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Representation of Fields in a Relativistic Plasma by a Surface Integral

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The collective motion of electrons in a relativistic isotropic plasma which is described by the linearized Maxwell–Boltzmann equations is treated with time harmonic dependence. It is shown that the electric intensity at any point in an enclosed volume, can be determined when the tangential components of the electric and magnetic field are prescribed along with the electronic distribution function, on the surface enclosing the volume.

I. INTRODUCTION

One of the most important results in electromagnetic theory for harmonic time dependence, is the Stratton–Chu[1] formulation which prescribes the electric or magnetic field at any point in a volume, in terms of the tangential field components on the surface enclosing the volume. From this, knowledge of the tangential fields on the surface is sufficient to determine the fields everywhere in the enclosed volume. Extensions of this formulation have been obtained by letting the point in the volume approach the surface, giving rise to an integral equation for the tangential components, the solution of which has given rise to many results.

It is shown here, that this formulation can be generalized to the collective motion of the electrons in a relativistic isotropic plasma which is described by the coupled Maxwell–Boltzmann equations. Time harmonic dependence is taken, and it is assumed that the frequency is sufficiently high such that the ions remain frozen in, i.e., their motion can be neglected. The prime result that is obtained here, is that the electric intensity at any point in a volume can be determined when the tangential components of the electric and magnetic field are prescribed along with the electronic distribution function, on the surface enclosing the volume. It naturally follows from Maxwell's equations that the magnetic field at any point in the volume is obtained immediately.

In developing the theory, a vector Green's function which corresponds physically to the field produced by an electric dipole on a relativistic plasma, is derived. Following that, a modification of the Green's function is developed which is then employed to obtain the sought-for representation.

II. VECTOR GREEN'S FUNCTION

The fundamental equations for Green's function (denoted by a subscript d) are Maxwell's equations [time dependence $\exp(-i\omega t)$ suppressed]

$$\nabla \times \mathbf{E}_d = i\omega\mu_0\mathbf{H}_d, \tag{1}$$

$$\nabla \times \mathbf{H}_d = -i\omega\epsilon_0\mathbf{E}_d + \mathbf{j} + \mathbf{a}\,\delta(\mathbf{x} - \mathbf{x}_0), \tag{2}$$

where $\mathbf{j}$ is the current due to the collective motion of the plasma and the term involving the delta function corresponds to the current of an electric dipole located at $\mathbf{x}_0$, with $\mathbf{a}$ being a unit vector of arbitrary direction. The collective current $\mathbf{j}$ is related to the distribution function by the relation

$$\mathbf{j} = -ec \int \beta \mathbf{u} f_d \, d\mathbf{u}, \tag{3}$$

where $-e$ is the electronic charge, $\mathbf{u}$ is the reduced velocity, and β is given by

$$\beta = (1 + u^2)^{-1/2}.$$

In combination with the above equations we have the relativistic Boltzmann equation (given by Clemmow and Willson[2])

$$(-i\omega + \nu)f_d + \beta c\mathbf{u}\cdot\nabla f_d - \frac{e}{mc}\mathbf{E}\cdot\nabla_u f_0 = 0, \tag{4}$$

where the unperturbed distribution function f_0 has the form

$$f_0 = nF_0, \tag{5}$$

$$F_0 = \frac{\exp\left[-\lambda(1 + u^2)^{1/2}\right]}{[(4\pi/\lambda)K_2(\lambda)]}, \quad \lambda = \frac{mc^2}{\kappa T}. \tag{6}$$

The solution of the above equations for the vector Green's function, can be obtained by taking a three-

[1] J. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, New York, 1941), p. 464.

[2] P. C. Clemmow and A. J. Willson, Proc. Roy. Soc. (London) **A237**, 117 (1956).

dimensional Fourier transform of the equations, with careful consideration being given to the interchange of integration and differentiation. The algebraic manipulations will not be presented here since some of it is very similar to the analysis presented by Kuehl[3]; instead, the results will be presented.

