and we have, after simplification,

$$\Delta c' = ic_i, \quad c_i = \frac{6\alpha R}{5} - \frac{\alpha(3 \cot \beta + \alpha^2 SR)}{3}. \qquad (37)$$

Comparison of this result with Benjamin's[3] Eq. (5.3) shows that the difference is only in a factor $\frac{3}{2}$, arising from the different choices of the reference velocity. In other words, using a power expansion in α, which is analogous in approach to the power expansion in αR used in I, results identical to Benjamin's are obtained by taking only two terms (ϕ_0 and ϕ_1) in the expansion. Higher approximations can be carried out if desired. The two approximations cannot, and did not, take more than two hours to accomplish at the most. The method is really very simple and useful.

The eigenfunction for the second approximation will now be discussed. The left-hand sides of (33), (34), (35), and (36) are not linearly independent. In fact, with c_i given by (37), these equations are not linearly independent. We can, then, assign any value to ΔA and calculate ΔB and ΔC, with ΔD given by (36), whatever the value of ΔA. The results are

$$\Delta C = -\frac{i2c_i}{3} + \Delta A,$$

$$\Delta B = \frac{i\alpha R}{20} - \Delta D + \frac{i2c_i}{3} - 2\Delta A.$$

Thus the terms in ϕ_1 involving ΔA are

$$\Delta A(1 - y)^2,$$

which is proportional to ϕ_0. We shall therefore take ΔA to be zero, for otherwise we should be starting another first approximation at the stage of the second approximation. Whereas this would not affect the c_i obtained, it is quite unnecessary. Another point of view is as follows. An eigenfunction is determined up to a constant multiplier. This multiplier was chosen once and for all when we chose A to be unity. In fact, the successive approximations, with $\Delta A = 0$, furnish more and more accurate eigenfunctions and eigenvalues which satisfy the differential system more and more closely, and that is all that is desired. In the following section dealing with small Reynolds numbers (but any wavenumber, large or small), the same approach will be used. We shall simply take ΔA to be zero without further explanation.

Equation (37) shows that, while c_i is zero at $\alpha = 0$, c_i will increase or decrease when α increases from zero, according as

$$R > \tfrac{5}{6} \cot \beta \quad \text{or} \quad R < \tfrac{5}{6} \cot \beta.$$

In other words, the neutral-stability curve has a bifurcation point at

$$\alpha = 0, \qquad R = \tfrac{5}{6} \cot \beta, \qquad (38)$$

at which a branch goes off from the axis $\alpha = 0$, which is itself a branch of the neutral-stability curve. This consideration contributes to the qualitative description of the c_i curves given in Fig. 2.

The value $R = \frac{5}{6} \cot \beta$ was given by Benjamin as the critical Reynolds number. This is, however, the Reynolds number above which some disturbances will be amplified. It is not the Reynolds number below which all disturbances will be damped. As has been seen, for $\alpha = 0$ neutral disturbances exist right up to $R = 0$, even if β is less than $\frac{1}{2}\pi$ (so that the plate is not vertical).

IV. SOLUTION FOR SMALL REYNOLDS NUMBERS

For the case of small Reynolds numbers and any α, the first approximation will be carried out by taking R to be zero to begin with. The justification of this procedure is similar to that given for the case of small α, and need not be repeated here.

To avoid confusion in regard to the term containing SR in the boundary condition (iv), we note that

$$SR = T/\mu\bar{u}_a,$$

which will be denoted by S', to avoid the impression that it must be zero when R is zero. Since $\bar{u}_a$ will, for any given μ, approach zero as R approach zero, this term may seem troublesome. From the second of (4) it can be seen that for given values of g, d, β, and ρ, $\mu\bar{u}_a$ is finite. If μ is very large, $\bar{u}_a$ will be very small, and the Reynolds number very small. In the limit it can be zero. Hence it is not meaningless to consider the case of small Reynolds numbers and zero Reynolds numbers, even if the surface tension T is not zero. Actually $\mu\bar{u}_a$ may be very small. So long as it is not zero, which it is not unless $d = 0$, it causes no trouble. If T is assumed to be zero, the term containing SR in (iv) will of course drop out.

Setting R equal to zero in the differential system, we have

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = 0, \tag{39}$$

with the boundary conditions (22), (23), (24), and

$$\text{(iv)} \quad -[i\alpha(3 \cot \beta + \alpha^2 S')/c']\phi(0) + 3\alpha^2\phi'(0) - \phi'''(0) = 0. \tag{40}$$

The solution of (39) is

$$\phi = e^{\alpha y} + Be^{-\alpha y} + Cye^{\alpha y} + Dye^{-\alpha y}, \tag{41}$$

in which the coefficient of the term $e^{\alpha y}$ has been taken to be unity, because the eigenfunction can be multiplied by any constant. Equation (24) demands

$$(2\alpha^2 - 3/c')(1 + B) + 2\alpha(C - D) = 0. \tag{42}$$

Equation (40) yields

$$-i(3 \cot \beta + \alpha^2 S')(1 + B) + 2\alpha^2 c'(1 - B) = 0,$$

or

$$B = \frac{-i(3 \cot \beta + \alpha^2 S') + 2\alpha^2 c'}{i(3 \cot \beta + \alpha^2 S') + 2\alpha^2 c'}. \tag{43}$$

