

Energy-Dependent Boltzmann Equation in the Fast Domain*

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(Received 17 March 1969)

This work presents some aspects of the static energy-dependent Boltzmann equation in plane geometry using a continuous-energy formulation. In a first part, solutions are found for a class of synthetic separable (but nondegenerate) energy-transfer kernels. Such kernels are representative, for instance, of neutron *inelastic* slowing down. In a second part, the same problem is considered with the addition of a projection kernel (typical of neutron fission); it is shown that the solutions split into space-energy separable components and nonseparable "slowing-down transients."

I. INTRODUCTION

Little progress has been made in the solution of the energy-dependent Boltzmann equation, as opposed to the status of the one-speed transport equation, where Case's method of singular normal modes¹ has yielded considerable success.

For a long time, energy-dependent investigations were specialized to finding exact solutions to the spatially dependent neutron slowing-down problem, with *elastic* scattering and without fission.²⁻⁷

More recent work has been applied to the neutron thermalization domain: The energy-transfer operator has been approximated by a sum of degenerate (projection) kernels, which, in turn, allows the reduction of the initial equation to a set of coupled one-speed transport equations.⁸⁻¹⁵ Also, some work has been done on the multigroup formulation,¹⁶⁻¹⁸ but the discretization of the energy variable distorts the spectrum of the Boltzmann operator: This is of prime

importance in the study of the time-dependent evolution.¹⁹

Works trying to extend Case's method to the most general energy-dependent equation are somewhat limited in scope.²⁰ (The completeness theorem involved relies upon the Fredholm alternative for the inversion of operator equations, which is correct only when the energy-transfer kernel, or some iterate, is compact²¹; it fails for unbounded and noncompact kernels such as are found in neutron slowing-down theory.)

This work presents some aspects of the energy-dependent, static Boltzmann equation, in plane geometry, with a continuous energy formulation. In a first part, solutions are found for a class of synthetic, separable, but not degenerate, energy-transfer kernels: Such kernels are representative, for instance, of neutron *inelastic* slowing down. A new energy transformation is developed, which reduces the initial equation to a simple form, and an asymptotic evaluation of the Green's function is given.

In a second part, the same problem is considered with the addition of a projection kernel (typical of neutron fission); it is shown that the solutions split into (1) space-energy separable components, representative of the neutron regeneration, and asymptotically dominant; and (2) nonseparable, "slowing-down transients" solution of the initial equation without the fission-projection kernel. This generalizes the results of a previous work²²: In order to achieve completeness for the normal modes of the Boltzmann equation with fission and slowing down, one must introduce auxiliary modes which are solutions of the ordinary slowing-down equation.

* Work supported by the National Science Foundation. Based in part on the author's Ph.D. thesis, Department of Nuclear Engineering, The University of Michigan.

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II. THE BOLTZMANN EQUATION¹ WITH A CLASS OF SYNTHETIC ENERGY-TRANSFER KERNELS

A. Introduction of a Synthetic Scattering Kernel

Many extensive solutions have been proposed to the problem of *spatial* neutron slowing down with *elastic* scattering.^{2-7,23,24} Yet, little attention has been paid to the fast domain, where *inelastic* scattering is overwhelming dominant, especially for heavy nuclei: Most calculation schemes use the multigroup (discrete energy) formulation. Such a multigroup formulation distorts the spectrum of the Boltzmann operator. The necessity for a continuous energy formulation has been widely recognized in the neutron thermalization domain, where a rigorous study of the spectrum of the Boltzmann operator is fundamental for the interpretation of the time-dependent evolution of the neutron field.¹⁹ This is also valid for the fast domain.

With the assumption of plane symmetry, we are interested in the following slowing-down transport equation:

$$\begin{aligned} \mu \frac{\partial \psi}{\partial x}(x, \mu, E) + \psi(x, \mu, E) \\ = \frac{c_i}{2} \int_E^{+\infty} \int_{-1}^{+1} K_{\text{in}}(E' \rightarrow E) \psi(x, \mu', E') dE' d\mu' \\ + \frac{c_e}{2} \int_{-1}^{+1} \psi(x, \mu', E) d\mu' + S(x, \mu, E), \end{aligned} \quad (1)$$

where $\psi(x, \mu, E)$ is the angular neutron density, c_i is the mean number of secondaries emitted after an inelastic scattering collision times the probability of inelastic scattering, c_e is the mean number of secondaries emitted after an elastic scattering collision times the probability of elastic scattering, x is the position variable measured in optical units, E is the neutron energy, μ is the cosine of the angle between the neutron velocity vector and the x axis, and $S(x, \mu, E)$ is the source term. The kernel $K_{\text{in}}(E' \rightarrow E)$ gives the probability that a neutron of energy E' will be slowed down to a unit energy interval about the energy E by inelastic collision. Inelastic scattering is assumed isotropic in the laboratory system.

Equation (1) contains two simplifying assumptions:

(a) The cross sections are supposed to be constant above the inelastic scattering threshold (first excited level $\simeq 30$ keV for heavy fissionable nuclei);

(b) Above the inelastic threshold, the energy transfer due to *elastic* scattering is considered as negligible

compared to the inelastic energy degradation (which is valid for heavy nuclei).

In no way does Eq. (1) assume constant cross sections throughout the whole energy range. Below the inelastic threshold energy E_0 , the inelastic scattering term becomes a *known* isotropic source term;

$$\frac{c_i}{2} \int_{-1}^{+1} d\mu' \int_{E_0}^{\infty} K_{\text{in}}(E' \rightarrow E) \psi(x, \mu', E) dE', \quad (2)$$

and we are left with the solution of a classical spatial *elastic* slowing-down problem.^{2-7,23,24} Therefore, we are interested in solutions of Eq. (1) for energies greater than E_0 .