The electric intensity can be represented in terms of a transverse potential $\phi_t(R)$ and a longitudinal potential $\phi_l(R)$, which are functions of the radial variable $R = |\mathbf{x} - \mathbf{x}_0|$ only, by the following:

$$\mathbf{E}_d = \nabla \times \nabla \times \phi_t \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}\phi_l). \tag{7}$$

Upon defining $\psi = -\nabla^2 \phi_t$, this can be placed in the form

$$\mathbf{E}_d = \mathbf{a}\psi(R) + \nabla[\nabla \cdot \mathbf{a}(\phi_t + \phi_l)]. \tag{8}$$

The potential functions ϕ_t, ϕ_l, and ψ have the explicit form

$$\phi_t(R) = \frac{i\omega\mu_0}{2\pi^2} \int_0^\infty \frac{\sin(pR)}{pR} \frac{1}{(p^2 - k_0^2\epsilon_t)}\, dp, \tag{9}$$

$$\phi_l(R) = \frac{i\omega\mu_0}{2\pi^2} \int_0^\infty \frac{\sin(pR)}{pR} \frac{1}{k_0^2\epsilon_l}\, dp, \tag{10}$$

$$\psi(R) = \frac{i\omega\mu_0}{2\pi^2} \int_0^\infty \frac{p \sin(pR)}{R} \frac{1}{(p^2 - k_0^2\epsilon_t)}\, dp. \tag{11}$$

In the above expressions $\epsilon_t(\omega, p)$ and $\epsilon_l(\omega, p)$ correspond, respectively, to the relative permittivity for a transverse and longitudinal wave propagating with frequency ω and wavenumber p. They have the explicit representations

$$\epsilon_t(\omega, p) = 1 + \frac{\omega_p^2}{\omega(\omega + i\nu)} \cdot \int \frac{\beta u_x^2}{u} \frac{dF_0}{du} \frac{1}{(1 - \mu\beta u_z)}\, d\mathbf{u}, \tag{12}$$

$$\epsilon_l(\omega, p) = 1 + \frac{\omega_p^2}{\omega(\omega + i\nu)} \cdot \int \frac{\beta u_z^2}{u} \frac{dF_0}{du} \frac{1}{(1 - \mu\beta u_z)}\, d\mathbf{u}, \tag{13}$$

where $\mu = pc/(\omega + i\nu)$.

Following Clemmow and Willson[2] and Taylor[4] these can be expressed in the alternative representations involving a single integration

$$\epsilon_t = 1 + \frac{\omega_p^2 2\pi}{\omega(\omega + i\nu)} \cdot \int_0^\infty \frac{[(1 + u^2)F_0 - G_0]}{(1 + u^2)^{3/2}[1 + u^2(1 - \mu^2)]}\, du, \tag{12'}$$

$$\epsilon_l = 1 - \frac{\omega_p^2 4\pi}{\omega(\omega + i\nu)} \cdot \int_0^\infty \frac{u^2 G_0}{(1 + u^2)^{3/2}[1 + u^2(1 - \mu^2)]}\, du, \tag{13'}$$

where

$$G_0(u) = \left((1 + u^2) + \frac{2}{\lambda}(1 + u^2)^{1/2} + \frac{2}{\lambda^2}\right) F_0(u).$$

From the above representation for the potential functions, it can be seen that when R approaches zero, ϕ_t is finite, but ϕ_l and ψ are singular. In fact, ϕ_l and ψ have the following behavior when $R \sim 0$:

$$\phi_l \sim \frac{i\omega\mu_0}{4\pi k_0^2} \frac{1}{R}, \tag{14}$$

$$\psi \sim \frac{i\omega\mu_0}{4\pi} \frac{1}{R}. \tag{15}$$

The distribution function can be decomposed into two parts corresponding to the transverse and longitudinal components as follows:

$$f_d = f_t + f_l,$$

$$f_t = \frac{ei}{mc} \frac{1}{u} \frac{df_0}{du} \mathbf{u} \cdot [-\mathbf{a}\nabla^2 g_t + \nabla(\nabla \cdot \mathbf{a} g_t)], \tag{16}$$

$$f_l = \frac{ei}{mc} \frac{1}{u} \frac{df_0}{du} \mathbf{u} \cdot \nabla(\mathbf{a} \cdot \nabla g_l),$$

where

$$g_t = \frac{i\omega\mu_0}{(2\pi)^3} \cdot \int \frac{\exp[i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)]}{p^2(p^2 - k_0^2\epsilon_t)(\omega + i\nu - \beta c\mathbf{u} \cdot \mathbf{p})}\, d\mathbf{p}, \tag{17a}$$

$$g_l = \frac{i\omega\mu_0}{(2\pi)^3} \cdot \int \frac{\exp[i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)]}{p^2 k_0^2\epsilon_l(\omega + i\nu - \beta c\mathbf{u} \cdot \mathbf{p})}\, d\mathbf{p}. \tag{17b}$$