Equations (22) and (23) have the forms

$$e^{\alpha} + Be^{-\alpha} + Ce^{\alpha} + De^{-\alpha} = 0, \tag{44}$$

and

$$\alpha e^{\alpha} - B\alpha e^{-\alpha} + C(1 + \alpha)e^{\alpha} + D(1 - \alpha)e^{-\alpha} = 0. \tag{45}$$

Since B is known, (44) and (45) can be solved for C and D. The results are

$$D = -(1/2\alpha)[e^{2\alpha} + B(1 + 2\alpha)], \tag{46}$$

with B given by (43), and

$$C = -1 - e^{-2\alpha}(B + D), \tag{47}$$

with B and D given by (43) and (46). On the other hand, C is given by (42) to be

$$C = D - (1/2\alpha)(2\alpha^2 - 3/c')(1 + B).$$

Equating the two expressions given for C, we can solve for c', since B and D are known. The result is, after some intermediate calculation which is entirely straightforward,

$$c' = \frac{1}{1 + \cosh 2\alpha + 2\alpha^2} \cdot \left[3 + \frac{i(2\alpha - \sinh 2\alpha)}{2\alpha^2}(3 \cot \beta + \alpha^2 S')\right], \tag{48}$$

Now, for small α,

$$(2\alpha - \sinh 2\alpha)/2\alpha^2 = O(\alpha).$$

Hence at $\alpha = 0$ we have again $c' = \frac{3}{2}$, in agreement with (29). Since $\sinh 2\alpha > 2\alpha$ for all $\alpha > 0$, the flow is always stable at $R = 0$ for all nonzero wavenumbers if $\cot \beta$ is positive (i.e., if $\beta < \frac{1}{2}\pi$), as can be seen from (48). As α increases the stabilizing effect of the slope becomes less and less important as compared with the surface-tension effect, as is to be expected. When $\alpha \to \infty$,

$$c' \to -\tfrac{1}{2}iS', \quad \text{or} \quad c \to \tfrac{3}{2} - \tfrac{1}{2}iS', \tag{49}$$

Thus, for $S' = 0$, $c = \frac{3}{2}$ at $\alpha = \infty$. This fact and Eq. (70), to be presented, contradict Benjamin's (estimated) neutral-stability curve for $S' = 0$ (corresponding to his $\zeta = 0$), drawn at $\alpha = 0.43$ approximately. Benjamin stated that the region lying above that curve represents stability. Benjamin's dashed line, meant to be a rough estimate only, was unfortunately taken seriously by some researchers, who have produced a neutral-stability curve of similar trend with a high-speed computer.

The neutral stability at $R = 0$ for zero surface tension T is of course a puzzling situation. Benjamin has carefully avoided the discussion of this limiting case, and restricted his discussion to values of R other than zero. In a note appended to his paper, the present author has sought to explain this puzzling situation. A fuller explanation will be given here. Since c is expressed in terms of a reference velocity (in the present paper and in I, in terms of $\bar{u}_a$), and this velocity is zero if d/ν is not zero and if R vanishes. Hence if c is equal to a finite number the real rate of damping, $-\alpha c_i \bar{u}_a/d$, is zero, and the disturbance will not be damped out. But now if d/ν is finite, $\bar{u}_a$ can be zero only if $g \sin \beta$ is zero, in which case gravity is not a source of energy. There is no pressure gradient in the X direction, and since T is zero the surface tension does not enter into the question. (Since the mean surface is flat, any corrugation would mean an *expenditure* of surface energy, even if S' is not zero.) How could a *disturbance motion* be maintained? The answer is that, for $T = 0$ and any α, in the limiting case of $R = 0$ there is no disturbance motion, but merely a surface corrugation, which has no reason to be damped. Benjamin's contribution is that in the neighborhood of this case of static corrugation there are unstable waves.

In the second approximation we shall concentrate on the case $\beta = \frac{1}{2}\pi$ and $S' = 0$, because if these conditions do not exist (47) already tells us that the flow will be stable for all nonzero values of α at $R = 0$. The first approximation indicates that the axis $R = 0$ is a part of the neutral stability curve if $\cot\beta$ and S' are all zero. It is important to see whether the flow will be stabilized or destabilized as R is increased slightly.

The development in I is exactly what is needed for the second approximation. Denoting by ϕ_0 the ϕ given by the first approximation, i.e., by (41), (43), (46), and (47), the ϕ for the second approximation must satisfy

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[-(\tfrac{3}{2}y^2 + c_0')(\phi_0'' - \alpha^2\phi_0) + 3\phi_0]. \quad (50)$$

Now, with $\cos\beta$ and S' both zero,

$$\phi_0 = e^{\alpha y} + e^{-\alpha y} - (1/2\alpha)(2\alpha - 1 - e^{-2\alpha})ye^{\alpha y} + (1/2\alpha)(-2\alpha - 1 - e^{2\alpha})ye^{-\alpha y}, \quad (51)$$

as can be seen from (41), (43), (46), and (47). We shall call the particular solutions of (50) $\alpha R(\phi_{11}, \phi_{12}, \phi_{13}, \phi_{14})$ when $e^{\alpha y}$, $e^{-\alpha y}$, $ye^{\alpha y}$, and $ye^{-\alpha y}$ are substituted for ϕ_0 in (50). Then the solution of (50) is

$$\phi = \phi_0 + i\alpha R\phi_1 + \Delta Be^{\alpha y} + \Delta Cye^{\alpha y} + \Delta Dye^{-\alpha y}, \quad (52)$$

with

$$\phi_1(y) = \phi_{11} + \phi_{12} - (1/2\alpha)(2\alpha - 1 - e^{-2\alpha})\phi_{13} - (1/2\alpha)(2\alpha + 1 + e^{2\alpha})\phi_{14}. \quad (53)$$