The exact shape of $K_{\text{in}}(E' \rightarrow E)$ is poorly known, and, as in thermalization theory, it is advantageous to introduce a synthetic kernel. The simplest approximation is to assume

$$\begin{aligned} K_{\text{in}}(E' \rightarrow E) &= f(E')g(E), \quad \text{for } E' > E, \\ &= 0, \quad \text{for } E' < E, \end{aligned} \quad (3)$$

where $f(E)$ and $g(E)$ are *a priori* arbitrary functions. The synthetic kernel (3) was first introduced by Okrent *et al.*,²⁵ in connection with Weisskopf's statistical evaporation model. Recently, it was proposed as a synthetic kernel *per se*, by Cadilhac *et al.*,²⁶ that is, a kernel adaptable to experimental data or more involved nuclear theory. Such a kernel has, in fact, only one arbitrary function, namely $g(E)$; this stems from the requirement of the conservation of the total inelastic cross section:

$$\int_0^{E'} K_{\text{in}}(E' \rightarrow E) dE = f(E') \int_0^{E'} g(E) dE = 1. \quad (4)$$

Defining

$$h(E) = 1/f(E), \quad (5)$$

this yields

$$g(E) = \frac{d}{dE} h(E). \quad (6)$$

In Weisskopf's statistical evaporation model, $g(E)$ assumes the shape²⁵

$$g(E) = Ee^{-E/T}, \quad (7a)$$

where T is the "nuclear temperature," and

$$h(E) = \int_0^E E' \exp(-E'/T) dE'. \quad (7b)$$

²³ B. Davison, *Neutron Transport Theory* (Oxford University Press, London, 1957).

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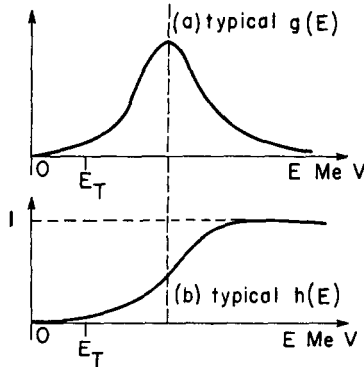


FIG. 1. Typical shapes for (a) $g(E)$ and (b) $h(E)$.

Cadilhac's approach²⁶ is more general; it consists in keeping $g(E)$ *a priori* arbitrary, and fitting it so that the approximate operator has the same action as the exact one on a particular reference energy spectrum. Mathematically, we need only know that

$$\begin{aligned} g(E) &\rightarrow 0, & \text{for } E &\rightarrow 0+, \\ & & E &\rightarrow +\infty, \\ h(E) &\text{ is bounded for } E &\rightarrow +\infty. \end{aligned}$$

The latter fact is not trivial, but holds for all suggested physical models (see Fig. 1). Similarly, one can assume that $g(E)$ is positive for $\forall E \in [0, +\infty]$.

B. Reduction of the Boltzmann Equation Through a New Energy Transformation

We consider Eq. (1) with a synthetic inelastic slowing-down kernel:

$$\begin{aligned} \mu \frac{\partial}{\partial x} \psi(x, \mu, E) + \psi(x, \mu, E) \\ = \frac{c_1}{2} \int_{-1}^{+1} d\mu' g(E) \int_E^{\infty} \frac{\psi(x, \mu', E')}{h(E')} dE' \\ + \frac{c_0}{2} \int_{-1}^{+1} \psi(x, \mu', E) d\mu' + S(x, \mu, E). \end{aligned} \quad (8)$$

A classical method in *elastic* slowing-down problems consists of looking for energy eigenfunctions of the slowing-down operator (namely, exponentials of the lethargy variable) and making an expansion of the neutron density in this set of eigenfunctions (that is, a Fourier-Laplace transformation of the lethargy variable, or a Mellin transformation of the velocity variable).²³

In the present case, the inelastic slowing-down operator has no eigenfunctions; the following equation,

$$\lambda \phi(E) = g(E) \int_E^{\infty} \frac{\phi(E')}{h(E')} dE', \quad (9)$$

has no solutions [Volterra integral equation with a bounded kernel; $h(\infty)$ is finite, different from zero].

However, consider the *adjoint* to the inelastic slowing-down operator; the adjoint eigenfunction equation is

$$\phi^+(E) = \frac{\lambda}{h(E)} \int_0^E g(E') \phi^+(E') dE'. \quad (10)$$

Equation (10) admits the following solutions:

$$\phi^+(E) = h(E)^{\lambda-1}, \quad \forall \lambda, \quad \text{Re } \lambda \geq 1 \quad (11)$$

[keeping in mind that $g(E) = dh(E)/dE$]. So, the adjoint operator (Volterra integral equation with an *unbounded* kernel) admits the set of eigenfunctions $\{h(E)^{\lambda-1}\}$. Therefore, let us make the "scalar product" of Eq. (8) by $h(E)^{\lambda-1}$; multiply both sides of the transport equation (8) by $h(E)^{\lambda-1}$ and integrate over the whole energy range; defining

$$\bar{\psi}(x, \mu, \lambda) = \int_0^{\infty} \psi(x, \mu, E) h(E)^{\lambda-1} dE, \quad (12)$$

one obtains

$$\begin{aligned} \mu \frac{\partial}{\partial x} \bar{\psi}(x, \mu, \lambda) + \bar{\psi}(x, \mu, \lambda) \\ = \frac{c_0}{2} \int_{-1}^{+1} \bar{\psi}(x, \mu', \lambda) d\mu' + \bar{S}(x, \mu, \lambda) \\ + \frac{c_1}{2} \int_{-1}^{+1} d\mu' \int_0^{\infty} g(E) h(E)^{\lambda-1} \int_E^{\infty} \frac{\psi(x, \mu', E')}{h(E')} dE' dE. \end{aligned} \quad (13)$$

In the last (inelastic scattering) term of Eq. (13), change the order of integrations, use relations (10) and (11), and obtain

$$\begin{aligned} \mu \frac{\partial}{\partial x} \bar{\psi}(x, \mu, \lambda) + \bar{\psi}(x, \mu, \lambda) \\ = \frac{c_0}{2} \int_{-1}^{+1} \bar{\psi}(x, \mu', \lambda) d\mu' \\ + \frac{c_1}{2\lambda} \int_{-1}^{+1} \bar{\psi}(x, \mu', \lambda) d\mu' + \bar{S}(x, \mu, \lambda). \end{aligned} \quad (14)$$

So, if one defines the transformation \mathcal{M} by

$$\mathcal{M}\psi(x, \mu, E) = \int_0^{\infty} \psi(x, \mu, E) h(E)^{\lambda-1} dE, \quad (15)$$

this transformation reduces the initial Boltzmann equation (8) to a pseudomonokinetic equation (14), where λ is only a parameter appearing in the "multiplication coefficient" $\{c_0 + (c_1/\lambda)\}$ —which can take complex values, as opposed to the classical one-speed situation.