The associated functions g_t and g_l can be placed in the following alternative form (see Appendix A) which displays their singularity properties to better advantage:

$$g_t = -i \int_0^\infty \exp[-(\nu - i\omega)q]\phi_t(\zeta)\, dq,$$
$$g_l = -i \int_0^\infty \exp[-(\nu - i\omega)q]\phi_l(\zeta)\, dq, \tag{18}$$

where

$$\zeta^2 = [\hat{\mathbf{u}} \cdot (\mathbf{x} - \mathbf{x}_0) - \beta cuq]^2 + R^2 - [\hat{\mathbf{u}} \cdot (\mathbf{x} - \mathbf{x}_0)]^2. \tag{19}$$

[3] H. H. Kuehl, J. Math. Phys. **43**, 218 (1964).
[4] E. C. Taylor, J. Res. Natl. Bur. Std. **69D**, 735 (1965).

Since $\phi_t(R)$ is finite at $R = 0$, g_t will be nonsingular at $R = 0$. However, g_l will be singular, not only at $R = 0$, but along the semi-infinite line given by $R^2 = [\hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0)]^2$ for $\hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0) \geq 0$. If we define

$$\rho^2 = R^2 - [\hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0)]^2,$$

we see that g_l behaves as $1/\rho$ as $\rho \to 0$, for $\hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0) > 0$. Employing the relation

$$\beta c\mathbf{u}\cdot\nabla\zeta = -\frac{\partial\zeta}{\partial q}, \tag{20}$$

it can be seen that

$$(-i\omega + \nu)g_l + \beta c\mathbf{u}\cdot\nabla g_l = -i\phi_l(R) \tag{21}$$

which implies that $\mathbf{u}\cdot\nabla g_l$ behaves like $1/\rho$ as $\rho \to 0$ for $\hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0) > 0$. A similar relation may be developed for g_t, i.e.,

$$(-i\omega + \nu)g_t + \beta c\mathbf{u}\cdot\nabla g_t = -i\phi_t(R). \tag{22}$$

Both relations (21) and (22) will be employed later on in the analysis. However, it follows from these two relations that expressions (16) and (7) satisfy the Boltzmann equation (4).

III. ADJOINT GREEN'S FUNCTION

The Green's function in the previous section corresponds to the solution of the coupled Maxwell–Boltzmann equations for an electric dipole source. As such it has direct physical meaning. However for the present analysis, the adjoint form will be required. The adjoint Green's function will be represented by a tilde, i.e., $(\tilde{\mathbf{E}}_d, \tilde{\mathbf{H}}_d, \tilde{f}_d)$. Recall that the Green's function satisfies the equations

$$i\omega\mu_0\mathbf{H}_d = \nabla\times\mathbf{E}_d, \tag{23}$$

$$\nabla\times\mathbf{H}_d = -i\omega\epsilon_0\mathbf{E}_d - ec\int\beta\mathbf{u}f_d\,d\mathbf{u} + \mathbf{a}\,\delta(\mathbf{x} - \mathbf{x}_0), \tag{24}$$

$$(-i\omega + \nu)f_d + \beta c\mathbf{u}\cdot\nabla f_d - \frac{e}{mc}\mathbf{E}_d\cdot\mathbf{u}\frac{1}{u}\frac{df_0}{du} = 0\,. \tag{25}$$

We shall now define $(\tilde{\mathbf{E}}_d, \tilde{\mathbf{H}}_d, \tilde{f}_d)$ by the following:

$$\tilde{\mathbf{E}}_d = -\mathbf{E}_d,$$
$$\tilde{\mathbf{H}}_d = \frac{-i}{\omega\mu_0}\nabla\times\mathbf{E}_d = \mathbf{H}_d, \tag{26}$$
$$\tilde{f}_d(\mathbf{x}, \mathbf{u}) = -f_d(\mathbf{x}, -\mathbf{u}).$$

