Propagation of Circularly Polarized Electromagnetic Waves in a Finite Temperature Electromagnetoplasma

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The dispersion relation for circularly polarized electromagnetic waves in a warm two-component plasma subject to parallel static electric and magnetic fields has been derived from the linearized coupled Boltzmann–Maxwell equations with the collision frequency assumed to be independent of the particle velocity. The effect of a weak longitudinal electrostatic field, $\mathbf{E}_0$, on the propagation characteristic of the right- and left-hand circularly polarized waves in an isothermal electron–proton plasma is examined in detail and illustrated numerically for a conveniently chosen set of the system parameters. For the right-hand polarized wave the electrostatic field effect is found to be significant for a wave with frequency ω in the vicinity of the electron cyclotron frequency $\omega_z \equiv (eB_0/m)$. For example, for a given ω and $\delta \equiv (eE_0/mc\omega) > 0$ an increase in δ, or in E_0, leads to the increase or decrease of the attenuation constant α, of the wave according to whether $Y \equiv (\omega_z/\omega) < 1$ or $Y > 1$. Moreover, for $Y = 1.10$, when $\delta < 0$ (i.e., when $\mathbf{E}_0$ and the wave vector $\mathbf{k}$ are oppositely directed) α increases with $|\delta|$. On the other hand, when $\delta > 0$ an increase in $|\delta|$ causes α to decrease and for a sufficiently large value of δ, α may become negative so that the wave may experience a spatial growth.

I. INTRODUCTION

A theory of growing electromagnetic waves was advanced some years ago by Bailey[1–3] in his electromagnetoionic theory, which is an extension of the well-known magnetoionic theory of Appleton and others. The basis of Bailey's theory consists of the following physical laws:

(1) Maxwell's law governing the behavior of electromagnetic fields.

(2) The conservation of electron and positive ions.

(3) Maxwell's law for the transfer of momentum in a mixture of different kinds of particles.

The analysis of the dispersion relation for the system, derived from the above macroscopic laws under the small-amplitude condition, led Bailey[1] to predict the amplification of a plane wave, within a certain frequency band, in an ionized medium pervaded by static electric and magnetic fields which are both parallel to the direction of wave propagation. Bailey then applied his theory to explain the excess noise radiation observed in sunspot.[4] However, Bailey's[3] theory of amplified circularly polarized waves in an ionized medium was first criticized by Twiss[5] who argued that the growing wave, which Bailey interprets as an amplified wave, can only be excited by reflection at the boundary. A critical analysis of Bailey's theory was also given later by Piddington,[6] who shows that the electromagnetoionic theory predicts spurious growing waves which do not correspond to any interchange of energy between the gas and the field, but are attributed to the movement of the observer and emitter relative to the gas particles. Even with the additional ion motion the mass drift of the electrons and ions together introduce no new wave form in the electromagnetoionic theory although drift does modify the existing waves. While these authors are concerned primarily with the amplification aspects, no attention has been given to the other aspects of the propagation characteristics.

On the other hand, studies of electromagnetic wave propagation based on a macroscopic small-signal theory have been made by several authors for a drifting cold magnetoplasma[7–10] and a stationary two-component warm plasma.[11]

Recently, in the course of examining the dispersion relations for a finite temperature electromagnetoplasma some interesting effects on the propagation of circularly polarized electromagnetic waves due to static electric fields have been observed.[12] For example, the presence of an applied transverse static electric field causes the cutoff frequency to shift[13] and in addition results in the coupling of the longitudinal mode to a transverse circularly polar-

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ized mode; these are discussed elsewhere. On the other hand, when the applied static electric field is directed parallel to the static magnetic field, the presence of a static electric field may significantly affect the amplitude and phase of the electromagnetic wave.

The purpose of this paper, therefore, is to discuss in detail the effect of an applied longitudinal static electric field upon the propagation characteristics of a transverse circularly polarized wave which travels along the static magnetic field in a finite temperature unbounded two-component plasma. In the present discussion the coupled Boltzmann–Maxwell equation is used.

II. BASIC EQUATION

The electron distribution function $f(\mathbf{r}, \mathbf{v}, t)$ and the ion distribution function $F(\mathbf{r}, \mathbf{v}, t)$ for this plasma are governed by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v f = \nu_e(f_0 - f) \quad (1a)$$

and

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F + \frac{e}{M}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_v F = \nu_i(F_0 - F), \quad (1b)$$

where m and M denote the electron and ion masses, respectively, and e is the electronic charge which is taken as a positive quantity. ν_e and ν_i are the frequencies of collision of electrons with positive ions and of ions with electrons, respectively. These collision frequencies are assumed to be independent of the particle velocity. f_0 and F_0 are the equilibrium distribution functions of the electron and ion, respectively.

The electromagnetic fields in the plasma are governed by the Maxwell equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t},$$
$$\nabla\cdot\mathbf{D} = \rho, \quad \text{and} \quad \nabla\cdot\mathbf{B} = 0. \quad (2)$$

The electric displacement vector $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$ are, respectively, related to the electric field intensity $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ in the following manner:

$$\mathbf{D} = \epsilon_0\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_0\mathbf{H}, \quad (3)$$

where ϵ_0 and μ_0 denote the dielectric constant and the permeability of vacua.

The convection current density $\mathbf{J}$ and the charge density ρ may be written in terms of the distribution function as

$$\mathbf{J} = e\int \mathbf{v}(F - f)\, d^3v \quad \text{and} \quad \rho = e\int (F - f)\, d^3v. \quad (4)$$

Consider all quantities of interest to be composed of two parts, a time-independent part and a time-varying part, which are denoted by the subscripts 0 and 1, respectively,

$$\mathbf{B} = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), \qquad \mathbf{E} = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}, t);$$
$$\mathbf{J} = \mathbf{J}_0(\mathbf{r}) + \mathbf{J}_1(\mathbf{r}, t), \qquad \rho = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t); \quad (5)$$
$$f = f_0(\mathbf{r}, \mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t), \quad F = F_0(\mathbf{r}, \mathbf{v}) + F_1(\mathbf{r}, \mathbf{v}, t).$$

In the present paper the following assumptions are made:

(1) Small-amplitude conditions are satisfied so that the terms involving the product of time-dependent quantities are negligible.