with

$$N_j(\lambda) = \frac{1}{2} \left(c_e + \frac{c_i}{\lambda} \right) \cdot v_j^3(\lambda) \cdot \left[\frac{\left(c_e + \frac{c_i}{\lambda} \right)}{v_j^3(\lambda) - 1} - \frac{1}{v_j^3(\lambda)} \right], \tag{21d}$$

$$N(v, \lambda) = v \left\{ \left(1 - \left(c_e + \frac{c_i}{\lambda} \right) v \tanh^{-1} v \right)^2 + \frac{\pi^2}{4} \left(c_e + \frac{c_i}{\lambda} \right)^2 v^2 \right\}, \tag{21e}$$

$$S(\lambda) = \mathcal{M}S(E). \tag{21f}$$

In Eq. (21b), the second term of the right-hand side corresponds to transport transients, and the first one is the (spatially) asymptotic component of the solution. The next problem is to find an asymptotic evaluation of the cumbersome contour integral in the complex λ plane. At first sight, use of the saddle-point method seems to be appropriate for the evaluation of such an integral; this is especially true if one introduces an auxiliary "lethargy" variable u associated with the inelastic slowing down and defined by

$$u = -\log [h(E)]. \tag{22a}$$

This is a one-to-one mapping of $E \in [0, +\infty]$ onto $u \in [+\infty, 0]$. Then, one gets contour integrals very similar to those encountered in spatially-dependent elastic slowing-down problems,^{3-6,28} and classically evaluated by saddle-point methods. The position of the saddle-point λ_0 is given by the following equation²⁸:

$$\left. \frac{d(1/v_0(\lambda))}{d\lambda} \right|_{\lambda=\lambda_0} = \frac{u}{|x|}, \tag{22b}$$

where $v_0(\lambda)$ is the solution of Eq. (21c) with the largest absolute value. Unfortunately, the solution of the set of coupled implicit equations (21c) and (22b) is impracticable, unless one resorts to numerical tabulation.

Therefore, a mathematical method has been developed which yields an explicit asymptotic evaluation of formula (21b) valid for large distances, and which is briefly outlined in the next paragraph.

One can first simplify the Green's function through the following remark: Consider the full-space Green's function $G(x, \mu, E)$ solution of the (\mathcal{M} -transformed) equation

$$\begin{aligned} \mu \frac{\partial}{\partial x} \bar{G}(x, \mu, \lambda) + \bar{G}(x, \mu, \lambda) \\ = \frac{1}{2} \left(c_e + \frac{c_i}{\lambda} \right) \int_{-1}^{+1} \bar{G}(x, \mu', \lambda) d\mu' + \frac{1}{2} S(\lambda) \delta(x). \end{aligned} \tag{23a}$$

Introduce the following associated equation, where the \mathcal{M} transform of the source is constant:

$$\begin{aligned} \mu \frac{\partial}{\partial x} F(x, \mu, \lambda) + F(x, \mu, \lambda) \\ = \frac{1}{2} \left(c_e + \frac{c_i}{\lambda} \right) \int_{-1}^{+1} F(x, \mu', \lambda) d\mu' + \frac{\delta(x)}{2}. \end{aligned} \tag{23b}$$

Then, in terms of the energy variable, one gets the following relation between $G(x, \mu, E)$, $F(x, \mu, E)$, and $S(E)$:

$$G(x, \mu, E) = g(E) \int_V^1 \{S/g\} \left(\frac{V}{w} \right) \cdot \frac{F(x, \mu, w) dw}{g(w) w}, \tag{24a}$$

where

$$\begin{aligned} V &= h(E), \\ w &= h(E'), \end{aligned} \tag{24b}$$

and where $\{S/g\}(V/w)$ is the value of the function $S(E)/g(E)$ for E such that $h(E) = V/w$. Relation (24) is nothing but the "Faltung theorem" for the inverse Mellin transform,²⁹ similar to the convolution theorem for Laplace and Fourier transforms. Such a theorem is immediately extended to the inverse \mathcal{M} transformation, which is closely related to Mellin transforms. In view of relation (24), it is sufficient to find an asymptotic evaluation of the energy Green's function $F(x, \mu, E)$ [Eq. (23b)].

The crux of the method consists in considering the inelastic scattering term in Eq. (23b) as an auxiliary source term:

$$\begin{aligned} \mu \frac{\partial}{\partial x} F(x, \mu, \lambda) + F(x, \mu, \lambda) \\ = \frac{c_e}{2} \int_{-1}^{+1} F(x, \mu', \lambda) d\mu' \\ + \frac{1}{2} \left\{ \delta(x) + \frac{c_i}{2\lambda} \int_{-1}^{+1} F(x, \mu', \lambda) d\mu' \right\}. \end{aligned} \tag{25}$$

Then one applies a spatial Fourier transformation to Eq. (25), and obtains a simple expression for the Fourier transform of $F(x, \mu, \lambda)$ in terms of the well-known solution of the one-speed Boltzmann equation associated to a multiplication coefficient c_e . At this stage, in contrast to the normal-modes approach, inverse \mathcal{M} transformation is immediate, and one is left with the asymptotic evaluation of $F(x, \mu, E)$ for large distances, knowing its x Fourier transform. Analytical details, being quite lengthy, are found in Appendix B, with the final asymptotic expression for the full-space, energy Green's-function solution of the inelastic slowing-down Boltzmann equation.

²⁸ Ref. 24, p. 498ff.

²⁹ Ref. 27, Vol. I, p. 308, relation 14.

III. ENERGY-DEPENDENT BOLTZMANN EQUATION WITH FISSION AND SLOWING-DOWN KERNELS

The addition of a fission projection kernel to the transport equation is of prime importance when it comes to studying fast multiplying systems. Physically, the classical slowing-down problem of thermal reactor theory is changed to a situation with simultaneous neutron degradation and regeneration; this may allow *self-sustaining modes*. Mathematically, this implies that the sum of a fission and slowing-down operators is likely to have a *discrete*, regular eigenfunction, which is not true for the plain slowing-down kernel.

