Relationships among Generalized Phase-Space Distributions*

G. C. SUMMERFIELD† AND P. F. ZWEIFEL†
Department of Nuclear Engineering, The University of Michigan, Ann Arbor

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The generalized phase-space distributions, including the Wigner distribution, are presented in terms of expected values of generating operators. A generalization of the Weyl correspondence is obtained to provide expressions for generalized Wigner equivalents. Finally, rather simple relationships are obtained connecting the generalized phase-space distributions to the Wigner distribution; and similar relationships are obtained for the generalized Wigner equivalents. In particular, it appears that, among the class considered, there is no reason to use any distribution other than the Wigner for performing any calculations.

I. INTRODUCTION

In 1932, Wigner[1] introduced a method of performing quantum-mechanical ensemble averages in terms of phase-space integrations over c-number variables. Since that time, a number of extensions, modifications, discussions, derivations, applications, etc., have appeared in the literature. We refer the reader to a review[2] in which further references can be found.

Actually, there exist an infinite number of quasi-distribution functions which can be used for the same purpose as the Wigner distribution function. In a recent paper,[3] Cohen described one method for generating such distributions, and showed how the Wigner function, the so-called "symmetric" function, and the Born–Jordan function could be generated. He also obtained equations of motion (quantum Liouville equations) for these distribution functions.

In the present paper we present a particularly simple and elegant manner for generating an infinite class of distribution functions which include, as special cases, the Wigner, symmetric, and Born–Jordan functions. We also show that all of these various distributions can be obtained from the Wigner distribution by a rather trivial transformation.

For the purposes of our later discussion, it is convenient for us to point out several general properties that all of these distributions have in common.

We represent the $6N$-dimensional phase space by the $3N$-dimensional momentum and position vectors r and p. A generalized phase-space distribution is a function of the variables r and p and time, $f(r, p, t)$. These functions satisfy the following conditions:

(A) *Classical Limit:* The function

$$f_c(r, p, t) = \lim_{\hbar \to 0} f(r, p, t) \tag{1}$$

must be the "correct" classical phase-space distribution. That is, $f_c(r, p, t)$ must satisfy the Liouville equation.

(B) *Marginal Distributions:* The integral of f over one of the variables r or p must give the correct distribution in the other variable:

$$\int dr\, f(r, p, t) = \langle \delta(P - p) \rangle, \tag{2}$$

$$\int dp\, f(r, p, t) = \langle \delta(R - r) \rangle, \tag{3}$$

where R and P are the position and momentum operators.

(C) *Generalized Wigner Equivalents:* For any given function $A(R, P)$ of the position and momentum operators, we must be able to determine a generalized Wigner equivalent $a(r, p)$ such that

$$\langle A(R, P) \rangle = \int dr\, dp\, f(r, p, t) a(r, p). \tag{4}$$

We might point out here that the distributions introduced by Cohen[3] do not, in general, provide for a generalized Wigner equivalent. In particular for Cohen's distribution [Eq. (6.2) of Ref. 3], an operator of the form $A(\theta \cdot R + \tau \cdot P)$ does not have a generalized Wigner equivalent.

The most convenient way of finding generalized Wigner equivalents is by first finding the generalized Weyl correspondence. That is, we find the operator $A_g(\theta, \tau, R, P)$ for which the generalized Wigner equivalent is

$$a_g(\theta, \tau, r, p) = e^{i(\theta \cdot r + \tau \cdot p)}. \tag{5}$$

Then if the operators A_g are complete, we can expand any operator as

$$A(R, P) = \int d\theta\, d\tau\, \alpha(\theta, \tau) A_g(\theta, \tau, R, P). \tag{6}$$

(We consider the completeness of the A_g's when we specify the details of the distribution.) Clearly we can

* Work supported by the National Science Foundation.
† Present address: Dept. of Physics, Virginia Polytechnic Institute, Blacksburg, Va.

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[2] K. İmre, E. Özizmir, M. Rosenbaum, and P. F. Zweifel, J. Math. Phys. **8**, 1097 (1967).

[3] L. Cohen, J. Math. Phys. **7**, 781 (1966).

determine the Wigner equivalent of $A(R, P)$ by knowing the generalized Wigner equivalent of the right-hand side of (6), that is, using Eq. (6):

$$\langle A(R, P)\rangle = \int d\theta\, d\tau\, \alpha(\theta, \tau)\langle A_g(\theta, \tau, R, P)\rangle,$$

or using (5) and (4):

$$a(r, p) = \int d\theta\, d\tau\, \alpha(\theta, \tau) e^{i(\theta\cdot r + \tau\cdot p)}. \qquad (7)$$

It is easily shown that the expected values of the following generating operator:

$$D(R, P, r, p) = \frac{1}{(2\pi)^{6N}} \int d\tau'\, d\theta'\, e^{-i(\theta'\cdot r + \tau'\cdot p)} A_g(\theta', \tau', R, P) \qquad (8)$$

gives a distribution for which (4) and (5) hold:

$$f_g(r, p, t) = \langle D(R, P, r, p)\rangle. \qquad (9)$$

We show that this distribution also satisfies the other conditions that we listed earlier. Our approach here is related to that followed by Cohen.[3]

II. THE DISTRIBUTIONS

We can specify a distribution by writing the operators $A_g(\theta, \tau, R, P)$. We take, generally,

$$A_g(\theta, \tau, R, P) = g(\hbar\theta\cdot\tau) e^{i(\theta\cdot R + \tau\cdot P)}, \qquad (10)$$

where $g(x)$ has a series expansion about zero of the form

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} g^{(2n)}(0). \qquad (11)$$

Clearly we must take g to be an even function of $\hbar\tau\cdot\theta$ to insure that D is Hermitian.