(2) A one-dimensional analysis is applicable, i.e., all quantities vary only with one spatial variable, and $\partial/\partial x = \partial/\partial y = 0$ in a rectangular Cartesian coordinates system.

(3) All time-varying quantities have harmonic dependence of the form $\exp[j(\omega t - kz)]$, where ω and k are the angular frequency and the propagation constant, respectively.

Based on the above assumptions, the substitution of Eq. (5) into Eqs. (1)–(4) results in two sets of differential equations, one of which governs the time-independent quantities and the other governs the time-varying quantities. The former is given by

$$v_z\frac{\partial f_0}{\partial v_z} - \frac{e}{m}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v f_0 = 0, \quad (6a)$$

$$v_z\frac{\partial F_0}{\partial v_z} + \frac{e}{M}(\mathbf{E}_0 + \mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v F_0 = 0, \quad (6b)$$

$$\frac{\partial E_{0x}}{\partial x} = 0, \qquad \frac{\partial E_{0y}}{\partial y} = 0, \qquad \frac{\partial E_{0z}}{\partial z} = \frac{\rho_0(z)}{\epsilon_0}, \quad (6c)$$

$$\frac{\partial B_{0y}}{\partial z} = -\mu_0 J_{0x}, \qquad \frac{\partial B_{0x}}{\partial z} = \mu_0 J_{0y}, \qquad \frac{\partial B_{0z}}{\partial z} = 0, \quad (6d)$$

$$\mathbf{J}_0 = e\int \mathbf{v}(F_0 - f_0)\, d^3v, \quad (6e)$$

and

$$\rho = e\int (F_0 - f_0)\, d^3v, \quad (6f)$$

where $d^3v = dv_x\, dv_y\, dv_z$ denotes the volume element in velocity space. On the other hand, in discussing the set of equations relating the time-varying quantities it is convenient to consider the following transformation of variables:

$$v_x = v_r\cos\varphi \quad \text{and} \quad v_y = v_r\sin\varphi, \quad (7a)$$

and

$$E_\pm = \tfrac{1}{2}(E_{1x} \pm jE_{1y}) \text{ and } B_\pm = \tfrac{1}{2}(B_{1x} \pm jB_{1y}). \quad (7b)$$

Then, by introducing the following parameters:

$$\boldsymbol{\omega}_c = \left(\frac{e\mathbf{B}_0}{m}\right), \quad \mathbf{a} = \left(\frac{e\mathbf{E}_0}{m}\right), \quad \boldsymbol{\Omega}_c = \left(\frac{-e\mathbf{B}_0}{M}\right),$$

and

$$\mathbf{A} = \left(\frac{-e\mathbf{E}_0}{M}\right),$$

the time-varying parts of Eqs. (2)–(4) can be combined to give

$$2\left(\frac{\omega^2}{c^2} - k^2\right)E_\pm = j\omega\mu_0 e \int_{-\infty}^{\infty}\int_0^\infty\int_0^{2\pi} e^{\pm i\varphi}(F_1 - f_1)v_r^2\, d\varphi\, dv_r\, dv_z \quad (8a)$$

and

$$E_{1z} = \frac{je}{\omega\epsilon_0}\int_{-\infty}^{\infty}\int_0^\infty\int_0^{2\pi}(F_1 - f_1)v_r v_z\, d\varphi\, dv_r\, dv_z. \quad (8b)$$

Moreover, the time-varying parts of Eq. (1a), with the aid of Eqs. (7), can be written as follows:

$$\begin{aligned}
&\left[j(\omega - kv_z) + \nu_e + \omega_z\frac{\partial}{\partial\varphi}\right]f_1 - a_z\frac{\partial f_1}{\partial v_z} \\
&\quad - \left[a_-\left(\frac{\partial f_1}{\partial v_r} + \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right) + \omega_-\frac{v_z}{v_r}\frac{\partial f_1}{\partial\varphi} + j\omega_- D(f_1)\right]e^{i\varphi} \\
&\quad - \left[a_+\left(\frac{\partial f_1}{\partial v_r} - \frac{j}{v_r}\frac{\partial f_1}{\partial\varphi}\right) + \omega_+\frac{v_z}{v_r}\frac{\partial f_1}{\partial\varphi} - j\omega_+ D(f_1)\right]e^{-i\varphi} \\
&= \frac{e}{m}M_-(f_0)E_-e^{i\varphi} + \frac{e}{m}M_+(f_0)E_+e^{-i\varphi} + \frac{e}{m}E_{1z}\frac{\partial f_0}{\partial v_z} \\
&\quad - \frac{e}{m}B_{1z}\frac{\partial f_0}{\partial\varphi}, \qquad (9a)
\end{aligned}$$

where the differential operator D is defined by

$$D(\ ) \equiv \left(v_r\frac{\partial}{\partial v_z} - v_z\frac{\partial}{\partial v_r}\right)$$

and

$$\omega_\pm = \tfrac{1}{2}(\omega_{cx} \pm j\omega_{cy}), \qquad a_\pm = \tfrac{1}{2}(a_x \pm ja_y),$$

$$M_\pm(f_0) \equiv \left[\left(1 - \frac{kv_z}{\omega}\right)\left(\frac{\partial f_0}{\partial v_r} \mp j\frac{1}{v_r}\frac{\partial f_0}{\partial\varphi}\right) + \frac{kv_r}{\omega}\frac{\partial f_0}{\partial v_z}\right]. \quad (9b)$$

On the other hand, the time-varying part of Eq. (1b) which governs the F_1 function, is easily obtained by replacing $a_\pm$, a_z, $\omega_\pm$, ω_z, ν_e, f_1, and f_0 with $A_\pm$, A_z, $\Omega_\pm$, Ω_z, ν_i, F_1, and F_0, respectively, in Eqs. (9).