Nevertheless, little work has been done up to now in studying simultaneous fission and slowing down in the *transport* equation. Diffusion approximations or multigroup (discrete-energy) schemes have been the rule²³—or the problem has been reduced to a plain slowing-down situation by assimilating the fission sources to a high-energy Dirac distribution.²³

We are interested in the following equation, where energy transfer occurs only through fission and *inelastic* scattering:

$$\begin{aligned} \mu \frac{\partial}{\partial x} \psi(x, \mu, E) + \psi(x, \mu, E) &= \frac{c_s}{2} \int_{-1}^{+1} \psi(x, \mu', E) d\mu' \\ &+ \frac{c_i}{2} \int_{-1}^{+1} d\mu' g(E) \int_E^\infty \frac{\psi(x, \mu', E') dE'}{h(E')} \\ &+ \frac{c_F}{2} \chi(E) \int_{-1}^{+1} d\mu' \int_0^\infty \psi(x, \mu', E') dE' \\ &+ \frac{1}{2} \delta(x - x_0) S(E). \end{aligned} \quad (26)$$

The notation is the same as in Eq. (1); $\chi(E)$ is the fission spectrum, and c_F is the mean number of secondaries emitted after a fission collision times the probability of fission. In a previous work,²² we solved an identical problem with *elastic* slowing down. In this work, quite similar results will be found for Eq. (26). Define the global energy-transfer operator \mathcal{O} by

$$\begin{aligned} \mathcal{O} \phi(E) = c_F \chi(E) \int_0^\infty \phi(E') dE' + c_e \phi(E) \\ + c_i g(E) \int_E^\infty \frac{\phi(E') dE'}{h(E')}. \end{aligned} \quad (27a)$$

Eigenfunctions of \mathcal{O} are such that

$$\mathcal{O} \phi_v(E) = \nu \phi_v(E). \quad (27b)$$

At this point, we note the conditions for a null integral.

Lemma: The necessary and sufficient condition for a function $\phi(E) \in L^1[0, \infty]$ to have a null integral,

$$\int_0^\infty \phi(E') dE' = 0,$$

is that

$$\bar{\phi}(1) \equiv 0,$$

where $\bar{\phi}(\lambda)$ is the \mathcal{M} transform of $\phi(E)$. [This stems from the set of reciprocal formulas (15) and (20b) for the \mathcal{M} transform; $\bar{\phi}(\lambda) = \int_0^\infty \phi(E) h(E) \lambda^{-1} dE$.]

Coming back to Eq. (27b), its \mathcal{M} transform is

$$\nu \bar{\phi}_v(\lambda) = \left(c_e + \frac{c_i}{\lambda} \right) \bar{\phi}_v(\lambda) + c_F \bar{\chi}(\lambda) \bar{\phi}_v(1). \quad (28)$$

Solutions of Eq. (28) belong to two categories:

1. For solutions such that $\bar{\phi}_v(1) \neq 0$, or

$$\int_0^\infty \phi_v(E) dE \neq 0,$$

there is a unique eigenvalue,

$$\nu = c_F + c_e + c_i, \quad (29a)$$

to which corresponds a single regular eigenfunction of the operator \mathcal{O} :

$$\bar{\mathcal{K}}(\lambda) = c_F \bar{\chi}(\lambda) / \left[c_F + c_i \left(1 - \frac{1}{\lambda} \right) \right] \quad (29b)$$

[keeping in mind that $\bar{\chi}(1) = \int_0^\infty \chi(E) dE = 1$].

From Eq. (29), or from direct solution of Eq. (27b) (by reduction to a differential equation), one gets the following expression for $\mathcal{K}(E)$:

$$\begin{aligned} \mathcal{K}(E) = \frac{c_F}{c_F + c_i} \chi(E) + \frac{c_i}{c_F + c_i} g(E) h(E)^{-c_i/(c_F+c_i)} \\ \times \int_E^\infty \frac{c_F}{c_F + c_i} \frac{\chi(E')}{h(E')^{c_F/(c_F+c_i)}} dE'. \end{aligned} \quad (30)$$

The regular eigenfunction $\mathcal{K}(E)$ of the energy-transfer operator in Eq. (26) corresponds to the *asymptotic neutron energy spectrum*.

2. For solutions of Eq. (28) such that $\bar{\phi}_v(1) = 0$ or $\int_0^\infty \phi_v(E) dE = 0$, then Eq. (28) reduces to

$$\nu \bar{\phi}_v(\lambda) = \left(c_e + \frac{c_i}{\lambda} \right) \bar{\phi}_v(\lambda), \quad (31a)$$

the solutions of which are singular³⁰:

$$\bar{\phi}_v(\lambda) = \delta(\lambda - \lambda_0), \quad \text{with } \lambda_0 \neq 1, \quad (31b)$$

$$\nu = c_e + (c_i/\lambda_0). \quad (31c)$$

³⁰ L. Schwartz, *Théorie des distributions* (Hermann & Cie, Paris, 1966).

These distributions correspond to the ordinary slowing-down operator.

The regular eigenfunction $\mathcal{K}(E)$, Eq. (30), and the continuum of eigendistributions defined in Eq. (31) form a *complete* set in $L^1[0, \infty]$; this is expressed by the following theorem.

Theorem II: $\forall \phi(E) \in L^1[0, \infty]$ has the *unique* decomposition

$$\phi(E) = \mathfrak{S} \cdot \mathcal{K}(E) + \frac{g(E)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{A}(\lambda) h(E)^{-\lambda} d\lambda$$

with $\bar{A}(1) \equiv 0$.

Proof: Since $\int_0^\infty \mathcal{K}(E) dE = 1$, one must have

$$\mathfrak{S} = \int_0^\infty \phi(E) dE.$$

Then, defining $\Gamma(E)$,

$$\Gamma(E) = \phi(E) - \mathfrak{S} \cdot \mathcal{K}(E),$$

one has

$$\int_0^\infty \Gamma(E) dE = 0;$$

and one must prove that $\Gamma(E)$ admits the representation

$$\Gamma(E) = \frac{g(E)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{A}(\lambda) h(E)^{-\lambda} d\lambda.$$

The existence and uniqueness of such a representation is immediate, since $\Gamma(E)$ is \mathcal{M} transformable:

$$\bar{A}(\lambda) = \int_0^\infty \Gamma(E) h(E)^{\lambda-1} dE.$$

Finally, from the lemma,

$$\bar{A}(1) \equiv 0. \quad \text{Q.E.D.}$$

Coming back to the transport equation (26), we see that if the source term were of the form

$$S(E) = \mathcal{K}(E),$$

then Eq. (26) would reduce to a one-speed equation, with a multiplication coefficient $c_F + c_e + c_i$, since the solution is separable into a function of space and angle times $\mathcal{K}(E)$.