In (52) ΔA has been taken to be zero, for reasons given in Sec. III. The functions ϕ_{11}, ϕ_{12}, ϕ_{13}, and ϕ_{14} were given in I, in which it is also stated that these can be expressed in terms of exponential functions. [The expression for ϕ_{11} contains a misprint of a sign. The parenthesis $(3 - 2\alpha y)$ should read $(3 + 2\alpha y)$. The expression given for $\phi_{11}(\phi_1^{(1)}$ in I) later in I is in agreement with the correct sign.] These are

$$\phi_{11}(\alpha, y) = (3/16\alpha^4)[(3 - 4\alpha y + 2\alpha^2y^2)e^{\alpha y} - (3 + 2\alpha y)e^{-\alpha y}], \quad (54)$$

$$\phi_{12}(\alpha, y) = \phi_{11}(-\alpha, y), \quad (55)$$

$$\phi_{13}(\alpha, y) = -(3c_0'/8\alpha^3)[(3 - 4\alpha y + 2\alpha^2y^2)e^{\alpha y} - (3 + 2\alpha y)e^{-\alpha y}] - (1/32\alpha^5)[(2\alpha^4y^4 - 12\alpha^3y^3 + 30\alpha^2y^2 - 42\alpha y + 27)e^{\alpha y} - (12\alpha y + 27)e^{-\alpha y}], \quad (56)$$

$$\phi_{14}(\alpha, y) = \phi_{13}(-\alpha, y). \quad (57)$$

Since the derivatives of these functions are also needed, they will also be given. These are

$$\phi_{11}'(\alpha, y) = (3/16\alpha^3)[(-1 + 2\alpha^2y^2)e^{\alpha y} + (1 + 2\alpha y)e^{-\alpha y}], \quad (58)$$

$$\phi_{12}'(\alpha, y) = \phi_{11}'(-\alpha, y), \quad (59)$$

$$\phi_{13}'(\alpha, y) = -(c_0'/8\alpha^2)[(-1 + 2\alpha^2y^2)e^{\alpha y} + (1 + 2\alpha y)e^{-\alpha y}] - (1/32\alpha^4)[(2\alpha^4y^4 - 4\alpha^3y^3 - 6\alpha^2y^2 + 18\alpha y - 15)e^{\alpha y} + (15 + 12\alpha y)e^{-\alpha y}], \quad (60)$$

$$\phi_{14}'(\alpha, y) = \phi_{13}'(-\alpha, y). \quad (61)$$

Although the second and third derivatives of ϕ occur in the boundary conditions (iii) and (iv), we need not give them for the functions given by (54), (55), (56), and (57). This is because these functions have been so arranged (by adding linear combinations of the complementary solutions) as to contain no power of y less than the fourth. Hence these functions and their derivatives up to the third ones all vanish at $y = 0$, where (iii) and (iv) are to be applied.

With ϕ given by (52), the boundary condition (iii) given by (24) assumes the form

$$(2\alpha^2 - 3/c_0')\Delta B + 2\alpha\Delta C - 2\alpha\Delta D + 6\Delta c/c_0'^2 = 0, \quad (62)$$

if terms quadratic in R and the Δ quantities are neglected. In (62) the last term comes from the variation of c'. In obtaining it the facts that $\phi_0(0) = 2$ and that $\Delta c' = \Delta c$ have been used. Also, $\phi_1(0) = 0$, as explained before, so that ϕ_1 does not appear in (62) at all. The c_0' in (62) is the c' given in the first approximation, by (48), with $\cot\beta$ and S' equal to zero. Hence

$$c_0' = \frac{3}{1 + \cosh 2\alpha + 2\alpha^2},$$

and

$$2\alpha^2 - \frac{3}{c_0'} = -(1 + \cosh 2\alpha). \quad (63)$$

The boundary condition (iv) has the form (since $\beta = \frac{1}{2}\pi$ and $S' = 0$)

$$\alpha(Rc' + 3\alpha i)\phi'(0) - i\phi'''(0) = 0.$$

Substitution of (52) into it yields, in the same way,

$$\Delta B = (iRc_0'/2\alpha^3)(2\alpha + \sinh 2\alpha). \quad (64)$$

The boundary conditions (i) and (ii), given by (22) and (23), have the forms

$$e^{-\alpha}\Delta B + e^{\alpha}\Delta C + e^{-\alpha}\Delta D = -iR\alpha\phi_1(1), \quad (65)$$

$$-\alpha e^{-\alpha}\Delta B + (1+\alpha)e^{\alpha}\Delta C + (1-\alpha)e^{-\alpha}\Delta D = -iR\alpha\phi_1'(1), \quad (66)$$

from which

$$2\alpha\Delta D = iR\alpha e^{\alpha}[\phi_1'(1) - (1+\alpha)\phi_1(1)] - (1+2\alpha)\Delta B, \quad (67)$$

and

$$-2\alpha\Delta C = iR\alpha e^{-\alpha}[\phi_1'(1) - (1-\alpha)\phi_1(1)] - e^{-2\alpha}\Delta B. \quad (68)$$