It should be noted that E_- and E_+ in Eq. (9a) are the left- and right-hand circularly polarized components of electric field, respectively.

III. DISPERSION RELATIONS

Suppose that the positive z direction is taken in the direction of the magnetostatic field $\mathbf{B}_0$, i.e., $B_{0x} = B_{0y} = 0$ so that $\omega_\pm = 0$. Since $\nabla\cdot\mathbf{B} = 0$, B_{1z} must be independent of z, and it is taken to be zero in the present discussion (which is reasonable for a longitudinal propagation). Moreover, consider that the time-varying electron distribution function f_1 is composed of three parts and may be written as

$$f_1 = f_-(z, t, v_r, v_z)e^{i\varphi} + f_+(z, t, v_r, v_z)e^{-i\varphi} + g(z, t, v_r, v_z). \quad (10)$$

Since Eq. (9a) must be valid for an arbitrary value of φ, the substitution of Eq. (10) into Eq. (9a) yields the following system of equations:

$$j(\omega - j\nu_e - kv_z + \omega_z)f_- - a_z\frac{\partial f_-}{\partial v_z} - a_-\frac{\partial g}{\partial v_r} = \frac{e}{m}M_-(f_0)E_-, \quad (11a)$$

$$j(\omega - j\nu_e - kv_z - \omega_z)f_+ - a_z\frac{\partial f_+}{\partial v_z} - a_+\frac{\partial g}{\partial v_r} = \frac{e}{m}M_+(f_0)E_+, \quad (11b)$$

and

$$j(\omega - j\nu_e - kv_z)g - a_z\frac{\partial g}{\partial v_z} - \frac{2a_-}{v_r}f_+ - \frac{2a_+}{v_r}f_- = \frac{e}{m}\frac{\partial f_0}{\partial v_z}E_{1z} \quad (11c)$$

which clearly suggests the possibility of coupling between the transverse mode and the longitudinal mode when a_+ and a_- are nonzero, which is the case when the transverse electrostatic field is present. This case has been examined and discussed elsewhere.[13]

In the present investigation it is assumed that $a_\pm = 0$, i.e., $E_{0x} = E_{0y} = 0$, since the effect of the longitudinal electrostatic field is of primary concern. Suppose that f_-, f_+, and g have the v_z dependence of the form $\exp(-\alpha_1 v_z^2)$, in which $\alpha_1 = m/(2KT_e)$; then f_-, f_+, and g can be explicitly expressed in terms of E_-, E_+, and E_{1z}, respectively, from Eqs. (11) as

$$f_\mp = \frac{(e/m)M_\mp(f_0)E_\mp}{j(\tilde{b}_1 \pm \omega_z)} \quad \text{and} \quad g = \frac{(e/m)(\partial f_0/\partial v_z)E_{1z}}{j\tilde{b}_1}, \quad (12)$$

where

$$\tilde{b}_1 = (\tilde{\omega}_1 - \tilde{k}_1 v_z),\ \tilde{\omega}_1 = (\omega - j\nu_e),\ \tilde{k}_1 = (k + jK_1),$$

and

$$K_1 = \left(\frac{eE_0}{KT_e}\right).$$

K and T_e denote the Boltzmann constant and the electron temperature, respectively. Thus combining Eqs. (10) and (12) the distribution function f_1 is expressed in terms of $E_\pm$ and E_{1z}. Similarly the time-varying ion distribution function F_1 can be written as

$$F_1 = F_-(z, t, v_r, v_z)e^{i\varphi} + F_+(z, t, v_r, v_z)e^{-i\varphi} + G(z, t, v_r, v_z), \tag{13}$$

where

$$F_\pm = \frac{-(e/M)M_\mp(F_0)E_\mp}{j(\tilde{b}_2 \pm \Omega_z)}$$

and

$$G = \frac{-(e/M)(\partial F_0/\partial v_z)E_{1z}}{j\tilde{b}_2}, \tag{14}$$

in which

$$\tilde{b}_2 = (\tilde{\omega}_2 - \tilde{k}_2 v_z),\ \tilde{\omega}_2 = (\omega - j\nu_i),\ \tilde{k}_2 = (k + jK_2),$$

and

$$K_2 = \left(\frac{-eE_0}{KT_i}\right).$$

M and T_i denote the mass and temperature of the ion, respectively,

Upon substituting F_1 and f_1, given by Eqs. (13) and (10), respectively, into Eqs. (8) the following set of equations is obtained:

$$1 + \frac{\pi(\omega e/\epsilon_0)}{(\omega^2 - c^2k^2)} \cdot \int_{-\infty}^{\infty}\int_0^{\infty}\left[\frac{(e/m)M_\mp(f_0)}{(\tilde{b}_1 \pm \omega_z)} + \frac{(e/M)M_\mp(F_0)}{(\tilde{b}_2 \pm \Omega_z)}\right]v_r^2\, dv_r\, dv_z = 0 \tag{15a}$$

and

$$1 + \frac{2\pi e}{\omega\epsilon_0} \cdot \int_{-\infty}^{\infty}\int_0^{\infty}\left(\frac{e}{m}\frac{1}{\tilde{b}_1}\frac{\partial f_0}{\partial v_z} + \frac{e}{M}\frac{1}{\tilde{b}_2}\frac{\partial F_0}{\partial v_z}\right)v_r v_z\, dv_r\, dv_z = 0. \tag{15b}$$

Equation (15a) represents the dispersion relation for the transverse circularly polarized modes; the upper sign is to be taken for the left-hand circularly polarized mode and the lower sign is for the right-hand circularly polarized mode. Equation (15b) is the dispersion relation for the longitudinal mode. In the following discussion, Eq. (15a) is examined in detail.