For a general source term, the method consists in applying the expansion of theorem II to $S(E)$:

$$S(E) = \mathfrak{S} \cdot \mathcal{K}(E) + \Gamma(E), \quad (32a)$$

where

$$\mathfrak{S} = \int_0^\infty S(E) dE, \quad (32b)$$

$$\Gamma(E) = \frac{g(E)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\lambda) h(E)^{-\lambda} d\lambda, \quad (32c)$$

$$\Gamma(\lambda) = \int_0^\infty h(E)^{\lambda-1} \Gamma(E) dE, \quad (32d)$$

$$\Gamma(1) = \int_0^\infty \Gamma(E) dE = 0. \quad (32e)$$

Since the transport equation (26) is linear, its solution can be expressed as the one-speed solution due to $\mathfrak{S} \cdot \mathcal{K}(E)$ plus the solution due to a source $\Gamma(E)$ of null integral. Call the former solution $\mathcal{K}(E) \cdot \phi_E(x, \mu)$ and the latter $\phi_{tr}(x, \mu, E)$:

$$\psi(x, \mu, E) = \mathcal{K}(E) \phi_E(x, \mu) + \phi_{tr}(x, \mu, E). \quad (33)$$

Explicitly, $\phi_E(x, \mu)$ is the solution of

$$\begin{aligned} \mu \frac{\partial}{\partial x} \phi_E(x, \mu) + \phi_E(x, \mu) \\ = \frac{c_e + c_i + c_F}{2} \int_{-1}^{+1} \phi_E(x, \mu') d\mu' + \frac{1}{2} \delta(x - x_0) \cdot \mathfrak{S}. \end{aligned} \quad (34)$$

As to $\phi_{tr}(x, \mu, E)$,

$$\begin{aligned} \mu \frac{\partial}{\partial x} \phi_{tr}(x, \mu, E) + \phi_{tr}(x, \mu, E) \\ = \frac{c_e}{2} \int_{-1}^{+1} \phi_{tr}(x, \mu', E) d\mu' \\ + \frac{c_i}{2} \int_{-1}^{+1} d\mu' g(E) \int_E^\infty \frac{\phi_{tr}(x, \mu', E')}{h(E')} dE' \\ + \frac{\Gamma(E)}{2} \delta(x - x_0). \end{aligned} \quad (35)$$

The crucial point is that $\phi_{tr}(x, \mu, E)$ is the solution of a *plain slowing-down* equation without any fission term; since the integral of $\Gamma(E)$ over the whole energy range is null, the same follows for $\phi_{tr}(x, \mu, E)$. To prove it, take the \mathcal{M} transform of Eq. (35):

$$\begin{aligned} \mu \frac{\partial}{\partial x} \bar{\phi}_{tr}(x, \mu, \lambda) + \bar{\phi}_{tr}(x, \mu, \lambda) \\ = \frac{1}{2} \left(c_e + \frac{c_i}{\lambda} \right) \int_{-1}^{+1} \bar{\phi}_{tr}(x, \mu', \lambda) d\mu' + \frac{\Gamma(\lambda)}{2} \delta(x - x_0). \end{aligned} \quad (36)$$

Keeping in mind that $\Gamma(1) \equiv 0$ and that the \mathcal{M} -transformed Eq. (36) is homogeneous in λ , it follows that

$$\bar{\phi}_{tr}(x, \mu, \lambda) \equiv 0 \quad \text{for } \lambda = 1.$$

The success of the decomposition of Eq. (26) into the associated Eqs. (34) and (35) had to be expected, since one has in fact made an expansion of the source $S(E)$ with the set of "eigenfunctions" of the global energy-transfer operator.

The solutions of the one-speed equation (34) are well known¹; Eq. (35) is a plain slowing-down equation solved in Sec. II. Thus, the present problem is completely solved.

The Milne problem and criticality problems involve only Eq. (34). But the full-space and half-space Green's functions involve both solutions of Eq. (34) (space-energy separable modes) and Eq. (35) (nonseparable transients). The space-energy separable components are proportional to the characteristic energy-mode $\mathcal{H}(E)$; they are representative of self-sustaining modes in the fast-multiplying medium, and are asymptotically dominant. The nonseparable modes are "slowing down transient," solutions of an ordinary slowing-down equation; they are *not* classical one-speed "singular transport transients" (they may decay more slowly than $e^{-|x|}$, as shown in Appendix B on the asymptotic evaluation of the slowing-down Green's function), and they represent the adjustment of the neutron field from the initial source-energy distribution to the final self-sustaining asymptotic spectrum. They are likely to delay the approach to equilibrium in integral experiments on fast systems (for instance, exponential experiments). The relative importance of space-energy separable modes and "slowing-down transients" is quite sensitive to the degree of criticality of the fast system. This, in turn, limits the validity of asymptotic transport theory for the energy-dependent Boltzmann equation, since its basic assumption is space-energy separability, which leads to the omission of all "slowing-down transients."

Similar results have been found for the case of fission with *anisotropic elastic* slowing down.²² In conclusion, in order to achieve completeness for the normal-modes solution of the Boltzmann equation with fission and slowing down, one must consider fundamental separable modes reflecting the multiplicative process, together with "slowing-down transients," solution of an ordinary slowing-down equation. In a further paper, numerical results will be presented on the relative importance of asymptotic separable modes and "slowing-down transients," in the approach to equilibrium in exponential experiments on fast-neutron-multiplying media.

APPENDIX A: PROOF OF THEOREM I

This theorem (Sec. II.B) states that $\bar{\psi}(x, \mu, \lambda)$, the \mathcal{M} transform of $\psi(x, \mu, E)$, is uniformly bounded in λ , for $\forall \lambda$ such that $\text{Re } \lambda \geq 1$.

Proof:

$$\bar{\psi}(x, \mu, \lambda) = \int_0^1 \Psi(x, \mu, V) V^{(\text{Re } \lambda) - 1} V^{i \text{Im } \lambda} dV \quad (\text{A1})$$

[cf. Eqs. (16)–(19)];

$$|\bar{\psi}(x, \mu, \lambda)| \leq \int_0^1 |\Psi(x, \mu, V)| \cdot |V^{(\text{Re } \lambda) - 1}| dV, \quad (\text{A2})$$

but

$$\left. \begin{array}{l} 0 \leq V \leq 1 \\ (\text{Re } \lambda) - 1 \geq 0 \end{array} \right\} \Rightarrow |V^{(\text{Re } \lambda) - 1}| \leq 1. \quad (\text{A3})$$

So

$$|\bar{\psi}(x, \mu, \lambda)| \leq \int_0^1 |\Psi(x, \mu, V)| dV \leq M, \quad (\text{A4})$$

where

$$M = \int_0^\infty |\psi(x, \mu, E)| dE. \quad (\text{Q.E.D.}) \quad (\text{A5})$$