The completeness of the operators $e^{i(\theta\cdot R + \tau\cdot P)}$ is shown in Ref. 2.

The Wigner distribution is obtained by taking

$$g(x) = 1.$$

Then,

$$f_w(r, p, t) = \frac{1}{(2\pi)^{6N}} \int d\tau'\, d\theta'\, e^{-i(\theta'\cdot r + \tau'\cdot p)} \langle e^{i(\theta'\cdot R + \tau'\cdot P)}\rangle. \qquad (12)$$

This form was obtained by Moyal.[4] If we recall that

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[B,A]}, \qquad (13)$$

for

$$[A, [B, A]] = [B, [B, A]] = 0,$$

and

$$[\theta\cdot R, \tau\cdot P] = i\hbar\theta\cdot\tau, \qquad (14)$$

we can write (12) as

$$\begin{aligned} f_w(r, p, t) &= \frac{1}{(2\pi)^{6N}} \int d\tau'\, d\theta'\, e^{-i(\theta'\cdot r + \tau'\cdot p)} \langle e^{i\tau'\cdot P/2} e^{i\theta'\cdot R} e^{i\tau'\cdot P/2}\rangle \\ &= \frac{1}{(2\pi)^{3N}} \int d\tau'\, e^{-\tau'\cdot p} \langle e^{i\tau'\cdot P/2} \delta(R - r) e^{i\tau'\cdot P/2}\rangle \qquad (15) \end{aligned}$$

or, alternatively, we can write

$$\begin{aligned} f_w(r, p, t) &= \frac{1}{(2\pi)^{6N}} \int d\tau'\, d\theta'\, e^{-i(\theta\cdot r + \tau'\cdot p)} \langle e^{i\theta'\cdot R/2} e^{i\tau'\cdot P} e^{i\theta'\cdot R/2}\rangle \\ &= \frac{1}{(2\pi)^{3N}} \int d\theta'\, e^{-i\theta'\cdot r} \langle e^{i\theta'\cdot R/2} \delta(P - p) e^{i\theta'\cdot R/2}\rangle. \qquad (16) \end{aligned}$$

Using (15) and (16), it is a straightforward matter to derive Eqs. (5a) and (5b) of Ref. 2.

It is clear that the generating operator for the generalized distribution is related to the generating operator for the Wigner distribution by commutators of R and P, since

$$g(\hbar\tau\cdot\theta) = g(-i[\theta\cdot R, \tau\cdot P]). \qquad (17)$$

As an example, let us consider the symmetric distribution introduced by Margenau and Hill.[5] As discussed by Cohen,[3] the appropriate $g(x)$ for this case is

$$g(x) = \cos(x/2).$$

In this case the distribution is

$$f_s(r, p, t) = \frac{1}{(2\pi)^{6N}} \int d\theta'\, d\tau' \cos(\hbar\tau'\cdot\theta'/2) \times e^{-i(\theta'\cdot r + \tau'\cdot p)} \langle e^{i(\theta'\cdot R + \tau'\cdot P)}\rangle. \qquad (18)$$

When we note that

$$\cos(\hbar\tau'\cdot\theta'/2) = \tfrac{1}{2}\{e^{\frac{1}{2}[\theta'\cdot R, \tau'\cdot P]} + e^{-\frac{1}{2}[\theta'\cdot R, \tau'\cdot P]}\} \qquad (19)$$

and use (13), we can write (18) as

$$\begin{aligned} f_s(r, p, t) &= \frac{1}{(2\pi)^{6N}} \int d\theta'\, d\tau'\, e^{-i(\theta'\cdot r + \tau'\cdot p)} \times \langle \tfrac{1}{2}\{e^{i\theta'\cdot R} e^{i\tau'\cdot P} + e^{i\tau'\cdot P} e^{i\theta'\cdot R}\}\rangle \\ &= \tfrac{1}{2}\langle \delta(R - r)\delta(P - p) + \delta(P - p)\delta(R - r)\rangle. \qquad (20) \end{aligned}$$

The remaining distributions commonly found in the literature can also be generated by an appropriate choice of $g(x)$.

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[4] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[5] H. Margenau and R. N. Hill, Progr. Theoret. Phys. (Kyoto) **26**, 722 (1961).

III. CONNECTIONS AMONG THE DISTRIBUTIONS

First let us show that the three properties of generalized phase-space distributions listed in Sec. I hold for the distributions generated by (8), (9), and (10).

Of course, our choice was made to provide a simple means of determining the generalized Wigner equivalents. Therefore we need not discuss this point further.