It should be noted that when $E_0 = 0$, $K_1 = K_2 = 0$, and when $\nu_e = \nu_i = 0$, $\tilde{\omega}_1 = \tilde{\omega}_2 = \omega$, so that $\tilde{b}_1 = \tilde{b}_2 = (\omega - kv_z)$, then Eq. (15a) is reduced to those given by Montgomery and Tidman.[14]

For a one-dimensional analysis in a Maxwellian plasma, f_0 and F_0 can be written as

$$f_0 = n\left(\frac{\alpha_1}{\pi}\right)^{\frac{3}{2}} \exp\left[-\alpha_1(v_r^2 + v_z^2) + \frac{e\Phi(z)}{KT_e}\right] \tag{16a}$$

and

$$F_0 = N\left(\frac{\alpha_2}{\pi}\right)^{\frac{3}{2}} \exp\left[-\alpha_2(v_r^2 + v_z^2) - \frac{e\Phi(z)}{KT_i}\right], \tag{16b}$$

in which the electric scalar potential $\Phi(z)$ is related to static electric field by $E_{0z} = -d\Phi/dz$. n and N are the electron and positive ion concentrations, respectively. It is easily verified that f_0 and F_0 given by Eqs. (16) satisfy Eqs. (6a) and (b), respectively. Furthermore, they yield that $J_{0x} = J_{0y} = J_{0z} = 0$ which implies that the static magnetic field is independent of z. The space-charge density ρ_0 is given by

$$\rho_0(z) = eN \exp\left(\frac{-e\Phi}{KT_i}\right) - en \exp\left(\frac{e\Phi}{KT_e}\right). \tag{17}$$

Suppose that Φ is sufficiently small so that

$$\left|\frac{e\Phi}{KT_e}\right| \ll 1 \quad \text{and} \quad \left|\frac{e\Phi}{KT_i}\right| \ll 1 \tag{18}$$

and the condition of electrical neutrality is also satisfied, i.e., $n = N$. Then ρ_0 vanishes and from Eq. (6c), E_{0z} will be independent of z.

In the present discussion E_{0z} is assumed to be constant and denoted by E_0.

Having assumed the form of the functions f_0 and F_0, the indicated integration in Eq. (15a) can be carried out to give

$$1 - \frac{c^2k^2}{\omega^2} = \frac{X_1}{W_1}\left(1 + \frac{V_1^2\tilde{k}_1^2}{2\omega^2}\frac{1}{W_1^2}\right) + \frac{X_2}{W_2}\left(1 + \frac{V_2^2\tilde{k}_2^2}{2\omega^2}\frac{1}{W_2^2}\right), \tag{19}$$

where

$$W_1 = [(1 \pm Y_1) - jZ_1], \qquad W_2 = [(1 \pm Y_2) - jZ_2],$$

$$\tilde{k}_1 = (k + jK_1), \qquad \tilde{k}_2 = (k + jK_2),$$

$$X_1 = \left(\frac{\omega_p}{\omega}\right)^2, \qquad Y_1 = \left(\frac{\omega_z}{\omega}\right),$$

$$Z_1 = \left(\frac{\nu_e}{\omega}\right), \qquad K_1 = \left(\frac{eE_0}{KT_e}\right),$$

[14] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill Book Company, Inc., New York, 1964), Chap. 10.

$$X_2 = \left(\frac{\Omega_p}{\omega}\right)^2, \qquad Y_2 = \left(\frac{\Omega_z}{\omega}\right), \qquad Z_2 = \left(\frac{\nu_i}{\omega}\right),$$

$$K_2 = \left(\frac{-eE_0}{KT_i}\right), \qquad \omega_z = \left(\frac{eB_{0z}}{m}\right), \qquad \omega_p = \left(\frac{e^2n}{m\epsilon_0}\right)^{\frac{1}{2}},$$

$$\Omega_z = \left(\frac{-eB_{0z}}{M}\right), \qquad \Omega_p = \left(\frac{e^2N}{M\epsilon_0}\right)^{\frac{1}{2}}.$$

It should be pointed out that the derivation of Eq. (19) involves an evaluation of an integral of the form

$$G_0(\chi) \equiv \frac{j}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} \frac{\exp\,(-\alpha v_z^2)}{(v_z - \chi)}\, dv_z, \tag{20a}$$

where χ may take on the value of $(\omega/\tilde{k}_{1,2})(1 - jZ_{1,2})$ or $(\omega/\tilde{k}_{1,2})W_{1,2}$. This integral has been discussed in detail by Stix[15] and his results are used here. When the term representing the Landau or cyclotron damping is neglected and taking only the first two terms of its asymptotic expansion, $G_0(\chi)$ can be given by

$$G_0(\chi) = \frac{-j}{\alpha^{\frac{1}{2}}\chi}\left(1 + \frac{1}{2\alpha\chi^2}\right), \tag{20b}$$

provided that $|\alpha^{\frac{1}{2}}\chi|^4 \gg 1$ is satisfied. It is of interest to note that when $V_1 = V_2 = 0$, Eq. (19) is reduced to the familiar dispersion equation in the cold-plasma magnetoionic theory. On the other hand, when $E_0 = 0$, $K_1 = K_2 = 0$, so that $\tilde{k}_1 = \tilde{k}_2 = k$, and when $\nu_1 = \nu_2 = 0$, $\tilde{\omega}_1 = \tilde{\omega}_2 = \omega$ so that Eq. (19) is reduced to those given by Heald and Wharton.[16]