APPENDIX B: ASYMPTOTIC EVALUATION OF THE FULL-SPACE GREEN'S FUNCTION

Using the notation of Sec. II.C, the full-space, energy Green's function $F(x, \mu, E)$ obeys the following \mathcal{M} -transformed equation:

$$\begin{aligned} \mu \frac{\partial}{\partial x} \bar{F}(x, \mu, \lambda) + \bar{F}(x, \mu, \lambda) \\ = \frac{c_e}{2} \int_{-1}^{+1} \bar{F}(x, \mu', \lambda) d\mu' \\ + \frac{1}{2} \left\{ \delta(x) + \frac{c_1}{\lambda} \int_{-1}^{+1} \bar{F}(x, \mu', \lambda) d\mu' \right\}, \end{aligned} \quad (\text{25})$$

where the inelastic scattering term is considered as an extraneous source. Define

$$\bar{F}(|x|, \lambda) = \int_{-1}^{+1} \bar{F}(x, \mu', \lambda) d\mu'. \quad (\text{B1})$$

Then call $G_e(|x|)$ the Green's function corresponding to the one-speed transport equation with the "multiplication coefficient" c_e (elastic scattering):

$$\mu \frac{\partial}{\partial x} G_e(x, \mu) + G_e(x, \mu) = \frac{c_e}{2} \int_{-1}^{+1} G_e(x, \mu') d\mu' + \frac{\delta(x)}{2}, \quad (\text{B2})$$

where

$$G_e(|x|) = \int_{-1}^{+1} G_e(x, \mu') d\mu'. \quad (\text{B3})$$

The exact expression of $G_e(|x|)$ is well known:

$$G_e(|x|) = \frac{1}{2} \frac{dK_e^2}{dc_e} \frac{e^{-K_e |x|}}{K_e} + \frac{1}{2} \int_0^1 \frac{e^{-|x|/\nu}}{\nu N(c_e, \nu)} d\nu, \quad (\text{B4a})$$

where K_e is the root of

$$1 = \frac{c_e}{K_e} \tanh^{-1} K_e, \quad (\text{B4b})$$

and

$$N(c_e, \nu^2) = \left\{ (1 - c_e \nu \tanh^{-1} \nu)^2 + \frac{\pi^2 \nu^2 c_e^2}{4} \right\}. \quad (\text{B4c})$$

Now, apply a *Fourier* spatial transformation to Eq. (25):

$$F(K^2, \lambda) = \int_{-\infty}^{+\infty} F(|x|, \lambda) e^{-iKx} dx, \quad (\text{B5a})$$

$$\tilde{G}_e(K^2) = \int_{-\infty}^{+\infty} G_e(|x|) e^{-iKx} dx. \quad (\text{B5b})$$

We get

$$(iK\mu + 1)F(K, \mu, \lambda) = \frac{c_e}{2} \int_{-1}^{+1} F(K, \mu', \lambda) d\mu' + \frac{1}{2} \left\{ 1 + \frac{c_1}{\lambda} F(K^2, \lambda) \right\}. \quad (\text{B5c})$$

Since Eq. (B5c) is *homogeneous in both variables K and λ*, this yields

$$F(K^2, \lambda) = \tilde{G}_e(K^2) \left\{ 1 + \frac{c_1}{\lambda} F(K^2, \lambda) \right\} \quad (\text{B6})$$

and

$$F(K^2, \lambda) = \frac{\tilde{G}_e(K^2)}{1 - (c_1/\lambda)\tilde{G}_e(K^2)}. \quad (\text{B7})$$

In Eq. (7), let us isolate the term $\tilde{G}_e(K^2)$:

$$F(K^2, \lambda) = \tilde{G}_e(K^2) + \frac{c_1 \{\tilde{G}_e(K^2)\}^2}{\lambda - c_1 \tilde{G}_e(K^2)}. \quad (\text{B8})$$

Recall that²³

$$\tilde{G}_e(K^2) = \frac{dK_e^2}{dc_e} \frac{1}{K_e^2 + K^2} + \int_0^1 \frac{d\nu}{(1 + K^2 \nu^2) N(c_e, \nu^2)}. \quad (\text{B9})$$

At this stage, the inverse \mathcal{M} transformation of Eq. (B8) is immediate; keeping in mind that

$$\mathcal{M}\{g(E)h(E)^{-\rho}\} = \int_0^\infty g(E)h(E)^{\lambda-\rho-1} dE = \frac{1}{\lambda - \rho}, \quad (\text{B10})$$

we obtain

$$F(K^2, E) = \tilde{G}_e(K^2) \delta(u) + c_1 g(E) \{\tilde{G}_e(K^2)\}^2 h(E)^{-c_1 \tilde{G}_e(K^2)}, \quad (\text{B11})$$

where $\delta(u)$ is a Dirac distribution, and u is the "lethargy" defined in Eq.(22a),

$$u = -\log [h(E)]. \quad (\text{22a})$$

In relation (B11), the *exact* inverse \mathcal{M} transformation has been successfully performed for the energy Green's function of the infinite medium; this expression is valid for *all energies*. The first term in Eq.

(B11) corresponds to the elastic scattering of the source.

The next step is to find an asymptotic expression for the spatial behavior of $F(|x|, E)$. Apply an inverse Fourier transformation to Eq. (B11):

$$F(|x|, E) = G_e(|x|) \cdot \delta(u) + \frac{c_1 \cdot g(E)}{2\pi} \times \int_{-\infty}^{+\infty} \{\tilde{G}_e(K^2)\}^2 e^{uc_1 \tilde{G}_e(K^2)} e^{+iKx} dK. \quad (\text{B12})$$

In Eq. (B12), the first term $G_e(|x|) \cdot \delta(u)$ is perfectly well known [Eq. (B4)]. One is left with the evaluation of

$$R(|x|, E) = \frac{c_1 g(E)}{2\pi} \int_{-\infty}^{+\infty} \{\tilde{G}_e(K^2)\}^2 e^{(uc_1 \tilde{G}_e(K^2) + iKx)} dK. \quad (\text{B13})$$

