Cartan–Gram determinants for the simple Lie groups

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The Cartan–Gram determinants for the simple root systems are evaluated for the simple Lie groups A_n, B_n, C_n, D_n, and E_k $(k = 6,7,8)$. The determinants satisfy a linear recursion relation which turns out to be the same for all these groups. For the E_n family, the Cartan–Gram determinant contains an explicit factor of $(9 - n)$ which vanishes for $n = 9$ and is negative for $n > 9$. This gives a simple explanation why the E_n family terminates at E_8. The Cartan–Gram determinant affords a systematic explanation for the nonexistence of the forbidden Dynkin diagrams.

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I. INTRODUCTION

The Cartan–Killing classification of simple Lie groups into the classical groups [namely, the unitary $A_n = \mathrm{SU}(n+1)$, orthogonal $B_n = \mathrm{SO}(2n+1)$, $D_n = \mathrm{SO}(2n)$, and symplectic $\mathrm{Sp}(2n)$] and the five exceptional ones $(G_2, F_4, E_6, E_7$, and $E_8)$ is well known.[1–6] We shall speak of Lie groups and Lie algebras interchangeably.

The purpose of this note is to advocate the use of the Gram determinant (or, apart from a scale, the determinant of the Cartan matrices[1–5]) as an unambiguous clean test for the linear independence of the set of simple root vectors. For the allowed Dynkin diagrams, the Cartan–Gram determinants are positive definite. For the forbidden configurations, the determinants are negative or zero. The Cartan–Gram determinants are explicitly evaluated for the simple Lie groups A_n, B_n, C_n, D_n, and E_n. The answers are remarkably simple. The determinants are found to satisfy the *same* recursion relation for all these groups. For the E_n family, the Cartan–Gram determinant contains an explicit factor of $(9 - n)$, which vanishes for $n = 9$ and is negative for $n > 9$. This gives a simple explanation why the E_n family terminates at E_8.

For the basic notion and terminology, the reader is referred to the literature.[1–6] The following two paragraphs provide a minimal setting.

An arbitrary Lie group is decomposable into a semi-simple one and a solvable one. A semi-simple group is decomposable into simple groups. The r parameters of a simple Lie group of rank n can be split into n commuting (i.e., simultaneously diagonalizable) operators H_i $(i = 1,...,n)$, $\frac{1}{2}(r-n)$ raising operators E_α and $\frac{1}{2}(r-n)$ lowering operators $E_{-\alpha}$ such that, among other things, the composition (commutator) relation reads

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{1}$$

This says that the commutator between H_i and E_α corresponds to an eigenvalue problem in the adjoint representation. The eigenvalues α_i $(i = 1,...,n)$ are called components of a root vector α in a n-dimensional Euclidean space. By judicious choice, the root space can be spanned by a set of basis vectors (not necessarily orthogonal) called the simple roots. The set of Dynkin diagrams correspond to all the admissible simple root vectors that satisfy the following three requirements:

(a) angular restriction between two vectors: $\theta = \pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$,

(b) relative length restriction: $-$, $1{:}\sqrt{2}{:}\sqrt{3}$, and

(c) linear independence.

II. THE CARTAN DETERMINANT; THE GRAM DETERMINANT

For a rank n Lie group, the Cartan matrix is a $n \times n$ matrix whose elements A_{ij} is defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \tag{2}$$

where α_i denotes the ith simple root and (α_i, α_i) is the inner product. The Cartan matrices for simple Lie groups are listed in the literature.[2,4,5]

On the other hand, the Gram matrix for a set of n vectors is defined as

$$G_{ij} = (\alpha_i, \alpha_j). \tag{3}$$

It is well known that the Gram determinant[7] (a) is positive for a set of linearly independent real Euclidean vectors, and (b) vanishes if and only if the set of vectors is linearly dependent.

The determinant of the Cartan matrix will be called the Cartan determinant here. It is simply proportional to the Gram determinant, the proportionality constant depends on the normalization of the simple roots.