IV. PROPAGATION CONSTANT

Equation (19) is a quadratic in k and can conveniently be written as

$$\tilde{A}\left(\frac{\tilde{k}}{\beta_0}\right)^2 + \tilde{B}\left(\frac{\tilde{k}}{\beta_0}\right) - \tilde{C} = 0, \tag{21a}$$

where

$$\tilde{A} = \left(1 + \frac{\tau_1 X_1}{\tilde{W}_1^3} + \frac{\tau_2 X_2}{\tilde{W}_2^3}\right),$$

$$\tilde{B} = j2\left(\frac{\delta_1 X_1}{\tilde{W}_1^3} + \frac{\delta_2 X_2}{\tilde{W}_2^3}\right), \tag{21b}$$

$$\tilde{C} = \left(1 - \frac{X_1}{\tilde{W}_1} - \frac{X_2}{\tilde{W}_2}\right),$$

in which

$$\tau_1 = \left(\frac{KT_e}{mc^2}\right), \qquad \tau_2 = \left(\frac{KT_i}{Mc^2}\right), \qquad \delta_1 = \left(\frac{eE_0}{mc\omega}\right),$$

$$\delta_2 = \left(\frac{-eE_0}{Mc\omega}\right) \quad \text{and} \quad \beta_0 = \left(\frac{\omega}{c}\right).$$

[15] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), Chap. 8.

[16] M. A. Heald and C. B. Wharton, *Plasma Diagnostics with Microwaves* (John Wiley & Sons, Inc., New York, 1965), Chap. 3.

The symbol $\sim$ appearing in Eqs. (21) is introduced to emphasize the fact that the quantity under consideration is a complex quantity. In the present discussion the wave angular frequency ω is regarded as real and the propagation constant $\tilde{k}$ is regarded as a complex quantity which can be written as

$$\tilde{k} = (\beta - j\alpha), \tag{22}$$

where α and β are the amplitude and phase constants, respectively. Since the time and spatial dependence is assumed to be in the form $\exp\,[j(\omega t - \tilde{k}z)] = \exp\,[-\alpha z + j(\omega t - \beta z)]$, the forward and backward waves are represented by positive and negative values of β, respectively. On the other hand, the attenuation and amplification of the wave are represented by a positive and negative value of α, respectively. Once the system parameters are specified, Eq. (21a) can be solved for $(\tilde{k}/\beta_0) = (\beta/\beta_0 - j\alpha/\beta_0)$ and the propagation characteristics can be examined.

It should be noted that for an electrically neutral plasma the electronic and ionic parameters are related by $X_2 = \xi X_1$, $Y_2 = -\xi Y_1$, $\tau_2 = (\xi/\theta)\tau_1$ and $\delta_2 = -\delta_1/\xi$, where $\xi = (m/M)$ and $\theta = (T_e/T_i)$. Furthermore, the collision frequencies ν_e and ν_i are, in general, dependent on the density and temperature. For a fully ionized gas ν_e is determined by the electron–ion encounter, while for a partially ionized gas ν_e is determined by electron–neutral encounter and electron–ion encounter.[17] For the purpose of illustration, a fully ionized gas, consisting of electrons and protons, is considered here. For a Maxwellian isothermal plasma (i.e., $T_e = T_i$) ν_{ei} can be given by[18]

$$\nu_{ei} = 3.63 \times 10^{-6}\left(\frac{n}{T_e^{\frac{3}{2}}}\right) \ln \Lambda, \tag{23}$$

where

$$\Lambda = 1.24 \times 10^7\left(\frac{T_e^3}{n}\right)^{\frac{1}{2}},$$

in which n is the electron concentration in mks units. On the other hand, the effective collision frequency for proton–electron encounters can be given by[19]

$$\nu_p = \left(\frac{m}{M}\right)^{\frac{1}{2}}\nu_e \simeq \frac{1}{43}\,\nu_e, \tag{24}$$

[17] B. S. Tanenbaum and D. Mintzer, Phys. Fluids **5**, 1226 (1962).