Make the following change of variable:

$$ik = K. \quad (\text{B14})$$

Equation (B13) becomes

$$R(|x|, E) = \frac{c_1 g(E)}{2\pi i} \int_{-i\infty}^{+i\infty} \{\tilde{G}_e(k^2)\}^2 e^{(uc_1 \tilde{G}_e(k^2) - kx)} dk, \quad (\text{B15})$$

where [see Eq. (B4) and (B9)]

$$\tilde{G}_e(k^2) = \frac{dK_e^2}{dc_e} \frac{1}{K_e^2 - k^2} + \int_0^1 \frac{d\nu}{(1 - k^2 \nu^2) N(c_e, \nu^2)} = \frac{dK_e^2}{dc_e} \frac{1}{K_e^2 - k^2} + \Omega(k^2). \quad (\text{B16})$$

Equation (B15) involves a contour integral along the imaginary axis for a function of the complex variable k ; the latter is analytic everywhere, except for

(1) the cuts $[+1, +\infty]$ and $[-\infty, -1]$ on the real axis, since these are cuts for $\Omega(k^2)$ in Eq. (B16);

(2) the *essential* singularities $k = \pm K_e$ on the real axis; this is due to the exponential blowup of the term

$$\exp uc_1 \tilde{G}_e(k^2),$$

which behaves as $e^{1/k \pm K_e}$ in the neighborhood of the essential singularities $\pm K_e$.

Let us now shift the integration contour in Eq. (B15) from the imaginary axis to the real one. For positive values of x , the corresponding Bromwich contour lies in the positive half-plane, since for $\text{Re}(k) > 0$ and $x > 0$, one has $|e^{-kx}| \rightarrow 0$. Then, for $x > 0$,

$$R(x, E) = \frac{c_1 g(E)}{2\pi i} \left\{ \int_D + \int_C \{\tilde{G}_e(k^2)\}^2 e^{(uc_1 \tilde{G}_e(k^2) - kx)} dk \right\} \quad (\text{B17})$$

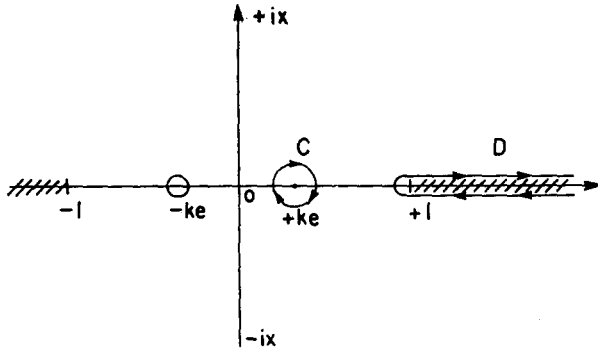


FIG. 3. The Bromwich contour for $R(x, E) - x > 0$ —in the k complex plane.

(see Fig. 3). It is clear that the contribution of the contour D along the cut $[+1, +\infty]$ involves modes all decaying faster than e^{-x} ; these are *transport-spatial transients*. All modes decaying more slowly than $e^{-|x|}$ are yielded by the contour integral C around the essential singularity $+K_e$; this is indeed the spatial asymptotic component one is looking for.

So, define

$$R_{as}(x, E) = \frac{c_1 g(E)}{2\pi i} \oint_c \{\bar{G}_e(k^2)\}^2 e^{\{uc_1 \bar{G}_e(k^2) - kx\}} dk. \quad (B18)$$

From Eq. (B16),

$$\{\bar{G}_e(k^2)\}^2$$

is

$$\{\bar{G}_e(k^2)\}^2 = \left(\frac{dK_e^2}{dc_e}\right)^2 \frac{1}{(K_e^2 - k^2)^2} + \frac{dK_e^2}{dc_e} \cdot \frac{2\Omega(k^2)}{K_e^2 - k^2} + \{\Omega(k^2)\}^2. \quad (B19)$$

$\Omega(K^2)$ is an analytic function on the contour C and within the domain surrounded by C . Using the three components of $\{\bar{G}_e(k^2)\}^2$ in Eq. (B19), we can split $R_{as}(x, E)$ into three parts:

$$R_{as}(x, E) = R_{as}^{(1)}(x, E) + R_{as}^{(2)}(x, E) + R_{as}^{(3)}(x, E).$$

Since the procedure of next calculations is quite similar for these three parts, we outline them for $R_{as}^{(1)}(x, E)$:

$$R_{as}^{(1)}(x, E) = \frac{c_1 g(E)}{2\pi i} \times \oint_c \left(\frac{dK_e^2}{dc_e}\right)^2 \frac{1}{(K_e^2 - k^2)^2} e^{\{uc_1 \bar{G}_e(k^2) - kx\}} dk; \quad (B20)$$

this can be rewritten as

$$R_{as}^{(1)}(x, E) = \frac{c_1 g(E)}{2\pi i} \oint_c \frac{L(k, u)}{(K_e - k)^2} \times \exp\left\{\frac{uc_1}{2K_e} \frac{dK_e^2}{dc_e} \frac{1}{K_e - k} - kx\right\} dk, \quad (B21)$$

where we have defined

$$L(k, u) = \left(\frac{dK_e^2}{dc_e}\right)^2 \frac{1}{(K_e + k)^2} \times \exp\left[\left(\frac{uc_1}{2K_e}\right) \left(\frac{dK_e^2}{dc_e}\right) \times (1/K_e + k) + uc_1 \Omega(k^2)\right]. \quad (B22)$$

$L(k, u)$ is an analytic function of k on the contour C and within the domain surrounded by C . This analyticity of $L(k, u)$ enables us to replace it by $L(K_e, u)$ in the contour integral (B21), and obtain an asymptotic evaluation of $R_{as}^{(1)}(x, E)$ in the same way, valid for large x :

$$L(K_e, u) = \frac{1}{4K_e^2} \left(\frac{dK_e^2}{dc_e}\right)^2 e^{uc_1 \{ (1/4K_e^2) (dK_e^2/dc_e) + \Omega(K_e^2) \}}, \quad (B23)$$

$$R_{as}^{(1)}(x, E) \sim \frac{c_1 g(E)}{2\pi i} L(K_e, u) \times \oint_c \frac{e^{\{uc_1 (dK_e^2/dc_e) \cdot 1/(K_e - k) - kx\}}}{(K_e - k)^2} dk. \quad (B24)$$

The idea is then to reduce the contour integral in (B24) to a *classical inverse Laplace transform in x* ; for this purpose, put

$$K_e - k = p, \quad (B25)$$

$$\alpha = \frac{uc_1}{2K_e} \cdot \frac{dK_e^2}{dc_e}. \quad (B26)$$

Then Eq. (B24) reduces to

$$R_{as}^{(1)}(x, E) \sim e^{-K_e x} \cdot L(K_e^2, u) \cdot \frac{c_1 g(E)}{2\pi i} \oint_c \frac{e^{\alpha/p}}{p^2} e^{px} dp \quad (B27)$$

(with counterclockwise integration, this time).