With the known simple root system for the simple Lie groups, their Cartan determinants $\Delta \equiv \det A_{ij}$ of (2) and the Gram determinants $g \equiv \det G_{ij}$ of (3) can be easily evaluated. With the nesting structure such that the determinant of the next subgroup corresponds to the first principal minor, we have for the $n \times n$ determinants:

$$\Delta_n(A_n) = \begin{vmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot \\ \cdot & & & \ddots & & & \\ \cdot & & & \cdot & -1 & 2 & -1 \\ \cdot & & & \cdot & 0 & -1 & 2 \end{vmatrix} = n+1, \tag{4a}$$

$$g_n(A_n) = 2^{-n}(n+1), \tag{4b}$$

$$\Delta_n(B_n) = \begin{vmatrix} 2 & -1 & 0 & \cdot & \cdot & & \cdot \\ -1 & 2 & -1 & 0 & \cdot & & \cdot \\ 0 & -1 & 2 & -1 & 0 & & \cdot \\ \cdot & & & \ddots & & & \\ \cdot & & & & & & \\ & & & \cdot & -1 & 2 & -2 \\ \cdot & & & \cdot & 0 & -1 & 2 \end{vmatrix} = 2, \tag{5a}$$

$$g_n(B_n) = 2^{2-n}, \tag{5b}$$

$$\Delta_n(C_n) = \begin{vmatrix} 2 & -1 & 0 & \cdot & \cdot & & \cdot \\ -1 & 2 & -1 & 0 & \cdot & & \cdot \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot & & \\ \cdot & & & \cdot & -1 & 2 & -1 \\ \cdot & & & \cdot & 0 & -2 & 2 \end{vmatrix} = 2, \tag{6a}$$

and

$$g_n(C_n) = 2^{-n}. \tag{6b}$$

Note that the matrices for C_n and B_n are the transpose of each other.

$$\Delta_n(D_n) = \begin{vmatrix} 2 & -1 & 0 & \cdot & \cdot & & \cdot \\ -1 & 2 & -1 & 0 & & & \cdot \\ 0 & -1 & 2 & -1 & 0 & & \cdot \\ \cdot & \cdot & \cdot & \ddots & & & \\ \cdot & \cdot & \cdot & \cdot & 2 & -1 & -1 \\ \cdot & & & \cdot & -1 & 2 & 0 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \end{vmatrix} = 4, \tag{7a}$$

$$g_n(D_n) = 2^{2-n}, \tag{7b}$$

$$\Delta_n(E_n) = \begin{vmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \ddots & & & & & & \\ \cdot & & -1 & 2 & -1 & 0 & & & \cdot \\ & & \cdot & -1 & 2 & 0 & -1 & 0 & 0 \\ \cdot & & & \cdot & 0 & 2 & -1 & 0 & 0 \\ \cdot & & & \cdot & -1 & -1 & 2 & -1 & 0 \\ & & & \cdot & 0 & 0 & -1 & 2 & -1 \\ \cdot & & & \cdot & 0 & 0 & 0 & -1 & 2 \end{vmatrix} = 9-n, \tag{8a}$$

$$[E_4 \equiv A_4, E_5 \equiv D_5], \quad g_n(E_n) = 2^{-n}(9-n). \tag{8b}$$

A simple recursion relation is seen to hold for all these cases. We have

$$\Delta_n = 2\Delta_{n-1} - \Delta_{n-2}, \tag{9a}$$

$$g_n = g_{n-1} - \tfrac{1}{4} g_{n-2}. \tag{9b}$$

The $(9-n)$ factor of the Cartan–Gram determinant for the E_n family gives a simple explanation why the family does not extend beyond E_8.

We state without elaboration that the Cartan–Gram determinant affords a clean systematic test for the admissibility of a simple root system.

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[7]See, e.g., R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley-Interscience, New York, 1953), Vol. 1, p.34.