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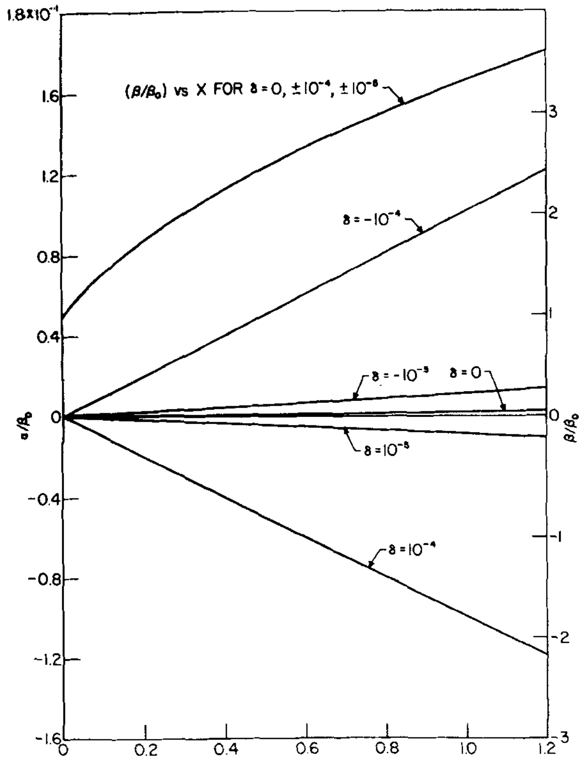
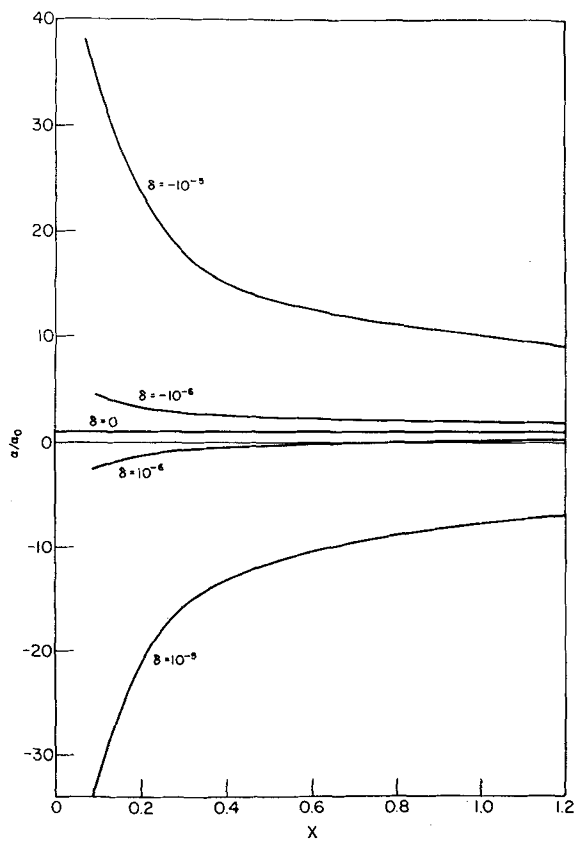
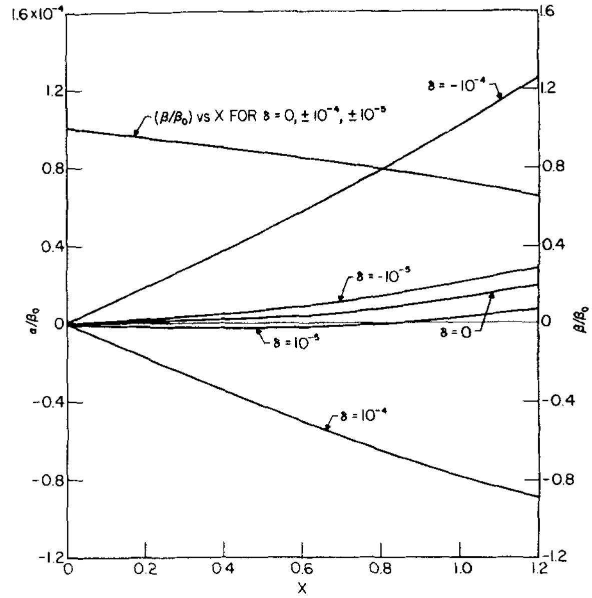


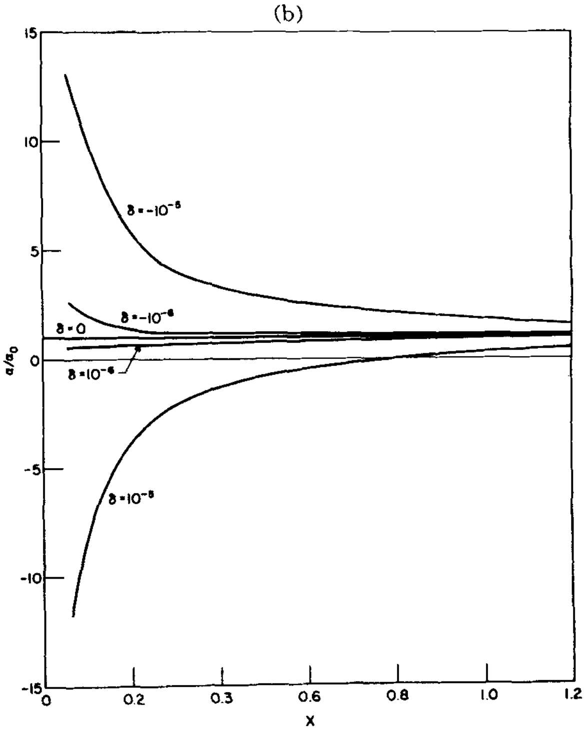
(a)



(c)



(b)



(d)

FIG. 1. (a) Variation of the amplitude coefficient (α/β_0) and the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave in an isothermal electron–proton plasma, with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (b) Variation of the amplitude coefficient (α/β_0) and the phase coefficient (β/β_0) with electron density (X) for left-hand circularly polarized wave in an isothermal electron–proton plasma, with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (c) Variation of the normalized amplitude constant (α/α_0) with the normalized electron density (X) for right-hand circularly polarized wave in an isothermal electron-proton plasma with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec. (d) Variation of the normalized amplitude constant (α/α_0) with the normalized electron density (X) for left-hand circularly polarized wave in an isothermal electron–proton plasma with $Y=1.10$, $\tau=2\times10^{-6}$, and $\omega=2\pi\times10^9$ rad/sec.

where m_p is the proton mass, therefore, $Z_2 = \xi^{\frac{1}{2}} Z_1$. The variations of (α/β_0) and (β/β_0) are illustrated numerically in Figs. 1–3 for an isothermal electron–proton plasma (i.e., $\xi = 1/1836$ and $\theta = 1$) with the aid of Eqs. (23) and (24).