Substituting (63), (64), (67), and (68) into (62), we have, after some simplifications,

$$\Delta c = \tfrac{1}{6}iRc_0'^2\{(c_0'/2\alpha^3)(2\alpha + \sinh 2\alpha) \cdot(\cosh 2\alpha - 2\alpha - e^{-2\alpha}) + 2\alpha[\cosh\alpha\,\phi_1'(1) - (\cosh\alpha + \alpha\sinh\alpha)\phi_1(1)]\}, \quad (69)$$

with ϕ_1 and ϕ_1' given by Eqs. (53) through (61). For very large α, a brief calculation shows that

$$\Delta c \sim \frac{iRc_0'^2}{6}\left(\frac{3e^{3\alpha}}{16\alpha^4}\right) \sim \frac{i9Re^{-\alpha}}{8\alpha^4} \quad (70)$$

and

$$c_i \to 0 \quad \text{as} \quad \alpha \to \infty$$

in the neighborhood of $R = 0$. (Indeed R must be so small that αR is kept small as α increases.) Equation (70) means that if α is very large, c_i is very small. For intermediate values of α it is well nigh impossible to determine the sign of the bracket in (69) algebraically. A numerical computation[7] has not turned up any negative or zero value for c_i for α from 0.2 to 4.0. The fact that dc/dR is nowhere zero for finite values of α indicates that a bifurcation point, such as the intersection of the dashed line and the axis $R = 0$ in Benjamin's Fig. 2, of the neutral-stability curve cannot exist on $R = 0$. This seems to rule out the possibility that the dashed line of Benjamin's Fig. 2 divides two regions of instability, and merely marks a valley of least c_i, which happens to be zero.

The second approximation carried out here shows that the expansion in I is useful and capable of producing definite results when only two terms in the expansion are used. Actually, Benjamin's expansion in power series of y and the expansion used in I and the present paper are identical and will produce the same results *provided* a very great number of terms are used in both expansions. If only a limited number of terms are taken, then the method presented here is more flexible and versatile. It produces identical results as Benjamin's for small α, and for small R (but any α) it produces results hitherto unavailable.

[7] The author is indebted to Philip Davis for the assistance rendered in this computation.

V. STABILITY OF VERY SHORT WAVES

If α is very large, then for *any finite* R, the Orr-Sommerfeld equation can be approximated by (39), provided c is small compared with α. This provision, of course, can only be verified *a posteriori*. Since the boundary conditions will then be exactly those used in the first approximation carried out in Sec. IV, i.e., (22), (23), (24), and (40), the result is known and exactly given by (48), which shows that $c' \to -\frac{1}{2}iS'$ as $\alpha \to \infty$. Since $S' = T/\mu\bar{u}_a$, very short waves are damped by surface tension T, and can only be neutral if T is zero, or if $\mu\bar{u}_a$ is infinite. For any given T the dimensionless rate of damping is decreased if $\mu\bar{u}_a$ is increased. Since the complex wave velocity c has been expressed in terms of $\bar{u}_a$, so that $\alpha c_i\bar{u}_a/d$ is the actual rate of growth (and if negative, of damping), the result $c_i = -\frac{1}{2}S'$ shows that the actual rate of damping of very short waves is reduced in magnitude if μ is increased, for any T. This result is entirely unexpected, and is a new example of the dual role of viscosity. Of course, increasing μ can never bring about instability. But it does reduce the degree of stability at large wavenumbers.

VI. TOPOLOGICAL FEATURES OF THE c_i CURVES

From the foregoing discussions the following facts have been established:

(1) The axis $\alpha = 0$ is always a part of the neutral-stability curve.

(2) There is a bifurcation point on $\alpha = 0$ for the neutral-stability curve, at $R = \frac{5}{6}\cot\beta$. This point is at the origin if $\beta = \frac{1}{2}\pi$.

(3) The axis $R = 0$ is part of the neutral-stability curve if $T = 0$ and $\beta = \frac{1}{2}\pi$.

(4) If $T = 0$ but $\beta \neq \frac{1}{2}\pi$, c_i varies on $R = 0$, from zero at $\alpha = 0$ through a minimum to zero again at $\alpha = \infty$.

(5) If $T \neq 0$ but $\beta = \frac{1}{2}\pi$, c_i varies on $R = 0$ from zero at $\alpha = 0$ monotonically to $-\frac{1}{2}S'$ at $\alpha = \infty$. The monotonicity can be established readily by differentiating c_i given by (48) with respect to α. The result is (for $R = 0$)

$$\frac{dc_i}{d\alpha} = -\frac{1}{2\alpha^4(1 + \cosh 2\alpha + 2\alpha^2)^2} \cdot[4\alpha^4(\cosh 2\alpha + 1)(3\cot\beta + \alpha^2 S') + 6\alpha(2\alpha - \sinh 2\alpha)(1 + \cosh 2\alpha + 2\alpha^2)\cot\beta],$$

which is evidently negative if $\beta = \frac{1}{2}\pi$.

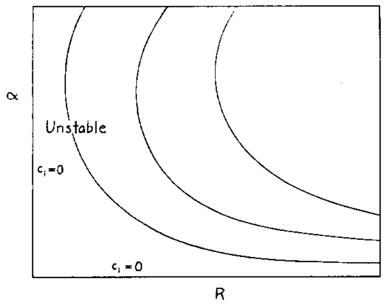


FIG. 2. Topological features of the c_i curves for the case $S' = 0$ and $\beta = \frac{1}{2}\pi$. The axes constitute the neutral-stability curve.