It follows that $(\tilde{\mathbf{E}}_d, \tilde{\mathbf{H}}_d)$ satisfy the equations

$$\nabla\times\tilde{\mathbf{E}}_d = -i\omega\mu_0\tilde{\mathbf{H}}_d, \tag{27}$$

$$\nabla\times\tilde{\mathbf{H}}_d = i\omega\epsilon_0\tilde{\mathbf{E}}_d - ec\int\beta\mathbf{u}\tilde{f}_d\,d\mathbf{u} + \mathbf{a}\,\delta(\mathbf{x} - \mathbf{x}_0). \tag{28}$$

In Eq. (25) replace $\mathbf{u}$ by $-\mathbf{u}$, yielding

$$(-i\omega + \nu - \beta c\mathbf{u}\cdot\nabla)f_d(\mathbf{x}, -\mathbf{u}) + \frac{e}{mc}\mathbf{E}_d\cdot\mathbf{u}\frac{1}{u}\frac{df_0}{du} = 0$$

which, upon employing the definition for $\tilde{f}_d$ and $\tilde{\mathbf{E}}_d$ becomes

$$(i\omega - \nu + \beta c\mathbf{u}\cdot\nabla)\tilde{f}_d - \frac{e}{mc}\tilde{\mathbf{E}}_d\cdot\mathbf{u}\frac{1}{u}\frac{df_0}{du} = 0. \tag{29}$$

Equations (27), (28), and (29) constitute the fundamental equations satisfied by the adjoint Green's function (26).

IV. EXTENSION OF THE STRATTON–CHU FORMULA

Let $(\mathbf{E}, \mathbf{u}, f)$ represent the unknown field quantities and distribution function satisfying the coupled Maxwell–Boltzmann equations

$$\nabla\times\mathbf{E} = i\omega\mu_0\mathbf{H}, \tag{30}$$

$$\nabla\times\mathbf{H} = -i\omega\epsilon_0\mathbf{E} - ec\int\beta\mathbf{u}f\,dy, \tag{31}$$

$$(-i\omega + \nu)f + \beta c\mathbf{u}\cdot\nabla f - \frac{e}{mc}\mathbf{E}\cdot\mathbf{u}\frac{1}{u}\frac{df_0}{du} = 0, \tag{32}$$

in a volume V enclosed by the surface S. We shall seek an expression for $\mathbf{E}$ at a point in V in terms of the field quantities on the surface S. To achieve this, first take the dot product of Eq. (30) with $\tilde{\mathbf{H}}_d$, and add to this the equation formed by taking the dot product of Eq. (27) with $\mathbf{H}$. We thus obtain

$$\tilde{\mathbf{H}}_d\cdot\nabla\times\mathbf{E} + \mathbf{H}\cdot\nabla\times\tilde{\mathbf{E}}_d = 0.$$

In a similar manner, it may be shown that Eqs. (28) and (31) may be combined to yield

$$\tilde{\mathbf{E}}_d\cdot\nabla\times\mathbf{H} + \mathbf{E}\cdot\nabla\times\tilde{\mathbf{H}}_d = -ec\int\beta(\mathbf{u}\cdot\tilde{\mathbf{E}}_d f + \mathbf{u}\cdot\mathbf{E}\tilde{f}_d)\,du + \mathbf{E}\cdot\mathbf{a}\,\delta(\mathbf{x} - \mathbf{x}_0).$$

In addition, Eqs. (25) and (32) may be combined to yield

$$\beta c\mathbf{u}\cdot\nabla[f\tilde{f}_d] = \frac{e}{mc}\frac{1}{u}\frac{df_0}{du}(\mathbf{E}\cdot\mathbf{u}\tilde{f}_d + \tilde{\mathbf{E}}_d\cdot\mathbf{u}f).$$