In these illustrations the frequency of the wave under consideration is taken as 1 GHz. The plots of amplitude coefficient (α/β_0) and phase coefficient (β/β_0) versus the electron density (X) for different values of the parameter $\delta \equiv (eE_0/mc\omega)$, in the case $Y = 1.10$, are shown in Figs. 1. Figure 1(a) deals with the right-hand circularly polarized wave, while Fig. 1(b) is for the left-hand circularly polarized wave. From both Figs. 1(a) and 1(b), it is observed, that for a given set of system parameters a change in δ has a profound effect on the amplitude, but it has practically no effect on the phase of the circularly polarized wave. Furthermore, when $\delta < 0$ (i.e., when the longitudinal electrostatic field $\mathbf{E}_0$ is directed opposite to the wave vector, $\mathbf{k}$) an increase in $|\delta|$ leads to an increase in the attenuation of the wave. On the other hand, when $\delta > 0$ (i.e., when $\mathbf{E}_0$ and $\mathbf{k}$ are in the same direction) an increase in $|\delta|$ leads to the reduction of the wave attenuation. In this case if δ is sufficiently large, α may become negative so that the wave may experience an amplification rather than an attenuation. However, it is difficult to give a comprehensive physical interpretation for gain in wave amplitude when $\delta > 0$, without a detailed analysis of the dynamic behavior of the charged particles or the studies of the energy conversion process between the particles and the electromagnetic wave. This is not done in the present paper; however, the question of energy conversion process will be considered in a future paper. The result of the present simple-minded theory appears to suggest that the effect of electrostatic field on the amplitude of the electromagnetic wave is evident. While the collision process in the plasma tends to randomize the order motion, the introduction of electrostatic field in the wave direction tends to reorganize the motion of the particle in such a way as to make the exchange of energy between the particles and the wave easier. In other words, it makes the extraction of particle energy in the plasma easier. Comparison of Figs. 1(a) and (b) shows that the amplitude coefficient for the right-hand circularly polarized wave is three orders of magnitude greater than that of the left-hand circularly polarized wave.

In the interest of emphasizing the change in the amplitude constant caused by the electrostatic field, the plots of α/α_0 vs X for the right-hand circularly polarized and the left-hand circularly polarized waves are shown in Fig. 1(c) and Fig. 1(d), respectively, where α_0 denotes the amplitude constant for the case of zero static electric field and also represents the rate of collision damping. In these figures, it is observed that for a given X, $|\alpha/\alpha_0|$ increases as $|\delta|$, which suggests that the effect of E_0 on the

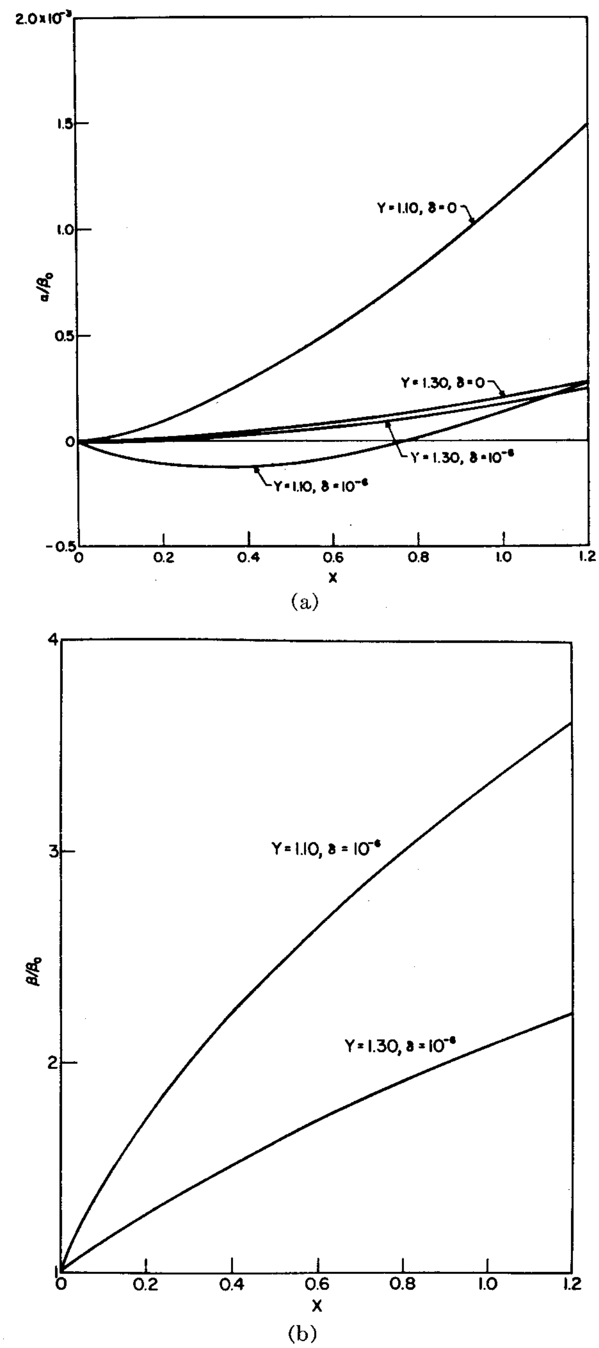


FIG. 2. (a) Variation of the amplitude coefficient (α/β_0) with electron density (X) for right-hand circularly polarized wave with $Y>1$, $\tau=2 \times 10^{-6}$, and $\omega=2\pi \times 10^9$ rad/sec. (b) Variation of the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave with $Y > 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec.