(6) If $T \neq 0$ and $\beta \neq \frac{1}{2}\pi$, the expression for $dc_i/d\alpha$ at $R = 0$ shows that $dc_i/d\alpha = -\cot\beta$ at $\alpha = 0$, so that, with $\beta < \frac{1}{2}\pi$, c_i will initially decrease on $R = 0$. But it is evident that as α increases (on $R = 0$) $dc_i/d\alpha$ will eventually be positive. Hence c_i will have at least one minimum on $R = 0$. (More than one minimum is very unlikely, although a conclusive algebraic proof that only one minimum exists is not obvious.) We shall assume that only one minimum exists.

(7) At large α, for any given β and R, $c_i = -\frac{1}{2}S'$.

From these facts we conclude that the topological features of the c_i curves are given in Figs. 2–5 for the four cases described in the figure captions. It can be seen that the neutral-stability curve given in I for the case $S' = 0$ and $\beta = \frac{1}{2}\pi$ is more a curve of constant nonzero c_i. The numerical calculation in I was, however, too inaccurate, and produced erroneous values for c_r, which did not seem to affect the conclusion that instability could happen at low Reynolds numbers.

Figures 2–5 are the simplest configurations that can be constructed with the information available. If new information should turn out to show that these figures are incomplete and oversimplified, the information upon which these figures have been drawn, wherever definite, should not be subjected to doubt.

VII. EFFECTS OF VARIABLE SURFACE TENSION

In case there is a thin film of contamination at the free surface, the surface tension may vary from place to place, giving rise to a variable shear stress just below the film. If the surface tension at the free surface when the film is unstretched is T_0, and the variation from this is proportional to the amount of stretching per unit length in the direction of stretching, then

$$T = T_0 + T_1\, \partial\xi/\partial X,$$

in which ξ is the distance in the direction of stretching (assumed to be the X direction) covered by a certain particle, and T_1 is a second constant specifying the variability of surface tension. This variability has no effect on the boundary condition (iv), but does have an effect on the boundary condition (iii). Since

$$D\xi/Dt = \bar{u} + \bar{u}_a u',$$

we have, on the free surface,

$$\xi = \bar{u}(0)t + \xi_1,$$

where

$$\frac{D\xi}{Dt} = \frac{\bar{u}_a}{d}\frac{D\xi}{D\tau} = \bar{u} + \bar{u}_a u' = \bar{u} + \bar{u}_a \phi'(0) e^{i\alpha(x-c\tau)}.$$

Thus

$$\xi_1 = (id/\alpha c')\phi'(0)e^{i\alpha(x-c\tau)}, \quad c' = c - U(0) = c - \tfrac{3}{2}.$$

Now the variation in τ_{xy} on the free surface is

$$-\frac{\partial T}{\partial X} = -\frac{iT_1}{\alpha d}\frac{\partial^2}{\partial x^2}\frac{\phi'(0)}{c'}e^{i\alpha(x-c\tau)}$$
$$= \frac{i\alpha T_1}{d}\frac{\phi'(0)}{c'}e^{i\alpha(x-c\tau)}.$$

Hence (iii) has the form

$$\text{(iii)} \quad \phi''(0) + (\alpha^2 - 3/c')\phi(0) + i\alpha S_1\, \phi'(0)/c' = 0,$$

in which

$$S_1 = T_1/\mu\bar{u}_a.$$

Since the additional term involving S_1 is associated with α, it has no effect on the first approximation for $\alpha = 0$. It does have an effect on the next approximation for c_i, which determines the stability condition, and also on the first approximation for $R = 0$. It is calculable by the method given in Sec. III. The author is indebted to Dr. T. B. Benjamin for pointing out the factor i/α in ξ_1, which was missing in the first draft.

VIII. VESTIGES OF SHEAR WAVES

The waves discussed so far in this paper are surface waves. Since they can occur at small Reynolds numbers and are only moderately damped

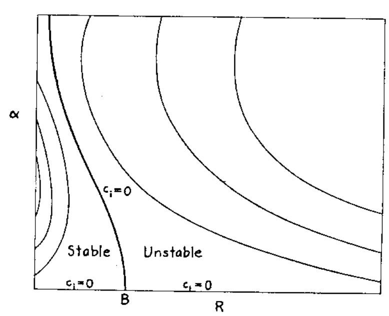


FIG. 3. Topological features of the c_i curves for the case $S' = 0$ and $\beta < \frac{1}{2}\pi$. $R = 5/6 \cot\beta$ at the bifurcation point B. $c_i = 0$ on $\alpha = 0$.

or amplified at small values of $R\alpha$, they may be called soft waves. These are distinct from the shear waves, or hard waves, in plane Poiseuille flow and other confined flows, which are highly damped at small values of $R\alpha$. But it does seem strange that the existence of a free surface should remove the possibility of shear waves altogether, particularly if the surface tension is large, so that the free surface is rather stiff against corrugation. Where are the vestiges of shear waves? In providing the answer that they are there, we shall also illustrate how to make the correct approximation to the mathematical system to be solved, gain a deeper insight into the fine structure of an arbitrary disturbance possessing different components (each represented by an eigenfunction of the differential system governing stability), and demonstrate that the differential system has infinitely many eigenvalues for c for given values of R and α.