Combining these last three equations, one can obtain the following expression:

$$\mathbf{E}\cdot\mathbf{a}\,\delta(\mathbf{x}-\mathbf{x}_0) = \nabla\cdot\left[\mathbf{H}\times\tilde{\mathbf{E}}_d + \tilde{\mathbf{H}}_d\times\mathbf{E} + mc^3 \cdot\int \beta^2\mathbf{u}f\tilde{f}_d\left(\frac{1}{u}\frac{df_0}{du}\right)^{-1}\right] du$$

which, upon integration over the volume V, and employment of the divergence theorem yields

$$\mathbf{E}(\mathbf{x}_0) = \int_S \mathbf{n}\cdot\left[\mathbf{H}\times\tilde{\mathbf{E}}_d + \tilde{\mathbf{H}}_d\times\mathbf{E} + mc^3 \cdot\int \beta^2\mathbf{u}f\tilde{f}_d\left(\frac{1}{u}\frac{df_0}{du}\right)^{-1} d\mathbf{u}\right] dS \qquad (33)$$

when $\mathbf{x}_0$ lies in V. When $\mathbf{x}_0$ lies on the surface S, a factor $\frac{1}{2}$ must be included on the left-hand side. When $\mathbf{x}_0$ lies outside V, the left-hand side must be zero. In the above expression $\mathbf{n}$ is the unit outward normal to the surface. The above expression gives a relation for the electric intensity $\mathbf{E}$ at a point $\mathbf{x}_0$ in the volume, in terms of the tangential fields and the distribution function. However, as such it is not expressed in the simplest form.

To reduce it, we make use of the relations,

$$\tilde{\mathbf{E}}_d = -\mathbf{a}\psi - \nabla[\nabla\cdot\mathbf{a}(\phi_t + \phi_l)],$$

$$\omega\mu_0\tilde{\mathbf{H}}_d = -i\nabla\times\psi\mathbf{a},$$

from which it can be shown that

$$\omega\mu_0\int_S \mathbf{n}\cdot\tilde{\mathbf{H}}_d\times\mathbf{E}\,ds = i\mathbf{a}\cdot\int\nabla\psi\times[\mathbf{E}\times\mathbf{n}]\,dS, \qquad (34)$$

and

$$\int_S \mathbf{n}\cdot\mathbf{H}\times\tilde{\mathbf{E}}_d\,ds = \mathbf{a}\cdot\int\mathbf{H}\times\mathbf{n}\psi\,ds - \mathbf{a}\cdot\nabla_0\int\mathbf{H}\times\mathbf{n}\cdot\nabla(\phi_l + \phi_t)\,dS, \qquad (35)$$

where the subscript zero on ∇_0 signifies differentiation with respect to coordinates of $\mathbf{x}_0$. Further reduction is achieved on employing the relation

$$\int_S \mathbf{n}\cdot\nabla\times[(\phi_l + \phi_t)\mathbf{H}]\,dS = 0$$

to give

$$\int_S \mathbf{H}\times\mathbf{n}\cdot\nabla(\phi_l + \phi_t)\,dS = \int_S \mathbf{n}\cdot(i\omega\epsilon_0\mathbf{E} - \mathbf{j})(\phi_l + \phi_t)\,dS. \qquad (36)$$

Combining Eqs. (34), (35), and (36) we have

$$\omega\mu_0\int_S \mathbf{n}\cdot[\tilde{\mathbf{H}}_d\times\mathbf{E} + \mathbf{H}\times\tilde{\mathbf{E}}_d]\,dS = i\mathbf{a}\cdot\int_S [i\omega\mu_0\mathbf{n}\times\mathbf{H}\psi + (\mathbf{n}\times\mathbf{E})\times\nabla\psi + \nabla(\phi_l + \phi_t)(k_0^2\mathbf{E} + i\omega\mu_0\mathbf{j})\cdot\mathbf{n}]\,dS. \qquad (37)$$