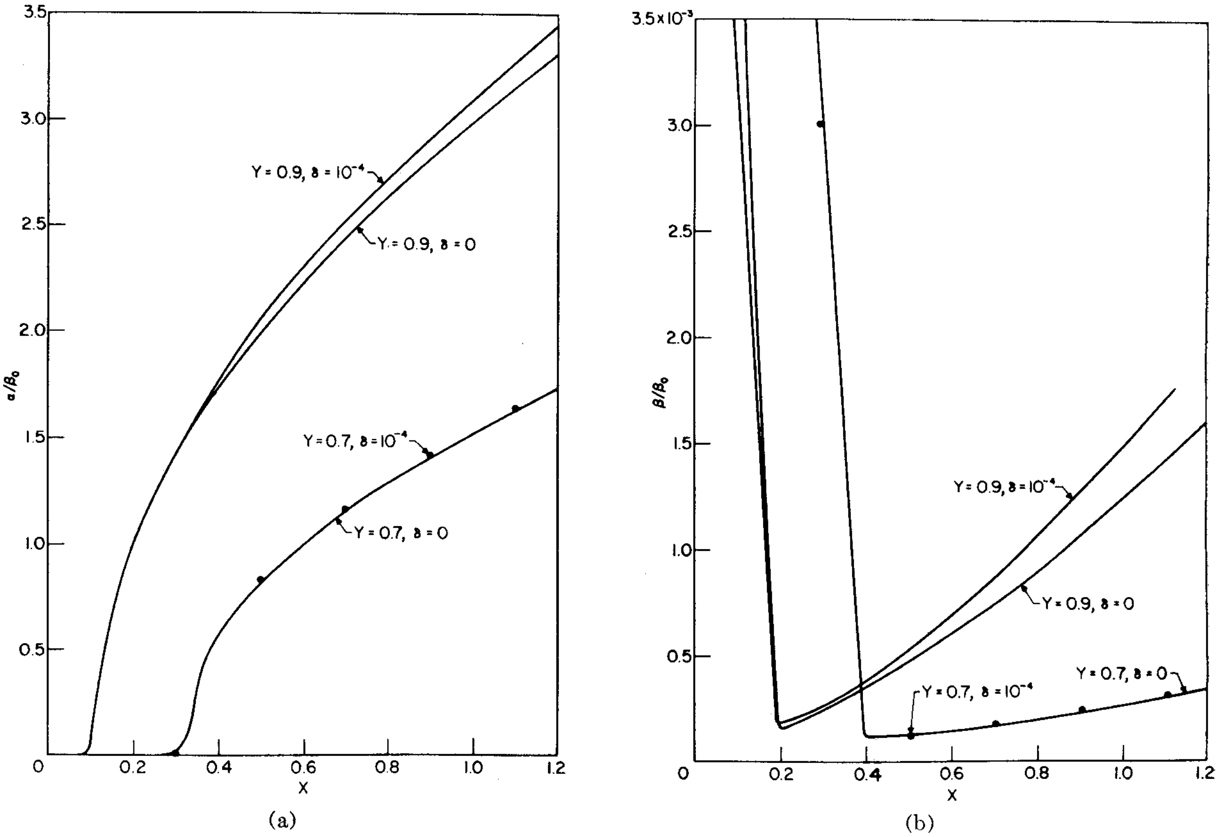


FIG. 3. (a) Variation of the amplitude coefficient (α/β_0) with electron density (X) for right-hand circularly polarized wave with $Y < 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec. (b) Variation of the phase coefficient (β/β_0) with electron density (X) for right-hand circularly polarized wave with $Y < 1$, $\tau = 2 \times 10^{-6}$, and $\omega = 2\pi \times 10^9$ rad/sec.

amplitude constant α increases as $|E_0|$. On the other hand, the change in the amplitude constant α is more drastic in the range of small X than in the range of large X; i.e., the effect of E_0 on the change in amplitude of the electromagnetic wave is greater in the region of low electron number density than in the region of high number density.

The effects of change in the strength of magnetostatic field $\mathbf{B}_0$ upon the plots of (α/β_0) and (β/β_0) vs X, for the right-hand circularly polarized wave, are illustrated in Figs. 2 for the case $Y > 1$ and in Figs. 3 for the case $Y < 1$. It is observed that in the case $Y > 1$ [see Figs. 2(a) and (b)], for a given value of X, and $\delta = 0$ an increase in $Y \equiv (eB_0/m\omega)$ causes both α and β to decrease, which suggests that an increase in $|\mathbf{B}_0|$ reduces the attenuation and increases the phase velocity of the wave. On the other hand, in the case $Y < 1$ [see Figs. 3(a) and (b)] for $X > 0.4$ an increase in Y causes both α and β to increase so that an increase in $|\mathbf{B}_0|$ leads to an increase of attenuation and reduction of phase velocity of the wave.

It is also of interest to note, by comparing Figs. 2(a) and 3(a), that when $\delta > 0$ an increase in δ decreases α for the case $Y > 1$, whereas it increases α for the case $Y < 1$. On the other hand, comparison of Figs. 2(b) and 3(b) suggests that the presence of E_0 does modify the phase of the wave somewhat for the case $Y < 1$, but it has no effect on the phase for the case $Y > 1$.

V. CONCLUDING REMARKS

In the present discussion the electrostatic electric field E_0 is assumed to be sufficiently weak and the drift velocity of the plasma is much smaller than the phase velocity of the wave under consideration. Thus, it is assumed that the medium through which the electromagnetic wave propagates is essentially stationary rather than drifting.

It is shown that the effect of a weak static electric field, directed along the direction of wave propagation, on the amplitude and phase of the right-hand circularly polarized wave is most significant when the wave frequency is in the vicinity of the electron cyclotron frequency, e.g., for cases $Y = 1.1$ or 0.9.

A constant collision frequency model has been used in the present discussion. For the system parameters chosen for illustration this assumption is reasonable. The effect of a velocity-dependent collision frequency on the Appleton–Hartree equation of magnetoionic theory has been discussed by Shkarofsky.[20]

It should be pointed out that in the present discussion a Maxwellian plasma has been considered, i.e., the time-independent distribution functions of electron and positive ions are assumed to be a Maxwellian. Furthermore, it is also assumed that the time-varying distribution function of electron and positive ions has a Maxwellian distribution in the direction of wave propagation. It should be noted that the latter assumption may not in general be valid. The reasonableness of this assumption might be tested by an experimental investigation which is to be considered in the future. However, if this assumption is not too unreasonable, then the result of the present theory suggests that the introduction of an electrostatic field in the direction of wave propagation may reduce the attenuation of the wave. It is of interest to note that with a proper strength of E_0 it may also lead to an amplification of the circularly polarized electromagnetic wave in a warm collisional two-component magnetoplasma.

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ACKNOWLEDGMENT

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