To find the classical limit we note[2] that $\langle e^{i(\theta\cdot R+\tau\cdot P)}\rangle$ has a series expansion in $\hbar$ and

$$\lim_{\hbar\to 0}\langle e^{i(\theta\cdot R+\tau\cdot P)}\rangle=\int dr'\,dp'\,f_c(r',p',t)e^{i(\theta\cdot r'+\tau\cdot p')}. \quad (21)$$

Also, we note from (11) that

$$\lim_{\hbar\to 0} g(\hbar\theta\cdot\tau)=1. \quad (22)$$

Then,

$$\begin{aligned}\lim_{\hbar\to 0} f_g(r,p,t)&=\frac{1}{(2\pi)^{6N}}\int d\tau'\,d\theta'\,dr'\,dp' f_c(r',p',t)e^{i[\theta'\cdot(r'-r)+\tau'\cdot(p'-p)]}\\&=f_c(r,p,t). \quad (23)\end{aligned}$$

Now let us consider the marginal distributions

$$\begin{aligned}\int dr\, f_g(r,p,t)&=\frac{1}{(2\pi)^{6N}}\int d\tau'\,d\theta'\,dr\, e^{-i(\theta'\cdot r+\tau'\cdot p)}g(\hbar\theta'\cdot\tau')\langle e^{i(\theta'\cdot R+\tau'\cdot P)}\rangle\\&=\frac{1}{(2\pi)^{3N}}\int d\tau'\,d\theta'\,\delta(\theta')e^{-i\tau'\cdot p}\langle e^{i\tau'\cdot P}\rangle,\end{aligned}$$

where we have taken $\theta'=0$ and noted that $g(0)=1$. The remaining integrations give Eq. (2) for f_g. It is obviously just as easy to show that Eq. (3) holds for f_g.

To establish the equivalence of the various distributions, we explicitly insert (8) and (10) in (9):

$$f_g(r,p,t)=\frac{1}{(2\pi)^{6N}}\int d\tau'\,d\theta'\,g(\hbar\theta'\cdot\tau')e^{-i(\theta'\cdot r+\tau'\cdot p)}\langle e^{i(\theta'\cdot R+\tau'\cdot P)}\rangle. \quad (24)$$

Using the property

$$g(x)=g(-x),$$

we note that

$$g(\hbar\theta'\cdot\tau)e^{-i(\theta'\cdot r+\tau'\cdot p)}=g(\hbar\nabla_r\cdot\nabla_p)e^{-i(\theta'\cdot r+\tau'\cdot p)}. \quad (25)$$

Recalling Eq. (12), we see that

$$f_g(r,p,t)=g(\hbar\nabla_r\cdot\nabla_p)f_w(r,p,t). \quad (26)$$

A form somewhat similar to this was used by von Roos[6] to obtain a distribution function for a molecular gas.

Now let us consider the generalized Wigner equivalent

$$a_g(r,p)=\int d\theta\,d\tau\,\alpha_g(\theta,\tau)e^{i(\theta\cdot r+\tau\cdot p)}, \quad (27)$$

where α_g is obtained from

$$A(R,P)=\int d\theta\,d\tau\,\alpha_g(\theta,\tau)g(\hbar\theta\cdot\tau)e^{i(\theta\cdot R+\tau\cdot P)}.$$

Since $g=1$ for the Wigner distribution, we must have

$$\alpha_w(\theta,\tau)=\alpha_g(\theta,\tau)g(\hbar\theta\cdot\tau). \quad (28)$$

Applying (28) and (25) in (27), we have

$$a_w(r,p)=g(\hbar\nabla_r\cdot\nabla_p)a_g(r,p). \quad (29)$$

IV. DISCUSSION

Clearly, the generalized phase-space distributions and the generalized Wigner equivalents are different for different choices of $g(x)$. However, the important conclusions regarding these distributions must be concerned with their connections with experiments in terms of Eq. (4). Consider, then,

$$\langle F(R,P)\rangle=\int dr\,dp\,f_g(r,p,t)a_g(r,p,t). \quad (30)$$

Using (26) we have

$$\langle F(R,P)\rangle=\int dr\,dp\,a_g(r,p,t)g(\hbar\nabla_r\cdot\nabla_p)f_w(r,p,t).$$

Integrating by parts gives

$$\langle F(R,P)\rangle=\int dr\,dp\,f_w(r,p,t)g(\hbar\nabla_r\cdot\nabla_p)a_g(r,p,t)$$

and, using (29),

$$\langle F(R,P)\rangle=\int dr\,dp\,f_w(r,p,t)a_w(r,p,t). \quad (31)$$

It is not surprising that both (30) and (31) hold, since we constructed the generalized phase-space distributions to satisfy just these equations. However, the rather trivial connections among the various distributions does not seem to have been pointed out in the literature, and leads one to wonder why more than the Wigner distribution need be considered for any calculations.

Using Eqs. (26) and (29), we can immediately relate the results already obtained for the Wigner distribution (as for example in Ref. 2) to the corresponding results for a generalized phase-space distribution.

[6] O. von Roos, J. Chem. Phys. 31, 1415 (1959).