In Eq. (B27), one recognizes the following inverse Laplace transform:

$$\frac{1}{2\pi i} \oint_c \frac{e^{\alpha/p}}{p^2} e^{px} dp = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\alpha/p}}{p^2} e^{px} dp. \quad (B28)$$

But, from Bateman's *Table of Integral Transforms*,³¹

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-\nu-1} e^{\alpha/p} e^{px} dp = \alpha^{-\nu/2} x^{\nu/2} I_\nu[2(\alpha x)^{1/2}], \quad (B29)$$

where I_ν is the hyperbolic Bessel function of order ν .²⁷ One then obtains the final expression for $R_{as}^{(1)}(x, E)$, for large $x > 0$:

$$R_{as}^{(1)}(x, E) \sim c_1 g(E) L(K_e^2, u) \cdot e^{-K_e x} (x/\alpha)^{1/2} I_1[2(\alpha x)^{1/2}]. \quad (B30)$$

³¹ Ref. 27, Vol. I, p. 245, relation 35.

A similar procedure can be applied to $R_{as}^{(2)}(x, E)$ and $R_{as}^{(3)}(x, E)$; one uses the following set of inverse Laplace transforms:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{p} e^{\alpha/n} e^{px} dp = I_0[2(\alpha x)^{\frac{1}{2}}], \quad (\text{B31a})$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\alpha/n} e^{px} dp = \delta(x) + \left(\frac{\alpha}{x}\right)^{\frac{1}{2}} I_1[2(\alpha x)^{\frac{1}{2}}]. \quad (\text{B31b})$$

So, omitting the detailed calculations, one can write the final expression for $F_{as}(|x|, E)$, the spatially asymptotic part of the infinite-space, energy Green's function, as

$$\begin{aligned} F_{as}(|x|, E) &= \frac{1}{2} \frac{dK_e^2}{dc_e} \cdot \frac{e^{-K_e|x|}}{K_e} \cdot \delta(u) + c_i g(E) \\ &\times \exp \{uc_i[(1/4K_e^2)(dK_e^2/dc_e) + \Omega(K_e^2)]\} e^{-K_e|x|} \\ &\times \left\{ \frac{1}{4} \left(\frac{1}{K_e} \frac{dK_e^2}{dc_e} \right)^2 \left(\frac{|x|}{\alpha} \right)^{\frac{1}{2}} I_1[2(\alpha|x|)^{\frac{1}{2}}] \right. \\ &+ \frac{1}{K_e} \cdot \frac{dK_e^2}{dc_e} \cdot \Omega(K_e^2) I_0[2(\alpha|x|)^{\frac{1}{2}}] \\ &\left. + (\Omega(K_e^2))^2 \left(\frac{\alpha}{|x|} \right)^{\frac{1}{2}} I_1[2(\alpha|x|)^{\frac{1}{2}}] \right\}. \quad (\text{B32}) \end{aligned}$$

$\Omega(K^2)$ is defined in Eq. (B16), K_e in Eq. (B4), α (which is a function of u) in Eq. (B26). The expression (B32) holds for intermediate and large distances; numerical calculations have shown it to be quite accurate at distances beyond 2–3 mean free paths, in typical fast systems. It is readily seen that $F_{as}(|x|, E)$ is split into two parts: one which decays as $e^{-K_e|x|}$ and corresponds to plain elastic scattering of the source term; the second one, $R_{as}(|x|, E)$, which includes all inelastic scattering effects. Recalling that

$$I_0(0) = 1, \quad I_1(0) = 0,$$

and that hyperbolic Bessel functions are monotonically increasing, we see that $R_{as}(|x|, E)$ decays more slowly

than $\exp(-K_e|x|)$. More precisely, making an asymptotic expansion of the hyperbolic Bessel functions,²⁷

$$I_n[2(\alpha|x|)^{\frac{1}{2}}]_{|x| \rightarrow \infty} \sim \frac{1}{(4\pi)^{\frac{1}{2}} (\alpha|x|)^{\frac{1}{2}}}, \quad (\text{B33})$$

we obtain the following expression for $R_{as}(|x|, E)$:

$$\begin{aligned} R_{as}(|x|, E)_{|x| \rightarrow \infty} &\sim c_i g(E) e^{uc_i[(1/4K_e^2)(dK_e^2/dc_e) + \Omega(K_e^2)]} \\ &\times 1/(4\pi)^{\frac{1}{2}} (\alpha|x|)^{\frac{1}{2}} e^{-K_e|x| + 2(\alpha|x|)^{\frac{1}{2}}} \\ &\times \left\{ \frac{1}{4} \left(\frac{1}{K_e} \frac{dK_e^2}{dc_e} \right)^2 \cdot \left(\frac{|x|}{\alpha} \right)^{\frac{1}{2}} + \frac{1}{K_e} \frac{dK_e^2}{dc_e} \Omega(K_e^2) \right. \\ &\left. + \{\Omega(K_e^2)\}^2 \left(\frac{\alpha}{|x|} \right)^{\frac{1}{2}} \right\}. \quad (\text{B34}) \end{aligned}$$

In Eq. (B34), the leading term is

$$\exp[-K_e|x| + 2(\alpha|x|)^{\frac{1}{2}}]. \quad (\text{B35})$$

Since α is linearly increasing with the "lethargy" u [see Eq. (B26)], $F_{as}(|x|, E)$ will decay with space more slowly for low energies than for high energies. This has been verified by numerical calculations, which agrees also with the measurements of apparent relaxation lengths in the natural uranium exponential experiment²²; in this case, one has nearly a pure slowing-down situation, and apparent relaxation lengths for high-energy neutrons (≥ 1.0 MeV) are systematically smaller than for low-energy neutrons (≤ 0.5 MeV).

ACKNOWLEDGMENTS

The author wishes to express his appreciation to the Michigan Memorial-Phoenix Project and to the National Science Foundation for financial support. Also, he expresses his appreciation to his thesis adviser, Professor P. F. Zweifel.

²² J. L. Campan, P. P. Clauzon, C. P. Zaleski, "Etude du flux de neutrons à l'équilibre dans l'Uranium naturel," *Proceedings of the International Symposium on Exponential Experiments* (IAEA, Vienna, 1961), Vol. I, p. 345