We deal with shear waves of plane Poiseuille flows for three cases:

(1) Small Reynolds number for any α,
(2) Small wavenumber for any finite R,
(3) Large wavenumbers.

In each case it is shown that the waves are strongly damped. We then take up the shear waves in the film flow under study in this paper, deal with the same three cases, and show that the waves are again strongly damped, thus demonstrating the vestiges of shear waves in film flow. After all, the film flow is one-half of a plane Poiseuille flow, and if these vestiges could not be found it would be very strange indeed.

For plane Poiseuille flow, the differential equation is still given by (21), with U given by (3), except that U is now defined in $-1 \leq y \leq 1$. The boundary conditions are

$$\phi(\pm 1) = 0, \qquad \phi'(\pm 1) = 0.$$

It is well known that if the differential system

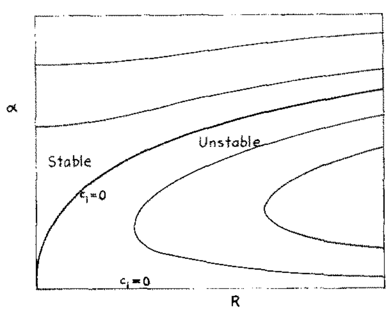


Fig. 4. Topological features of the c_i curves for the case $S' \neq 0$ and $\beta = \frac{1}{2}\pi$. $c_i = 0$ on $\alpha = 0$. c_i curves flatten and c_i approaches $-\frac{1}{2}S'$ as α increases indefinitely.

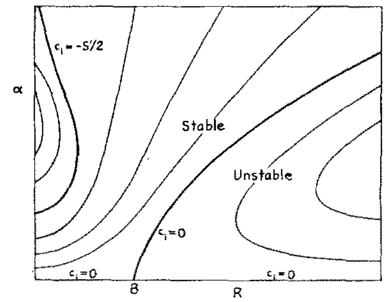


Fig. 5. Topological features of the c_i curves for the case $S' \neq 0$ and $\beta < \frac{1}{2}\pi$. $c_i = 0$ on $\alpha = 0$. $c_i \to -\frac{1}{2}S'$ as $\alpha \to \infty$. $R = 5/6 \cot \beta$ at the bifurcation point B.

governing stability is entirely symmetric, so that it admits of an odd function of y or an even function of y as a solution, the secular equation for any solution, which is neither even or odd, is factorizable into two factors, one of which is for the odd part of the solution and the other the even part. Hence even and odd solutions can be treated separately. In general, they have different eigenvalues.

Case 1. For small R, assume that Rc is not small. This is important, for otherwise c would drop out of the differential system altogether in the limiting case of $R = 0$, and the differential system could not have any nontrivial solution. For any α, the differential equation can then be written as

$$(D^2 - \beta^2)(D^2 - \alpha^2)\phi = 0, \tag{71}$$

in which $D = d/dy$, and

$$\beta^2 = \alpha^2 - i\alpha Rc. \tag{72}$$

The even solution is

$$\phi = A \cosh \alpha y + B \cosh \beta y. \tag{73}$$

The satisfaction of the boundary conditions demands

$$A \cosh \alpha + B \cosh \beta = 0,$$

$$A\alpha \sinh \alpha + B\beta \sinh \beta = 0,$$

so that the secular equation is

$$\beta \tanh \beta = \alpha \tanh \alpha. \tag{74}$$

The solution $\beta = \alpha$ is to be excluded, because in the first place it violates the assumption $Rc \neq 0$, and in the second place the solution (73) would be invalid if $\beta = \alpha$. But there are infinitely many solutions of (74). Let $\beta = \gamma i$. Then (74) becomes

$$\gamma \tan \gamma = -\alpha \tanh \alpha, \tag{75}$$

the solutions of which are the abscissas of the intersections of the curves

$$\Gamma = \tan \gamma \quad \text{and} \quad \Gamma = -\alpha \tanh \alpha/\gamma$$

in the γ–Γ plane, and are infinite in number, at a distance of π apart asymptotically. Let these real solutions of γ be denoted by γ_n ($n = 1, 2, \cdots$). Then

$$i\alpha Rc_n = \alpha^2 + \gamma_n^2,$$

and

$$c_n = -(i/\alpha R)(\alpha^2 + \gamma_n^2), \tag{76}$$

giving infinitely many eigenvalues of c, all corresponding to high rate of damping since R is small. The rate increases in magnitude as n increases. Note that as n increases γ_n also does, so that $\cosh \beta_n y$ ($= \cos \gamma_n y$), hence ϕ, is more and more oscillatory. That a more and more oscillatory disturbance will be damped faster and faster is quite to be expected.

The odd solution is

$$\phi = A \sinh \alpha y + B \sinh \beta y,$$

with the secular equation

$$\beta \coth \beta = \alpha \coth \alpha.$$

Again the solution $\beta = \alpha$ is to be excluded. The other solutions are found again by putting β equal to γi and solving

$$\gamma \cot \gamma = \alpha \coth \alpha \tag{77}$$

for γ. The solutions can be obtained by finding the intersections of

$$\Gamma = \cot \gamma \quad \text{and} \quad \Gamma = \alpha \coth \alpha/\gamma.$$

Then again c_n is given by (76). The γ's are infinite in number, asymptotically at distance π apart, and asympotically at the distance $\frac{1}{2}\pi$ from the roots of (72). The γ_1 for even ϕ is smaller than γ_1 for odd ϕ if α is large (by a little less than $\frac{1}{2}\pi$ in fact), but the reverse is true if α is small. The infinity of the number of eigenvalues has now been amply demonstrated. Note that the assumption that Rc is not small has been justified *a posteriori*.