The last term of Eq. (33) may be simplified upon using the relation

$$mc\tilde{f}_d\left(\frac{1}{u}\frac{df_0}{du}\right)^{-1} = ei\mathbf{a}\cdot\{-\mathbf{u}\nabla^2\tilde{g}_t + \nabla[\mathbf{u}\cdot\nabla(\tilde{g}_l + \tilde{g}_t)]\}, \qquad (38)$$

where $\tilde{g}_t$ corresponds to g_t with $\mathbf{u}$ replaced by $-\mathbf{u}$. A similar connotation exists for $\tilde{g}_l$. By replacing $\mathbf{u}$ by $-\mathbf{u}$ in Eqs. (21) and (22), and combining the resulting equations we obtain

$$\beta c\mathbf{u}\cdot\nabla(\tilde{g}_l + \tilde{g}_t) = (-i\omega + \nu)[\tilde{g}_t + \tilde{g}_l] + i[\phi_l(R) + \phi_t(R)]. \qquad (39)$$

From Eqs. (38) and (39) it can be seen that

$$\int_S \mathbf{n}\cdot\int\beta^2 mc^3\mathbf{u}f\tilde{f}_d\left(\frac{1}{u}\frac{df_0}{du}\right)^{-1} d\mathbf{u}\,dS = \mathbf{a}\cdot\int_S\nabla(\phi_l + \phi_t)\mathbf{n}\cdot\mathbf{j}\,ds + ei\mathbf{a}\cdot\chi, \qquad (40)$$

where

$$\chi = \int_S dS\int du\,(\mathbf{u}\cdot\mathbf{n})f[-\mathbf{u}\beta^2c^2\nabla^2\tilde{g}_t + \beta c(i\omega - \nu)\nabla_0(\tilde{g}_l + \tilde{g}_t)]. \qquad (41)$$

Thus, it follows from Eq. (33), (37), and (40) that

$$\mathbf{E}(\mathbf{x}_0) = \frac{i}{\omega\mu_0}\int_S [i\omega\mu_0\mathbf{n}\times\mathbf{H}\psi + (\mathbf{n}\times\mathbf{E})\times\nabla\psi + k_0^2(\mathbf{n}\cdot\mathbf{E})\nabla(\phi_l + \phi_t)]\,dS + ei\chi. \qquad (42)$$

Expression (42) is the sought-for representation. It relates the electric intensity at a point $\mathbf{x}_0$ in V, in terms of the components of the electric and magnetic field, and the electronic distribution on the enclosed surface. As it stands, it appears that the normal component of $\mathbf{E}$ on the surface is required. However, at the initial representation given by Eq. (33), only the tangential field components were required. The expressions for ϕ_t, ϕ_l, and ψ are given by Eqs. (9), (10), and (11). $\tilde{g}_t$ and $\tilde{g}_l$ are given by either Eqs. (17a) and (17b) or (18) with $\mathbf{u}$ replaced by $-\mathbf{u}$.

APPENDIX

The alternative representation for g_t can be obtained from Eq. (17a) by employing spherical polar coordinates for the variables of integration (p, θ_p, ϕ_p) where $\hat{\mathbf{u}}\cdot\mathbf{p} = p \cos \theta_p$; giving

$$g_t = \frac{i\omega\mu_0}{(2\pi)^2} \int_0^\infty \frac{dp}{(p^2 - k_0^2\epsilon_t)}$$

$$\cdot \int_0^\pi \frac{\exp(ipz' \cos\theta_p) J_0(p\rho \sin\theta_p) \sin\theta_p \, d\theta_p}{(\omega + i\nu - \beta cup \cos\theta_p)}, \quad \text{(A1)}$$

where

$$z' = \hat{\mathbf{u}}\cdot(\mathbf{x} - \mathbf{x}_0),$$

$$\rho^2 = R^2 - (z')^2.$$

The integral with respect to the θ_p variable can be written in the form

$$-i \int_0^\infty \exp[-(\nu - i\omega)q]$$

$$\cdot \int_0^\pi \exp[ip \cos\theta_p(z - \beta cuq)]$$

$$\cdot J_0(\rho p \sin\theta_p) \sin\theta_p \, d\theta_p \, dq$$

$$= -i2 \int_0^\infty \exp[-(\nu - i\omega)q] \frac{\sin(p\zeta)}{p\zeta} dq, \quad \text{(A2)}$$

where $\zeta^2 = (z' - \beta\, cuq)^2 + \rho^2$.

Inserting this back in (A1), and using Eq. (9), it follows that

$$g_t = -i \int_0^\infty \exp[-(\nu - i\omega)q]\phi_t(\zeta)\, dq. \quad \text{(A3)}$$