Case 2. If α is small, we shall assume that αc is not small. The differential equation can then be written as

$$\phi'''' + i\alpha Rc\phi'' = 0. \tag{78}$$

The even solution is

$$\phi = A + B \cosh \beta y,$$

with β now given by

$$\beta^2 = -i\alpha Rc. \tag{79}$$

The boundary conditions $\phi'(\pm 1) = 0$ give, without the conjunction of the other two boundary conditions,

$$\sinh \beta = 0, \quad \text{or} \quad \beta = n\pi i.$$

This gives

$$c = -(i/\alpha R)(n\pi)^2. \tag{80}$$

The odd solution is

$$\phi = Ay + B \sinh \beta y.$$

The boundary conditions demand

$$A + B \sinh \beta = 0,$$

$$A + B\beta \cosh \beta = 0,$$

so that

$$\beta \coth \beta = 1.$$

There are no real solutions of this equation except $\beta = 1$, which is to be excluded. But, with $\beta = i\gamma$,

$$\gamma \cot \gamma = 1, \tag{81}$$

the solutions of which are[8]

$\gamma_1 = 4.4934, \quad \gamma_2 = 7.7253, \quad \gamma_3 = 10.9041,$

$\gamma_4 = 14.0662, \quad \gamma_5 = 17.2208.$

Further roots are at distance π apart, approximately. In this case it is evident that antisymmetric disturbances (even ϕ) are less stable than symmetric ones (odd ϕ).

Case 3. If α is large, we assume c is at least of the order of α. The governing equation is again (71), and the eigenvalues are given by (76), with γ_n given by (75) for antisymmetric disturbances and by (77) for symmetric disturbances. Since α is large, the γ_n for antisymmetric disturbances (even ϕ) is approximately

$$\gamma_n = \tfrac{1}{2}(2n + 1)\pi, \tag{82}$$

and the γ_n for symmetric disturbances (odd ϕ) is

$$\gamma_n = n\pi, \tag{83}$$

so long as n is not too large for either case. When n is large, the eigenvalues γ for even ϕ are still at distance π apart approximately, but there is a forward "phase shift" of $\frac{1}{2}\pi$ from the values given by (82), whereas those for odd ϕ, also at distance π apart approximately, undergo a backward "phase shift" of $\frac{1}{2}\pi$ from those given by (83).

We have now dealt with plane Poiseuille flow,

[8] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford University Press, New York, 1959), 2nd ed., p. 492.

and shall turn our attention to the free-surface flow under study.

Cases 1 and 3. Due to the complexity of the free-surface boundary conditions, a simple, explicit solution for the eigenvalues is not possible for these two cases, although the solution of the differential equation is itself simple and explicit. For small R or large α, we shall use the method of Synge.[9] It turns out that a single demonstration is sufficient for these two cases.

Again, if R is small, we assume Rc is not small, and if α is large, we assume c is at least of the order of α. These assumptions can be verified *a posteriori*. In either case the governing differential equation is (71), or

$$(D^2 - \alpha^2)^2\phi = -i\alpha Rc(D^2 - \alpha^2)\phi. \tag{84}$$

The boundary conditions are (22), (23),

$$\text{(iii)}\quad \phi''(0) + \alpha^2\phi(0) = 0, \tag{85}$$

and (25). In (25) the first term can be neglected. If R is small and Rc not small, c and hence c' must be large in magnitude, hence the first term can be neglected. (Otherwise the term SR is not necessarily negligible, even if R is small; see Sec. IV for explanation.) If α is large and if c (hence c') is at least of the order of α, $|\phi'|$ is at least of the order $\alpha\,|\phi|$ and $|\phi'''|$ at least of the order $\alpha^3\,|\phi|$. Hence again the first term in (25) can be neglected. However, since it causes no trouble to keep it, we shall do so, and verify the negligibility of the term after the final result has been obtained. Since in both Case 1 and Case 3 c is very large, c' in (25) will be replaced by c.

Multiplying (84) by ϕ^*, the complex conjugate of ϕ, and integrating the result (by parts if necessary), we have, with the aid of the boundary conditions (22) and (23),

$$\begin{aligned} -\phi^*(0)\phi'''(0) + \phi'^*(0)\phi''(0) + I_2 \\ + 2\alpha^2\phi^*(0)\phi'(0) + 2\alpha^2 I_1 + \alpha^4 I_0 \\ = -i\alpha Rc[-\phi^*(0)\phi'(0) - I_1 - \alpha^2 I_0], \end{aligned} \tag{86}$$

in which

$$I_0 = \int_0^1 |\phi|^2\,dy,\quad I_1 = \int_0^1 |\phi'|^2\,dy,\quad I_2 = \int_0^1 |\phi''|^2\,dy.$$

By the use of (85), (86) becomes

$$\begin{aligned} i\alpha Rc(I_1 + \alpha^2 I_0) &= I_2 + 2\alpha^2 I_1 + \alpha^4 I_0 \\ &- i\phi^*(0)\{\alpha Rc\phi'(0) - i\phi'''(0) \\ &+ i3\alpha^2\phi'(0) + [\alpha(3\cot\beta + \alpha^2 SR)/c]\phi(0)\} \\ &- \alpha^2[\phi'^*(0)\phi(0) + \phi'(0)\phi^*(0)] \\ &+ [i\alpha(3\cot\beta + \alpha^2 SR)/c]\,|\phi(0)|^2. \end{aligned} \tag{87}$$

Since c' may be replaced by c (because both are large, so that the difference $\frac{3}{2}$ is unimportant), the first curly bracket in (87) is the left-hand side of (25), and is zero. Hence

$$\begin{aligned} i\alpha Rc(I_1 + \alpha^2 I_0) - [i\alpha(3\cot\beta + \alpha^2 SR)/c]\,|\phi(0)|^2 \\ = I_2 + 2\alpha^2 I_1 + \alpha^4 I_0 - \alpha^2[\phi'^*(0)\phi(0) + \phi'(0)\phi^*(0)]. \end{aligned}$$

But

$$\begin{aligned} \phi'^*(0)\phi(0) + \phi'(0)\phi^*(0) \\ = 2\int_0^1 |\phi'|^2\,dy + \int_0^1 (\phi''^*\phi + \phi''\phi^*)\,dy \\ = 2I_1 + \int_0^1 (\phi''^*\phi + \phi''\phi^*)\,dy. \end{aligned}$$

Hence

$$\begin{aligned} i\alpha Rc(I_1 + \alpha^2 I_0) - [\alpha(3\cot\beta + \alpha^2 SR)/c]\,|\phi(0)|^2 \\ = I_2 - \alpha^2\int_0^1 (\phi''^*\phi + \phi''\phi^*)\,dy + \alpha^4 I_0 \\ = \int_0^1 |\phi'' - \alpha^2\phi|^2\,dy. \end{aligned} \tag{88}$$

The right-hand side of (88) cannot be equal to zero, for otherwise

$$\phi'' - \alpha^2\phi = 0,$$

and

$$\phi = A\cosh\alpha y + B\sinh\alpha y,$$

which cannot nontrivially satisfy $\phi(1) = \phi'(1) = 0$. Hence the last integral of (88) is positive definite. Taking the real part of (88), we have

$$-\alpha Rc_i\left[I_1 + \alpha^2 I_0 + \frac{(3\cot\beta + \alpha^2 SR)}{|c|^2 R}\,|\phi(0)|^2\right] = \int_0^1 |\phi'' - \alpha^2\phi|^2\,dy > 0,$$

which proves that

(1) c_i is negative;

(2) if R is small, the term containing $|\phi(0)|^2$ is negligible, as expected, if Rc is assumed not small;

(3) if that term is neglected then Rc is not small,

[9] C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, New York, 1955).

as was assumed, hence, there is at least consistency in the assumption;

(4) if α is large, the right-hand side is of the order $\alpha^4 |\phi|^2$, the left-hand side of the order $Rc_i\alpha^3 |\phi|^2$, hence $c_i = O(\alpha)$, as assumed, and the term containing $|\phi(0)|^2$ is negligible, as expected.

The development shows rather convincingly that for small R and large α, there are highly damped modes. These modes are the vestiges of shear waves. They may be overshadowed by surface waves, but they are there.

Case 2. For small α we can obtain sharper results. Under the assumption that αc is not small, the differential equation can be written as

$$\phi'''' + i\alpha Rc\phi'' = 0,$$

the solution of which is

$$\phi = A + By + Ce^{\beta y} + De^{-\beta y},$$

with

$$\beta^2 = -i\alpha Rc.$$

The boundary conditions (iii) and (iv) are

$$\phi''(0) = 0, \qquad \beta^2\phi'(0) - \phi'''(0) = 0,$$

from which

$$C = -D, \qquad B = -4\beta D.$$

The boundary condition $\phi'(1) = 0$ gives

$$B = 2\beta \cosh \beta D.$$

Since β has been assumed to be different from zero, the two evaluations of B give

$$\cosh \beta = -2.$$

The boundary condition $\phi(1) = 0$ can be used to evaluate A in terms of D (say), and is not needed to arrive at the secular equation. Thus

$$\beta = \cosh^{-1} 2 + (2n + 1)\pi i$$

and

$$-i\alpha Rc = (\cosh^{-1} 2)^2 - (2n + 1)^2\pi^2 + i2(2n + 1)\pi \cosh^{-1} 2,$$

with n any integer, positive or negative. Breaking this into its real and imaginary parts, we have

$$\alpha Rc_i = (\cosh^{-1} 2)^2 - (2n + 1)^2\pi^2,$$

$$\alpha Rc_r = -2(2n + 1)\pi \cosh^{-1} 2.$$

Since $\cosh^{-1} 2 = 1.32$ approximately, c_i is always negative, and very large in magnitude when α is small. Since α is always assumed positive (because the results are unchanged if α is negative) and n may be positive or negative, the waves can propagate both upstream and downstream, and if they propagate with the same speed they are damped equally fast. Note again that as $|2n + 1|$ increases the eigenfunction has more and more oscillations in $0 \leq y \leq 1$. It is therefore not surprising that the pertinent waves are damped faster and faster. We have now shown that the shear waves are indeed there, though it is the surface waves that govern stability.

We conclude with the warning that although surface waves have been shown to govern stability at small Reynolds numbers, it has not been conclusively shown that for very small β (so that $\frac{5}{6} \cot \beta$ is large) shear waves do not, after all, govern the stability of the flow.

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